Continuous subgroups of the generalized Schrödinger groups

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Some general results on the subalgebras of the Lie algebra $ASch(n)$ of the generalized Schrödinger group $Sch(n)$ and on the subalgebras of the Lie algebra $A\widetilde{Sch}(n)$ of the generalized extended Schrödinger group $\widetilde{Sch}(n)$ have been obtained. The subalgebra structure of the algebras $ASch(n)$ and $A\widetilde{Sch}(n)$ are studied with respect to inner automorphisms of the groups $Sch(n)$ and $\widetilde{Sch}(u)$, respectively. The maximal Abelian subalgebras and the one-dimensional subalgebras of the algebras $ASch(n)$ and $A\widetilde{Sch}(n)$ have been explicitly found. The full classification of the subalgebras of the algebras $ASch(3)$ and $A\widetilde{Sch}(n)$, which are nonconjugate to the subalgebras of $ASch(2)$, $A\widetilde{Sch}(2)$, respectively, has been carried out.

1. Introduction

To construct exact solutions of both linear and nonlinear Schrödinger and heat equations it is important to know the subgroup structure of the extended Schrödinger group $\widetilde{Sch}(3)$ (see [1]). Other important applications of subgroup structure of this group were discussed in [2, 3]. It is natural to generalize the notions of the three-dimensional Schrödinger group for the case of arbitrary n-dimensional Euclidean space and to solve the problem of subgroup classification for these generalized groups. If we restrict ourselves by continuous subgroups, then the problem will be reduced to classification of subalgebras of correspondent Lie algebras. This classification was realized for $n=1$ in [4] and for $n=2$ in [2].

In the present paper we study subalgebra structure of both the Lie algebra $ASch(n)$ of the Schrödinger group $Sch(n)$ and the Lie algebra $A\widetilde{Sch}(n)$ of the extended Schrödinger group $\widetilde{Sch}(n)$ with respect to inner automorphisms of the group $Sch(n)$ and the group $\widetilde{Sch}(n)$, respectively. This paper is a continuation of investigations that were carried out in [5–9]. The applied general method of Patera, Winternitz, and Zassenhaus [10] gets further development for classes of groups under consideration.

In Sec. 2 we give definitions of the generalized Schrödinger groups and algebras and introduce some other concepts and basis notation used in the whole paper. In Sec. 3, completely reducible subalgebras of the algebra $AO(n)\oplus ASL(2,R)$ are derived, and all subalgebras of this algebra are described for $n=3$. In Sec. 4 a number of general results about splitting subalgebras of the algebra $ASch(n)$ are obtained. Abelian subalgebras of the extended Schrödinger algebra $A\widetilde{Sch}(n)$ are described in Sec. 5. Classification of subalgebras of the algebras $ASch(3)$ and $A\widetilde{Sch}(3)$ is carried out in Sec. 6. The conclusions are summarized in Sec. 7.

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2. Definitions of Schrödinger groups and algebras. Main notation

Let R be the real number field, R an arithmetical n-dimensional Euclidean space, and AG the Lie algebra of the Lie group G. The Schrödinger group $Sch(n)$ is the multiplicative group of matrices

$$\begin{pmatrix} W & \boldsymbol{v} & \boldsymbol{a} \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix},$$

where $W \in O(n)$, $\boldsymbol{a}, \boldsymbol{v} \in R^n$, and $\alpha\delta - \beta\gamma = 1$ $(\alpha, \beta, \gamma, \delta \in R)$. If $\alpha = \delta = 1$, $\gamma = 0$, we obtain matrices that are elements of the Galilei group $G(n)$. If at the same time $\beta = 0$, we have elements of the isochronous Galilei group $G^0(n)$. Besides, the Schrödinger group $Sch(n)$ can be realized as the transformation group

$$\boldsymbol{x} \to \frac{W\boldsymbol{x} + t\boldsymbol{v} + \boldsymbol{a}}{\gamma t + \delta}, \quad t \to \frac{\alpha t + \beta}{\gamma t + \delta},$$

where t is time and $\boldsymbol{x}$ is a variable vector of the space R^n.

The Lie algebra $ASch(n)$ of the group $Sch(n)$ consists of real matrices

$$\begin{pmatrix} X & \boldsymbol{v} & \boldsymbol{a} \\ 0 & \alpha & \beta \\ 0 & \gamma & -\alpha \end{pmatrix},$$

where $X \in AO(n)$, $\alpha, \beta, \gamma \in R$, and $\boldsymbol{a}, \boldsymbol{v} \in R^n$. Let I_{ab} be a matrix of degree $n+2$ having unity at the intersection of the ath line and the bth column and zeros at the other places $(a, b = 1, \ldots, n+2)$. Then the basis of the algebra $ASch(n)$ is formed by the matrices

$$\begin{aligned} &J_{ab} = I_{ab} - I_{ba}, \quad G_a = I_{a,n+1}, \quad P_a = I_{a,n+2}, \\ &D = -I_{n+1,n+1} + I_{n+2,n+2}, \quad S = -I_{n+2,n+1}, \quad T = I_{n+1,n+2} \end{aligned}$$

$(a < b,\ a, b = 1, \ldots, n)$. They satisfy the following commutation relations:

$$\begin{aligned} &[J_{ab}, J_{cd}] = \delta_{ad}J_{bc} + \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac}, \quad [P_a, J_{bc}] = \delta_{ab}P_c - \delta_{ac}P_b, \\ &[P_a, P_b] = 0, \quad [G_a, J_{bc}] = \delta_{ab}G_c - \delta_{ac}G_b, \quad [G_a, G_b] = 0, \quad [G_a, P_b] = 0, \\ &[D, J_{ab}] = [S, J_{ab}] = [T, J_{ab}] = 0, \quad [D, P_a] = -P_a, \quad [D, G_a] = G_a, \\ &[S, P_a] = G_a, \quad [S, G_a] = 0, \quad [T, P_a] = 0, \quad [T, G_a] = -P_a, \\ &[D, S] = 2S, \quad [D, T] = -2T, \quad [T, S] = D, \quad (a, b, c, d = 1, 2, \ldots, n). \end{aligned}$$

The extended Schrödinger algebra $A\widetilde{Sch}(n)$ is obtained from the algebra $ASch(n)$ by adding the central element M, and, moreover, $[G_a, P_b] = \delta_{ab}M$ and other commutation relations do not change. The factor algebra $A\widetilde{Sch}(n)/\langle M\rangle$ is identified with $ASch(n)$. We shall denote the generators of algebras $ASch(n)$ and $A\widetilde{Sch}(n)$ by the same symbols.

The algebra $A\tilde{G}^0(n) = AO(n) \mathbin{\supset\!\!\!\!+} \langle M, P_1, \ldots, P_n, G_1, \ldots, G_n\rangle$ is called the extended isochronous Galilei algebra, and the algebra $AG^0(n) = A\tilde{G}^0(n)/\langle M\rangle$ is called the isochronous Galilei algebra.

Since the Lie algebra $L = \langle M, P_1, \ldots, P_n, G_1, \ldots, G_n\rangle$ is nilpotent, L is a Lie algebra of some connected and simply connected nilpotent Lie group H. As H is an exponential group, any of its elements can be denoted as $\exp(\theta M)\exp(\boldsymbol{vG}+\boldsymbol{aP})$, where $\theta\in R$, $\boldsymbol{vG}=v_1G_1+\cdots+v_nG_n$, and $\boldsymbol{aP}=a_1P_1+\cdots+a_n,P_n$ ($a_i,v_i\in R$, $i=1,\dots,n$). The multiplication law is derived by the Campbell–Hausdorf formula. Let

$$\Delta\begin{pmatrix} W & 0 & 0\\ 0 & \alpha & \beta\\ 0 & \gamma & -\delta\end{pmatrix}$$

be an element of $O(n)\times SL(2,R)$. It is not difficult to show that in $Sch(n)$ we have

$$\Delta\cdot\exp(\boldsymbol{vG}+\boldsymbol{aP})=\exp((\delta W\boldsymbol{v}-\gamma W\boldsymbol{a})\boldsymbol{G}+(-\beta W\boldsymbol{v}+\alpha W\boldsymbol{a})\boldsymbol{P})\cdot\Delta. \qquad (1)$$

An arbitrary element of the group $\widetilde{Sch}(n)$ has the form

$$\exp(\theta M)\cdot\exp(\boldsymbol{vG}+\boldsymbol{aP})\cdot\Delta.$$

By definition, $\exp(\theta M)\cdot\Delta=\Delta\cdot\exp(\theta M)$, and the equality (1) holds true for $\Delta\cdot\exp(\boldsymbol{vG}+\boldsymbol{aP})$. Using these equalities andmultiplication laws in H and $O(n)\times SL(2,R)$ we shall establish multiplication in $\widetilde{Sch}(n)$ in the usual way. Evidently, $\widetilde{Sch}(n)=H\lambda(O(n))\times SL(2,R))$.

Subalgebras L_1 and L_2 of the algebra $A\widetilde{Sch}(n)$ are called $\widetilde{Sch}(n)$ conjugated if $gL_1g^{-1}=L_2$ for some element $g\in\widetilde{Sch}(n)$. Mapping: $\varphi_g: X\to gXg^{-1}$, $X\in A\widetilde{Sch}(n)$, is called an automorphism corresponding to the element g. If $g=\operatorname{diag}[W,1,1]$, where $W\in O(n)$, then φ_g is called an $O(n)$ automorphism correspondi-ng to the matrix W. We shall identify the automorphism φ_g with the element g.

Henceforth we shall use the following notations: $\langle X_1,\dots,X_s\rangle$ is a vector space or Lie algebra over R with the generators $X_1,\dots,X_s$; $V[k,l]=\langle G_k,\dots,G_l\rangle$ ($k\le l$) is a Euclidean space having the orthonormal basis $G_k,\dots,G_l$, $V[k]=V[k,k]$; $W[k,l]=\langle P_k,\dots,P_l\rangle$ ($k\le l$), $W[k]=W[k,k]$; $\mathfrak{M}[r,t]=\langle M,P_r,\dots,P_t,G_r,\dots,G_t\rangle$ ($r\le t$), $\mathfrak{M}[r]=\mathfrak{M}[r,r]$, $\overline{\mathfrak{M}}[r,t]=\mathfrak{M}[r,t]/\langle M\rangle$; π, ω, τ, ϵ, and ξ are projections of the algebras $A\widetilde{Sch}(n)$ and $ASch(n)$ onto $AO(n)\oplus ASL(2,R)$, $AO(n)$, $V[1,n]$, and $W[1,n]$, respectively.

Let U be a subspace of $\overline{\mathfrak{M}}[1,n]$ and $\hat{F}$ be a subalgebra of $ASch(n)$ such that $\pi(\hat{F})=F$. The notation $\hat{F}+U$ means that $[F,U]\subset U$ and $\hat{F}\cap\overline{\mathfrak{M}}[1,n]\subset U$. Considering algebras $(\hat{F}+U_1),\dots,(\hat{F}+U_s)$ we shall use the notation $\hat{F}:U_1,\dots,U_s$. In the case of the algebra $A\widetilde{Sch}(n)$ this notation has the same meaning.

Let L be the direct sum of Lie algebras $L_1,\dots,L_s$, K a Lie subalgebra of L, and π_i the projection of L onto L_i. If $\pi_i(K)=L_i$, for all $i=1,\dots,s$, then K is called the subdirect sum of algebras $L_1,\dots,L_s$. In this case we shall use the notation $K=L_1\dotplus\cdots\dotplus L_s$. The subdirect sum of modules over a Lie algebra is defined in a similar way.

3. On the subalgebras of the algebra $AO(n)\oplus ASL(2,R)$

In this section a number of auxiliary results to be used in following sections are obtained.

Lemma 3.1. *Subalgebras of the algebra $ASL(2,R)$ are exhausted with respect to $SL(2,R)$ conjugation by the following algebras: O, $\langle D\rangle$, $\langle T\rangle$, $\langle S+T\rangle$, $\langle D,T\rangle$, $ASL(2,R)$. The written algebras are not conjugated mutually.*

Later on, when we speak about subalgebras of the algebra $ASL(2,R)$ we shall mean the subalgebras given by Lemma 3.1.

By direct calculations we are convinced that the normalizer of $\langle D\rangle$ in the group $SL(2,R)$ consists of matrices

$$\begin{pmatrix} 0 & \alpha \\ -\alpha^{-1} & 0 \end{pmatrix}, \quad \begin{pmatrix} \sigma & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

where $\alpha \in R$, $\alpha \neq 0$. The normalizer of $\langle T\rangle$ and the normalizer of $\langle D,T\rangle$ in the group $SL(2R)$ consist of matrices $\pm\exp(\theta_1 D)\cdot\exp(\theta_2 T)$, where $\theta_1,\theta_2 \in R$. The normalizer of $\langle S+T\rangle$ coincides with the group

$$SO(2) = \left\{ \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \middle| \varphi \in R \right\}.$$

Proposition 3.1. *Let $AH(n)$ be the Cartan subalgebra of the algebra $AO(n)$. Up to conjugacy under $O(n)\times SL(2,R)$ the algebra $AO(n)\oplus ASL(2,R)$ has two maximal solvable subalgebras $AH(n)\oplus\langle S+T\rangle$, $AH(n)\oplus\langle D,T\rangle$.*

Proposition 3.1 follows immediately from Lemma 3.1 and the fact that $AO(n)$ has, with respect to $O(n)$ conjugation, the only maximal solvable subalgebra $AH(n)$.

Proposition 3.2. *Up to conjugacy under $O(n)\times AL(2,R)$ the algebra $AO(n)\oplus ASL(2,R)$ has the following subalgebras: (i) $F\oplus K$, where $F\subset AO(n)$, $K\subset ASL(2,R)$; (ii) $F\oplus\langle X+Y\rangle$, where $F\oplus\langle X\rangle\subset AO(n)$, $Y\in ASL(2,R)$; and (iii) $\langle X+D\rangle \Subset (F\oplus\langle T\rangle)$, where $F\oplus\langle X\rangle\subset AO(n)$.*

Proposition 3.2 is proved by the Goursat twist method [11].

Corollary. *Subalgebras of the algebra $AO(3)\oplus ASL(2,R)$ are exhausted with respect to $O(3)\times SL(2,R)$ conjugation by the following algebras:*

O; $\langle J_{12}\rangle$; $\langle D\rangle$; $\langle T\rangle$; $\langle S+T\rangle$; $\langle J_{12}+\alpha D\rangle$ $(\alpha>0)$;
$\langle J_{12}+T\rangle$; $\langle J_{12}+\alpha(S+T)\rangle$ $(\alpha>0)$; $\langle J_{12}+\alpha D, T\rangle$ $(\alpha>0)$;
$\langle D,T\rangle$; $\langle J_{12},D\rangle$; $\langle J_{12},T\rangle$; $\langle J_{12},S+T\rangle$; $\langle J_{12},D,T\rangle$;
$AO(3)$; $ASL(2,R)$; $\langle J_{12}\rangle\oplus ASL(2,R)$; $AO(3)\oplus\langle D\rangle$;
$AO(3)\oplus\langle T\rangle$; $AO(3)\oplus\langle S+T\rangle$; $AO(3)\oplus\langle D,T\rangle$; $AO(3)\oplus ASL(2,R)$.

The written algebras are not conjugated mutually.

The space can $\overline{\mathfrak{M}}[1,n]$ be considered as an exact module the Lie algebra $AO(n)\oplus ASL(2,R)$. Let L be a subalgebra of this algebra. If $\overline{\mathfrak{M}}[1,n]$ is a completely reducible L module, then the algebra L will be called completely reducible.

Theorem 3.1. *A subalgebra L of the algebra $AO(n)\oplus ASL(2,R)$ is completely reducible if and only if $\tau(L)$ does not coincide with $\langle T\rangle$ and $\langle D,T\rangle$.*

Proof. If $\tau(L)=0$, then L is a completely reducible algebra. If $\tau(L)=\langle D,T\rangle$, then $L=L_1\oplus L_2$, where $L_1\subset AO(n)$, $L_2=\langle X+D,T\rangle$, $X\in AO(n)$. Since the algebra L_2 is solvable and non-Abelian, then L is not a completely reducible algebra [12]. Let $\tau(L)=ASL(2,R)$. Since direct decomposition of $F\subset AO(n)$ can be realized through every ideal, and since every subalgebra of the algebra $AO(n)$ is not compact, then $L=\omega(L)+\tau(L)$. That is why [12] L is completely reducible.

Let us assume that $\tau(L)=\langle D\rangle$. Since $[D,P_a]=-P_a$, $[D,G_a]=G_a$, then $\overline{\mathfrak{M}}[1,n]$ can be decomposed into a direct sum of L-irreducible spaces. Consequently L is a completely reducible algebra.

As $[S+T, P_a] = G_a$ and $[S+T, G_a] = -P_a$, then the skew-symmetric matrix

$$\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$$

corresponds to the operator $S+T$ in a basis $P_1, \ldots, P_n$, $G_1, \ldots, G_n$ of the space $\overline{\mathfrak{M}}[1,n]$. Hence it follows that if $\tau(L) = \langle S+T \rangle$, then in the basis mentioned above every element of an algebra L is represented by a skew-symmetric matrix of degree $2n$, and that is why L is a completely reducible algebra.

Let $\tau(L) = \langle T \rangle$, and $V[k,l]$ be an irreducible $\omega(L)$ module. Evidently $V[k,l] + W[k,l]$ is an L module. Since by Lemma 4.2 of [9] this module can not be decomposed into a direct sum of irreducible L modules, an algebra L is not completely reducible. The theorem is proved.

4. The structure of splitting subalgebras of the Schrödinger algebra

The aim of this section is to study up to conjugation the subspaces of the space $\overline{\mathfrak{M}}[1,n]$ invariant under subalgebras of the algebra $AO(n) \oplus ASL(2,R)$. The main results are Theorems 4.1 and 4.2.

Let F be a subalgebra of $AO(n) \oplus ASL(2,R)$, and $\hat{F}$ be a subalgebra of the algebra $ASch(n)$ such that $\pi(\hat{F}) = F$. If algebra $\hat{F}$ is $Sch(n)$ conjugated to the algebra $F \in\!\!\!\!+ \mathfrak{N}$, where $\mathfrak{N}$ is an F-invariant subspace of the space $\overline{\mathfrak{M}}[1,n]$, then $\hat{F}$ is called a splitting in the algebra $ASch(n)$. The notion of a splitting subalgebra of the algebra $\widetilde{ASch}(n)$ is defined in an analogous way. If every subalgebra $\hat{F}$ is a splitting, we shall say that F has only splitting extensions in the algebra $ASch(n)$ (resp. in the algebra $\widetilde{ASch}(n)$).

We shall find all subalgebras F, which possess only splitting extensions.

Let

$$\begin{aligned}
&J(a,b) = J_{2a-1,2a} + \cdots + J_{2b-1,2b} \quad (a \le b),\\
&J(a) = J(a,a), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\\
&X = S + T + \alpha_1 J_{12} + \cdots + \alpha_t J_{2t-1,2t}, \quad 0 \le \alpha_1 \le \cdots \le \alpha_t,\\
&Y_{2a-1} = G_{2a-1} + P_{2a}, \quad Y_{2a} = G_{2a} - P_{2a-1}, \quad Z_{2a-1} = G_{2a-1} - P_{2a},\\
&Z_{2a} = G_{2a} + P_{2a-1}, \quad \mathfrak{L}_a = \langle Y_{2a-1}, Y_{2a} \rangle, \quad \mathfrak{N}_a = \langle Z_{2a-1}, Z_{2a} \rangle.
\end{aligned}$$

Obviously, $\mathfrak{L}_a + \mathfrak{N}_a = \overline{\mathfrak{M}}[2a-1, 2a]$. If $1 \le a \le t$, then

$$\begin{aligned}
&[X, Y_{2a-1}] = -(\alpha_{a-1}) Y_{2a}, \quad [X, Y_{2a}] = (\alpha_a - 1) Y_{2a-1},\\
&[X, Y_{2a-1}] = -(\alpha_a + 1) Z_{2a}, \quad [X, Z_{2a}] = (\alpha_a + 1) Z_{2a-1},
\end{aligned} \tag{2}$$

Thus $(\alpha_1 - 1)J$ is the matrix of ad X in the basis Y_{2a-1}, Y_{2a} of the space $\mathfrak{L}_a$, and $(\alpha_a + 1)J$ is the matrix of ad X in the basis Z_{2a-1}, Z_{2a} of the space $\mathfrak{N}_a$ $(1 \le a \le t)$. If $\alpha_a = 0$, we obtain a matrix corresponding to ad $(S+T)$.

Let $\alpha_a \ne 0$, $\alpha_a \ne 1$. The $\langle X \rangle$ module $\mathfrak{N}$ is called an elementary module of the first kind, and the $\langle X \rangle$ module $\mathfrak{N}_a$ is called an elementary module of the second kind. A subdirect sum of elementary modules of the first kind is called a module of the first kind, and a subdirect sum of elementary modules of the second kind is called a module of the second kind.

Lemma 4.1. *Let C be a matrix obtained from the identily matrix of degree n as a result of fulfilling a permutation over its columns*

$$\begin{pmatrix} 2k-1 & 2k & 2l-1 & 2l \\ 2l & 2l-1 & 2k & 2k-1 \end{pmatrix} \quad (k<l),$$

followed by the multiplication on (-1) columns which have number $2k$ and $2l$. The $O(n)$ automorphism φ of the algebra $ASch(n)$ which corresponds to the matrix C has the following properties:

$$\begin{aligned}
&(1) \quad \varphi(J_{2d-1,2d}) = J_{2d-1,2d}, \quad \textit{if} \;\; d \neq k,\; d \neq l, \\
& \quad \varphi(J_{2k-1,2k}) = J_{2l-1,2l}, \quad \varphi(J_{2l-1,2l}) = J_{2k-1,2k}; \\
&(2) \quad \varphi(G_{2k-1}) = G_{2l}, \quad \varphi(G_{2k}) = -G_{2l-1}, \\
& \quad \varphi(G_{2l-1}) = G_{2k}, \quad \varphi(G_{2l}) = -G_{2k-1}; \\
&(3) \quad \varphi(\mathfrak{L}_k) = \mathfrak{L}_l, \quad \varphi(\mathfrak{L}_l) = \mathfrak{L}_l, \\
& \quad \varphi(\mathfrak{N}_k) = \mathfrak{N}_l, \quad \varphi(\mathfrak{N}_l) = \mathfrak{N}_k.
\end{aligned}$$

Proof. For simplicity we can take $n = 4$ and

$$C = \begin{pmatrix} 0 & -J \\ -J & 0 \end{pmatrix}.$$

Then

$$C(\alpha_{12} + \beta J_{34})C^{-1} = \beta J_{12} + \alpha J_{34}, \quad C \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -y_4 \\ y_3 \\ -y_2 \\ y_1 \end{pmatrix}.$$

Using the last equality we conclude that $\varphi(G_1) = G_4$, $\varphi(G_2) = -G_3$, $\varphi(G_3) = G_2$, and $\varphi(G_4) = -G_1$. The lemma is proved.

Lemma 4.2. *Letting $n > 4$, $1 \le q \le [n/2] - 1$, and E_a be the identity matrix of degree a,*

$$C_1(\lambda) = \begin{pmatrix} \dfrac{1}{\sqrt{1+\lambda^2}} & \dfrac{\lambda}{\sqrt{1+\lambda^2}} \\ \dfrac{\lambda}{\sqrt{1+\lambda^2}} & \dfrac{-1}{\sqrt{1+\lambda^2}} \end{pmatrix} \oplus E_2,$$

$$C_1(\lambda) = \begin{pmatrix} \dfrac{1}{\sqrt{1+\lambda^2}} & 0 & \dfrac{\lambda}{\sqrt{1+\lambda^2}} \\ 0 & E_{k-1} & 0 \\ \dfrac{\lambda}{\sqrt{1+\lambda^2}} & 0 & \dfrac{-1}{\sqrt{1+\lambda^2}} \end{pmatrix} \oplus E_2 \quad (k \ge 2);$$

$$\begin{aligned}
&\Delta(1,k;\lambda) = \operatorname{diag}[C_k(\lambda), E_{n-2,(k+1)}], \quad \textit{if} \;\; 2(k+1) < n, \\
&\Delta(1,k;\lambda) = C_k(\lambda), \quad \textit{if} \;\; 2(k+1) = n; \\
&\Delta(q,k;\lambda) = \operatorname{diag}[E_{2q-2}, C_k(\lambda), E_{n-2(k+q)}], \quad \textit{if} \;\; q > 1, \;\; 2(k+q) < n, \\
&\Delta(q,k;\lambda) = \operatorname{diag}[E_{2q-2}, C_k(\lambda)], \quad \textit{if} \;\; q > 1, \;\; 2(k+q) = n;
\end{aligned}$$

and $\varphi(q,k;\lambda)$ is an $O(n)$ automorphism of the algebra $ASch(n)$ which corresponds to a matrix $\Delta(q,k;\lambda)$. Then

$$\varphi(q,k;\lambda)(J(q,q+k)) = J(q,q+k),$$
$$\varphi(q,k;\lambda)(G_{2q-1} + \lambda G_{2(q+k)-1}) = \sqrt{1+\lambda^2}\cdot G_{2q-1},$$
$$\varphi(q,k;\lambda)(G_{2q} + \lambda G_{2(q+k)}) = \sqrt{1+\lambda^2} G_{2q},$$

Proof. We may restrict ourselves only to the case when $n = 4$, $q = 1$, $k = 1$. Since

$$C_1(\lambda)\cdot\begin{pmatrix} X' & 0 \\ 0 & X' \end{pmatrix} = \begin{pmatrix} X' & 0 \\ 0 & X' \end{pmatrix}\cdot C_1(\lambda),$$

for every matrix X' of degree 2, and

$$C_1(\lambda)\cdot\begin{pmatrix} y_1 \\ y_2 \\ \lambda y_1 \\ \lambda y_2 \end{pmatrix} = \sqrt{1+\lambda^2}\begin{pmatrix} y_1 \\ y_2 \\ 0 \\ 0 \end{pmatrix}$$

then

$$\varphi(1,1;\lambda)(J(1,2)) = J(1,2),$$
$$\varphi(1,1;\lambda)(G_1 + \lambda G_3) = \sqrt{1+\lambda^2} G_1,$$
$$\varphi(1,1;\lambda)(G_2 + \lambda G_4) = \sqrt{1+\lambda^2} G_2,$$

The lemma is proved.

Proposition 4.1. *Let $X = S + T + \alpha J(k,l)$, where $\alpha > 0$, $\alpha \neq 1$. If U is an $\langle X\rangle$ submodule of the first (the second) king of the module $\overline{\mathfrak{M}}[2k-1,2l]$, then U is conjugated to the module*

$$\sum_{a=k}^{t}\mathfrak{L}_a\left(\sum_{a=k}^{t}\mathfrak{N}_a\right) \quad (t \le l).$$

Proof. Let us assume that U is a module of the first king. By Lemma 4.1 we shall suppose that a projection of U onto $\mathfrak{L}_k$ differs from 0. As

$$\exp(\theta J_{2a-1,2a})(\gamma_a Y_{2a-1} + \delta_a Y_{2a})\exp(-\theta J_{2a-1,2a}) =$$
$$= (\gamma_a\cos\theta + \delta_a\sin\theta)Y_{2a-1} + (\delta_a\cos\theta - \gamma_a\sin\theta)Y_{2a},$$

putting $\delta_a\cos\theta - \gamma_a\sin\theta = 0$, we may assume that if a projection of an element $Y \in U$ onto $\mathfrak{L}_a$ is equal to $\gamma_a Y_{2a-1} + \delta_a Y_{2a}$, then $\delta_a = 0$. Hence it follows that U has the element

$$Y = Y_{2k-1} + \lambda_{k+1}Y_{2k+1} + \cdots + \lambda_l Y_{2l-1} =$$
$$= (G_{2k-1} + \lambda_{k+1}G_{2k+1} + \cdots + \lambda_l G_{2l-1}) + (P_{2k} + \lambda_{k+1}P_{2k+2} + \cdots + \lambda_l P_{2l})$$

In view of Lemma 4.2, for some $O(n)$ automorphism $\varphi = \varphi(k,1;\mu_1)\cdot\varphi(k,2;\mu_2)\cdot\dots\cdot\varphi(k,l-k;\mu_{l-k})$ of the algebra $ASch(n)$ the following equalities hold true: $\varphi(X) = X$, $\varphi(Y) = \gamma(G_{2k-1} + P_{2k})$ ($\gamma \in R$, $\gamma \neq 0$). Therefore we may assume that $Y_{2k-1} \in U$. Then $Y_{2k} \in U$, and thus $\mathfrak{L}_k \subset U$. Using induction we conclude that $U = \sum\mathfrak{L}_a$.

The case when U is a module of the second king is treated similarly. The proposition is proved.

Theorem 4.1. *Let F be a subalgebra of the algebra $AO(n) \oplus ASL(2,R)$. Then F has only splitting extensions in $ASch(n)$ if and only if one of the following conditions is satisfied: (1) $D \in \tau(F)$; (2) $\tau(F) = \langle S+T \rangle$ and F is not conjugated to $\langle J_{12}+S+T \rangle \dot{+} K$, where K is a subalgebra of the algebra $\langle J_{ab} \mid a,b = 3,\ldots,n$; (3) $\tau(F) \subset \langle T \rangle$ and $\omega(F)$ is not conjugated to any subalgebra of the algebra $AO(n-1)$; or (4) $\tau(F) = 0$ and $\omega(F)$ is a semisimple algebra.*

Proof. Let $D \in \tau(F)$. If $\tau(F) = ASL(2,R)$, then by Theorem 3.1 F is a completely reducible algebra. Since in this case F annuls only zero subspace in $\overline{\mathfrak{M}}[1,n]$, then by Proposition 2.1 of [9] the algebra F has only splitting extensions in $ASch(n)$. If $\tau(F) = \langle D,T \rangle$, then $T \in F$. Algebra $F/\langle T \rangle$ acts completely reducible in $\overline{\mathfrak{M}}[1,n]$ and annuls only zero subspace in this space. From this, using Proposition 2.1 and Lemma 3.1, we conclude that F has only splitting extensions in $ASch(n)$. At the same time the case $\tau(F) = \langle D \rangle$ is considered.

Let $\tau(F) = \langle S+T \rangle$. If $S+T \in F$, then F annuls only zero subspace in $\overline{\mathfrak{M}}[1,n]$. Because of Theorem 3.1 the algebra F is completely reducible; then by Proposition 2.1 of [9] any algebra $\hat{F}$ in the algebra $ASch(n)$, then F contains $X = S+T+\alpha_1 J_{12} + \cdots + \alpha_t J_{2t-1,2t}$, where $0 < \alpha_1 \le \cdots \le \alpha_t$. We may assume that prolections of other basis elements of the algebra F onto $\langle S+T \rangle$ are equal to 0. In view of Proposition 2.1 of [9] the algebra F annuls in $\overline{\mathfrak{M}}[1,n]$ a certain nonzero subspace U. It follows from this and formula (2) that $U \subset \langle Y_1, Y_2, \ldots, Y_{2k} \rangle$ and $X = S+T+J(1,t)$ $(k \le t)$ or

$$X = S + T + J(1,k) + \beta_{k+1} J(k+1) + \cdots + \beta_t J(t) \quad (t > k),$$

where $\beta_{k+1} > 0$, …, $\beta_t > 0$, $\beta_{k+1} \neq 1$, …, $\beta_t \neq 1$. Arguing as in the proof of Proposition 4.1 we obtain that $Y_1 \in U$ up to conjugacy. Hence it follows that $F = \langle S+T+J_{12} \rangle \dot{+} K$, where $K \subset \langle J_{ab} \mid a,b = 3,\ldots,n \rangle$. By Lemma 2.1 of [9] the algebra $\hat{F}$ which is obtained from F by replacing $S+T+J_{12}$ by $S+T+J_{12}+Y_1$, is nonsplitting.

Let $\tau(F) = \langle T \rangle$, $F_1 = \omega(F)$, and $\hat{F}$ be a subalgebra of the algebra $ASch(n)$ such that $\pi(\hat{F}) = F$. If F_1 is not conjugated to a subalgebra of the algebra $AO(n-1)$, then by Proposition 2.1 and Lemma 3.1 of [9] an algebra $\hat{F}$ is splitting. If F_1 is conjugated to a subalgebra of the algebra $AO(n-1)$, then $F = \langle X \rangle \oplus F_2$, where $X \neq 0$, and $\langle X \rangle$ and F_2 are subalgebras of the algebra $AO(n-1) \oplus \langle T \rangle$. An algebra $F_2 \Subset \langle P_n X + G_n \rangle$ is nonsplitting.

The case $\tau(F) = 0$ is considered in [5, 7]. The theorem is proved.

Proposition 4.2. *A subalgebra F of the algebra $AO(n) \oplus ASL(2,R)$ possesses only splitable extensions in $\widetilde{ASch}(n)$ if and only if F is a semisimple algebra.*

The proof of Proposition 4.2 is similar to the proof of Theorem 4.1.

Let $\Gamma : X \to X$ be the trivial representation of a subalgebra F of the algebra $AO(n)$. Then Γ is $O(n)$ equivalent to $\operatorname{diag}[\Gamma_1,\ldots,\Gamma_m]$, where Γ_i is an irreducible subrepresentation $(i = 1,\ldots,m)$. It is well known that if representations Δ and Δ' of Lie algebra L by skew-symmetric matrices are equivalent over R, then $C\Delta(X)C^{-1} = \Delta'(X)$ for some orthogonal matrix C $(X \in L)$, hence we conclude that if Γ_i and Γ_j

are equivalent representations, then we can assume that for every $X \in F$ the equality $\Gamma_i(X) = \Gamma_j(X)$ takes place. Uniting equivalent nonzero irreducible subrepresentations we shall get nonzero disjunctive primary subrepresentations $\Delta_1, \ldots, \Delta_q$ of the representation Γ. An algebra

$$K_i = \{\mathrm{diag}\,[0, \ldots, \Delta_i(X), \ldots, 0] \mid X \in F\} \quad (1 \le i \le q)$$

is called a primary part of the algebra F. Evidently F is a subdirect sum of its primary parts.

We shall say that the splitting subalgebra F of the algebra $ASch(n)$ or of the algebra $\widetilde{ASch}(n)$ has a splitting factor algebra in the case $\pi(\hat{F}) = F_1 \oplus F_2$, where $F_1 \subset AO(n)$, $F_2 \subset ASL(2, R)$. If this condition does not hold, then the factor algebra $\pi(\hat{F})$ of an algebra $\hat{F}$ is called nonsplitting.

Theorem 4.2. *Let $K_1, K_2, \ldots, K_q$ be primary parts of the nonzero subalgebra L' of the algebra $AO(n)$, L'' be a subalgebra of the algebra $ASL(2, R)$ differing from $\langle S + T\rangle$, and L be a subdirect sum of L' and L''. If U is a subspace of $\overline{\mathfrak{M}}[1, n]$, being invariant under L, then $U = U_1 \oplus \cdots \oplus U_q \oplus \tilde{U}$, where $U_i = [K_i, U] = [K_i, U_i]$; $[L'', U_i] \subset U_i$; $[K_j, U_i] = 0$ in the case $j \ne i$; $[K_i, \tilde{U}] = 0$, $[L'', \tilde{U}] \subset \tilde{U}$ $(i, j = 1, \ldots, q)$.*

Proof. If $L'' = ASL(2, R)$, then $L'' \subset L$. Therefore from $[L, U] \subset U$ it follows that $[L'', U] \subset U$. Since $\overline{\mathfrak{M}}[a]$ is invariant under $ASL(2, R)$ for any a, $1 \le a \le n$, then each subspace $U_i = [K_i, U]$ is invariant under this algebra. Let $\tilde{U}$ be a maximal subspace of the space U annulled by L', $U' = [L', U]$. Since L' is a completely reducible algebra, $U = U' \oplus \tilde{U}$ and $[L', U'] = U'$. Applying Lemma 3.1 of [9] we conclude that $U' = U_1 \oplus \cdots \oplus U_q$, where $U_i = [K_i, U] = [K_i, U_i]$ $(i = 1, \ldots, q)$.

Let $L'' = \langle T, D\rangle$. Since $\langle T, D\rangle$ is a non-Abelian solvable algebra and every subalgebra of the algebra $AO(n)$ is reductive, then applying the Goursat twist method [11] we obtain that $T \in L$. Therefore it is enough to consider the case $L'' = \langle D\rangle$. By Lemma 4.2 of [9], $[D, U] \subset U$, it follows that $[D, U_i] \subset U_i$, $[D, \tilde{U}] \subset \tilde{U}$ $(i = 1, \ldots, q)$.

The case $L'' = \langle T\rangle$ is considered in [5, 7]. The theorem is proved.

Because of Theorem 4.2, the study of splitting subalgebras $\hat{F}$ of the algebra $ASch(n)$, for which $\tau(\hat{F}) \ne \langle S+T\rangle$, is reduced to the study of splitting subalgebras $\hat{K}$ of the algebra $ASch(n)$ having the splitting factor algebra $\pi(\hat{K})$ and zero or primary projection onto $AO(n)$. Such subalgebras have been described in [13].

Proposition 4.3. *Nonzero subspaces of the space $\overline{\mathfrak{M}}[1, n]$ invariant under $\langle S + T\rangle$ are exhausted with respect to $O(n)$ conjugalion by the following spaces: $\overline{\mathfrak{M}}[1, d]$ $(d = 1, \ldots, n)$; U_q $(q = 1, \ldots, [n/2])$; $U_m + \overline{\mathfrak{M}}[2m + 1, t]$ $(m = 1, \ldots, [(n - 1)/2];$ $t = 2m + 1, \ldots, n)$, where U_q is a subdirect sum of $V[1, 2q]$ and $W[1, 2q]$ having zero intersections with there spaces. If*

$$\{G_j + \alpha_{1j}P_1 + \cdots + \alpha_{2q,j}P_{2q} \mid j = 1, \ldots, 2q\}$$

is a basis of U_q, then with respect to $O(2q)$ conjugation a matrix (α_{kj}) $(k, j = 1, \ldots, 2q)$ coincides with $\mathrm{diag}\,[\Gamma(\lambda_1), \ldots, \Gamma(\lambda_q)]$, where $0 < \lambda_1 \le \cdots \le \lambda_q \le 1$ and

$$\Gamma(\lambda) = \begin{pmatrix} 0 & \lambda \\ -\lambda^{-1} & 0 \end{pmatrix}.$$

The numbers $\lambda_1, \ldots, \lambda_q$ are defined by the space U_q uniquely.

Proposition 4.3 is proved along with Proposition 2.4 and Theorem 3.4 in [13].

Proposition 4.4. *Let*

$$\Lambda_b(a) = \langle P_1 + \lambda_1 P_{a+1}, \ldots, P_b + \lambda_b P_{a+b}\rangle,$$

where $0 < \lambda_1 \leq \cdots \leq \lambda_b$, $b \leq a$, $a + b \leq n$. *A subalgebra* $\hat{F}$ *of the algebra* $ASch(n)$ *such that* $\omega(\hat{F}) = 0$, $D \in \tau(\hat{F})$, *is* $Sch(n)$ *conjugated to* $L \Subset U$, *where* $L \subset ASL(2,R)$ *and* U *is a subspace of the space* $\overline{\mathfrak{M}}[1,n]$. *Let* $U \neq 0$. *If* $L = ASL(2,R)$, *then* U *is conjugated to* $\overline{\mathfrak{M}}[1,d]$ $(1 \leq d \leq n)$. *If* $L = \langle D, T\rangle$, *then* U *is conjugated to one following spaces* $W[1,d]$, $\overline{\mathfrak{M}}[1,d]$, $(1 \leq d \leq n)$; $V[1,d] + W[1,t]$ $(1 \leq d \leq n-1,\ d+1 \leq t \leq n)$. *If* $L = \langle D\rangle$, *then* U *is conjugated to one of the following spaces:*

$W[1,d]$, $\overline{\mathfrak{M}}[1,d]$ $(1 \leq d \leq n)$; $V[1,d]+W[d+1,d+t]$ $(1 \leq d \leq [n/2];\ d \leq t \leq n-d)$;

$V[1,d] + W[1,c] + W[d+1,d+t]$ $(1 \leq d \leq n-1;\ 1 \leq c \leq d;\ d-c \leq t \leq n-d$, if $c \neq d$; $1 \leq t \leq n-d$, if $c = d)$;

$V[1,d] + \Lambda_d(d)$ $(1 \leq d[n/2])$; $V[1,d] + \Lambda_t(d) + W[t+1,d]$ $(2 \leq d \leq n-1;\ 1 \leq t \leq \min\{d-1, n-d\})$;

$V[1,d] + \Lambda_t(d) + W[d+t+1, b+t+s] + W[d+t+1, d+t+s]$ $(1 \leq d \leq [n/2];\ 1 \leq t \leq \min\{d, n-d-1\};\ 1 \leq s \leq n-d-t;\ s+t \geq d)$;

$V[1,d] + \Lambda_t(d) + W[t+1,b] + W[d+t+1, d+t+s]$ $(2 \leq d \leq n-2;\ 1 \leq t \leq \min\{d-1, n-d-1\};\ t+1 \leq b \leq d;\ 1 \leq s \leq n-d-t; b+s \geq d)$.

The proof of Proposition 4.4 is similar to the proof of Theorem 3.3 [13].

Using Theorem 4.2 to investigte splitting subalgebras wilh nonsplitting factor algebra, it is enough to consider the algebras $\hat{F} \subset ASch(n)$ for which $\tau(\hat{F}) = \langle S+T\rangle$ and $\tau(\hat{F}) \not\subset \hat{F}$. In this case $\pi(\hat{F}) = F' \oplus \langle X\rangle$, where F' is a subalgebra of $AO(n)$ and $X = S + T + \alpha_i J_{12} + \cdots + \alpha_k J_{2k-1,2k}$. We may suppose that $0 < \alpha_1 \leq \cdots \leq \alpha_k$. Henceforth we shall discuss subspaces of the space $\overline{\mathfrak{M}}[1,n]$ that are invariainl under X.

Lemma 4.3. *Let* $1 \leq a$, $b \leq k$. *Then* $\mathfrak{L}_a \cong \mathfrak{L}_b$ *if and only if* $\alpha_a = \alpha_b$ *or* $\alpha_a + \alpha_b = 2$; $\mathfrak{N}_a \cong \mathfrak{N}_b$ *if and only if* $\alpha_a = \alpha_b$; $\mathfrak{L}_a \cong \mathfrak{N}_b$ *if and only if* $\alpha_a = 2 + \alpha_b$ $(a \neq b)$. *Modules* $\mathfrak{L}_a$ *and* $\mathfrak{N}_a$ *are not isomorphic.*

Proof. The matrices λJ, μJ are similar if and only if $\lambda^2 = \mu^2$. It follows thal $\mathfrak{L}_a \cong \mathfrak{L}_b$ if and only if $(\alpha_a - 1)^2 = (\alpha_b - 1)^2$. In the case $\alpha_a - \alpha_b \neq 0$, $\alpha_a + \alpha_b = 2$.

If $\mathfrak{N}_a \cong \mathfrak{N}_b$ then $(\alpha_a + 1)^2 = (\alpha_b + 1)^2$, whence $2(\alpha_a - \alpha_b) = -(\alpha_a - \alpha_b)(\alpha_a + \alpha_b)$. In the case $\alpha_a - \alpha_b \neq 0$, $2 = -(\alpha_a + \alpha_b)$. But this contradicts the fact that $\alpha_a, \alpha_b > 0$.

Let $\mathfrak{L}_a \cong \mathfrak{N}_b$. Then $(\alpha_a - 1)^2 = (\alpha_b + 1)^2$, whence $2(\alpha_a + \alpha_b) = (\alpha_a - \alpha_b)(\alpha_a + \alpha_b)$. Thus if $\alpha \neq 0$, then $\alpha_a - \alpha_b = 2$. The lemma is proved.

Let us remark that if $\alpha_a \neq 1$, then the $\langle X\rangle$ modules $\mathfrak{L}_a$ and $\mathfrak{N}_a$ are irreducible, and any $\langle X\rangle$ submodule of the module $\overline{\mathfrak{M}}[1,n]$ is completely reducible.

Proposition 4.5. *Let*

$$X = S + T + \sum_{i=1}^{s} \beta_i J(k_{i-1}+1, k),$$

where $s \geq 2$, $k_0 = 0$, $\beta_i > 0$, $\beta_i \neq 1$, $\beta_i \neq \beta_j$ *if* $i \neq j$. *There exists an indecomposable* $\langle X\rangle$ *submodule with nonzero projections onto* $\overline{\mathfrak{M}}[1,2k_1]$, $\overline{\mathfrak{M}}[2k_1 + 1, 2k_2]$, ..., $\overline{\mathfrak{M}}[2k_{s-1}+1, 2k_s]$ *of the* $\langle X\rangle$ *module* $\overline{\mathfrak{M}}[1,2k_s]$ *if and only if* $s = 2$ *and* *one of the following conditions is satisfied:* (i) $\beta_1 = 2 + \beta_2$; (ii) $\beta_2 = 2 + \beta_1$; (iii) $\beta_1 + \beta_2 = 2$. *If U is ademanded indecomposable $\langle X\rangle$ module and U, is the projection of U onto $\overline{\mathfrak{M}}[2k_{i-1}+1, 2k_i]$ $(i = 1, 2)$, then in case (i) U_1 is a module of the first kind and U_2 is a module of the second kind; in the case (ii) U_1 is a module of the second kind; and U_2 is a module of the first kind; and in cas (iii) U_1 and U_2 are modules of the first kind.*

Proof. By Lemma 3.1 of [9], in the $\langle X\rangle$ module $\overline{\mathfrak{M}}[1, 2k_s]$, there exists an indecomposable submodule demanded if and only if the $\langle X\rangle$ modules $\overline{\mathfrak{M}}[1, 2k_1]$, $\overline{\mathfrak{M}}[2k_1+1, 2k_2]$, $\ldots$, $\overline{\mathfrak{M}}[2k_{s-1}, +1, 2k_s]$, have isomorphic composition factors. If $\mathfrak{L}_{k_i} \cong \mathfrak{L}_{k_j}$, and $\mathfrak{L}_{k_j} \cong \mathfrak{L}_{k_r}$ then by Lemma 4.3 $\beta_i + \beta_j = 2$ and $\beta_j + \beta_r = 2$. From this it follows that $\beta_i = \beta_r$ and that is why $i = r$. If $\mathfrak{L}_{k_i} \cong \mathfrak{N}_{k_j}$ and $\mathfrak{N}_{k_j} \cong \mathfrak{N}_{k_r}$, then $\beta_i = 2 + \beta_j$ and $\beta_r = 2 + \beta_j$, whence $i = r$. Thus $s \le 2$ and one of the following conditions is satisfied: (1) $\beta_1 = 2 + \beta_2$; (2) $\beta_2 = 2 + \beta_1$; (3) $0 < \beta_1 < 2$, $\beta_2 = 2 - \beta_1$. Statements about the kinds of projections follow from Lemma 4.3. The proposition is proved.

Proposition 4.6. *Let $X = S + T + \beta J(1, k)$ $(\beta > 0)$. In the $\langle X\rangle$ module $\overline{\mathfrak{M}}[1, n]$ there exists an indecomposable $\langle X\rangle$ submodule with nonzero projections onto $\overline{\mathfrak{M}}[1, 2k]$ and $\overline{\mathfrak{M}}[2k+1, n]$ if and only if $\beta = 2$. If U is such a submodule and U_1 is the projection U onto $\overline{\mathfrak{M}}[1, 2k]$, then U_1 is a module of the first kind.*

5. Abelian subalgebras of the extended Schrödinger algebra

The main results of this section are Theorem 5.1 and its two corollaries.

Let us use the following notation:

$$X_t = \alpha_1 J_{12} + \alpha_2 J_{34} + \cdots + \alpha_t J_{2t-1,2t},$$

where $\alpha_1 = 1$, $0 < \alpha_2 \le \cdots \le \alpha_t \le 1$ if $t \ge 2$;

$$AH(0) = 0, \quad AH(2d) = AH(2d+1) = \langle J_{12}, J_{34}, \ldots, J_{2d-1,2d}\rangle;$$
$$\Delta_0[r, t] = \langle G_r + \alpha_r P_r, \ldots, G_t + \alpha_t P_t\rangle, \quad \Delta[r, t] = \Delta_0[r, t] + \langle M\rangle,$$

where $r \le t$, $\alpha_r \le \cdots \le \alpha_t$, $\alpha_r = 0$, and $\alpha_t = 1$ if $\alpha_t \ne 0$;

$$\Pi(a, b) = \langle Y_{2a-1}, Y_{2a+1}, \ldots, Y_{2b-1}\rangle \quad (a \le b).$$

We recall that $Y_{2c-1} = G_{2c-1} + P_{2c}$ and $Y_{2c} = G_{2c} - P_{2c-1}$.

The algebra $AH(n)$ is a maximal Abelian subalgebra of the algebra $AO(n)$. It is well known that any maximal Abelian subalgebra of the algebra $AO(n)$ is conjugated $AH(n)$ with respect to inner automorphisms of the algebra $AO(n)$. Henceforth when speaking about Abelian subalgebras of the algebra $AO(n)$ we shall mean subalgebras of the algebra $AH(n)$.

Lemma 5.1. *Let L be an Abelian subalgebra of the algebra $\langle J(a, b) + S + T\rangle \subset +\mathfrak{M}[2a-1, 2b]$ such that its projection onto $\langle J(a, b) + S + T\rangle$ is nonzero and its projection onto $\langle M\rangle$ is equal to 0. Then L is conjugated to one of the following algebras:*

$$\langle J(a, b) + S + T + \alpha Y_{2b-1}\rangle \quad (\alpha \ge 0);$$
$$\Pi(a, c) \oplus \langle J(a, b) + S + T + \alpha Y_{2b-1}\rangle \quad (\alpha \ge 0,\ c \le b).$$

The written algebras are pairwise nonconjugated.

Proof. The maximal subspace of the space $\mathfrak{M}[2a-1, 2b]$ anulled by $\langle J(a,b)+S+T\rangle$ and having zero projection onto $\langle M\rangle$ coincides with

$$\sum_{c=a}^{b} \mathfrak{L}_c.$$

Let $U = L \cap \mathfrak{M}[2a-1, 2b]$. By the same arguments as in the proof of Proposition 4.1 we can establish that if $U \neq 0$, then U contains Y_{2a-1}. As $[Y_{2a-1}, Y_{2a}] = -2M$, so $U = \langle Y_{2a-1}\rangle + U^1$, where U^1 is a subspace of the space

$$\sum_{c=a+1}^{b} \mathfrak{L}_c.$$

Continuing these arguments we obtain that $U = \Pi(a,c)$ $(c \leq b)$ and L contains $J(a,b)+S+T+\alpha Y_{2b-1}$ $(\alpha \geq 0)$. The lemma is proved.

Theorem 5.1. *Let L be a nonzero Abelian subalgebra of algebra $A\widetilde{Sch}(n)$. If $\tau(L) = \langle D\rangle$, then L is conjugated to the subdirect sum $L_1 \dot{+} L_2 \dot{+} L_3$ of algebras L_1, L_2, L_3, where $L_1 \subset AH(2d)$, $L_2 = \langle D\rangle$, $L_3 \subset \langle M\rangle$ $(0 \leq d \leq [n/2]$. If $\tau(L) = \langle T\rangle$, then L is conjugated to $L_1 \dot{+} L_2 \dot{+} L_3 \dot{+} L_4$, where $L_1 \subset AH(2d)$, $L_2 = \langle T + \alpha G_{2d+1}\rangle$ $(\alpha \in 0, 1)$, $L_3 = 0$ or $L_3 = W[r,t]$, $L_4 \subset \langle M\rangle$ $(0 \leq d \leq [n/2]$; $r = 2d+1$ if $\alpha = 0$, $2d+1 \leq n$; $r = 2d+2$ if $\alpha = 1$, $2d+2 \leq n)$; if $\tau(L) = \langle S+T\rangle$, then L is conjugated to $L_1 \dot{+} L_2 \dot{+} L_3 \dot{+} L_4$, where $L_1 \subset \langle M\rangle$, $L_2 \subset AH(2d)$ $(0 \leq d \leq [n/2])$, and the algebras L_3 and L_4 satisfy one of the following conditions:*

(1) $L_3 = \langle S+T\rangle$, $\quad L_4 = 0$;

(2) $L_3 = \langle J(d+1,t)+S+T+\alpha Y_{2t-1}\rangle$, $\quad L_4 = 0$ $\ (\alpha > 0)$;

(3) $L_3 = \langle J(d+1,t)+S+T+\alpha Y_{2t-1}\rangle$, $\quad L_4 = \Pi(d+1,s)$ $\ (s \leq t;\ \alpha \geq 0)$.

If $L \subset A\tilde{G}^0(n)$, then L is conjugated to $L_1 \dot{+} L_2 \dot{+} L_3 \dot{+} L_4$, where $L_1 \subset AH(2d)$, $L_2 = 0$ or $L_2 = \Delta_0[2d+1, s]$, $L_3 = 0$ or $L_3 = W[k,l]$, $L_4 = 0$ or $L_4 = \langle M\rangle$ $(0 \leq d \leq [n/2]$; $k = s+1$ if $L_2 \neq 0$; $k = 2d+1$ if $L_2 = 0$; $l \leq n)$.

Proof. If $\tau(L) = \langle D\rangle$, then by Theorem 4.1 the algebra L is conjugated to the algebra $U+F$, where $U \subset \mathfrak{M}[1,n]$ and $F \subset AH(n) \oplus \langle D, M\rangle$. Since D annuls only $\langle M\rangle$ in $\mathfrak{M}[1,n]$ and by Theorem 4.2 $[D,U] \subset U$, then $U \subset \langle M\rangle$. Thus L is conjugated to some subalgebra of the algebra $AH(2d) \oplus \langle D, M\rangle$ $(0 \leq d \leq [n/2])$.

If $\tau(L) = \langle T\rangle$, then by Theorem 4.1 the algebra L is conjugated to the algebra $U+F$ satisfying one of the following conditions: $U \subset \mathfrak{M}[1,n]$ and F is a subalgebra of $AH(n) + \langle M, T\rangle$; or $U \subset \mathfrak{M}[1,2d]$ and F is a subalgebra of $AH(2d) + \mathfrak{M}[2d+1, n] + \langle T\rangle$ $(d \geq 1)$. Let us consider the last case. Let us suppose that the projection K of the algebra F onto $AO(n)$ is not conjugated to any subalgebra of the algebra $AH(2d-2)$. Since K annuls only the zero subspace of $V[1,2d]$, then $U \subset \langle M\rangle$. Therefore we shall assume that $U = 0$. As $[T, G_a] = -P_a$, so by Witt's mapping theorem [14] $\epsilon(F) = 0$, or $\epsilon(F) = \langle G_{2d+1}\rangle$. Since

$$\exp(\theta T)(T + \alpha G_{2d+1} + \beta P_{2d+1})\exp(-\theta T) = T + \alpha G_{2d+1} + (\beta - \theta\alpha)P_{2d+1}$$

and

$$\exp(\lambda D)(T+\alpha G_{2d+1})\exp(-\lambda D)=\exp(-2\lambda)(T+\alpha\exp(3\lambda)\cdot G_{2d+1}),$$

then if $\epsilon(F)\neq 0$, the projection of F onto $\langle T\rangle \mathbin{\oplus\!\!\!\!+} \mathfrak{M}[2d+1,n]$ contains $T+G_{2d+1}$. In this case, by Witt's theorem $\xi(F)$ coincides with 0 or $W[2d+2,t]$. If $\epsilon(F)=0$, then $\xi(F)=0$ or $\xi(F)=W[2d+1,t]$.

Let $\tau(L)=\langle S+T\rangle$. If $S+T\in L$ then $\epsilon(L)=0$ and $\xi(L)=0$. If $S+T\notin L$, then an algebra L contains

$$Y=S+T+\sum_{a=1}^{[n/2]}\alpha_a J_{2a-1,2a}+\sum_{i=1}^{n}(\beta_i G_i+\gamma_i P_i)+\delta M.$$

We shall suppose that projections of the at rest basis elements of the algebra L onto $\langle S+T\rangle$ are equal to zero, and $\alpha_a\geq 0$ for all a. If $\alpha_c\neq 1$, then $\langle S+T+\alpha_c J_{2c-1,2c}\rangle$ is a completely reducible algebra of linear iransformalions of the vector space $\mathfrak{M}[2c-1,2c]$ and annuls only the zero subspace of this space, whence by Proposition 2.1 of [9] we conclude that the projection of L onto $\mathfrak{M}[2c-1,2c]$ is equal to zero. Therefore we may assume that

$$Y=J(d+1,t)+S+T+\sum_{i=2d+1}^{2t}(\beta_i G_i+\gamma_i P_i).$$

From Proposition 2.1 of [9] it also follows that

$$\sum_{i=2d+1}^{2t}(\beta_i G_i+\gamma_i P_i)\in\sum_{j=d+1}^{i}\mathfrak{L}_j.$$

Applying Theorem 4.1 and Lemma 5.1 we conclude that, with respect to the conjugation $\omega(L)\subset AH(2d)+\langle J(d+1,t)\rangle$,

$$Y=J(d+1,t)+S+T+\alpha Y_{2t-1},$$

and $L\cap\overline{\mathfrak{M}}[1,n]=0$ or $L\cap\overline{\mathfrak{M}}[1,n]=\Pi(d+1,s)$ $(\alpha\geq 0;\ s\leq t)$.

Let us assume thet $L\subset A\tilde{G}^0(n)$. By Theorem 2 of [7] the algebra L is conjugated to an algebra $U+F$, which statisfies one of the following conditions: $U\subset\mathfrak{M}[1,n]$ and F is a subalgebra of $AH(n)+\langle M\rangle$; or $U\subset\mathfrak{M}[1,2d]$ and F is a subalgebra of $AH(2d)+\mathfrak{M}[2d+1,n]$ $(1\leq d\leq[n-1/2])$. Let us restrict ourselves to the last case. Let the projection K of the algebra $AH(2d-2)$. Since K annuls only the zero subspace of the space $\overline{\mathfrak{M}}[1,2d]$, $U\subset\langle M\rangle$. Therefore we suppose that $U=0$.

Let N be the projection of F onto $\mathfrak{M}[2d+1,n]$ and $\epsilon(N)=V[2d+1,2d+q]$. By Witt's mapping theorem the algebra N is a subdirect sum of the algebras N_1, N_2, N_3, where $N_1\subset\overline{\mathfrak{M}}[2d+1,2d+q]$ (as a space), $N_2=0$ or $N_2=W[2d+q+1,t]$, and $N_3\subset\langle M\rangle$. Let

$$Z_i=G_1+\beta_{2d+1,i}P_{2d+1}+\cdots+\beta_{2d+q,i}P_{2d+q}\quad (i=(2d+1),\ldots,(2d+q)),$$

$N_q=\langle Z_{2d+1},\ldots,Z_{2d+q}\rangle$. Evidently $[Z_i,Z_j]=(\beta_{ij}-\beta_{ji})M$. Since N_1 is an Abelian algebra, $\beta_{ij}=\beta_{ji}$. Hence it foliows that the matrix $B=(\beta_{ij})$ is symmetric. Therefore there exists a matrix $Q\in O(q)$ such that $QBQ^{-1}=\operatorname{diag}[\lambda_1,\ldots,\lambda_q]$. From this it follows that with respect to automorphisms from the group $O(2d)\times O(q)\times O(n-2d-q)$ we may assume that $Z_{2d+j} = G_{2d+j} + \lambda_j P_{2d+j}$ $(j = 1, \ldots, q)$, where $\lambda_1 \leq \cdots \leq \lambda_q$. Applying the automorphism $\exp(\lambda_1 T)$ we obtain the generators $G_{2d+j} + \mu_j P_{2d+j}$ $(j = 1, \ldots, q)$, where $\mu_1 = 0$, $0 \leq \mu_2 \leq \cdots \leq \mu_q$. If $\mu_q > 0$, then $\mu_q = \exp(-2\theta)$. Obviously

$$\exp(\theta D)(G_{2d+j} + \mu_j P_{2d+j}) \exp(-\theta D) = \exp\theta(G_{2d+j} + \mu_j \exp(-2\theta) P_{2d+j}).$$

Therefore if $\mu_q > 0$, we may suppose that $\mu_q = 1$. This proves that the algebra N_1 is conjugated to $\Delta_0[2d+1, 2d+1, 2d+q]$. The theorem is proved.

Corollary 1. *The maximal Abelian subalgebras of the algebra $A\widetilde{Sch}(n)$ are exhausted with respect to the $\widetilde{Sch}(n)$ conjugation by the following algebras:*

$AH(n) \oplus \langle T, M\rangle$ $(n \equiv 0 \pmod 2)$; $AH(n) \oplus \langle S+T, M\rangle$;
$AH(n) \oplus \langle D, M\rangle$; $AH(n-1) \oplus \langle G_n + T, M\rangle$; $(n \equiv 1 (\mathrm{mod}\ 2))$;
$AH(2d) \oplus \Delta[2d+1, n]$ $(d = 0, 1 \ldots, [(n-1)/2])$;
$AH(2d) \oplus \Delta[2d+1, t] \oplus W[t+1, n]$ $(d = 0, 1, \ldots, [(n-2)/2]$; $t = 2d+1, \ldots, n-1)$;
$AH(2d) \oplus \langle T, M\rangle \oplus W[2d+1, n]$ $(d = 0, 1, \ldots, [(n-1)/2])$;
$AH(2d) \oplus \langle G_{2d+1} + T\rangle \oplus W[2d+2, n] \oplus \langle M\rangle$ $(d = 0, 1, \ldots, [(n-2)/2])$;
$AH(2d) \oplus \langle J(d+1, r) + S + T\rangle \oplus \langle M\rangle \oplus \Pi(d+1, r)$ $(d = 0, 1, \ldots, [(n-2)/2])$; $r = d+1, \ldots, [n/2])$.

Corollary 2. *Let $\alpha, \beta \in R$, $\alpha > 0$, $\beta > 0$; $t = 1, \ldots, [(n-1)/2]$; $n \geq 3$. One-dimensional subalgebras of the algebra $A\widetilde{Sch}(n)$ are exhausted with respect to the $\widetilde{Sch}(n)$ conjugation by the following algebras:*

$\langle D\rangle$; $\langle T\rangle$; $\langle S+T\rangle$; $\langle M\rangle$; $\langle D + \alpha M\rangle$; $\langle T \pm M\rangle$; $\langle S + T \pm \alpha M\rangle$; $\langle P_1\rangle$;
$\langle G_1 + P_2\rangle$; $\langle G_1 + T\rangle$; $\langle X_t\rangle$; $\langle X_t + \alpha D\rangle$; $\langle X_t + \alpha D + \beta M\rangle$; $\langle X_t + T\rangle$;
$\langle X_t + \alpha(S+T)\rangle$; $\langle X_t + \alpha M\rangle$; $\langle X_t + \alpha(S+T) \pm \beta M\rangle$; $\langle X_s + P_{2s+1}\rangle$;
$\langle X_r + G_{2r+1} + \alpha P_{2r+2}\rangle$ $(r = 1, \ldots, [(n-2)/2])$; $\langle X_s + T + \alpha G_{2s+1}\rangle$;
$\langle X_t + S + T + \alpha(G_1 + P_2)\rangle$.

Remark. One-dimensional subalgebras of the algebra $A\widetilde{Sch}(n)$ are exhausted with respect to the $\widetilde{Sch}(n)$ conjugation by one-dimensional subalgebras, of the algebra $A\widetilde{Sch}(n)$ whose generators do not contain λM as an addend ($\lambda \neq 0$.

Theorem 5.2. *Let L be a nonzero Abelian subalgebra of the algebra $ASch(n)$. If $\tau(L) = \langle D\rangle$, then L is conjugated to a subdirect sum of $\langle D\rangle$ and the subalgebra of the algebra $AH(2d)$ $(0 \leq d \leq [n/2])$. If $\tau(L) = \langle T\rangle$, then L is conjugated to $L_1 \dot{+} L_2 \dot{+} L_3$, where $L_1 \subset AH(2d)$, $L_2 = \langle T + \alpha G_{2d+1}\rangle$, and L_3 is one of the following algebras:*

0; $W[2d+2, t]$; $\langle P_{2d+1} + \lambda P_{2d+2}\rangle + \gamma W[2d+2] + \delta W[2d+3, t]$ $(0 \leq d \leq [n/2]$; $t \leq n$; $\alpha, \gamma, \delta \in \{0, 1\}$; $\lambda \leq 0)$.

If $\tau(L) = \langle S+T\rangle$, then L is conjugated to $L_1 \dot{+} L_2 \dot{+} L_3$, where $L_1 \subset AH(2d)$ $(0 \leq d \leq [n/2])$ and the algebras L_2, L_3 satisfy one of the following conditions: (1) $L_2 = \langle S+T\rangle$ and $L_3 = 0$; or (2) $L_2 = \langle J(d+1, t) + S + T + \alpha Y_{2t-1}\rangle$ $(\alpha \geq 0)$ and L_3 is a subalgebra of the algebra

$$\sum_{a=d+1}^{t} \mathfrak{L}_a$$

If $L \subset AG^0(n)$*, then* L *is conjugated to* $L_1 \dot{+} L_2$*, where* $L_1 \subset AH(2d)$ *and* $L_2 \subset \overline{\mathfrak{M}}[2d+1,n]$ $(0 \le d \le [n/2])$.

The theorem is proved along the same lines as Theorem 5.1.

Corollary. *The maximal Abelian subalgebras of the algebra* $ASch(n)$ *are exhausted with respect to the* $Sch(n)$ *conjugation by the following algebras:*

$AH(n) \oplus \langle D\rangle$; $AH(n) \oplus \langle S+T\rangle$; $AH(n)\oplus\langle T\rangle$ $[n \equiv 0 \pmod 2]$;
$AH(2d) \oplus \langle T\rangle \oplus W[2d+1,n]$ $(d = 0,1,\ldots,[(n-1)/2])$;
$AH(2d) \oplus \overline{\mathfrak{M}}[2d+1,n]$ $(d = 0,1,\ldots,[(n-1)/2])$;
$AH(2d) \oplus \langle G_{2d+1}+T\rangle + W[2d+1,n]$ $(d = 0,1,\ldots[(n-1)/2])$;
$AH(2d) \oplus \langle J(d+1,r)+S+T\rangle \oplus \sum\limits_{a=d+1}^{r} \mathfrak{L}_a$ $(d = 0,1,\ldots[(n-2)/2];\ r = d+1,\ldots,[n/2])$.

6. Classification of subalgebras of the algebras $ASCh(3)$ and $A\widetilde{Sch}(3)$

In this section we make use of the previous results to provide a classification of all subalgebras of the algebras $ASch(3)$ and $A\widetilde{Sch}(3)$.

Let $A\tilde{G}(3) = (AO(3)\oplus\langle T\rangle) \Subset \mathfrak{M}[1,3]$ and $AG(3) = A\tilde{G}(3)/\langle M\rangle$. Subalgebras of the algebras $AG(3)$ and $A\tilde{G}(3)$ were classified up to conjugacy under $G(3)$ and $\tilde{G}(3)$, respectively, in [5]. Further simplification of these subalgebras is being realized by $SL(2,R)$ automorphisms.

Theorem 6.1. *Let* $\alpha,\beta,\gamma,\lambda,\mu \in R$*, and* $\alpha > 0$*,* $\beta > 0$*,* $\gamma \ne 0$*. The splitting subalgebras of the algebra* $AG(3)$ *are exhausted with respect to* $Sch(3)$ *conjugation by the splitting subalgebras of the algebra* $AG(2)$ *(see [2]) and by the following algebras (the subalgebras preceded by the sign* $\sim$ *are subalgebras of* $ASch(3)$*):*

$\sim\langle G_1+P_2,P_3\rangle$; $\langle G_1+P_2,P_1+\alpha P_3\rangle$; $\sim\langle G_1+\gamma P_1+P_3,G_2+\alpha P_3\rangle$;
$\langle G_1+\lambda P_1+P_3,G_2+\gamma P_1+\alpha P_3\rangle$; $\langle G_1+\lambda P_1+P_3,G_2+\alpha P_1\rangle$;
$\langle G_1+P_2+\alpha P_3,G_2-P_1+\beta P_2+\lambda P_3\rangle$; $\langle G_1+P_2+\alpha P_3,G_2-P_1\rangle$;
$\langle G_1+P_2,G_2-P_1+\alpha P_2+\beta P_3\rangle$; $\sim\langle P_1,P_2,P_3\rangle$; $\sim\langle G_1,P_2,P_3\rangle$; $\langle G_1+P_2,P_1,P_3\rangle$;
$\langle G_1,P_1+\alpha P_2,P_3\rangle$; $\langle G_1+P_3,G_2+\alpha P_3,P_1\rangle$; $\langle G_1+P_3,G_2,P_1\rangle$; $\langle G_1,G_2+P_3,P_1\rangle$;
$\langle G_1+\lambda P_1,G_2+P_1,P_3\rangle$; $\sim\langle G_1+\gamma P_1,G_2,P_3\rangle$; $\langle G_1+\lambda P_1,G_2+P_1,P_1+\alpha P_3\rangle$;
$\langle G_1+\lambda P_1,G_2,P_1+\alpha P_3\rangle$; $\langle G_1+P_2+\alpha P_3,G_2+\lambda P_3,P_1\rangle$; $\langle G_1+P_2,G_2+\alpha P_3,P_1\rangle$;
$\langle G_1+P_2,G_2-P_1,P_3\rangle$; $\langle G_1+P_2,G_2-P_1+\alpha P_2,P_3\rangle$;
$\langle G_1+P_2+\lambda P_3,G_2+\mu P_3,P_1+\alpha P_3\rangle$;
$\langle G_1-P_2+\alpha P_3,G_2+P_1+\beta P_2+\lambda P_3,G_3+\alpha P_1+\lambda P_2+\mu P_3\rangle$ $(\mu-\alpha^2\beta\ne 0)$;
$\langle G_1-P_2,G_2+P_1+\beta P_2+\alpha P_3,G_3+\alpha P_2+\gamma P_3\rangle$;
$\langle G_1-P_2+\alpha P_3,G_2+P_1,G_3+\alpha P_1+\gamma P_3\rangle$; $\langle G_1,P_1,P_2,P_3\rangle$; $\langle G_1,G_2,P_1,P_3\rangle$;
$\langle G_1+P_2,G_2,P_1,P_3\rangle$; $\langle G_1,G_2+P_3,P_1+\alpha P_3,P_2\rangle$; $\langle G_1,G_2+P_3,P_1,P_2\rangle$;
$\langle G_1,G_2,P_1+\alpha P_3,P_2\rangle$; $\langle G_1+P_2,G_2-P_1+\alpha P_2,G_3+\beta P_1+\lambda P_2,P_3\rangle$;
$\langle G_1+P_2,G_2-P_1+\alpha P_2,G_3+\beta P_2,P_3\rangle$; $\langle G_1+P_2,G_2-P_1+\alpha P_2,G_3,P_3\rangle$;
$\langle G_1,G_2+P_2,G_3+\alpha P_1+\beta P_2,P_3\rangle$; $\langle G_1+P_2,G_2-P_1,G_3+\alpha P_1,P_3\rangle$;
$\langle G_1+P_2,G_2-P_1,G_3,P_3\rangle$; $G_1,G_2,P_1,P_2,P_3\rangle$;
$\langle G_1,G_2+P_1,G_3,P_2,P_3\rangle$; $\langle G_1,G_2,G_3,P_1,P_2,P_3\rangle$;
$\langle T\rangle$: $\sim W[1,3]$, $\langle G_1+P_2,P_1,P_3\rangle$, $\langle G_1,P_1,P_2,P_3\rangle$, $\langle G_1+P_3,G_2,P_1,P_2\rangle$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle J_{12}\rangle$: $\sim W[3]$, $\overline{\mathfrak{M}}[3]$, $\sim W[1,3]$, $\sim W[1,2]+V[3]$, $\mathfrak{L}_1+W[3]$, $V[3]+W[1,3]$, $\mathfrak{L}_1+\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;

$\langle J_{12}+T\rangle$: $\sim W[3]$, $\overline{\mathfrak{M}}[3]$, $\sim W[1,3]$, $V[3]+W[1,3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle J_{12},T\rangle$: $\sim W[3]$, $\overline{\mathfrak{M}}[3]$, $\sim W[1,3]$, $V[3]+W[1,3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$AO(3)$: ~ 0, $\sim W[1,3]$, $\overline{\mathfrak{M}}[1,3]$; $AO(3)\oplus\langle T\rangle$: ~ 0, $\sim W[1,3]$, $\overline{\mathfrak{M}}[1,3]$.

Theorem 6.2. *The nonsplitting subalgebras of the algebra $AG(3)$ are exhausted with respect to $Sch(3)$ conjugation by the nonsplitting subalgebras of the algebra $AG(2)$ [2] and by the following algebras:*

$\langle T+G_1\rangle$: $\sim W[2,3]$, $\langle P_1+\alpha P_2,P_3\rangle$, $\langle G_2+\alpha P_3,P_2\rangle$, $\langle G_2+\alpha P_1+\beta P_3,P_2\rangle$; $W[1,3]$; $\langle G_2,P_2,P_3\rangle$, $\langle G_2+\alpha P_1,P_2,P_3\rangle$, $\overline{\mathfrak{M}}[2]+\langle P_1+\alpha P_3\rangle$, $\langle G_2+\alpha P_3,P_1+\beta P_3,P_2\rangle$, $\langle G_2+\alpha P_3,P_1,P_2\rangle$, $V[2]+W[1,3]$, $\overline{\mathfrak{M}}[2,3]$, $\langle G_2+\alpha P_1,G_3,P_2,P_3\rangle$, $\overline{\mathfrak{M}}[2,3]+W[1]$ $(\alpha>0,\ \beta>0)$;
$\langle J_{12}+G_3\rangle$: ~ 0, $W[3]$, $\sim W[1,2]$, $\sim V[1,2]$, $W[1,3]$, $V[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,2]$, $\overline{\mathfrak{M}}[1,2]+W[3]$;
$\langle J_{12}+T+\alpha G_3\rangle$: ~ 0, $W[3]$, $\sim W[1,2]$, $W[1,3]$, $\overline{\mathfrak{M}}[1,2]$, $\overline{\mathfrak{M}}[1,2]+W[3]$;
$\langle J_{12}+\alpha G_3\rangle$: $\mathfrak{L}_1$, $\mathfrak{L}_1+W[3]$ $(\alpha>0)$;
$\langle J_{12},T+G_3\rangle$: ~ 0, $W[3]$, $\sim W[1,2]$, $W[1,3]$, $\overline{\mathfrak{M}}[1,2]$, $\overline{\mathfrak{M}}[1,2]+W[3]$;
$\langle J_{12}+\alpha G_3,T+G_3\rangle$: $W[3]$, $W[1,3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$ $(\alpha>0)$;
$\langle J_{12}+G_3,T\rangle$: $W[3]$, $W[1,3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$;
$\langle J_{12}+P_3,T\rangle$: ~ 0, $\sim W[1,2]$, $\overline{\mathfrak{M}}[1,2]$;
$\langle J_{12}+\alpha P_3,T+G_3\rangle$: 0, $W[1,2]$, $\overline{\mathfrak{M}}[1,2]$.

The written algebras are not mutually conjugated.

Theorem 6.3. *The subalgebras of the algebra $A\tilde{G}(3)$ are exhausted with respect to $Sch(3)$ conjugation by the subalgebras of the algebra $AG(2)$ (see [2]), by the algebras preceded by the sign $\sim$ in Theorems 6.1 and 6.2, by algebras obtained from algebras written in Theorems 6.1 and 6.2 by adding the generator M, and by the following algebras:*

$\langle T\pm M,P_1,P_2,P_3\rangle$;
$\langle J_{12}+\alpha M\rangle$: $W[3]$, $W[1,3]$, $W[1,2]+V[3]$ $(\alpha>0)$;
$\langle J_{12}+T\pm\alpha M\rangle$: $W[3]$, $W[1,3]$ $(\alpha>0)$;
$\langle J_{12}+\alpha M,T\rangle$: $W[3]$, $W[1,3]$ $(\alpha>0)$;
$\langle J_{12}+\alpha M,T+G_3\rangle$: 0, $W[1,2]$ $(\alpha>0)$;
$\langle J_{12}+\alpha P_3+\beta M,T\pm M\rangle$: 0, $W[1,2]$ $(\alpha>0,\ \beta>0)$;
$\langle J_{12}+\alpha P_3,T\pm M\rangle$: 0, $W[1,2]$ $(\alpha>0)$;
$\langle J_{12}+P_3+\alpha M,T\rangle$: 0, $W[1,2]$ $(\alpha>0)$;
$A0(3)\oplus\langle T\pm M\rangle$: 0, $W[1,3]$.

The written algebras are not mutually conjugated.

Theorem 6.4. *Let $\alpha\in R$, $\alpha>0$. The subalgebras of the algebra $ASch(3)$ which are nonconjugated to subalgebras of the algebras $AG(3)$ and $ASch(2)$ are exhausted with respect to $Sch(3)$ conjugation by the following algebras:*

$\langle D\rangle$: $\sim W[1,3]$, $\sim\langle G_1,P_2,P_3\rangle$, $\langle G_1,P_1+\alpha P_2,P_3\rangle$, $\langle G_1,G_2,P_1+\alpha P_3,P_2\rangle$, $\langle G_1,G_2,P_1,P_3\rangle$, $V[1]+W[1,3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle S+T,G_1-\lambda^{-1}P_2,G_2+\lambda P_1,G_3,P_3\rangle$ $(0<\lambda\leq 1)$; $\langle S+T\rangle \Subset \overline{\mathfrak{M}}[1,3]$;
$\langle J_{12}+\alpha D\rangle$: $\sim W[3]$, $\overline{\mathfrak{M}}[3]$, $\sim W[1,3]$, $W[1,2]+V[3]$, $W[1,2]+\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle S+T+\alpha J_{12}\rangle$: $\overline{\mathfrak{M}}[3]$, $\mathfrak{L}_1+\overline{\mathfrak{M}}[3]$, $\mathfrak{N}_1+\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle S+T+2J_{12},G_1+P_2+\alpha P_3,G_2-P_1-\alpha G_3\rangle$;

$\langle S+T+J_{12}\rangle$: $\langle G_1+P_2\rangle+\overline{\mathfrak{M}}[3]$, $\langle G_1+P_2\rangle+\mathfrak{N}_1+\overline{\mathfrak{M}}[3]$;
$\langle D,T\rangle$: $\sim W[1,3]$, $V[1,j]+W[1,3]$ $(j=1,2,3)$;
$\langle J_{12}+\alpha D,T\rangle$: $\sim W[3]$, $\overline{\mathfrak{M}}[3]$, $\sim W[1,3]$, $W[1,2]+\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle J_{12},D\rangle$: $\sim W[3]$, $\overline{\mathfrak{M}}[3]$, $\sim W[1,3]$, $\sim W[1,2]+V[3]$, $W[1,2]+\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle J_{12},S+T\rangle$: $\overline{\mathfrak{M}}[3]$, $\mathfrak{L}_1+\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,3]$;
$\langle J_{12},D,T\rangle$: $\sim W[3]$, $\overline{\mathfrak{M}}[3]$, $\sim W[1,3]$, $W[1,2]+\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,2]+W[3]$, $\overline{\mathfrak{M}}[1,3]$;
$ASL(2,R)\Subset\overline{\mathfrak{M}}[1,3]$; $\langle J_{12}\rangle\oplus ASL(2,R)$: $\overline{\mathfrak{M}}[3]$, $\overline{\mathfrak{M}}[1,3]$;
$AO(3)\oplus\langle D\rangle$: ~ 0, $\sim W[1,3]$, $\overline{\mathfrak{M}}[1,3]$; $AO(3)\oplus\langle S+T\rangle$: ~ 0, $\overline{\mathfrak{M}}[1,3]$;
$A)(3)\oplus\langle D,T\rangle$: ~ 0, $\sim W[1,3]$, $\overline{\mathfrak{M}}[1,3]$; $AO(3)\oplus ASL(2,R)$: ~ 0, $\overline{\mathfrak{M}}[1,3]$;
$\langle S+T+J_{12}+\alpha(G_1+P_2)\rangle$: $\overline{\mathfrak{M}}[3]$, $\langle G_2-P_1\rangle+\overline{\mathfrak{M}}[3]$, $\mathfrak{N}_1+\overline{\mathfrak{M}}[3]$, $\langle G_2-P_1\rangle+\mathfrak{N}_1+\overline{\mathfrak{M}}[3]$.

The written algebras are not mutually conjugated.

Theorem 6.5. *Let $\alpha,\beta,\gamma\in R$, and $\alpha>0$, $\beta\neq 0$. The subalgebras of the algebra $ASch(3)$ are exhausted with respect to $\widetilde{Sch}(3)$ conjugation by subalgebras of the algebra $A\tilde{G}(3)$, by subalgebras of the algebra $A\widetilde{Sch}(2)$ (see [2]), by algebras preceded by the sign $\sim$ in Theorem 6.4, by algebras obtained from algebras written in Theorem 6.4 by adding the generator M, and by the following algebras:*

$\langle D+\beta M\rangle$: $W[1,3]$, $V[1]+W[2,3]$; $\langle J_{12}+\alpha D+\beta M\rangle$: $W[3]$, $W[1,3]$, $W[1,2]+V[3]$;
$\langle S+T+2J_{12}+\beta M, G_1+P_2+\sqrt{2}P_3, G_2-P_1-\sqrt{2}G_3\rangle$;
$\langle D+\beta M,T\rangle\Subset W[1,3]$; $\langle J_{12}+\alpha M,D\rangle$: $W[3]$, $W[1,3]$, $W[1,2]+V[3]$;
$\langle J_{12}+\alpha M,D+\beta M\rangle$: $W[3]$, $W[1,3]$, $W[1,2]+V[3]$;
$\langle J_{12},D+\beta M\rangle$: $W[3]$, $W[1,3]$, $W[1,2]+V[3]$;
$\langle J_{12}+\alpha D+\beta M,T\rangle$: $W[3]$, $W[1,3]$; $\langle J_{12}+\alpha M,D+\gamma M,T\rangle$: $W[3]$, $W[1,3]$;
$\langle J_{12},D+\beta M,T\rangle$: $W[3]$, $W[1,3]$; $AO(3)\oplus\langle D+\beta M\rangle$: 0, $W[1,3]$;
$AO(3)\oplus\langle S+T+\beta M\rangle$; $AO(3)\oplus\langle D+\beta M,T\rangle$: 0, $W[1,3]$.

The written algebras are not mutually conjugated.

7. Conclusions

The results of the present paper may be summarized in the following way.

(1) The completely reducible subalgebras of the algebra $AO(n)\oplus ASL(2,R)$ have been identified (Theorem 3.1).

(2) The subalgebras of $AO(n)\oplus ASL(2,R)$ which possess only splitting extensions in the algebra $ASch(n)$ have been described (Theorem 4.1).

(3) We have established that the description of the splitting subalgebras of the algebra $ASch(n)$ whose projections onto $ASL(2,R)$ are not equal to $\langle S+T\rangle$ is reduced to the description of the splitting subalgebras of $ASch(n)$ whose projections onto $AO(n)$ are equal to zero or to primary algebras (Theorem 4.2).

(4) The maximal Abelian subalgebras and the one-dimensional subalgebras of the algebras $ASch(n)$ and $A\widetilde{Sch}(n)$ have been explicitly found (the corollaries to Theorems 5.1 and 5.2).

(5) The classification of the subalgebras of $ASch(3)$ and $A\widetilde{Sch}(3)$ with respect to $Sch(3)$ conjugation and $\widetilde{Sch}(3)$ conjugation, respectively, has been carried out (Theorems 6.1–6.5). This classification gives the possibility to construct the wide

classes of exact solutions of the nonlinear, Schrödinger-type equations in [15–18],

$$i\frac{\partial\Psi}{\partial t} - \Delta\Psi + \lambda|\Psi|^{3/4}\Psi = 0,$$
$$i\frac{\partial\Psi}{\partial t} - \Delta\Psi + \lambda\frac{\partial(\Psi^*\Psi)}{\partial X_a}\frac{\partial(\Psi^*\Psi)}{\partial X_a}(\Psi^*\Psi)^{-2}\cdot\Psi = 0,$$

which are invariant under $Sch(3)$.

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