Merons and instantons as products of self-interaction of the Dirac–Gürsey spinor field

W.I. FUSHCHYCH, W.M. SHTELEN

In this letter we show that the most physically interesting solutions of the $SU(2)$ Yang–Mills equations, the well known meron and instanton solutions (and some others), are generated by corresponding solutions of nonlinear Dirac–Gürsey spinor equations.

The idea of describing particles (fields) of spin $0, 1, \frac{3}{2}, 2, \ldots$ by means of a field of spin 1/2 was put forward by Louis de Broglie in the 1930s. Later, in the 1950s it was developed by Heisenberg and Pauli in their unified field theory. Here we consider another realisation of this idea based on the possibility of constructing from a spinor field ψ, which satisfies some given equation, different bispinor densities, say scalar $u = \bar{\psi}\psi$, vector $A_\mu = \bar{\psi}\gamma_\mu\psi$, and so on. The fruitfulness of such an approach is demonstrated by examples of meron and instanton solutions of $SU(2)$ Yang–Mills (YM) equations. We hope that it is not simply a mathematical trick but possibly reveals some important intrinsic features of merons and instantons.

It is well known that a vast class of solutions of $SU(2)$ YM equations

$$\begin{aligned}\Box\boldsymbol{Y}_\mu - \partial_\mu\partial_\nu\boldsymbol{Y}_\nu + e[(\partial_\nu\boldsymbol{Y}_\nu)\times\boldsymbol{Y}_\mu - 2(\partial_\nu\boldsymbol{Y}_\mu)\times\boldsymbol{Y}_\nu + (\partial_\mu\boldsymbol{Y}_\nu)\times\boldsymbol{Y}_\nu]+ \\ + e^2\boldsymbol{Y}_\nu\times(\boldsymbol{Y}_\nu\times\boldsymbol{Y}_\mu) = 0,\end{aligned} \tag{1}$$

where $\boldsymbol{Y}_\nu = \boldsymbol{Y}_\nu(x) = \{Y_\nu^1, Y_\nu^2, Y_\nu^3\}$ is the YM potential, $\mu,\nu = \overline{0,3}$, $x \in R(4)$ (Euclidean space), can be constructed by means of a scalar field φ which satisfies the nonlinear wave equation

$$\Box\varphi + \lambda_1\varphi^3 = 0 \tag{2}$$

(λ_1 is an arbitrary constant). In order to do this one has to use the 't Hooft–Corrigan–Fairlie–Wilczek ansatz (see, for example, [1])

$$\begin{aligned}&eY_0^a = \mp\partial_a\ln\varphi, \quad a = 1,2,3,\\ &eY_j^a = (\varepsilon_{jan}\partial_n \pm \delta_{ja}\partial_0)\ln\varphi.\end{aligned} \tag{3}$$

The following solutions of equation (2) are of special interest

$$\varphi = \left(\lambda_1 x^2\right)^{-1/2}, \tag{4}$$

$$\varphi = \left(\frac{(a-b)^2}{\lambda_1(x-a)^2(x-b)^2}\right)^{1/2}, \tag{5}$$

$$\varphi = \sqrt{\frac{8}{\lambda_1}}\frac{\alpha}{x^2+\alpha^2}, \tag{6}$$

J. Phys. A: Math. Gen., 1990, **23**, L517–L520.

where $(x-a)\equiv(x_\nu-a_\nu)(x_\nu-a_\nu)$ and a_ν, b_ν, α are arbitrary constants, because they give rise to the one-meron [2]

$$eY_0^a=\pm\frac{x_a}{x^2},\quad eY_j^a=-\varepsilon_{jan}\frac{x_n}{x^2}\mp\delta_{aj}\frac{x_0}{x^2} \tag{7}$$

to the two-meron [2]

$$\begin{gathered}eY_0^a=\pm\left(\frac{(x-a)_a}{(x-a)^2}+\frac{(x-b)_a}{(x-b)^2}\right),\\ eY_j^a=-\varepsilon_{jan}\left(\frac{(x-a)_n}{(x-a)^2}+\frac{(x-b)_n}{(x-b)^2}\right)\mp\delta_{aj}\left(\frac{(x-a)_0}{(x-a)^2}+\frac{(x-b)_0}{(x-b)^2}\right)\end{gathered} \tag{8}$$

and to the instanton [3]

$$eY_0^a=\mp\frac{2x_a}{x^2+\alpha^2},\quad eY_j^a=-\varepsilon_{jan}\frac{2x_a}{x^2+\alpha^2}\pm\delta_{aj}\frac{2x_0}{x^2+\alpha^2} \tag{9}$$

solutions of YM equations (1), respectively. We shall show that scalar fields (4)–(6) can be constructed in turn from the spinor field ψ which satisfies the Dirac–Gürsey equation

$$\left[i\gamma\partial+\lambda(\bar\psi\psi)^{1/3}\right]\psi=0, \tag{10}$$

where γ_ν are 4×4 Dirac matrices, $\psi=\psi(x)$ is a four-component complex function (column), $\bar\psi=\psi^+\gamma_0$ and λ is an arbitrary constant. Equation (10) is conformally invariant as well as (1) and (2), but it has conformal degree 3/2 while the conformal degree of the scalar field from (2) is 1. (Detailed analysis of conformal symmetry is given in [4] where, in particular, it was pointed out that the conformal degree is an important intrinsic characteristic of a field.) So, to construct the scalar field φ from the spinor field ψ properly, we should not simply put $\varphi=\bar\psi\psi$ but

$$\varphi=(\bar\psi\psi)^{1/3}. \tag{11}$$

Further, we consider the following two solutions of equation (10) obtained by Kortel [5] and Merwe [6]

$$\psi(x)=\frac{1}{4}\left(\frac{3}{\lambda}\right)^{3/2}\frac{i\gamma x+\sqrt{x^2}}{(x^2)^{5/4}}\chi \tag{12}$$

and

$$\psi(x)=\left(\frac{4\alpha}{\lambda}\right)^{3/2}\frac{i\gamma x+\alpha}{(x^2+\alpha^2)^2}\chi, \tag{13}$$

where χ is an arbitrary constant spinor and one can choose, without loss of generality, $\bar\chi\chi=1$; α is an arbitrary constant. These solutions were obtained by means of the Heisenberg ansatz [7]

$$\psi(x)=[f(w)+i\gamma xg(w)]\chi, \tag{14}$$

where f and g are real scalar functions, $w=\sqrt{x^2}$ and χ is a constant spinor. One can make sure that the substitution of (12) and (13) into (11) gives rise to (4) and (6) provided $\lambda = \frac{3}{2}\sqrt{\lambda_1}$ and $\lambda = \sqrt{2\lambda_1}$ respectively. It should be noted that, generally speaking, scalar field (11), constructed from spinor field (14) and satisfying equation (10), does not satisfy equation (2), but we do not know the explicit form of such solutions of equation (10).

Further we note that solution (5) of equation (2) can be obtained as a result of the following procedure of group multiplication of solutions. Applying to (4) the formulae of generating solutions by conformal transformations [4]

$$\varphi_{II}(x) = \frac{1}{\sigma(c,x)}\varphi_I(x), \quad x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(c,x)}, \quad \sigma(c,x) = 1 - 2cx + c^2x^2, \tag{15}$$

where $\varphi_{II}(x)$ means a new solution, $\varphi_I(x)$ means an old one, and c_μ are arbitrary constants, then by translational transformations

$$\varphi_{II}(x) = \varphi_I(x'), \quad x'_\mu = x_\mu - a_\mu \tag{16}$$

get, letting

$$c_\mu = \frac{b_\mu - a_\mu}{(a-b)^2} \tag{17}$$

the solution (5) of equation (2). For the case of Dirac spinor field ψ, formulae analogous to those given in (15), (16) are [4, 8]

$$\psi_{II}(x) = \frac{1 - \gamma x \gamma c}{\sigma^2(x,c)}\psi_I(x'), \quad x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(c,x)} \tag{18}$$

and

$$\psi_{II}(x) = \psi_I(x'), \quad x'_\mu = x_\mu - a_\mu. \tag{19}$$

Having applied formulae (18), (19) and (17) to (12) we get a new solution of equation (10):

$$\begin{aligned}\psi = {}& \frac{1}{4}\left(\frac{3}{\lambda}\right)^{3/2}\left(\frac{(a-b)^2}{(x-a)^2(x-b)^2}\right)^{3/4} \times \\ & \times \left[i\frac{\gamma x - \gamma a}{\sqrt{(x-a)^2}} + \left(1 - \frac{(\gamma x - \gamma a)(\gamma b - \gamma a)}{(a-b)^2}\right)\left(\frac{(a-b)^2}{(x-b)^2}\right)^{1/2}\right]\chi, \quad \bar{\chi}\chi = 1\end{aligned} \tag{20}$$

and it is the solution which gives rise (by means of (11)) to (5) when $\lambda = \frac{3}{2}\sqrt{\lambda_1}$.

So, we have shown that one-meron (7), two-meron (8) and instanton (9) solutions of YM equation (1) are actually generated by the spinor fields (12), (20) and (13), respectively, which satisfy the Dirac–Gürsey equation (10).

The procedure for obtaining a two-meron solution from the one-meron solution described above can be applied to instanton solutions (6) and (13). So, in this case, making use of formulae (15), (16), (18) and (19), choosing c_ν as in (17) and $\alpha^2 = (a-b)^2$ we get from (6) and (13) a new solution of equation (2)

$$\varphi = \left(\frac{8(a-b)^2}{\lambda_1}\right)^{1/2}\left(\frac{1}{(x-a)^2 + (x-b)^2}\right) \tag{21}$$

and a new solution of the Dirac–Gürsey equation (10)

$$\psi = \left(\frac{4\alpha}{\lambda}\right)^{3/2} \left(\frac{1}{[(x-a)^2+(x-b)^2]^2}\right) \times \\ \times \left[i(\gamma x - \gamma a) + \left(1 - \frac{(\gamma x - \gamma a)(\gamma b - \gamma a)}{(a-b)^2}\right)\sqrt{(a-b)^2}\right]\chi, \quad \bar{\chi}\chi = 1. \tag{22}$$

The corresponding solution of YM equations (1) has the form

$$eY_0^i = \pm 2\frac{(x-a)_i + (x-b)_i}{(x-a)^2+(x-b)^2}, \\ eY_j^i = -2\varepsilon_{jin}\frac{(x-a)_n + (x-b)_n}{(x-a)^2+(x-b)^2} \mp 2\delta_{ij}\frac{(x-a)_0+(x-b)_0}{(x-a)^2+(x-b)^2}. \tag{23}$$

This new solution of YM equations is also generated by spinor field ψ satisfying the Dirac–Gürsey equation (10); now it is given by (22), according to (11) ($\lambda = \sqrt{2\lambda_1}$) and (3).

In conclusion we would like to note that all solutions of the Dirac–Gürsey equation (10) considered above (see (12), (13), (20), (22)) are non-analytic in the coupling constant λ. A great number of solutions of this equation which are analytic in λ are obtained in [4, 8]. These solutions also generate scalar fields φ which satisfy equation (2) (and therefore give rise to solutions of YM equations), but in these cases we lose the connection between coupling constants λ and λ_1, and, generally speaking, $\varphi = (\bar{\psi}\psi)^k$, $k \neq 1/3$.

It will also be noted that solutions of YM equations can be looked for in the form $Y_\mu^a = \bar{\psi}^a\gamma_\mu\psi^a$ (no sum over $a = 1, 2, 3$), where ψ^a satisfy some nonlinear spinor equation. In the same spirit solutions of other field equations can be constructed.

We would like to express our gratitude to the referees for their useful suggestions.

1. Actor A., *Rev. Mod. Phys.*, 1979, **51**, 461–525.
2. de Alfaro V., Fubini S., Furlan G., *Phys. Lett. B*, **65**, 163–166.
3. Belavin A.A., Polyakov A.M., Schwartz A.S., Tyupkin Yu.S., *Phys. Lett. B*, 1975, **59**, 85–87.
4. Fushchych W.I., Shtelen W.M., Serov N.I., Symmetry analysis and exact solutions of nonlinear equations of mathematical physics, Kyiv, Naukova Dumka, 1989.
5. Kortel F., *Nuovo Cimento*, 1956, **4**, 211–215.
6. Merwe P.T., *Phys. Lett. B*, 1981, **106**, 485–486.
7. Heisenberg W., *Z. Naturf. A*, 1954, **9**, 292–303.
8. Fushchych W.I., Shtelen W.M., *J. Phys. A: Math. Gen.*, 1983, **16**, 271–277.