Reduction and exact solutions of the Navier–Stokes equations

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We construct a complete set of $\tilde{G}(1,3)$-inequivalent ansätze of codimension 1 for the Navier–Stokes (NS) field which reduce the ns equations to systems of ordinary differential equations (ODE). Having solved these ODEs we thereby obtain solutions of the NS equations. Formulae of group multiplication of solutions are given. Several non-Lie ansätze are discussed.

1. Introduction

The NS equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \Delta\boldsymbol{u} + \nabla p = 0, \quad \operatorname{div}\boldsymbol{u} = 0, \tag{1.1}$$

where $\boldsymbol{u} = \boldsymbol{u}(x) = \{u^1, u^2, u^3\}$ is the velocity field of a fluid, $p = p(x)$ is the pressure, $x = \{t, \boldsymbol{x}\} \in R(4)$, $\nabla = \{\partial/\partial x_a\}$, $a = 1,2,3$, Δ is Laplacian, are basic equations of hydrodynamics which describe motion of an incompressible viscous fluid. The problem of finding exact solutions of nonlinear equations (1.1) is an important but rather complicated one. Considerable progress in solving this problem can be achieved by making use of a symmetry approach. Equations (1.1) have non-trivial symmetry properties; it is well known (see, e.g. Birkhoff [3]) that they are invariant under the extended Galilei group $\tilde{G}(1,3)$ generated by operators

$$\begin{gathered}\partial_t \equiv \frac{\partial}{\partial t}, \quad \partial_a \equiv \frac{\partial}{\partial x_a}, \quad G_a = t\partial_a + \partial_{u^a}, \\ J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}, \quad D = 2t\partial_t + x_a\partial_a - u^a\partial_{u^a} - 2p\partial_p,\end{gathered} \tag{1.2}$$

where $\partial_{u^a} \equiv \partial/\partial u^a$, $\partial_p \equiv \partial/\partial p$. Recently it was shown (Ovsyannikov [12], Lloyd [11]) that the maximal, in the sense of Lie invariance algebra, of the NS equations (1.1) is the direct sum of eleven-dimensional $A\tilde{G}(1,3)$ (1.2) and infinite-dimensional algebra A^∞ with basis elements

$$Q = f^a\partial_a + \dot{f}^a\partial_{u^a} - x_a\ddot{f}^a\partial_p, \quad R = g\partial_p, \tag{1.3}$$

where $f^a = f^a(t)$ and $g = g(t)$ are arbitrary differentiable functions of t; dot means differentiation with respect to t.

In this paper we systematically use symmetry properties of (1.1) to find their exact solutions. In section 2 we describe the complete set of $\tilde{G}(1,3)$-inequivalent ansätze of codimension 1

$$u^a(t,\boldsymbol{x}) = f^{ab}(x)\varphi^b(\omega) + g^a(x), \quad p(x) = F(x)\varphi(\omega), \tag{1.4}$$

where the functions f^{ab}, g^a and F, and new variable $\omega = \omega(x)$ are determined by means of operators of three-dimensional subalgebras of $A\tilde{G}(1,3)$ (1.2). We consider

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three-dimensional subalgebras of $A\tilde{G}(1,3)$ because an ansatz of the form (1.4), invariant under such a subalgebra, reduces (1.1) to a system of ODE immediately. As a rule reduced systems of ode can be solved by a standard method. (In most cases we find the general solutions of these reduced systems of ODE). Ansätze of the type (1.4), which are obtained by means of Lie symmetry operators, we shall call Lie ansätze. The method of finding exact solutions of PDE used here is based on Lie's ideas of invariant solutions and it is described in full detail in Fushchych et al [9].

Starting from solutions of the reduced systems of ODE (which are, of course, solutions of the NS equations) one can construct multiparameter families of solutions for the NS equations. To do this one has to use formulae of group multiplication of solutions which are given at the end of section 2.

In section 3 we consider some non-Lie ansätze for the NS field. These ansätze cannot be obtained within the framework of the local Lie approach used in section 2.

2. $\tilde{G}(1,3)$-inequivalent ansätze of codimension 1 for the NS field and exact solutions of the NS equations (1.1)

Let $\langle Q_j\rangle \equiv \langle Q_1, Q_2, Q_3\rangle$ be a three-dimensional subalgebra of $A\tilde{G}(1,3)$ (1.2). It follows from (1.2) that the general form of operator Q_j is

$$Q_j = \xi_j^\nu(x)\partial_\nu + \eta_j^a(\boldsymbol{u})\partial_{u^a} + \tilde{\eta}_j(p)\partial_p, \tag{2.1}$$

where $\nu = \overline{0,3}$, $\partial_0 \equiv \partial/\partial t$; ξ_j^ν, η_j^a, $\tilde{\eta}_j$ are linear functions of x, $\boldsymbol{u}$, p. The explicit form of an ansatz (1.4) is determined as the solution of the following equations

$$\begin{aligned} &\xi_j^\nu(x)\partial_\nu\omega(x) = 0, \\ &Q_j[u^a - f^{ab}(x)\varphi^b(\omega) - g^a(x)] = 0, \\ &Q_j[p - F(x)\varphi(\omega)] = 0. \end{aligned} \tag{2.2}$$

Equations (2.2) can be solved rather easily. All three-dimensional $\tilde{G}(1,3)$-inequivalent subalgebras of $A\tilde{G}(1,3)$ are found in Fushchych et al [6] and Barannik and Fushchych [1] with the help of the method developed by Patera et al [13]. In table 1 we list these three-dimensional subalgebras and give corresponding invariant ansätze of the form (1.4) obtained as solutions of equations (2.2).

In this table f, g, h, φ are differentiable functions of corresponding invariant variable ω; $\alpha \neq 0$ is an arbitrary constant.

Let us substitute ansätze from table 1 into the ns equations (1.1). As a result we obtain the following systems of ODE:

1°. $\dot{f} = 0, \quad \dot{g} = 0, \quad \dot{h} = 0.$

2°. $h\dot{f} - \ddot{f} = 0, \quad h\dot{g} - \ddot{g} = 0, \quad h\dot{h} - \ddot{h} + \dot{\varphi} = 0, \quad \dot{h} = 0.$

3°. $g + h\dot{f} - \ddot{f} = 0, \quad h\dot{g} - \ddot{g} = 0, \quad h\dot{h} - \ddot{h} + \dot{\varphi} = 0, \quad \dot{h} = 0.$

4°. $\dot{f}h + 2\ddot{f} = 0, \quad \dot{g}h + 2\ddot{g} = 0, \quad 1 - 2h\dot{h} - 4\ddot{h} - 2\dot{\varphi} = 0, \quad \dot{h} = 0.$

5°. $1 + h\dot{f} - \ddot{f} = 0, \quad g\dot{h} - \ddot{g} = 0, \quad h\dot{h} - \ddot{h} + \dot{\varphi} = 0, \quad \dot{h} = 0.$

6°. $g - 2h\dot{f} - 4\ddot{f} = 0, \quad h\dot{g} + 2\ddot{g} = 0, \quad 1 - 2h\dot{h} - 4\ddot{h} - 2\dot{\varphi} = 0, \quad \dot{h} = 0.$

7°. $(\alpha f - h)\dot{f} - 2(\alpha^2+1)\ddot{f} + \alpha\dot{\varphi} = 0, \quad (\alpha f - h)\dot{g} - 2(\alpha^2+1)\ddot{g} = 0,$
$(\alpha f - h)\dot{h} - 2(\alpha^2+1)\ddot{h} - \dot{\varphi} + \frac{1}{2} = 0, \quad \alpha\dot{f} - \dot{h} = 0.$

8°. $-\dot{f}(h - \alpha g) + g - (\alpha^2+1)\ddot{f} = 0, \quad -\dot{g}(h - \alpha g) + \alpha\dot{\varphi} - (\alpha^2+1)\ddot{g} = 0,$
$1 - \dot{h}(h - \alpha g) - \dot{\varphi} - (\alpha^2+1)\ddot{h} = 0, \quad \dot{h} - \alpha\dot{g} = 0.$

Table 1. $\tilde{G}(1,3)$-inequivalent ansätze of codimension 1 for the NS field

N	Algebra	Invariant variable ω	Anzatz
1	$\partial_1,\ \partial_2,\ \partial_3$	t	$u^1=f(\omega),\ u^2=g(\omega),\ u^3=h(\omega),\ p=\varphi(\omega)$
2	$\partial_t,\ \partial_1,\ \partial_2$	x_3	$u^1=f(\omega),\ u^2=g(\omega),\ u^3=h(\omega),\ p=\varphi(\omega)$
3	$\partial_t,\ \partial_1,\ G_1+G_2$	x_3	$u^1=x_2+f(\omega),\ u^2=g(\omega),\ u^3=h(\omega),$ $p=\varphi(\omega)$
4	$\partial_1,\ \partial_2,\ \partial_t+G_3$	t^2-2x_3	$u^1=f(\omega),\ u^2=g(\omega),\ u^3=t+h(\omega),\ p=\varphi(\omega)$
5	$\partial_1,\ \partial_2,\ \partial_t+G_1$	x_3	$u^1=t+f(\omega),\ u^2=g(\omega),\ u^3=h(\omega),\ p=\varphi(\omega)$
6	$\partial_1,\ \partial_2+G_1,$ ∂_t+G_3	t^2-2x_3	$u^1=x_2+f(\omega),\ u^2=g(\omega),\ u^3=t+h(\omega),$ $p=\varphi(\omega)$
7	$\partial_1+\alpha\partial_3,\ \partial_2,$ ∂_t+G_3	$t^2+2\alpha x_1-2x_3$	$u^1=f(\omega),\ u^2=g(\omega),\ u^3=t+h(\omega),\ p=\varphi(\omega)$
8	$\partial_1,\ \partial_t+G_3,$ $G_1+\partial_2+\alpha\partial_3$	$\alpha x_2-x_3+(t^2/2)$	$u^1=x_2+f(\omega),\ u^2=g(\omega),\ u^3=t+h(\omega),$ $p=\varphi(\omega)$
9	$\partial_t,\ \partial_3,\ J_{12}$	$(x_1^2+x_2^2)^{1/2}$	$u^1=x_1f(\omega)-x_2g(\omega),\ u^2=x_1g(\omega)+x_2f(\omega),$ $u^3=h(\omega),\ p=\varphi(\omega)$
10	$\partial_t+G_3,\ \partial_3,\ J_{12}$	$(x_1^2+x_2^2)^{1/2}$	$u^1=x_1f(\omega)-x_2g(\omega),\ u^2=x_1g(\omega)+x_2f(\omega),$ $u^3=t+h(\omega),\ p=\varphi(\omega)$
11	$\partial_t,\ \partial_3,\ D$	x_1/x_2	$u^1=(1/x_2)f(\omega),\ u^2=(1/x_2)g(\omega),$ $u^3=(1/x_2)h(\omega),\ p=(1/x_2^2)\varphi(\omega)$
12	$\partial_t,\ \partial_3,\ J_{12}+\alpha D$	$\ln(x_1^2+x_2^2)+$ $2\alpha\tan^{-1}(x_1/x_2)$	$u^1=(x_1^2+x_2^2)^{-1}(x_1f(\omega)-x_2g(\omega)),$ $u^2=(x_1^2+x_2^2)^{-1}(x_1g(\omega)+x_2f(\omega)),$ $u^3=(x_1^2+x_2^2)^{-1/2}h(\omega),\ p=(x_1^2+x_2^2)^{-1}\varphi(\omega)$
13	$\partial_t,\ J_{12},\ D$	$(x_1^2+x_2^2)^{1/2}/x_3$	$u^1=(x_1^2+x_2^2)^{-1}(x_1f(\omega)-x_2g(\omega)),$ $u^2=(x_1^2+x_2^2)^{-1}(x_1g(\omega)+x_2f(\omega)),$ $u^3=(x_1^2+x_2^2)^{-1/2}h(\omega),\ p=(x_1^2+x_2^2)^{-1}\varphi(\omega)$
14	$\partial_3,\ J_{12},\ D$	$(x_1^2+x_2^2)^{1/2}/t$	$u^1=(1/t)(x_1f(\omega)-x_2g(\omega)),$ $u^2=(1/t)(x_1g(\omega)+x_2f(\omega)),$ $u^3=(1/\sqrt{t})h(\omega),\ p=(1/t)\varphi(\omega)$
15	$G_3,\ J_{12},\ D$	$(x_1^2+x_2^2)^{1/2}/t$	$u^1=(1/t)(x_1f(\omega)-x_2g(\omega)),$ $u^2=(1/t)(x_1g(\omega)+x_2f(\omega)),$ $u^3=(1/\sqrt{t})h(\omega)+(x_3/t),\ p=(1/t)\varphi(\omega)$
16	$\partial_t,\ \partial_2,\ D$	$x_3/\sqrt{t}$	$u^1=(1/\sqrt{t})f(\omega),\ u^2=(1/\sqrt{t})g(\omega),$ $u^3=(1/\sqrt{t})h(\omega),\ p=(1/t)\varphi(\omega)$
17	$\partial_t,\ D,\ G_2+\alpha G_1$	$x_3/\sqrt{t}$	$u^1=(1/\sqrt{t})f(\omega)+(\alpha x_2/t),$ $u^2=(1/\sqrt{t})g(\omega)+(x_2/t),$ $u^3=(1/\sqrt{t})h(\omega),\ p=(1/t)\varphi(\omega)$
18	$G_1,\ G_2,\ D$	$x_3/\sqrt{t}$	$u^1=(1/\sqrt{t})f(\omega)+(x_1/t),$ $u^2=(1/\sqrt{t})g(\omega)+(x_2/t),$ $u^3=(1/\sqrt{t})h(\omega),\ p=(1/t)\varphi(\omega)$
19	$\partial_1,\ G_2,\ D$	$x_3/\sqrt{t}$	$u^1=(1/\sqrt{t})f(\omega),\ u^2=(1/\sqrt{t})g(\omega)+(x_2/t),$ $u^3=(1/\sqrt{t})h(\omega),\ p=(1/t)\varphi(\omega)$

$$
\begin{aligned}
&9^\circ. \quad f^2 - g^2 + \omega f\dot f + \frac{1}{\omega}\dot\varphi = \frac{3}{\omega}\dot f + \ddot f, \quad 2fg + \omega f\dot g = \frac{3}{\omega}\dot g + \ddot g,\\
&\qquad \omega f\dot h = \ddot h + \frac{1}{\omega}\dot h, \quad 2f + \omega\dot f = 0.\\
&10^\circ. \quad f^2 - g^2 + \omega f\dot f + \frac{1}{\omega}\dot\varphi = \frac{3}{\omega}\dot f + \ddot f, \quad 2fg + \omega f\dot g = \frac{3}{\omega}\dot g + \ddot g,\\
&\qquad 1 + \omega f\dot h = \ddot h + \frac{1}{\omega}\dot h, \quad 2f + \omega\dot f = 0.\\
&11^\circ. \quad f\dot f - g(f + \omega\dot f) + \dot\varphi = 2(1+\omega)f + \omega(2\dot f + \omega\ddot f),\\
&\qquad f\dot g - g(g + \omega\dot g) - \omega\dot\varphi = 2(1+\omega)g + \omega(2\dot g + \omega\ddot g),\\
&\qquad f\dot h - g(h + \omega\dot h) = 2(1+\omega)h + \omega(2\dot h + \omega\ddot h), \quad \dot f - (g + \omega\dot g) = 0.\\
&12^\circ. \quad -\tfrac{1}{2}(f^2 + g^2) + (f - \alpha g)\dot f - \varphi + \dot\varphi = 2(-f - \dot f + \alpha\dot g + (\alpha^2+1)\ddot f),\\
&\qquad -(f - \alpha g)\dot g + \alpha\dot\varphi = 2[g + \dot g + \alpha\dot f - (\alpha^2+1)\ddot g],\\
&\qquad -fh + 2(f - \alpha g)\dot h = h - 4\dot h + 4(\alpha^2+1)\ddot h, \quad \dot f - \alpha\dot g = 0.\\
&13^\circ. \quad -f^2 - g^2 + \omega f\dot f - \omega^2 h\dot f - 2\varphi + \omega\dot\varphi = \omega(-f + \omega\ddot f) + \omega^3(2\dot f + \omega\ddot f),\\
&\qquad f\dot g - \omega^2 h\dot g = \omega(-g + \omega\ddot g) + \omega^3(2\dot g + \omega\ddot g),\\
&\qquad f(-h + \omega\dot h) - \omega^2 h\dot h - \omega^2\dot\varphi = h - \omega\dot h + \omega^2\ddot h + \omega^3(2\dot h + \omega\ddot h),\\
&\qquad \dot f - \omega\dot h = 0. \qquad\qquad (2.3)\\
&14^\circ. \quad f^2 - g^2 + 2\omega f\dot f + 2\dot\varphi = 4(2\dot f + \omega\ddot f),\\
&\qquad g + \omega\dot g - 2f(g + \omega\dot g) = -(2\dot g + \omega\ddot g),\\
&\qquad -\left(\tfrac{1}{2}h + \omega\dot h\right) + 2\omega f\dot h = 4(\dot h + \omega\ddot h), \quad f + \omega\dot f = 0.\\
&15^\circ. \quad f^2 - g^2 + 2\omega f\dot f + 2\dot\varphi = 4(2\dot f + \omega\ddot f),\\
&\qquad g + \omega\dot g - 2f(g + \omega\dot g) = -4(2\dot g + \omega\ddot g),\\
&\qquad -\left(\tfrac{1}{2}h + \omega\dot h\right) + 2\omega f\dot h + h = 4(\dot h + \omega\ddot h), \quad f + \omega\dot f + \tfrac{1}{2} = 0.\\
&16^\circ. \quad -\tfrac{1}{2}(f + \omega\dot f) + h\dot f = \ddot f, \quad -\tfrac{1}{2}(g + \omega\dot g) + h\dot g = \ddot g,\\
&\qquad -\tfrac{1}{2}(h + \omega\dot h) + h\dot h + \dot\varphi = \ddot h, \quad \dot h = 0.\\
&17^\circ. \quad -\tfrac{1}{2}(f + \omega\dot f) + h\dot f + \alpha g = \ddot f, \quad -\tfrac{1}{2}(g + \omega\dot g) + h\dot g + g = \ddot g,\\
&\qquad -\tfrac{1}{2}(h + \omega\dot h) + h\dot h + \dot\varphi = \ddot h, \quad \dot h + 1 = 0.\\
&18^\circ. \quad \tfrac{1}{2}(f - \omega\dot f) + h\dot f = \ddot f, \quad \tfrac{1}{2}(g - \omega\dot g) + h\dot g = \ddot g,\\
&\qquad -\tfrac{1}{2}(h + \omega\dot h) + h\dot h = \ddot h, \quad \dot h + 2 = 0.\\
&19^\circ. \quad -\tfrac{1}{2}(f + \omega\dot f) + h\dot f = \ddot f, \quad \tfrac{1}{2}(g - \omega\dot g) + h\dot g = \ddot g,\\
&\qquad -\tfrac{1}{2}(h + \omega\dot h) + h\dot h + \dot\varphi = \ddot h, \quad \dot h + 1 = 0.
\end{aligned}
$$

Equations 1°–19° in (2.3) correspond to that of ansatze in table 1; dot means differentiation with respect to corresponding ω.

Equations 1°–10° (2.3) can easily be solved and their general solutions are as follows:

$$1^\circ. \quad f = c_1, \quad g = c_2, \quad h = c_3, \quad \varphi = \varphi(\omega)$$

(here and in what follows, c with a subscript denotes an arbitrary constant; $\varphi = \varphi(\omega)$ means that φ is an arbitrary differentiable function of ω).

$$
\begin{aligned}
2^\circ.\quad & f=\begin{cases}\dfrac{c_1}{c_3}e^{c_3\omega}+c_2, & c_3\neq 0,\\ c_1\omega+c_2, & c_3=0,\end{cases}\\
& g=\begin{cases}\dfrac{c_4}{c_3}e^{c_3\omega}+c_5, & c_3\neq 0,\\ c_4\omega+c_5, & c_3=0,\end{cases}\\
& h=c_3,\quad \varphi=c_6.\\
3^\circ.\quad & f=\begin{cases}c_1+c_2e^{c_3\omega}+\dfrac{c_4}{c_3^2}\left(\omega-\dfrac{1}{c_3}\right)e^{c_3\omega}-\dfrac{c_5}{c_3}\omega, & c_3\neq 0,\\ c_1+c_2\omega+\dfrac16 c_4\omega^3+\dfrac12 c_5\omega^2, & c_3=0,\end{cases}\\
& g=\begin{cases}\dfrac{c_4}{c_3}e^{c_3\omega}+c_5, & c_3\neq 0,\\ c_4\omega+c_5, & c_3=0,\end{cases}\\
& h=c_3,\quad \varphi=c_6.\\
4^\circ.\quad & f=\begin{cases}\dfrac{c_1}{c_3}\exp\left(-\dfrac12 c_3\omega\right)+c_2, & c_3\neq 0,\\ c_1+c_2\omega, & c_3=0,\end{cases}\\
& g=\begin{cases}\dfrac{c_4}{c_3}\exp\left(-\dfrac12 c_3\omega\right)+c_5, & c_3\neq 0,\\ c_4\omega+c_5, & c_3=0,\end{cases}\\
& h=c_3,\quad \varphi=\frac12\omega+c_6.\\
5^\circ.\quad & f=\begin{cases}-\dfrac{1}{c_3}\omega+\dfrac{c_1}{c_3^2}e^{c_3\omega}+c_2, & c_3\neq 0,\\ \dfrac12\omega^2+c_1\omega+c_2, & c_3=0,\end{cases}\\
& g=\begin{cases}\dfrac{c_4}{c_3}e^{c_3\omega}+c_5, & c_3\neq 0,\\ c_4\omega+c_5, & c_3=0,\end{cases}\\
& h=c_3,\quad \varphi=c_6.\\
6^\circ.\quad & f=\begin{cases}c_1+c_2\exp\left(-\dfrac12 c_3\omega\right)+\dfrac{c_5}{2c_3}\omega- & \\ \quad -\dfrac{c_4}{c_3^2}\left(\dfrac{\omega}{2}+\dfrac{1}{c_3}\right)\exp\left(-\dfrac12 c_3\omega\right), & c_3\neq 0,\\ \dfrac14\left(c_1+c_2\omega+\dfrac12 c_5\omega^2+\dfrac16 c_4\omega^3\right), & c_3=0,\end{cases}\\
& g=\begin{cases}\dfrac{c_4}{c_3}\exp\left(-\dfrac12 c_3\omega\right)+c_5, & c_3\neq 0,\\ c_4\omega+c_5, & c_3=0,\end{cases}\\
& h=c_3,\quad \varphi=\frac12\omega+c_6.
\end{aligned}
\tag{2.4}
$$

7°. $$f=\begin{cases} c_1\exp\left(\dfrac{c\omega}{2(\alpha^2+1)}\right)+c_2-\dfrac{\alpha\omega}{2(\alpha^2+1)c}, & c\neq 0,\\ \dfrac{\alpha\omega^2}{2[2(\alpha^2+1)]^2}+c_1\omega+c_2, & c=0,\end{cases}$$

$$g=\begin{cases} c_3\exp\left(\dfrac{c\omega}{2(\alpha^2+1)}\right)+c_4, & c\neq 0,\\ c_3\omega+c_4, & c=0,\end{cases}$$

$$h=\alpha f-c,\quad \varphi=\frac{\omega}{2(\alpha^2+1)}+c_5.$$

8°. $$f=\begin{cases} \dfrac{\alpha\omega^2}{2c^2(\alpha^2+1)}+\dfrac{\omega}{c}\left(\dfrac{\alpha}{c^2}-c_4\right)+\left[\dfrac{c_3}{c}\left(\omega-\dfrac{\alpha^2+1}{c}\right)+c_1\right]\times & \\ \quad\times\exp\left(\dfrac{c\omega}{\alpha^2+1}\right)+c_2, & c\neq 0,\\ (\alpha^2+1)^{-1}\left(\dfrac{\alpha\omega^4}{24(\alpha^2+1)^2}+\dfrac{c_3}{6}\omega^3+\dfrac{c_4}{2}\omega^2+c_1\omega+c_2\right), & c=0,\end{cases}$$

$$g=\begin{cases} \dfrac{-\alpha\omega}{c(\alpha^2+1)}+c_3\exp\left(\dfrac{c\omega}{\alpha^2+1}\right)+c_4, & c\neq 0,\\ \dfrac{\alpha}{2(\alpha^2+1)}\omega^2+c_3\omega+c_4, & c=0,\end{cases}$$

$$h=\alpha g-c,\quad \varphi=\frac{\omega}{\alpha^2+1}+c_6.$$

9°. $$f=\frac{c}{\omega^2},\quad =c_1\omega^c+\frac{c_2}{\omega^2},\quad h=c_3\omega^c+c_4,$$

$$\varphi=\begin{cases} \dfrac{c_1^2}{2(c+1)}\omega^{2(c+1)}+\dfrac{2c_1c_2}{c}\omega^c-\dfrac{c^2+c_2^2}{2\omega^2}+c_5, & c\neq -1,\ 0,\\ c_1^2\ln\omega-\dfrac{2c_1c_2}{\omega}-\dfrac{c_2^2+1}{2\omega^2}+c_5, & c=-1,\\ \dfrac{1}{2}c_1^2\omega^2+2c_1c_2\ln\omega-\dfrac{c_2^2}{2\omega^2}+c_5, & c=0.\end{cases}$$

10°. f, g and φ are the same as in the previous case 9°,

$$h=\begin{cases} \dfrac{\omega^2}{2(2-c)}+c_3\omega^c+c_4, & c\neq 2,\ 0,\\ \dfrac{\omega}{4}-c_3\ln\omega+c_4, & c=0,\\ \dfrac{\omega^2}{2}\ln\omega-\dfrac{\omega^2}{4}+c_3\omega^2+c_4, & c=2.\end{cases}$$

For 11° (2.3) we did not find solutions. A particular solution of 12° (2.3) is

12°. $$f=c,\quad g=0,\quad \varphi=2c-\frac{c^2}{2},$$

$$h=\begin{cases} c_1e^{\lambda_1\omega}+c_2e^{\lambda_2\omega}, & \dfrac{c^2}{4}>\alpha^2(1+c),\\ e^{\lambda\omega}(c_1+c_2\omega), & \dfrac{c^2}{4}=\alpha^2(1+c),\\ e^{\lambda\omega}(c_1\cos\beta\omega+c_2\sin\beta\omega), & \dfrac{c^2}{4}<\alpha^2(1+c),\end{cases}$$

$$\lambda_{1,2} = \frac{1+(c/2) \pm \sqrt{(c^2/4) - \alpha^2(1+c)}}{2(1+\alpha^2)}, \quad \lambda = \frac{1+(c/2)}{2(1+\alpha^2)},$$
$$\beta = \frac{\sqrt{\alpha^2(1+c) - (c^2/4)}}{2(1+\alpha^2)}.$$

A particular solution of 13° (2.3) is

$$13^\circ. \quad f = c_1, \quad g = c_2, \quad h = 0, \quad \varphi = -\frac{1}{2}(c_1^2 + c_2^2). \tag{2.4}$$

Consider system 14° (2.3). The last equation of 14° (2.3) immediately gives

$$f = c/\omega \tag{2.5}$$

(as before, c is an arbitrary constant). Substituting (2.5) into the remaining equations of 14° (2.3) we get

$$4\frac{d^2}{d\omega^2}(\omega g) + \left(1 - \frac{2c}{\omega}\right)\frac{d}{d\omega}(\omega g) = 0 \tag{2.6}$$

and

$$4\omega\ddot{h} + (\omega + 4 - 2c)\dot{h} + \frac{1}{2}h = 0. \tag{2.7}$$

Equation (2.6) can be easily integrated and the result is

$$g(\omega) = \frac{c_1}{\omega}\int^\omega x^{c/2}e^{-x/4}dx + \frac{c_2}{\omega}. \tag{2.8}$$

In particular, when $c = 0$, the general solution of equation (2.6) takes the form

$$g(\omega) = \frac{c_1}{\omega}e^{-\omega/4} + \frac{c_2}{\omega}. \tag{2.9}$$

Equation (2.7) is in itself an equation for a degenerate hypergeometric function and it can be rewritten in standard Whittaker form

$$4x^2\ddot{w} - (x^2 - 4kx + 4m^2 - 1)w = 0, \tag{2.10}$$

where $w = w(k, m, x)$; k, m are parameters, by the substitution

$$h(\omega) = \omega^{(c-2)/4}e^{-\omega/8}w\left(\frac{c}{4}, -\frac{c}{4}, \frac{\omega}{4}\right). \tag{2.11}$$

When $c = 0$, the substitution

$$h(\omega) = e^{-\tau}\tilde{Z}_0(\tau), \quad \tau = \frac{\omega}{8} \tag{2.12}$$

reduces (2.7) to the modified Bessel equation of null order, that is

$$\tau\ddot{\tilde{Z}}_0 + \dot{\tilde{Z}}_0 - \tau\tilde{Z}_0 = 0. \tag{2.13}$$

Summarizing results (2.5)–(2.12) we can write down the general solution of 14° (2.3) as follows

$$\begin{aligned} 14^\circ.\quad & f=\frac{c}{\omega},\quad g=\frac{c_1}{\omega}\int^{\omega}x^{c/2}e^{-x/4}dx+\frac{c_2}{\omega},\\ & h=\omega^{(c-2)/4}e^{-\omega/8}w\left(\frac{c}{4},-\frac{c}{4},\frac{\omega}{4}\right),\quad \varphi=-\frac{c^2}{2\omega}+\frac{1}{2}\int^{\omega}g^2(y)dy+c_3. \end{aligned} \tag{2.4}$$

(We continue to numerate solutions of reduced NS equations 1°–19° (2.3) as n° (2.4), where $n^\circ = 1^\circ$–19° indicates the corresponding ansatz of table 1.) When $c=0$ we get from 14° (2.4) the following particular solution of 14° (2.3)

$$\begin{aligned} 14^{\circ\circ}.\quad & f=0,\quad g=\frac{c_1}{\omega}e^{-\omega/4}+\frac{c_2}{\omega},\quad h=e^{-\omega/8}\tilde{Z}_0(\omega/8),\\ & \varphi=-\frac{c_2^2}{2\omega}+\frac{c_1^2}{2}\int^{\omega}\frac{e^{-y/2}}{y^2}dy+c_1c_2\int^{\omega}\frac{e^{-y/2}}{y^2}dy+c_3, \end{aligned} \tag{2.4}$$

where $\tilde{Z}_0$ is modified Bessel function satisfying equation (2.12).

Consider system 15° (2.3). The last equation in it gives

$$f=\frac{c}{\omega}-\frac{1}{2}. \tag{2.13}$$

The rest equations of 15° (2.3) take the form

$$2\frac{d^2}{d\omega^2}(\omega g)+\left(1-\frac{c}{\omega}\right)\frac{d}{d\omega}(\omega g)=0, \tag{2.14}$$

$$2\dot{\varphi}=\left(\frac{c}{\omega}\right)^2+g^2-\frac{1}{4}, \tag{2.15}$$

$$\omega\ddot{h}+\left(\frac{1}{2}\omega+1-\frac{c}{2}\right)\dot{h}-\frac{1}{8}h=0. \tag{2.16}$$

Equations (2.14), (2.15) can be easily integrated and the result is as follows

$$g=\frac{c_1}{\omega}\int^{\omega}x^{c/2}e^{-x/2}dx+\frac{c_2}{\omega}, \tag{2.17}$$

$$\varphi=\frac{1}{2}\int^{\omega}g^2(y)dy-\frac{c^2}{2\omega}-\frac{1}{8}\omega. \tag{2.18}$$

Equation (2.16) is reduced to the Whittaker equation (2.10) by the substitution

$$h(\omega)=\omega^{(c-2)/4}e^{-\omega/4}w\left(\frac{c-3}{4},-\frac{c}{4},\frac{\omega}{2}\right). \tag{2.19}$$

Note, when $c=3$, function $w\left(0,-\frac{3}{4},\frac{\omega}{2}\right)$ is reduced to the modified Bessel function $\tilde{Z}_{-3/4}(\omega/4)$. The general relation is (Bateman and Erdelyi [2])

$$w(0,m,x)=\sqrt{x}\tilde{Z}_m(x/2). \tag{2.20}$$

So, we can write down the general solution of reduced ns equations 15° (2.3) in the form

$$
\begin{aligned}
15^\circ.\quad & f=\frac{c}{\omega}-\frac{1}{2},\quad g=\frac{c_1}{\omega}\int^{\omega}x^{c/2}e^{-x/2}dx+\frac{c_2}{\omega},\\
& h=\omega^{(c-2)/4}e^{-\omega/4}w\left(\frac{c-3}{4},-\frac{c}{4},\frac{\omega}{2}\right),\quad \varphi=\frac{1}{2}\int^{\omega}g^2(y)dy-\frac{c^2}{2\omega}-\frac{1}{8}\omega,
\end{aligned}
\tag{2.4}
$$

where w satisfies the Whittaker equation (2.10).

Consider system 16° (2.3). The two last equations of it give rise to

$$
h=c,\quad \varphi=\frac{c\omega}{2}+c_1. \tag{2.21}
$$

Taking into account (2.21) we can rewrite the rest equations of system 16° (2.3) as follows

$$
\ddot f+\left(\frac{1}{2}\omega-c\right)\dot f+\frac{1}{2}f=0, \tag{2.22}
$$

$$
\ddot g+\left(\frac{1}{2}\omega-c\right)\dot g+\frac{1}{2}g=0. \tag{2.23}
$$

By substituting

$$
f(\omega)=F(\tau),\quad \tau=\frac{1}{2}\omega-c \tag{2.24}
$$

into (2.22), we obtain the following equation:

$$
\frac{d^2F}{d\tau^2}+2\tau\frac{dF}{d\tau}+2F=0. \tag{2.25}
$$

The general solution of (2.25) is

$$
F(\tau)=e^{-\tau^2}\left(c_2+c_3\int^{\tau}e^{y^2}dy\right). \tag{2.26}
$$

Summarizing results (2.21)–(2.26) we write down the general solution of equations 16° (2.3):

$$
\begin{aligned}
16^\circ.\quad & f=\exp\left[-\left(\frac{\omega}{2}-c\right)^2\right]\left(c_2+c_3\int^{(\omega/2)-c}e^{y^2}dy\right),\\
& g=\exp\left[-\left(\frac{\omega}{2}-c\right)^2\right]\left(c_4+c_5\int^{(\omega/2)-c}e^{y^2}dy\right),\\
& h=c,\quad \varphi=\frac{c\omega}{2}+c_1.
\end{aligned}
\tag{2.4}
$$

In the same way we find solutions of reduced equations 17°–19° (2.3). The solutions are as follows

$$
17^\circ.\quad \alpha = 1,
$$

$$
\begin{aligned}
f = g &= \left(\frac{3}{2}\omega - c\right)^{-1/2} \exp\left[-\frac{1}{6}\left(\frac{3}{2}\omega - c\right)^2\right] \times \\
&\times w\left[-\frac{5}{12}, \frac{1}{4}, \frac{1}{3}\left(\frac{3}{2}\omega - c\right)^2\right], \\
h &= \omega + c, \quad \varphi = \frac{3}{2}c\omega - \omega^2 + c_1,
\end{aligned} \tag{2.4}
$$

where $w(\cdot,\cdot,\cdot)$ is solution of the Whittaker equation (2.10). The above solution 17° (2.4) is a particular solution of equations 17° (2.3) with $\alpha = 1$. When α is an arbitrary constant, the general solution of 17° (2.3) has the form

$$
\begin{aligned}
17^{\circ\circ}.\quad g &= \left(\frac{3}{2}\omega - c\right)^{-1/2} \exp\left[-\frac{1}{6}\left(\frac{3}{2}\omega - c\right)^2\right] \times \\
&\times w\left[-\frac{5}{12}, \frac{1}{4}, \frac{1}{3}\left(\frac{3}{2}\omega - c\right)^2\right], \\
h &= \omega + c, \quad \varphi = \frac{3}{2}c\omega - \omega^2 + c_1
\end{aligned} \tag{2.4}
$$

and f satisfies the ODE

$$
\ddot{f} + \left(\frac{3}{2}\omega - c\right)\dot{f} + \frac{1}{2}f - \alpha g = 0.
$$

The general solution of 18° (2.3) is

$$
\begin{aligned}
18^\circ.\quad f = g &= \left(\frac{5}{2}\omega - c\right)^{-1/2} \exp\left[-\frac{1}{10}\left(\frac{5}{2}\omega - c\right)^2\right] \times \\
&\times w\left[-\frac{27}{20}, \frac{1}{4}, \frac{1}{5}\left(\frac{5}{2}\omega - c\right)^2\right], \\
h &= -2\omega + c, \quad \varphi = \frac{5}{2}c\omega - 3\omega^2 + c_1.
\end{aligned} \tag{2.4}
$$

The general solution of 19° (2.3) is

$$
\begin{aligned}
19^\circ.\quad f = g &= \left(\frac{3}{2}\omega - c\right)^{-1/2} \exp\left[-\frac{1}{6}\left(\frac{3}{2}\omega - c\right)^2\right] \times \\
&\times w\left[-\frac{1}{12}, \frac{1}{4}, \frac{1}{3}\left(\frac{3}{2}\omega - c\right)^2\right], \\
h &= -\omega + c, \quad \varphi = \frac{3}{2}c\omega - \omega^2 + c_1.
\end{aligned} \tag{2.4}
$$

In 17°–19° (2.4) $w(\cdot,\cdot,\cdot)$ is an arbitrary solution of the Whittaker equation (2.10).

Remark 1. The solutions of reduced ns equations 1°–19° (2.3) given in 1°–19° (2.4) should be considered together with the corresponding ansatze of table 1; then one gets solutions of the NS equations (1.1).

The solutions of the ns equations (1.1) obtained above can be used in a basic way to construct multiparameter families of solutions. A procedure for generating new solutions from a known one is based on the well known fact of Lie theory according to which symmetry transformations transform any solution of a given differential equation into another solution. For example, if transformations

$$x_\mu \to x'_\mu = f_\mu(x,\theta), \quad (\mu = \overline{0, n-1}\,),$$
$$u(x) \to u'(x') = R(x,\theta)u(x) + B(x,\theta),$$

where the θ are parameters, $u = \text{column}\,(u^1, u^2, \dots, u^k)$, $R(x,\theta)$ is a non-singular matrix $k \times k$, $R(x,0) = I$, f_μ, B (column) are some smooth functions, $f_\mu(x,0) = x_\mu$, $B(x,0) = 0$ leave considered PDEs invariant, then the function

$$u_{\mathrm{II}}(x) = R^{-1}(x,\theta)[u_{\mathrm{I}}(x') - B(x,\theta)] \tag{2.27}$$

will be a new solution of the equation provided $u_{\mathrm{I}}(x)$ is any given solution. Formulae like (2.27) we call formulae of group multiplication of solutions (GMS) (Fushchych et al [9]). So, to construct the formulae of GMS for the NS equations one has to find, first of all, the final transformations generated by symmetry operators (1.2), (1.3) and then, according to (2.27), construct the formulae. The results of this is given in the table 2.

Note that in 1–11 $p'(x') = p(x)$ and therefore $p_{\mathrm{II}} = p_{\mathrm{I}}(x')$. In this table δ_0, δ_a, α_a, θ_a, β, ε, κ are arbitrary constants, $\alpha = (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)^{1/2}$; $\boldsymbol{f}$ and g are arbitrary differentiable functions of t. The formulae of GMS stated above allow to construct new solutions $\boldsymbol{u}_{\mathrm{II}}(x)$ of the NS equations (1.1) starting from a known one $\boldsymbol{u}_{\mathrm{I}}(x)$.

Table 2. Final symmetry transformations and the corresponding formulae of GMS for the NS equations (1.1)

N	Operator	Final transformations			Formulas of GMS
			$x \to x'$	$u(x) \to u'(x')$	
1	∂_t	$t' = t + \delta_0$	$\boldsymbol{x}' = \boldsymbol{x}$	$\boldsymbol{u}'(x') = \boldsymbol{u}(x)$	$\boldsymbol{u}_{\mathrm{II}}(x) = \boldsymbol{u}_{\mathrm{I}}(x')$
2–4	∂_a	$t' = t$	$x'_a = x_a + \delta_a$	$\boldsymbol{u}'(x') = \boldsymbol{u}(x)$	$\boldsymbol{u}_{\mathrm{II}}(x) = \boldsymbol{u}_{\mathrm{I}}(x')$
5–7	J_{ab}	$t' = t$	$\boldsymbol{x}' = \boldsymbol{x}\cos\alpha + (\boldsymbol{x}\times\boldsymbol{\alpha})\frac{\sin\alpha}{\alpha} + \boldsymbol{\alpha}(\boldsymbol{\alpha}\cdot\boldsymbol{x})\frac{1-\cos\alpha}{\alpha^2}$	$u'^a(x') = \left(\delta_{ab}\cos\alpha + \varepsilon_{abc}\alpha_c\frac{\sin\alpha}{\alpha} + \alpha_a\alpha_b\frac{1-\cos\alpha}{\alpha^2}\right)u^b(x)$	$u^a_{\mathrm{II}}(x) = \left(\delta_{ab}\cos\alpha + \varepsilon_{abc}\alpha_c\frac{\sin\alpha}{\alpha} + \alpha_a\alpha_b\frac{1-\cos\alpha}{\alpha^2}\right)u^b_{\mathrm{I}}(x')$
8–10	G_a	$t' = t$	$\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{\theta}t$	$\boldsymbol{u}'(x') = \boldsymbol{u}(x) + \boldsymbol{\theta}$	$\boldsymbol{u}_{\mathrm{II}}(x) = \boldsymbol{u}_{\mathrm{I}}(x') - \boldsymbol{\theta}$
11	D	$t' = e^{2\beta}t$	$\boldsymbol{x}' = e^{\beta}\boldsymbol{x}$	$\boldsymbol{u}'(x') = e^{-\beta}\boldsymbol{u}(x)$ $p'(x') = e^{-2\beta}p(x)$	$\boldsymbol{u}_{\mathrm{II}}(x) = e^{\beta}\boldsymbol{u}_{\mathrm{I}}(x')$ $p_{\mathrm{II}}(x) = e^{2\beta}p_{\mathrm{I}}(x)$
12	Q	$t' = t$	$\boldsymbol{x}' = \boldsymbol{x} + \varepsilon\boldsymbol{f}(t)$	$\boldsymbol{u}'(x') = \boldsymbol{u}(x) + \varepsilon\dot{\boldsymbol{f}}(t)$ $p'(x') = p(x) - \varepsilon\boldsymbol{x}\cdot\ddot{\boldsymbol{f}}(t)$	$\boldsymbol{u}_{\mathrm{II}}(x) = \boldsymbol{u}_{\mathrm{I}}(x') - \varepsilon\dot{\boldsymbol{f}}(t)$ $p'_{\mathrm{II}}(x) = p_{\mathrm{I}}(x') + \varepsilon\boldsymbol{x}\cdot\ddot{\boldsymbol{f}}(t)$
13	R	$t' = t$	$\boldsymbol{x}' = \boldsymbol{x}$	$\boldsymbol{u}'(x') = \boldsymbol{u}(x)$ $p'(x') = p(x) + \kappa g(t)$	$\boldsymbol{u}_{\mathrm{II}}(x) = \boldsymbol{u}_{\mathrm{I}}(x')$ $p'_{\mathrm{II}}(x) = p_{\mathrm{I}}(x') - \kappa g(t)$

Remark 2. It will be noted that operator Q given in (1.3) generates transformations (N 12 in table 2) which can be considered as an invariant transition to a frame of reference which is moved arbitrarily: $\boldsymbol{x}_{\mathrm{ref}} = \varepsilon \boldsymbol{f}(t)$.

Let us give some examples of the application of formulae of GMS. Having applied formulae 5–7 of table 2 to solution 16° (2.4) we get a new multiparameter solution for the NS equations (1.1)

$$
\begin{gathered}
\boldsymbol{u}(x) = \frac{1}{\sqrt{t}}\left\{ e^{-\tau^2}\left[\boldsymbol{a}\left(\alpha_1 + \alpha_2\int^{\tau} e^{s^2}ds\right) + \boldsymbol{b}\left(\alpha_3 + \alpha_4\int^{\tau} e^{s^2}ds\right)\right] + \boldsymbol{c}\right\}, \\
\tau = \frac{\boldsymbol{c}\cdot\boldsymbol{x}}{2\sqrt{t}} - 1, \quad p(x) = \frac{1}{t}\left(\frac{\boldsymbol{c}\cdot\boldsymbol{x}}{2\sqrt{t}} + \alpha_5\right),
\end{gathered} \tag{2.28}
$$

where $\alpha_1, \ldots, \alpha_5$ are arbitrary constants, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are arbitrary orthonormal constant vectors

$$
\boldsymbol{a}^2 = \boldsymbol{b}^2 = \boldsymbol{c}^2 = 1, \quad \boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{c} = \boldsymbol{b}\cdot\boldsymbol{c} = 0. \tag{2.29}
$$

Further application of the formulae of GMS N 8–10 to (2.28) gives rise to the following solution of the NS equations

$$
\begin{gathered}
\boldsymbol{u}(x) = \frac{1}{\sqrt{t}}\left\{ e^{-y^2}\left[\boldsymbol{a}\left(\alpha_1 + \alpha_2\int^{y} e^{s^2}ds\right) + \boldsymbol{b}\left(\alpha_3 + \alpha_4\int^{y} e^{s^2}ds\right)\right] + \boldsymbol{c}\right\} - \boldsymbol{\theta}, \\
y = \frac{\boldsymbol{c}\cdot(\boldsymbol{x} + \boldsymbol{\theta}t)}{2\sqrt{t}} - 1, \quad p(x) = \frac{1}{t}\left(\frac{\boldsymbol{c}\cdot(\boldsymbol{x} + \boldsymbol{\theta}t)}{2\sqrt{t}} + \alpha_5\right),
\end{gathered} \tag{2.30}
$$

where the $\boldsymbol{\theta}$ are arbitrary constants, the rest are the same as in (2.28).

The procedure of generating solutions by means of symmetry transformations can be continued until one gets an ungenerative family of solutions, that is the family which is invariant (up to transformation of constant parameters) with respect to the total GMS procedure. Without doubt, the reader can carry out this procedure by analogy with the above examples, for any solution 1°–19° (2.4) of the NS equations.

3. Examples of non-Lie ansätze for the NS field

Ansätze collected in table 1, of course, do not exhaust all possible ansätze which reduce the NS equations. Here we consider several examples of ansätze which do not have the form (1.4). More complete consideration of this question will be given in our next paper.

Because all ansätze obtained within the framework of the Lie approach have form (1.4), it is natural to call other ansätze non-Lie. Our first example of this is the well known ansatz

$$
\boldsymbol{u} = \nabla\varphi, \tag{3.1}
$$

where $\varphi = \varphi(x)$ is a scalar function. If satisfies the Hamilton–Jacobi and Laplace equations

$$
\varphi_t + (\nabla\varphi)^2 + p = 0, \quad \Delta\varphi = 0 \tag{3.2}
$$

then the function $\boldsymbol{u}$ (3.1) automatically satisfies the NS equations (1.1). It is an example of non-local component reduction.

Ansatz

$$\boldsymbol{u} = \boldsymbol{a}\varphi(t, \boldsymbol{b}\cdot\boldsymbol{x}, \boldsymbol{c}\cdot\boldsymbol{x}), \tag{3.3}$$

where $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ are constant vectors satisfying (2.29), reduces (1.1) to the two-dimensional heat equation

$$\varphi_t - \Delta_2\varphi = 0, \quad \Delta_2 \equiv \frac{\partial^2}{\partial\omega_1^2} + \frac{\partial^2}{\partial\omega_2^2}, \quad \omega_1 = \boldsymbol{b}\cdot\boldsymbol{x}, \quad \omega_2 = \boldsymbol{c}\cdot\boldsymbol{x}, \tag{3.4}$$

Ansatz

$$\boldsymbol{u} = \boldsymbol{x}\varphi(x), \quad p = p(x) \tag{3.5}$$

reduces equations (1.1) to the system of pde for two scalar functions φ and p

$$\boldsymbol{x}(\varphi_t + \Delta\varphi) + \nabla(\varphi + p) = 0, \quad \varphi + (\boldsymbol{x}\cdot\nabla)\varphi = 0. \tag{3.6}$$

New ansätze and solutions of the NS equations (1.1) obtained within the framework of conditional symmetry will be given in our next paper. The concept and the term conditional invariance was firstly introduced by Fushchych [5] (see also Fushchych and Nikitin [7]). Further development and applications of this concept are contained in Fushchych et at [9], Fushchych and Serov [8], Levi and Winternitz [10].

Let us make some concluding remarks. It will be noted that the question of what spin is carried by the NS field has a rather strange answer (Fushchych [4]): the NS field carries not only spin 1 but all possible integer spins $s = 0, 1, 2, \ldots$. It is due to the fact that the space of solutions of the ns equations can be decomposed into an infinite direct sum of subspaces invariant under operators $S_{ab} = u^a\partial_{u^b} - u^b\partial_{u^a}$ from algebra $AO(3)$, and these subspaces are not invariant under operators G_a from (1.2) because of the unboundedness of operators ∂_{u^a}.

In hydrodynamics the linearized NS equations are sometimes used

$$\boldsymbol{u}_t - \Delta\boldsymbol{u} = 0, \quad \operatorname{div}\boldsymbol{u} = 0. \tag{3.7}$$

The maximal invariance algebra of (3.7) is the seven-dimensional Lie algebra with basis elements

$$\begin{gathered} \partial_t, \quad \partial_a, \quad D = 2t\partial_t + x_a\partial_a, \quad I = u^a\partial_{u^a}, \\ J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a}. \end{gathered} \tag{3.8}$$

It should be pointed out that (3.7) are not Galilei invariant and therefore they fail in adequately describing real hydrodynamics processes.

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