On vector and pseudovector Lagrangians for electromagnetic field

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A Lagrange function in terms of electromagnetic field strengths is constructed which is a 4-vector with respect to the total Poincaré group $\tilde{P}(1,3)$ and whose Euler–Lagrange equivalent to the Maxwell equations. The advantages of the known pseudovector with respect to the $\tilde{P}(1,3)$ group Lagrange function are shown. The conservation quantities on the basis of hte corresponding generalization of Noether theorem are found.

A development of Lagrange approach (L-approach) to electrodynamics in terms of field strength tensor $F = (F^{\mu\nu}) = (\boldsymbol{E}, \boldsymbol{H})$ of the elcectromagnetik field, without using the potentials A^μ, was discussed in [1–4]. It is easy to show that in terms of $(\boldsymbol{E}, \boldsymbol{H})$ there is no scalar, with respect to the Poincaré group $P(1,3)$, Lagrange function, for which the Euler–Lagrange (EL) equations coincide with the Maxwell equations.

The aim of this paper is a construction of a $\tilde{P}(1,3)$ vector Lagrangian in terms of $(\boldsymbol{E}, \boldsymbol{H})$, i.e. a Lagrange function $\mathcal{L}_\mu$ which is the vector with respect to the total Poincaré group $\tilde{P}(1,3)$ (including both $P(1,3)$ and the spase-time reflections) and for which the EL equations are exactly equivalent to the original Maxwell equations. In what follows such a Lagrangian $\mathcal{L}_\mu$ will be called a Lagrange vector.

Let us represent the Maxwell equations

$$\partial_0 \boldsymbol{E} = \operatorname{curl} \boldsymbol{H} - \boldsymbol{j}, \quad \operatorname{div} \boldsymbol{E} = \rho, \quad \partial_0 \boldsymbol{H} = -\operatorname{curl} \boldsymbol{E}, \quad \operatorname{div} \boldsymbol{H} = 0 \tag{1}$$

in a manifestly covariant form

$$Q^\mu = j^\mu, \quad R^\mu = 0, \quad \mu = \overline{0,3} \equiv 0, 1, 2, 3. \tag{2}$$

Here

$$Q^\mu \equiv F^{\mu\nu}_{,\nu} = \partial_\nu F^{\mu\nu}(x), \quad R^\mu \equiv \varepsilon F^{\mu\nu}_{,\nu}, \quad \varepsilon F^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{3}$$

$F = (F^{\mu\nu})$ is the field strength tensor:

$$F = (F^{\mu\nu}) = (\boldsymbol{E}, \boldsymbol{H}): \quad F^{0i} = E^i, \quad F^{ij} = \varepsilon^{ijk} H^k, \quad F^{\mu\nu} = -F^{\nu\mu}, \tag{4}$$

j is a current density:

$$j \equiv (j^\mu) = (\rho, \boldsymbol{j}), \quad j^0 = \rho, \quad \boldsymbol{j} = (j^i), \quad i = 1, 2, 3, \tag{5}$$

$\varepsilon^{\mu\nu\rho\sigma}$ is the completely antisymmetric unit tensor, $\varepsilon^{0123} = 1$, and

$$x = (x^\mu) \equiv (x^0, x^1, x^2, x^3), \quad \partial_\mu \equiv \partial / \partial x^\mu. \tag{6}$$

Nuovo Cimento B, 1989, **103**, № 4, 423–429.

The componets Q^μ, R^μ of the vectors $Q \equiv (Q^\mu)$ and $R \equiv (R^\mu)$ are connected with the field strengths $\boldsymbol{E} \equiv (E^i)$ and $\boldsymbol{H} \equiv (H^i)$ as

$$Q^0 = \operatorname{div} \boldsymbol{E}, \quad Q^i = (-\partial_0 \boldsymbol{E} + \operatorname{curl} \boldsymbol{H})^i \equiv -\partial_0 E^i + \varepsilon^{ijk}\partial_j H^k, \tag{7}$$

$$R^0 = \operatorname{div} \boldsymbol{H}, \quad R^i = (-\partial_0 \boldsymbol{H} - \operatorname{curl} \boldsymbol{E})^i \equiv -\partial_0 H^i - \varepsilon^{ijk}\partial_j E^k. \tag{8}$$

Now consider the 3rd-rank tensor $T_{\mu\rho\sigma}$ and pseudotensor $T'_{\mu\rho\sigma}$ (with respect to the $\tilde{P}(1,3)$ group), which are constructed with the help of the 4-vectors Q^μ, R^μ (3):

$$T_{\mu\rho\sigma} \equiv a[g_{\mu\rho}(Q_\sigma - j_\sigma) - g_{\mu\sigma}(Q_\rho - j_\rho)] + b\varepsilon_{\mu\nu\rho\sigma}R^\nu, \tag{9}$$

$$T'_{\mu\rho\sigma} \equiv a'(g_{\mu\rho}R_\sigma - g_{\mu\sigma}R_\rho) + b'\varepsilon_{\mu\nu\rho\sigma}(Q^\nu - j^\nu), \tag{10}$$

where a, b, a', b' are arbitrary constants.

Theorem 1. *For any $a, b, a', b' \neq 0$ each of the sets of equations*

$$T_{\mu\rho\sigma} = 0, \tag{11}$$

$$T'_{\mu\rho\sigma} = 0 \tag{12}$$

is equivalent to the original Maxwell equations (2).

One can easily verify the validity of this assertion by rewriting the components of tensors T, T' (11), (12) in the explicit form.

Just the $\tilde{P}$-tensor set of equations (11) and $\tilde{P}$-pseudotensor set of equation (12) will be used in this work for the construction of $\tilde{P}$-vector L-approach to the electromagnetic field $F = (\boldsymbol{E}, \boldsymbol{H})$.

Let us introduce in addition to the Lagrange variable for tensor eletromagnetic field the new Lagrange variables $\bar{F}$, $\bar{F}_{,\mu}$ which are dually conjugated to F, $F_{,\mu}$ (on the manifold Φ_0 of the solutions of Maxwell's equations $\bar{F} = \varepsilon F$, see (3)). The general form of $\tilde{P}$-vector Lagrangian

$$\mathcal{L}_\mu = \mathcal{L}_\mu(F, F_{,\nu}, \bar{F}, \bar{F}_{,\nu}), \quad \mathcal{L}_\mu : R^{60} \to R^1 \tag{13}$$

up to a total 4-divergence terms is the following:

$$\begin{aligned} \mathcal{L}_\mu = {} & a_1 F_{\mu\nu}Q^\nu + a_2 F_{\mu\nu}\bar{R}^\nu + a_3\varepsilon F_{\mu\nu}R^\nu + a_4\varepsilon F_{\mu\nu}\bar{Q}^\nu + a_5\bar{F}_{\mu\nu}\bar{Q}^\nu + \\ & + a_6\bar{F}_{\mu\nu}R^\nu + a_7\varepsilon\bar{F}_{\mu\nu}\bar{R}^\nu + a_8\varepsilon\bar{F}_{\mu\nu}Q^\nu + (q_1 F_{\mu\nu} + q_2\varepsilon\bar{F}_{\mu\nu})j^\nu. \end{aligned} \tag{14}$$

Here we have used also notations

$$\bar{Q}^\mu \equiv \bar{F}^{\mu\nu}_{,\nu}, \quad \bar{R}^\mu \equiv \bar{F}^{\mu\nu}_{,\nu}, \quad \varepsilon\bar{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{F}_{\rho\sigma}. \tag{15}$$

Theorem 2. *The EL equations for $\tilde{P}$-vector $\mathcal{L} = (\mathcal{L}^\mu)$ are equivalent to the Maxwell equations if and only if the coefficients in (14) obey the conditions*

$$\begin{aligned} & a_8 - a_2 = a = -b' = -q_1 \equiv -q = 0, \\ & a_6 - a_4 = a' = -b \neq 0, \quad a_1 - a_3 - a_6 - a_8 = a_2 + a_4 + a_5 - a_7 = 0. \end{aligned} \tag{16}$$

Proof. The straightforward calculations of Lagrange derivatives for the Lagrangian $\mathcal{L}_\mu$ (14) lead to the result

$$\delta\mathcal{L}_\mu/\delta F_{\rho\sigma} = T_{\mu\rho\sigma} = 0, \quad \delta\mathcal{L}_\mu/\delta\bar{F}_{\rho\sigma} = T'_{\mu\rho\sigma} = 0, \tag{17}$$

if and only if conditions (16) are fulfilled.

The four components of the Lagrange vector (14) generate four actions

$$W^\mu(F,\bar{F}) = \int d^3x\, \mathcal{L}^\mu(F(x),\bar{F}(x),\partial_\nu F(x),\partial_\nu \bar{F}(x)), \quad F,\bar{F}\in\Phi, \tag{18}$$

where F, $\bar{F}$ belong to the set Φ of twice differentiable functions, and Φ_0^μ defines the set of extremals of the action (18) with a fixed μ.

Theorem 3. *The intersection $\Phi_0 = \bigcap\limits_\mu \Phi_0^\mu$ of the sets Φ_0^μ of extremals of four actions (18), given by the Lagrangian $\mathcal{L}^\mu$ (14) whose coefficients obey conditions (16) coincides with the set of solutions of Maxwell equations (1).*

Proof. The validity of this assertion follows from the derivation of the explicit form of EL equations for (14), i.e. from (17) and the Theorem 1 about the equivalence of the set of eqs. (11) or (12) and the Maxwell equations (2), i.e. (1).

The $\tilde{P}$-vector Lagrangian (14), proposed here, has several advantages in comparison with the $\tilde{P}$-pseudovector Lagrangian from [3], which in our notations has the form

$$\mathcal{L}^\mu = \mathcal{L}^\mu(F,F_{,\nu}) = F^{\mu\nu}R_\nu - \varepsilon F^{\mu\nu}(Q_\nu - j_\nu). \tag{19}$$

Firstly, Lagrangian (19) leads only to the pseudotensor system of eqs. (12), i.e. it gives rise to the pseudotensor system of eqs. (12) in favour of the tensor system of eqs. (11). That is a direct consequence of the pseudovector character of Lagrangian (19). Let us note that without appealing to the additional Lagrange variable $\bar{F}\equiv(\bar{F}^{\mu\nu})$ it is impossible to construct a $\tilde{P}$-vector Lagrangian: the demand of function $\mathcal{L}^\mu(F,F_{,\nu})$ being a $\tilde{P}$-vector leads to the expression

$$\mathcal{L}^\mu = \mathcal{L}^\mu(F,F_{,\nu}) = F^{\mu\nu}Q_\nu + \varepsilon F^{\mu\nu}R_\nu, \tag{20}$$

for which the EL equations are the identities.

Secondly, as is seen from the terms with the current in (19), the interaction Lagrangian in [3] also is a $\tilde{P}$-pseudovector one:

$$\mathcal{L}_{\rm I}^\mu = \varepsilon F^{\mu\nu}j_\nu, \quad \mathcal{L}_{\rm I}^0 = \boldsymbol{j}\cdot\boldsymbol{H}, \quad \mathcal{L}_{\rm I}^i = (\boldsymbol{j}\times\boldsymbol{E} - \rho\boldsymbol{H})^i. \tag{21}$$

A physical unsatisfactoriness of such an infraction is evident already from the fact that the density of electric charge in (21) is connected not with the electric-field stengths $\boldsymbol{E}$ but with the magnetic-field strength $\boldsymbol{H}$.

Finally, the calculation of conserved quantities on the basis of Lagrangian (19) gives the result that a $\tilde{P}$-tensor generator of the Poincaré group is corresponded by $\tilde{P}$-pseudotensor conserved currents. This defect, together with the above-mentioned ones, is eliminated by using the $\tilde{P}$-vector Lagrangian (14).

Derivation or conserved quantities in the framework of the L-approach formulated here demands a generalization of Noether theorem for the case of vector Lagrangian.

Theorem 4. *Let*

$$\hat{q}:\; F(x)\to F'(x) = \hat{q}F(x) \tag{22}$$

be an arbitrary invariance transformation of eqs. (2) with $j=0$. Then the conserved current θ^μ_ν, constructed on the basis of the $\tilde{P}$-vector Lagrangian $\mathcal{L}_\mu$ (14) (of course with $j=0$) with the help of the formula

$$\hat{q}\to\theta^\mu_\nu \stackrel{\rm df}{=} \left(\frac{\partial\mathcal{L}_\nu}{\partial F^{\rho\sigma}_{,\mu}}F'^{\rho\sigma} + \frac{\partial\mathcal{L}_\nu}{\partial \bar{F}^{\rho\sigma}_{,\mu}}\bar{F}'^{\rho\sigma}\right), \quad F'\equiv\hat{q}F, \quad \bar{F}'\equiv\hat{q}\bar{F} = \varepsilon\hat{q}F, \tag{23}$$

is symmetric and its divergence vanishes for any solutions of eqs. (2) whith $j = 0$:

$$\partial_\mu \theta^\mu_\nu = 0. \tag{24}$$

Proof. Derivation of currents (23) for $\mathcal{L}_\mu$ (14) with $j = 0$ leads to the result

$$\begin{aligned} &\hat{q} \to \theta^\mu_\nu = A\left(F^{\mu\alpha}F'_{\alpha\nu} + F'^{\mu\alpha}F_{\alpha\nu} + \frac{1}{2}\delta^\mu_\nu F^{\alpha\beta}F'_{\alpha\beta}\right), \\ &A = a_1 - a_2 + a_7 - a_8 = a_3 + a_4 + a_5 + a_6. \end{aligned} \tag{25}$$

Summetry of tensor (25) is evident and eq. (24) is a consequence of the Maxwell equations (2) with $j = 0$.

Note that in the vector L-approach the correspond (according to the Noether theorem) to one generator of invariance transformation.

Let us give a short discussions of conserved quantities which are the concequences of (25). We obtain, taking $A = 1$, that generators of 4-translations ∂_μ according to the formula (25) give the trivial current

$$\partial_\mu \to \theta^{\mu\nu}(\hat{q} = \partial_\rho) = (\partial_\rho)^{\mu\nu} \equiv \partial_\rho T^{\mu\nu}, \tag{26}$$

where $T^{\mu\nu}$ is standard energv-momentum tensor for the field $F = (\boldsymbol{E}, \boldsymbol{H})$:

$$T^\mu_\nu = F^{\mu\alpha}T_{\alpha\nu} + \frac{1}{4}\delta^\mu_\nu F^{\alpha\beta}F_{\alpha\beta}, \quad T^0_\mu = \mathcal{L}_\mu, \tag{27}$$

$$\mathcal{L}_0 \equiv \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{H}^2), \quad \mathcal{L}_j \equiv (\boldsymbol{E} \times \boldsymbol{H})_j. \tag{28}$$

For the analysis of integral conserved quantities

$$\bar{\theta}^\mu = \int d^3x\, \theta^{0\mu}(x) = \text{const}, \quad \theta^{0\mu}(x) = \theta^{0\mu}(\hat{q}) \equiv (\hat{q}^{0\mu}) \tag{29}$$

it is sufficient to write down the densities $\theta^{0\mu}$, omitting the terms with spacelike derivatives, which are not contributed in integral $\bar{\theta}^\mu$. We obtain from formula (25) for the densities $\theta^{0\mu}$, corresponding to the rest of generators of conformal algebra $C(1,3)$ (for the definition of algebra $C(1,3)$ see, for example, [5]), the following expressions:

$$\hat{J}_{\rho\sigma} \to J^{0\mu}_{\rho\sigma} = \delta^\mu_\rho \mathcal{L}_\sigma - \delta^\mu_\sigma \mathcal{L}_\rho, \quad \hat{d} \to D^{0\mu} = \mathcal{L}^\mu, \tag{30}$$

$$\hat{K}_\rho \to \mathcal{K}^{0\mu}_\rho = 2(\delta^\alpha_\rho \mathcal{D} + \mathcal{J}_{\rho\sigma} g^{\sigma\mu}), \tag{31}$$

where

$$\mathcal{D} \equiv x^\mu \mathcal{L}_\mu, \quad \mathcal{J}_{\rho\sigma} \equiv x_\rho \mathcal{L}_\sigma - x_\sigma \mathcal{L}_\rho. \tag{32}$$

As one can see from (30)–(32), the $C(1,3)$-generators $\hat{q} = (\hat{\partial}, \hat{j}, \hat{d}, \hat{k})$ lead here to the conserved quantities, which are expressed in terms of well-known series of main conserved quantities for the electromagnetic field $F = (\boldsymbol{E}, \boldsymbol{H})$, found by Bessel-Hagen [6] on the basis of the L-approach for vector field $A = (A^\mu)$ of potentials, namely

$$\begin{aligned} &P_\rho = \int d^3x\, \mathcal{L}_\rho(x), \quad J_{\rho\sigma} = \int d^3x\, (x_\rho \mathcal{L}_\sigma(x) - x_\sigma \mathcal{L}_\rho(x)), \\ &D = \int d^3x \mathcal{D}(x), \quad K_\rho = \int d^3x\, (2x_\rho \mathcal{D}(x) - x^2 \mathcal{L}_\rho(x)). \end{aligned} \tag{33}$$

It is interesting to note that formula (25) gives the identical zero for the generator $\hat{q} = \varepsilon$ of duality transformations. In order to obtain nontrivial conservation laws with the help of ε let us remind ref. [1], where new invariance algebra $A_{32} \supset C(1,3)$ of free Maxwell's equations was found. A subset of the generators of the algebra A_{32} has the form of composition $\hat{q} = \varepsilon\hat{q}$ of $C(1,3)$ generators $\hat{q} = (\hat{\partial}, \hat{j}, \hat{d}, \hat{k})$ and the generator ε. Formula (25) gives nontrivial conservation laws just for the generators $\hat{q}' = (\varepsilon\hat{\partial}, \varepsilon\hat{j}, \varepsilon\hat{d}, \varepsilon\hat{k})$. The corresponding integral conservation laws expressed in terms of series

$$\begin{aligned} &Z^\mu_\rho = \int d^3x\, \mathcal{Z}^\mu_\rho(x), \quad Z^\mu_{\rho\sigma} = \int d^3x\, (x_\rho \mathcal{Z}^\mu_\sigma - x_\sigma \mathcal{Z}^\mu_\rho), \\ &Z^\mu = \int d^3x\, x^\nu \mathcal{Z}^\mu_\nu(x), \quad \overset{c}{Z}{}^\mu_\rho = \int d^3x\, (2x_\rho x^\sigma \mathcal{Z}^\mu_\sigma - x^2 \mathcal{Z}^\mu_\rho) \end{aligned} \tag{34}$$

of conserved quantities having polarization nature. In (34) the densities $\mathcal{Z}$ of conserved quantities are expressed in the terms of Lipkin's Zilch tensor [7] (in Kibble's notation [8])

$$\mathcal{Z}^\mu_\rho \equiv Z^{0|\mu}_\rho, \quad Z^{0|\mu}_\rho = F^{\nu\alpha} \varepsilon F^{\ ;\mu}_{\alpha\rho} - \varepsilon F^{\nu\alpha} F^{\ ;\mu}_{\alpha\rho}. \tag{35}$$

The conservation laws (34) were found in [4–10] without using the L-approach and Noether theorem (except in ref. [10], where a parameter-dependent Lagrangian in terms of potentials was used).

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