Symmetry and exact solutions of nonlinear spinor equations

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This review is devoted to the application of algebraic-theoretical methods to the problem of constructing exact solutions of the many-dimensional nonlinear systems of partial differential equations for spinor, vector and scalar fields widely used in quantum field theory. Large classes of nonlinear spinor equations invariant under the Poincaré group $P(1,3)$, Weyl group (i.e. Poincaré group supplemented by a group of scale transformations), and the conformal group $C(1,3)$ are described. Ansätze invariant under the Poincaré and the Weyl groups are constructed. Using these we reduce the Poincaré-invariant nonlinear Dirac equations to systems of ordinary differential equations and construct large families of exact solutions of the nonlinear Dirac–Heisenberg equation depending on arbitrary parameters and functions. In a similar way we have obtained new families of exact solutions of the nonlinear Maxwell–Dirac and Klein–Gordon–Dirac equations. The obtained solutions can be used for quantization of nonlinear equations.

1. Introduction

The Maxwell equations for the electromagnetic field and the Dirac equation for the spinor field,

$$(\gamma_\mu p^\mu - m)\psi = 0, \tag{1.1}$$

discovered 60 years ago, are the fundament of modern physics. In eq. (1.1) $\psi = \psi(x)$ is a four-component complex-valued function, $x = (x_0 \equiv t, x_1, x_2, x_3) \in \mathbb{R}(1,3)$, four-dimensional Minkowski space, γ_μ are 4×4 matrices satisfying the Clifford–Dirac algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \tag{1.2}$$

where $g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$, m is the particle mass. We use two equivalent representations of the γ-matrices,

$$\gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \tag{1.2a}$$

or

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma_a = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \quad a = 1,2,3, \tag{1.2b}$$

σ_a are the 2×2 Pauli matrices.

Fifteen years ago D. Ivanenko [1] made an attempt to obtain a nonlinear generalization of the Dirac equation, and suggested the following equation:

$$[\gamma_\mu p^\mu - m + \lambda(\bar\psi\psi)]\psi(x) = 0, \quad \bar\psi = \psi^+\gamma_0. \tag{1.3}$$

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In the early fifties W. Heisenberg [2–5] put forward a vast program to construct a unified field theory based on the nonlinear spinor equation

$$[\gamma_\mu p^\mu + \lambda(\bar{\psi}\gamma_\mu\gamma_4\psi)\gamma^\mu\gamma_4]\psi(x) = 0. \tag{1.4}$$

Heisenberg and his collaborators [2–5] did their best to construct the quantum field theory, to establish the quantization rules, and to calculate the mass spectrum of the elementary particles.

In two papers by R. Finkelstein and collaborators [6, 7] published in the early fifties, nonlinear spinor fields were investigated from the classical point of view, i.e., approximate and exact solutions of partial differential equations (PDE) were studied. From the classical point of view scalar field was studied by L. Schiff [8] and B. Malenka [9].

Like the general theory of relativity nonlinear spinor field theory is a mathematical model of physical reality based on a complicated multi-dimensional nonlinear system of PDE.

Up to now there exists a vast literature on exact solutions of the equations for the gravitational field. It is well-known which important role has been played in gravitation theory by Schwarzschild's, Friedman's and Kerr's exact solutions. So far many of the obtained solutions have no adequate physical interpretation. Nevertheless the number of exact solutions of the Einstein equations grows rapidly.

Nothing of the kind takes place in nonlinear field theory. There are few enough classical solutions of nonlinear spinor equations [10–18] although these equations are essentially simpler than those of gravitation theory. This surprising situation seems to be explained by the fact that many investigators underestimate the importance of exact solutions in the theory of quantized fields and expect the great successes in other domains of quantum field theory.

We think that a thorough investigation of nonlinear spinor equations and a construction of exact solutions for them sooner or later will lead to important physical results and to new physical ideas and methods. Let us recall that in this way the theory of solitons was created.

We will not adduce a concrete physical interpretation to the solutions of nonlinear spinor equations because we think that they speak for themselves. Nevertheless we will show how to construct nonlinear scalar fields (equations) using exact solutions of nonlinear spinor equations. In other words, we have a dynamical realization of de Broglie's idea to construct an arbitrary field by using a field with spin $s = \frac{1}{2}$ [19]. The kinematical realization of this idea is well known. It is reduced to a decomposition of a direct product of linear irreducible representations of the Lorentz and Poincare groups (with spin $s = \frac{1}{2}$).

It will be shown that the interaction of spinor and scalar fields gives rise to some mass spectrum (section 4). It is of interest that discrete relations connecting the masses of spinor and scalar fields are determined by the geometry of the solutions.

Exact solutions obtained by us can be used as a pattern to check the already known approximate methods and to create new ones. For example, solutions which depend on the coupling constant A in a singular way cannot be obtained by standard methods of perturbation theory.

Solutions (classes of solutions) with the same symmetry as the initial equation of motion seem to be of particular importance. These solutions (not the equation) can be used in the quantization procedure. From the set of solutions one can pick out ones that do not lead to an indefinite metric. This review is based on our papers [20–30], and the symmetry properties of PDE are used extensively. That is, we apply the classical ideas and methods of S. Lie to nonlinear spinor equations. The symmetry properties of nonlinear field equations make it possible to reduce multi-dimensional spinor equations to systems of ordinary differential equations (ODE). Integration of these ODE gives rise to exact solutions of the initial equation. Let us note that all the exact solutions of the nonlinear Dirac equation known to us are included in the set of solutions obtained in such a way.

The structure and content of the review are as follows. In section 2 we investigate the symmetry of the nonlinear Dirac equation

$$[\gamma_\mu p^\mu + F(\bar{\psi}, \psi)]\psi(x) = 0, \tag{1.5}$$

where $F(\bar{\psi}, \psi)$ is an arbitrary four-component matrix depending on eight field variables $\bar{\psi}$, ψ. All the matrices $F(\bar{\psi}, \psi)$ ensuring invariance of eq. (1.5) under the Poincaré group $P(1,3)$, extended Poincaré group $\tilde{P}(1,3)$ and conformal group $C(1,3)$ are described.

In section 3 we take the ansatz

$$\psi(x) = A(x)\varphi(\boldsymbol{\omega}), \tag{1.6}$$

suggested in ref. [30] and described systematically in refs. [23, 24, 29], which reduces the system of equations (1.5) to systems of equations for the four functions φ^0, φ^1, φ^2, φ^3 depending on three new invariant variables $\boldsymbol{\omega} = \{\omega_1(x), \omega_2(x), \omega_3(x)\}$. In (1.6) $A(x)$ is a variable nonsingular 4×4 matrix, whose explicit form is given in section 3. If φ depends on one independent variable then ansatz (1.6) reduces eq. (1.5) to a system of ODE. Most of them prove to be integrable. Integrating these and substituting the obtained results into the ansatz (1.6) one obtains particular solutions of eq. (1.5). Using this approach we have constructed large classes of exact solutions of the nonlinear Dirac–Heisenberg equation (DHE) for a spinor field.

In sections 4 and 5 multi-parameter families of exact solutions of the Dirac–Klein–Gordon and the Maxwell–Dirac systems, describing the interaction of a spinor field with scalar and electromagnetic fields are constructed.

2. Nonlinear spinor equations invariant under the Poincaré group $P(1,3)$ and its extensions, the groups $\tilde{P}(1,3)$ and $C(1,3)$

It is clear that arbitrary equations of the form (1.5) can not be taken as a physically acceptable generalization of the linear Dirac equation. A natural restriction of the form of the nonlinearity $F(\bar{\psi}, \psi)$ is imposed by demanding relativistic invariance. This condition ensures independence of the physical processes described by eq. (1.5) of the choice of inertial reference system (i.e., the nonlinear equation in question has to satisfy the Poincaré–Einstein relativity principle). It is common knowledge that the Dirac equation with zero mass admits the conformal group $C(1,3)$ (see e.g. ref. [31] and the literature cited there). Therefore it is of interest to choose from the set of Poincaré-invariant equations of the form (1.5) equations that are invariant under the conformal group.

In this section we describe all equations of the form (1.5) that are invariant under the Poincaré group $P(1,3)$ and its extensions, the group $\tilde{P}(1,3)$ and the conformal group $C(1,3)$. Let us recall that the extended Poincaré group $\tilde{P}(1,3)$ (or Weyl group) is an 11-parameter group of transformations $\{P(1,3), D(1)\}$, where $D(1)$ is a one-parameter group of scale transformations,

$$x'_\mu = e^\theta x_\mu, \quad \psi'(x') = e^{-k\theta}\psi(x), \quad k,\theta = \text{const}. \tag{2.1}$$

The 15-parameter conformal group $C(1,3)$ consists of the group $\tilde{P}(1,3)$ and the four-parameter group of special conformal transformations

$$x'_\mu = (x_\mu - \theta_\mu x\cdot x)\sigma^{-1}(x), \quad \psi'(x') = \sigma(x)[1-(\gamma\cdot\theta)(\gamma\cdot x)]\psi(x), \tag{2.2}$$

where $\sigma(x) = 1 - 2\theta\cdot x + (\theta\cdot\theta)(x\cdot x)$, θ_μ are parameters of the group, $\mu = 0,1,2,3$. Hereafter we use the following notation for the scalar product in Minkowski space $\mathbb{R}(1,3)$:

$$a\cdot b \equiv a_\mu b^\mu \equiv g^{\mu\nu}a_\mu b_\nu, \quad \mu,\nu = 0,1,2,3,$$

where $g_{\mu\nu} = \operatorname{diag}(1,-1,-1,-1)$ is the metric tensor of Minkowski space.

Theorem 1. *Equation (1.5) is Poincaré invariant iff*

$$F(\bar\psi,\psi) = F_1 + F_2\gamma_4, \tag{2.3}$$

where $\bar\psi = \psi^+\gamma_0$, $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$, F_1 and F_2 are arbitrary scalar functions of $\bar\psi\psi$ and $\bar\psi\gamma_4\psi$.

We give only a sketch of the proof, which is based on the infinitesimal Lie method [32–34]. Expanding the matrix $F(\bar\psi,\psi)$ in a linear combination of γ-matrices, the coefficients of the expansion depending $\bar\psi$ and ψ,

$$F = aI + b_\mu\gamma^\mu + c^{\mu\nu}S_{\mu\nu} + d^\mu\gamma_4\gamma_\mu + e\gamma_4, \quad S_{\mu\nu} = \frac{1}{4}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \tag{2.4}$$

and using the invariance criterion, one obtains the following necessary and sufficient conditions for the Poincaré invariance of eq. (1.5):

$$\begin{aligned}
&Q_{0k}a = 0, \quad Q_{0k}e = 0,\\
&Q_{0k}b_\mu + b^\alpha(g_{\alpha 0}g_{\mu k} - g_{\alpha k}g_{\mu 0}) = 0, \quad Q_{0k}d_\mu + d^\alpha(g_{\alpha 0}g_{\mu k} - g_{\alpha k}g_{\mu 0}) = 0,\\
&Q_{0k}c_{\mu\nu} + c^{\alpha\beta}(g_{\alpha k}\delta^{\mu\nu}_{\beta 0} + g_{\beta 0}\delta^{\mu\nu}_{\alpha k} - g_{\alpha 0}\delta^{\mu\nu}_{\beta k} - g_{\beta k}\delta^{\mu\nu}_{\alpha 0}) = 0,
\end{aligned} \tag{2.5}$$

where

$$\begin{aligned}
&iQ_{0k} = -(S_{0k}\psi)^\alpha\partial/\partial\psi^\alpha + (\bar\psi S_{0k})^\alpha\partial/\partial\bar\psi^\alpha, \quad k = 1,2,3,\\
&\delta^{\mu\nu}_{\alpha\beta} = \delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha, \quad \alpha,\beta,\mu,\nu = 0,1,2,3,
\end{aligned}$$

and δ^α_β is the Kronecker symbol.

After some cumbersome calculations we obtain the following general solution of the system of PDE (2.5):

$$\begin{aligned}
&a = A, \quad e = E, \quad b_\mu = B_1\bar\psi\gamma_\mu\psi + B_2\bar\psi\gamma_\mu\gamma_4\psi,\\
&c_{\mu\nu} = C_1\bar\psi S_{\mu\nu}\psi + C_2\bar\psi\gamma_4 S_{\mu\nu}\psi, \quad d_\mu = D_1\bar\psi\gamma_\mu\psi + D_2\bar\psi\gamma_4\gamma_\mu\psi,
\end{aligned} \tag{2.6}$$

where $A, B_1, \ldots, E$ are arbitrary smooth functions of $\bar\psi\psi$ and $\bar\psi\gamma_4\psi$.

Substituting the above formulae into (2.4) one obtains the following expression for the nonlinear term $F(\bar{\psi},\psi)\psi$:

$$F(\bar{\psi},\psi)\psi = \{AI + [B_1\bar{\psi}\gamma_\mu\psi + B_2\bar{\psi}\gamma_4\gamma_\mu\psi]\gamma^\mu + [C_1\bar{\psi}S_{\mu\nu}\psi + C_2\bar{\psi}\gamma_4 S_{\mu\nu}\psi]S^{\mu\nu} + \\ + [D_1\bar{\psi}\gamma_\mu\psi + D_2\bar{\psi}\gamma_4\gamma_\mu\psi]\gamma_4\gamma^\mu + E\gamma_4\}\psi.$$

This formula can be essentially simplified with the help of the identity [35]

$$(\bar{\psi}_1\gamma_\mu\psi_2)\gamma^\mu\psi_2 = (\bar{\psi}_1\psi_2)\psi_2 + (\bar{\psi}_1\gamma_4\psi_2)\gamma_4\psi_2,$$

where ψ_1 and ψ_2 are arbitrary four-component spinors, and as a result the nonlinearity $F(\bar{\psi},\psi)$ takes the form (2.3). This completes the proof.

Note. In the same way one can prove that the second-order spinor equation

$$p_\mu p^\mu\psi = F(\bar{\psi},\psi)\psi \tag{2.7}$$

is invariant under the Poincaré group iff $F(\bar{\psi},\psi)$ has the form (2.3).

Theorem 2 [29]. *Equation (1.5) is invariant under the Weyl group $\tilde{P}(1,3)$ iff $F(\bar{\psi},\psi)$ has the form (2.3), F_i being determined by the formulae*

$$F_i = (\bar{\psi}\psi)^{1/2k}\tilde{F}_i, \quad i = 1,2, \tag{2.8}$$

where $\tilde{F}_1$, $\tilde{F}_2$ are arbitrary functions of $\bar{\psi}\psi/\bar{\psi}\gamma_4\psi$.

Theorem 3 [29]. *Equation (1.5) is invariant under the conformal group $C(1,3)$ iff $F(\bar{\psi},\psi)$ has the form (2.3), (2.8) with $k = 3/2$.*

The proof of the last two statements is obtained with the help of the Lie method [32–34]; it is omitted here. Let us note that the sufficiency in theorem 3 can be established by direct verification. To do this we denote by G the following expression:

$$G(\bar{\psi},\psi) = \gamma_\mu p^\mu\psi + (\tilde{F}_1 + \tilde{F}_2\gamma_4)(\bar{\psi}\psi)^{1/3}\psi.$$

One can verify that the following identities hold:

$$G(\bar{\psi}',\psi') = e^{-5\theta/2}G(\bar{\psi},\psi),$$

if $\bar{\psi}'$, ψ' have the form (2.1) with $k = 3/2$,

$$G(\bar{\psi}',\psi') = \sigma^2(x)[1 - (\gamma\cdot\theta)(\gamma\cdot x)]G(\bar{\psi},\psi),$$

if $\bar{\psi}'$, ψ' have the form (2.2). Consequently, the equation $G = 0$ is invariant under the groups of transformations (2.1), (2.2).

Note 1. Unlike eq. (1.5), the class of equations (2.7) does not include conformally invariant ones. Therefore it seems reasonable to consider as an equation of motion for a spinor field the following second-order equation:

$$\begin{aligned} &p_\mu p^\mu\psi = \Phi(\bar{\psi},\psi,\bar{\psi}_1,\psi_1), \\ &\psi_1 = \{\partial\psi/\partial x_\mu,\ \mu = 0,1,2,3\}, \quad \bar{\psi}_1 = \{\partial\bar{\psi}/\partial x_\mu,\ \mu = 0,1,2,3\}. \end{aligned} \tag{2.9}$$

The problem of a complete group-theoretical classification of eqs. (2.9) will be considered in a future paper. Here we restrict ourselves to an example of a conformally invariant equation of the form (2.9),

$$p_\mu p^\mu\psi - (3\bar{\psi}\psi)^{-1}\gamma_\mu[p^\mu(\bar{\psi}\psi)]\gamma_\nu p^\nu\psi = 0. \tag{2.10}$$

It is worth noting that each solution of the nonlinear Dirac–Gürsey equation [36] satisfies the PDE (2.10).

Note 2. There exist Poincaré-invariant first-order equations which differ principally from the Dirac equation. An example is [29, 37]

$$(\bar{\psi}\gamma^{\mu}\psi)p_{\mu}\psi = 0. \tag{2.11}$$

On the set of solutions of the system (2.11) a representation of an infinite-dimensional Lie algebra is realized. This fact enables us to construct the general solution of eq. (2.11) in implicit form,

$$f^{\alpha}(x_{\mu}(j\cdot j) - j_{\mu}(j\cdot x), \bar{\psi}, \psi) = 0, \quad \alpha = 0, 1, 2, 3,$$

where $j_{\mu} = \bar{\psi}\gamma_{\mu}\psi$, $f^{\alpha} : \mathbb{R}\times\mathbb{C}^8 \to \mathbb{C}^1$ are arbitrary smooth functions.

3. Exact solutions of the nonlinear Dirac equation

According to refs. [23, 24, 37] a solution of eq. (1.5) is looked for as a solution of the following overdetermined system of PDE:

$$\begin{aligned} &\gamma_{\mu}p^{\mu}\psi + F(\bar{\psi},\psi)\psi = 0, \\ &\xi_a^{\mu}\psi_{x_{\mu}} + \eta_a(x,\bar{\psi},\psi)\psi = 0, \quad a = 1,2,3, \end{aligned} \tag{3.1}$$

where $\eta_a(x,\bar{\psi},\psi)$ are arbitrary 4×4 matrices, $\xi_a^{\mu}(x,\bar{\psi},\psi)$ are scalar functions satisfying the condition

$$\operatorname{rank}\{\xi_a^{\mu}(x,\bar{\psi},\psi)\} = 3. \tag{3.2}$$

The PDE (3.1) is a system of sixteen equations for four functions ψ^0, ψ^1, ψ^2, ψ^3. Therefore one has to investigate its compatibility (see also refs. [31, 39, 40]).

Theorem 4. *System (3.1) is compatible iff it is invariant under the one-parameter Lie groups generated by the operators*

$$Q_a = \xi_a^{\mu}\partial/\partial x_{\mu} - (\eta_a\psi)^{\alpha}\partial/\partial\psi^{\alpha}, \quad a = 1,2,3.$$

The main steps of the proof are as follows. Firstly, using condition (3.2) one reduces the system (3.1) to the equivalent system (to simplify the calculations we suppose that $\partial\xi_{\mu}^{a}/\partial\psi^{\alpha} = \partial\eta_a^{\alpha\beta}/\partial\psi^{\mu} = 0$)

$$\begin{aligned} &\gamma_{\mu}p^{\mu}\psi + F(\bar{\psi},\psi)\psi = 0, \\ &\tilde{Q}_a\psi \equiv (\partial/\partial x_a + \xi_{\alpha}\partial/\partial x_0 + \tilde{\eta}_a)\psi = 0. \end{aligned} \tag{3.1$'$}$$

It is not difficult to verify that system (3.1$'$) admits groups generated by the operators $\tilde{Q}_a$ iff the initial system admits groups generated by the operators Q_a, while the following relations hold:

$$[\tilde{Q}_a, \tilde{Q}_b] = 0, \quad a, b = 1,2,3. \tag{3.3}$$

It follows from the general theory of Lie groups that there exists the change of variables

$$\Psi(z) = \eta(x)\psi(x), \quad z_{\mu} = f_{\mu}(x), \quad \mu = 0,1,2,3$$

which reduces the operators $\tilde{Q}_a$ satisfying conditions (3.3) to the form

$$\tilde{Q}_a \to \tilde{\tilde{Q}}_a = \partial/\partial z_a. \tag{3.4}$$

System (3.1′) is rewritten in the following way:

$$\begin{aligned} &\partial\Psi/\partial z_0 = F_1(z,\bar{\Psi},\Psi)\Psi, \\ &\partial\Psi/\partial z_a = 0. \end{aligned} \tag{3.1″}$$

The necessary and sufficient conditions for the compatibility of the system (3.1″) are as follows:

$$\partial^2\Psi/\partial z_\mu\partial z_\nu = \partial^2\Psi/\partial z_\nu\partial z_\mu. \tag{3.5}$$

Applying these conditions to (3.1″) one has

$$\partial F_1/\partial z_a = 0, \quad a = 1,2,3, \tag{3.6}$$

whence the invariance of system (3.1″) under the operators $\tilde{\tilde{Q}}_a = \partial/\partial z_a$ follows. The reverse statement is also true — if system (3.1″) is invariant under the groups generated by the operators $\tilde{\tilde{Q}}_a$, then conditions (3.6) hold. Consequently, the initial system is invariant under the operators Q_a. The theorem is proved.

Consequence. *Substitution of the ansatz*

$$\psi(x) = A(x)\varphi(\omega), \tag{3.7}$$

where the 4×4 *matrix* $A(x)$ *and the scalar function* $\omega(x)$ *satisfy the system of PDE*

$$\xi_a^\mu\partial\omega/\partial x_\mu = 0, \tag{3.8}$$

$$Q_aA(x) = [\xi_a^\mu\partial/\partial x_\mu + \eta_a(x)]A(x) = 0, \tag{3.9}$$

into eq. (1.5) gives rise to a system of ODE for $\varphi = \varphi(\omega)$.

Proof. Integration of the last three equations of (3.1″) yields

$$\Psi = \Psi(z_0).$$

Returning to the original variables x and $\psi(x)$, one has

$$\psi(x) = [\eta(x)]^{-1}\Psi(z_0).$$

Choosing $A(x) = [\eta(x)]^{-1}$, $\omega = z_0(x)$, one obtains the statement required, the ODE for $\varphi(\omega)$ having the form

$$d\varphi/d\omega = F_1(\omega,\bar{\varphi},\varphi)\varphi.$$

Note. If (Q_1, Q_2, Q_3) is a three-dimensional invariance algebra of PDE (1.5), then the conditions of theorem 4 are evidently satisfied. Therefore the classical result on the reduction of PDE to ODE via Q_a-invariant solutions [32–34, 41] follows from theorem 4 as a particular case. If Q_a are not the symmetry operators then the reduction is done via conditionally Q_a-invariant solutions [31, 39, 40, 42].

3.1. Ansätze for the spinor field. In the following we shall consider spinor equations (1.5) with the nonlinearity (2.3), i.e. Poincaré-invariant systems of the form

$$\gamma_\mu p^\mu \psi = \Phi(\bar{\psi}, \psi) = (F_1 + F_2\gamma_4)\psi. \tag{3.10}$$

On the set of solutions of system (3.10) the following representation of the Poincaré algebra $AP(1,3)$ is realized:

$$P_\mu = p_\mu, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}. \tag{3.11}$$

Using theorem 4 and the group-theoretical properties of eq. (3.10) one can formulate the following algorithm for the reduction of the PDE (3.10) to systems of ODE.

At the first step one has to describe (to classify) all inequivalent three-dimensional algebras which are subalgebras of the Poincaré algebra (3.11). As a result we obtain a set of triplets of operators (Q_1, Q_2, Q_3), each of which determines an ansatz of the form (3.7).

At the second step the system of equations (3.8), (3.9) is integrated. According to the consequence of theorem 4 substitution of the obtained ansatze into the initial equation yields systems of ODE for the unknown function $\varphi = \varphi(\omega)$.

The efficiency of group-theoretical methods is ensured, first of all, by the fact that intermediate problems to be solved are linear. At the first step linear systems of algebraic equations are solved [43], at the second step systems of linear PDE having the same principal part.

The problem of classification of all inequivalent subalgebras of the Poincaré algebra $P(,3)$ was solved in refs. [43–45]. Integration of the PDE (3.8), (3.9) is carried out by standard methods but the calculations are rather cumbersome. We give here the final result in table 1.

As an example we consider the case $Q_1 = J_{03}$, $Q_2 = P_1$, $Q_3 = P_2$, i.e.,

$$(x_0p_3 - x_3p_0)\omega = 0, \quad p_1\omega = p_2\omega = 0; \tag{3.12}$$

$$\left(x_0p_3 - x_3p_0 + \frac{1}{2}i\gamma_0\gamma_3\right) A(x) = 0, \quad p_1A(x) = p_2A(x) = 0. \tag{3.13}$$

From the last two equations of the system (3.12) it follows that $\omega = \omega(x_0, x_3)$. Substituting this result into the first equation one obtains that $\omega(x_0, x_3)$ is a first integral of the Euler–Lagrange system

$$\frac{dx_0}{x_3} = \frac{dx_3}{x_0},$$

which can be chosen in the form $\omega = x_0^2 - x_3^2$.

A solution of the system (3.13) is looked for in the form

$$A(x) = \exp[\gamma_0\gamma_3 f(x)]$$

whence it follows that the scalar function $f(x)$ satisfies the following equation:

$$x_0 f_{x_3} + x_3 f_{x_0} = \frac{1}{2},$$

whose particular solution has the form

$$f(x) = \frac{1}{2}\ln(x_0 + x_3).$$

Table 1. $P(1,3)$-invariant anzätze for the spinor field

No.	Algebra	$A(x)$	$\omega(x)$
1	$P_0,\ P_1,\ P_2$	I	x_3
2	$P_1,\ P_2,\ P_3$	I	x_0
3	$P_0+P_3,\ P_1,\ P_2$	I	x_0+x_3
4	$J_{03},\ P_1,\ P_2$	$\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right]$	$x_0^2-x_3^2$
5	$J_{03},\ P_0+P_3,\ P_1$	$\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right]$	x_2
6	$J_{03}+\alpha P_2,\ P_0,\ P_3$	$\exp\left[\frac{x_2}{2\alpha}\gamma_0\gamma_3\right]$	x_1
7	$J_{03}+\alpha P_2,\ P_0+P_3,\ P_1$	$\exp\left[\frac{x_2}{2\alpha}\gamma_0\gamma_3\right]$	$\alpha\ln(x_0+x_3)-x_2$
8	$J_{12},\ P_0,\ P_3$	$\exp\left[-\frac{1}{2}\gamma_1\gamma_2\operatorname{arctg}\frac{x_1}{x_2}\right]$	$x_1^2+x_2^2$
9	$J_{12}+\alpha P_0,\ P_1,\ P_2$	$\exp\left[-\frac{x_0}{2\alpha}\gamma_1\gamma_2\right]$	x_3
10	$J_{12}+\alpha P_3,\ P_1,\ P_2$	$\exp\left[\frac{x_3}{2\alpha}\gamma_1\gamma_2\right]$	x_0
11	$J_{12}+P_0+P_3,\ P_1,\ P_2$	$\exp\left[\frac{1}{4}(x_3-x_0)\gamma_1\gamma_2\right]$	x_0+x_3
12	$G_1,\ P_0+P_3,\ P_2$	$\exp\left(\frac{x_1}{2(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1\right)$	x_0+x_3
13	$G_1,\ P_0+P_3,\ P_1+\alpha P_2$	$\exp\left(\frac{\alpha x_1-x_2}{2\alpha(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1\right)$	x_0+x_3
14	$G_1+P_2,\ P_0+P_3,\ P_1$	$\exp\left[\frac{x_2}{2}(\gamma_0+\gamma_3)\gamma_1\right]$	x_0+x_3
15	$G_1+P_0,\ P_0+P_3,\ P_2$	$\exp\left[-\frac{1}{2}(x_0+x_3)(\gamma_0+\gamma_3)\gamma_1\right]$	$2x_1+(x_0+x_3)^2$
16	$G_1+P_0,\ P_1+\alpha P_2,$ P_0+P_3	$\exp\left[-\frac{1}{2}(x_0+x_3)(\gamma_0+\gamma_3)\gamma_1\right]$	$2(x_2-\alpha x_1)-$ $-\alpha(x_0+x_3)^2$
17	$J_{03}+\alpha J_{12},\ P_0,\ P_3$	$\exp\left(-\frac{1}{2\alpha}(\gamma_0\gamma_3+\alpha\gamma_1\gamma_2)\operatorname{arctg}\frac{x_1}{x_2}\right)$	$x_1^2+x_2^2$
18	$J_{03}+\alpha J_{12},\ P_1,\ P_2$	$\exp\left[\frac{1}{2}(\gamma_0\gamma_3+\alpha\gamma_1\gamma_2)\ln(x_0+x_3)\right]$	$x_0^2-x_3^2$
19	$G_1,\ G_2,\ P_0+P_3$	$\exp\left(\frac{\gamma_0+\gamma_3}{2(x_0+x_3)}(\gamma_1x_1+\gamma_2x_2)\right)$	x_0+x_3
20	$G_1+P_2,$ $G_2+\alpha P_1+\beta P_2,$ P_0+P_3	$\exp\left(\frac{\gamma_0+\gamma_3}{2[(x_0+x_3)(x_0+x_3+\beta)-\alpha]}\times\right.$ $\times\{\gamma_1[(x_0+x_3+\beta)x_1-\alpha x_2]+$ $\left.+\gamma_2[(x_0+x_3)x_2-x_1]\}\right)$	x_0+x_3
21	$G_1,\ G_2+P_1+\beta P_2,$ P_0+P_3	$\exp\left(\frac{x_1}{2(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1+\right.$ $+\frac{x_2}{2(x_0+x_3)(x_0+x_3+\beta)}\times$ $\left.\times(\gamma_0+\gamma_3)[\gamma_2(x_0+x_3)-\gamma_1]\right)$	x_0+x_3
22	$G_1,\ G_2+P_2,$ P_0+P_3	$\exp\left[(\gamma_0+\gamma_3)\left(\frac{x_1}{2(x_0+x_3)}\gamma_1+\right.\right.$ $\left.\left.+\frac{x_2}{2(x_0+x_3+1)}\gamma_2\right)\right]$	x_0+x_3
23	$G_1,\ J_{03},\ P_2$	$\exp\left(\frac{x_1}{2(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1\right)\times$ $\times\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right]$	$x_0^2-x_1^2-x_3^2$
24	$J_{03}+\alpha P_1+\beta P_2,$ $G_1,\ P_0+P_3$	$\exp\left(\frac{x_1-\alpha\ln(x_0+x_3)}{2(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1\right)\times$ $\times\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right]$	$x_2-\beta\ln(x_0+x_3)$
25	$J_{12}+P_0+P_3,\ G_1,\ G_2$	$\exp\left(\frac{\gamma_0+\gamma_3}{2(x_0+x_3)}(\gamma_1x_1+\gamma_2x_2)\right)\times$ $\times\exp\left(-\frac{x\cdot x}{4(x_0+x_3)}\gamma_1\gamma_2\right)$	x_0+x_3

Table 1 (Continued)

No.	Algebra	$A(x)$	$\omega(x)$
26	$J_{03}+\alpha J_{12}$, G_1, G_2	$\exp\left(\frac{\gamma_0+\gamma_3}{2(x_0+x_3)}(\gamma_1 x_1+\gamma_2 x_2)\right)\times$ $\times\exp\left[\frac{1}{2}(\gamma_0\gamma_3+\alpha\gamma_1\gamma_2)\ln(x_0+x_3)\right]$	$x\cdot x$

Notation:

$G_k = J_{0k}+J_{3k}\equiv(x_0+x_3)p_k - x_k(p_0+p_3)+\frac{1}{2}i(\gamma_0+\gamma_3)\gamma_k$, $\exp\{R\}\equiv I+\sum\limits_{n=1}^{\infty}\frac{1}{n!}R^n$,
I is the 4×4 matrix, $\alpha,\beta\in\mathbb{R}^1$.

Finally one has

$$\omega(x)=x_0-x_3^2,\quad A(x)=\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right].$$

Other triplets Q_1, Q_2, Q_3 from table 1 are treated in an analogous way (we show in table 1 only algebras $\langle Q_1,Q_2,Q_3\rangle$ giving nontrivial ansätze (3.7)).

It is important to note that the ansätze listed in the table 1 do not exhaust all possible substitutions of the form (3.7) reducing the PDE (3.10) to ODE. Principally different ansätze are obtained when the conditions of theorem 4 are valid and some operators Q_a are not symmetry operators of eq. (3.10) (conditional invariance).

To investigate the conditional invariance of a differential equation one can also apply the infinitesimal Lie algorithm [32–34]. However, the determining equations to be solved are nonlinear (see refs. [39, 40]). To avoid this difficulty the following method was suggested [24, 30]: firstly, the dimension of the PDE is decreased by one using its group-theoretical properties and then the maximal symmetry of the reduced equations is investigated. Under certain circumstances this procedure yields such operators Q_a that the initial equation is conditionally invariant under Q_a.

We realize the above scheme for the PDE (1.5) invariant under the group $\tilde{P}(1,3)$, i.e. for equations of the form

$$\gamma_\mu p^\mu\psi=\Phi_2(\bar\psi,\psi)\equiv[(\tilde F_1+\tilde F_2\gamma_4)(\bar\psi\psi)^{1/2k}]\psi. \tag{3.14}$$

$\tilde{P}(1,3)$-invariant ansätze for the spinor field reducing (3.14) to three-dimensional PDE were constructed in refs. [24,29]. The general form is

$$\psi(x)=A(x)\varphi(\omega_1,\omega_2,\omega_3), \tag{3.15}$$

where φ is a new unknown spinor; the 4×4 matrix $A(x)$ and the scalar functions $\omega_i(x)$ are determined from table 2 (each ansatz in table 2 corresponds to some one-dimensional subalgebra of the algebra $AP(1,3)$; for more detail see ref. [24]). Substitution of the ansätze (3.15), with $A(x)$ and $\omega(x)$ as listed in the table 2, into the PDE (3.14) results in a reduction by one of the number of independent variables, i.e., the equations obtained depend on the three independent variables ω_1, ω_2, ω_3 only. Omitting intermediate calculations we write down the reduced equations for $\varphi(\omega_1,\omega_2,\omega_3)$.

(1) $k(\gamma_2-\gamma_0)\varphi+[(\gamma_0-\gamma_2)(\omega_1+a^{-2}\omega_2^2\omega_3^2)+(\gamma_0+\gamma_2)\omega_2^2-2a^{-1}\gamma_1\omega_3\omega_2^2-$
$-2\gamma_3\omega_1\omega_2]\varphi_{\omega_1}+[(\gamma_0-\gamma_2)\omega_2-\gamma_3\omega_2^2]\varphi_{\omega_2}+$
$+[a\gamma_1+(\gamma_2-\gamma_0)(\omega_3+1)]\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);$

Table 2. $\tilde{P}(1,3)$-invariant anzätze for the spinor field

No.	$A(x)$	$\omega_1(x)$	$\omega_2(x)$	$\omega_3(x)$
1	$(x_0^2-x_2^2)^{-k}\exp\left(\frac{1}{2a}\gamma_1(\gamma_2-\gamma_0)\ln(x_0-x_2)\right)$	$(x_0^2-x_1^2-x_2^2)x_3^{-2}$	$(x_0-x_2)x_3^{-1}$	$ax_1(x_0-x_2)^{-1}-\ln(x_0-x_2)$
2	$\exp\left[-\frac{1}{2}x_1(x_0-x_2)^{-1}\gamma_1(\gamma_0-\gamma_2)\right]$	x_0-x_2	x_3	$x_0^2-x_1^2-x_2^2$
3	$\exp\left[\frac{1}{2}(x_2-x_0)\gamma_1(\gamma_2-\gamma_0)\right]$	$x_1+\beta(x_0-x_2)$	$2x_1+(x_0-x_2)^2$	$3x_3+3x_1(x_0-x_2)+(x_0-x_2)^3$
4	$\exp\left(\frac{1}{2\beta}x_3\gamma_1(\gamma_2-\gamma_0)\right)$	x_0-x_2	$x_1^2-x_1^2-x_2^2$	$\beta x_1-(x_0-x_2)x_3$
5	$(x_2^2+x_3^2)^{-k/2}\exp\left[-\frac{1}{2}\gamma_2\gamma_3\operatorname{arctg}\frac{x_2}{x_3}\right]$	$x_0x_1^{-1}$	$\ln(x_2^2+x_3^2)+2\operatorname{arctg}\frac{x_2}{x_3}$	$(x_2^2+x_3^2)(x_0^2-x_1^2)^{-1}$
6	$(x_0^2-x_1^2)^{-k/2}\exp\left(\frac{1}{2(a+1)}\gamma_0\gamma_1\ln(x_0+x_1)-\right.$ $\left.-\frac{1}{2}\gamma_2\gamma_3\operatorname{arctg}\frac{x_2}{x_3}\right)$	$(x_0^2-x_1^2)^{-a-1}(x_0+x_1)^{2a}$	$(x_0^2-x_1^2)(x_2^2+x_3^2)^{-1}$	$b\ln(x_2^2+x_3^2)+2a\operatorname{arctg}\frac{x_2}{x_3}$
7	$(x_0^2-x_1^2)^{-k/2}\exp\left[-\frac{1}{4}\gamma_0\gamma_1\ln(x_0-x_1)-\right.$ $\left.-\frac{1}{2}\gamma_2\gamma_3\operatorname{arctg}\frac{x_2}{x_3}\right]$	x_0+x_1	$(x_0^2-x_1^2)(x_2^2+x_3^2)^{-1}$	$b\ln(x_2^2+x_3^2)-2\operatorname{arctg}\frac{x_2}{x_3}$
8	$\exp\left[\frac{1}{2}\gamma_0\gamma_1\ln(x_0+x_1)-\frac{1}{2}\gamma_2\gamma_3\operatorname{arctg}\frac{x_2}{x_3}\right]$	$x_0^2-x_1^2$	$x_2^2+x_3^2$	$b\ln(x_0+x_1)+\operatorname{arctg}\frac{x_2}{x_3}$
9	$(2x_0+2x_1+\beta)^{-k/2}\exp\left[\frac{1}{4}\gamma_0\gamma_1\ln(2x_0+\right.$ $\left.+2x_1+\beta)-\frac{1}{2}\gamma_2\gamma_3\operatorname{arctg}\frac{x_2}{x_3}\right)$	$(2x_0+2x_1+\beta)\times$ $\times\exp\left(\frac{2}{\beta}(x_1-x_0)\right)$	$(2x_0+2x_1+\beta)\times$ $\times(x_2^2+x_3^2)^{-1}$	$b\ln(x_2^2+x_3^2)+2\operatorname{arctg}\frac{x_2}{x_3}$
10	$\exp\left[\frac{1}{2}\gamma_0\gamma_1\ln(x_0+x_1)\right]$	$x_0^2-x_1^2$	$\ln(x_0+x_1)-x_2$	x_3
11	$\exp\left[-\frac{1}{2}\gamma_2\gamma_3\operatorname{arctg}\frac{x_2}{x_3}\right]$	$x_2^2+x_3^2$	$\operatorname{arctg}\frac{x_2}{x_3}+\beta_1x_0+\beta_2x_1$	$\alpha_2x_0+\alpha_1x_1$
12	$x_0^{-k}I$	$x_1x_0^{-1}$	$x_2x_0^{-1}$	$x_3x_0^{-1}$
13	I	x_0+x_3	x_1	x_2
14	I	x_1	x_2	x_3
15	I	x_0	x_1	x_2

Notation: I is the unit 4×4 matrix; a, b, α, α_1, α_2, β, β_1, β_2 are arbitrary real numbers.

$$
\begin{aligned}
&(2)\quad \frac{1}{2}\omega_1^{-1}(\gamma_0-\gamma_2)\varphi+(\gamma_0-\gamma_2)\varphi_{\omega_1}+\gamma_3\varphi_{\omega_2}+\\
&\qquad +[(\gamma_0+\gamma_2)\omega_1+(\gamma_0-\gamma_2)\omega_3\omega_1^{-1}]\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(3)\quad [\gamma_3+\beta(\gamma_0-\gamma_2)]\varphi_{\omega_1}+2\gamma_1\varphi_{\omega_2}+\frac{3}{2}(2\gamma_2+(\gamma_0-\gamma_2)\omega_2)\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(4)\quad \frac{1}{2}\beta^{-1}\gamma_4(\gamma_0-\gamma_2)\varphi+(\gamma_0-\gamma_2)\varphi_{\omega_1}+[(\gamma_0+\gamma_2)\omega_1-2\beta^{-1}\omega_3\gamma_1+\\
&\qquad +(\gamma_0-\gamma_2)(\beta^{-2}\omega_3^2+\omega_2)\omega_1^{-1}]\varphi_{\omega_2}+(\beta\gamma_1-\gamma_3\omega_1)\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(5)\quad \frac{1}{2}(1-2k)\gamma_3\varphi+(\omega_1\omega_3)^{1/2}(\gamma_0-\gamma_1\omega_1)\varphi_{\omega_1}+2(\gamma_3+a\gamma_2)\varphi_{\omega_2}+\\
&\qquad +[2\gamma_3-(\gamma_0+\gamma_1\omega_1)\omega_3^{1/2}\omega_1^{-1/2}]\omega_3\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(6)\quad \Big[-k(\gamma_0\cosh\ln\omega_1^{1/2(a+1)}-\gamma_1\sinh\ln\omega_1^{1/2(a+1)})+\\
&\qquad +\frac{1}{2(a+1)}(\gamma_0+\gamma_1)\omega_1^{-1/2(a+1)}+\frac{1}{2}\gamma_3\omega_2^{1/2}\Big]\varphi-\\
&\qquad -2(a+1)(\gamma_0\cosh\ln\omega_1^{1/2(a+1)}-\gamma_1\sinh\ln\omega_1^{1/2(a+1)})\omega_1\varphi_{\omega_1}+\\
&\qquad +2[\omega_2(\gamma_0\cosh\ln\omega_1^{1/2(a+1)}-\gamma_1\sinh\ln\omega_1^{1/2(a+1)})-\omega_2^{3/2}\gamma_3]\varphi_{\omega_2}+\\
&\qquad +2(a\gamma_2+b\gamma_3)\omega_2^{1/2}\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(7)\quad \left[-k(\gamma_0\cosh\ln\omega_1^{1/2}-\gamma_1\sinh\ln\omega_1^{1/2})+\frac{1}{4}(\gamma_0-\gamma_1)\omega_1^{1/2}+\frac{1}{2}\gamma_3\omega_2^{1/2}\right]\varphi+\\
&\qquad +(\gamma_0+\gamma_1)\omega_1^{1/2}\varphi_{\omega_1}+2\omega_2(\gamma_0\cosh\ln\omega_1^{1/2}-\gamma_1\sinh\ln\omega_1^{1/2}-\gamma_3\omega_2^{1/2})\varphi_{\omega_2}+\\
&\qquad +2\omega_2^{1/2}(b\gamma_3-\gamma_2)\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(8)\quad \frac{1}{2}(\gamma_0+\gamma_1+\gamma_3\omega_3^{-1/2})\varphi+[\omega_1(\gamma_0+\gamma_1)+\gamma_0-\gamma_1)\varphi_{\omega_1}+2\gamma_3\omega_2^{-1/2}\varphi_{\omega_2}+\\
&\qquad +[b(\gamma_0+\gamma_1)+\gamma_2\omega_2^{-1/2}]\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(9)\quad \frac{1}{2}[(1-2k)(\gamma_0+\gamma_1)+\gamma_3\omega_2^{1/2}]\varphi+2\beta^{-1}\omega_1[\beta(\gamma_0+\gamma_1)-\gamma_0+\gamma_1]\varphi_{\omega_1}+\\
&\qquad +2\omega_2(\gamma_0+\gamma_1-\gamma_3\omega_2^{1/2})\varphi_{\omega_2}+2\omega_2^{1/2}(\gamma_2+b\gamma_3)\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(10)\quad \frac{1}{2}(\gamma_0+\gamma_1)\varphi+[\omega_1(\gamma_0+\gamma_1)+\gamma_0-\gamma_1]\varphi_{\omega_1}+\\
&\qquad +(\gamma_0+\gamma_1-\gamma_2)\varphi_{\omega_2}+\gamma_3\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(11)\quad \frac{1}{2}\gamma_3\omega_1^{-1/2}\varphi+2\gamma_3\omega_1^{1/2}\varphi_{\omega_1}+(\beta_1\gamma_0+\beta_2\gamma_1+\gamma_2\omega_1^{-1/2})\varphi_{\omega_2}+\\
&\qquad +(\alpha_2\gamma_0+\alpha_1\gamma_1)\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(12)\quad -k\gamma_0\varphi+(\gamma_a-\gamma_0\omega_a)\varphi_{\omega_a}=-i\Phi_2(\bar\varphi,\varphi);\\
&(13)\quad (\gamma_0+\gamma_3)\varphi_{\omega_1}+\gamma_1\varphi_{\omega_2}+\gamma_2\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(14)\quad \gamma_1\varphi_{\omega_1}+\gamma_2\varphi_{\omega_2}+\gamma_3\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi);\\
&(15)\quad \gamma_0\varphi_{\omega_1}+\gamma_1\varphi_{\omega_2}+\gamma_2\varphi_{\omega_3}=-i\Phi_2(\bar\varphi,\varphi),
\end{aligned}
\tag{3.16}
$$

where $\varphi_{\omega_a}\equiv\partial\varphi/\partial\omega_a$, $a=1,2,3$,

$$\Phi_2\equiv[(\tilde F_1+\tilde F_2\gamma_4)(\bar\varphi\varphi)^{1/2k}]\varphi,\quad \tilde F_i=\tilde F_i(\bar\varphi\varphi/\bar\varphi\gamma_4\varphi).$$

The group-theoretical properties of eqs. (3.16) were investigated in ref. [29]. We consider in more detail the PDE (3) and (13)–(15) of (3.16). Using the Lie method [32–34] one can prove the following statements.

Proposition 1. *PDE (13) of (3.16) is invariant under the infinite-parameter Lie group, its generators being of the form*

$k=1$*:*

$$
\begin{aligned}
&Q_1=\phi_1(\omega_1)\partial_{\omega_2}+\phi_2(\omega_1)\partial_{\omega_3}+\frac{1}{2}[\dot{\phi}_1(\omega_1)\gamma_1+\dot{\phi}_2(\omega_1)\gamma_2](\gamma_0+\gamma_3),\\
&Q_2=-\omega_2\partial_{\omega_3}+\omega_3\partial_{\omega_2}+\frac{1}{2}\gamma_1\gamma_2,\\
&Q_3=\phi_0(\omega_1)\partial_{\omega_1}+\dot{\phi}_0(\omega_1)(\omega_2\partial_{\omega_2}+\omega_3\partial_{\omega_3})+\dot{\phi}_0(\omega_1)+\\
&\qquad+\frac{1}{2}\ddot{\phi}_0(\omega_1)(\gamma_1\omega_2+\gamma_2\omega_3)(\gamma_0+\gamma_3),\\
&Q_4=\phi_3(\omega_1)\gamma_4(\gamma_0+\gamma_3);
\end{aligned}
\tag{3.17}
$$

$k\neq 1$*:*

$$
\begin{aligned}
&Q_1=\partial_{\omega_1},\quad Q_2=-\omega_2\partial_{\omega_3}+\omega_3\partial_{\omega_2}+\frac{1}{2}\gamma_1\gamma_2,\\
&Q_3=\phi_1(\omega_1)\partial_{\omega_2}+\phi_2(\omega_1)\partial_{\omega_3}+\frac{1}{2}[\dot{\phi}_1(\omega_1)\gamma_1+\dot{\phi}_2(\omega_1)\gamma_2](\gamma_0+\gamma_3),\\
&Q_4=\omega_1\partial_{\omega_1}+\omega_2\partial_{\omega_2}+\omega_3\partial_{\omega_3}+k,\quad Q_5=\phi_3(\omega_1)\gamma_4(\gamma_0+\gamma_3),
\end{aligned}
$$

where $\phi_0,\ldots,\phi_3$ are arbitrary smooth functions, a dot means differentiation with respect to ω_1.

Proposition 2. *For $k=1$ PDE (14) and (15) of (3.16) are invariant under the conformal groups $C(3)$ and $C(1,2)$, respectively.*

It is important to note that for $k\neq 3/2$ the initial equation (3.14) is not conformally invariant. The same statement holds for the infinite-parameter group with generators (3.17). Consequently for $k=1$ the PDE (3.14) is conditionally invariant under the algebras (3.17), $AC(3)$ and $AC(1,2)$. Using this fact we have constructed ansatze which are principally different from ones listed in table 1:

$k=1$:

$$
\begin{aligned}
\psi(x)=\;&\phi_0^{-1}\exp\Big\{\phi_3\gamma_4(\gamma_0+\gamma_3)-\frac{1}{2}(\dot{\phi}_1\gamma_1+\dot{\phi}_2\gamma_2)(\gamma_0+\gamma_3)-\\
&-\frac{1}{2}\dot{\phi}_0\phi_0^{-1}[\gamma_1(x_1+\phi_1)+\gamma_2(x_2+\phi_2)](\gamma_0+\gamma_3)\Big\}\times\\
&\times\begin{cases}\varphi_1((x_1+\phi_1)/\phi_0),\\[4pt] \exp\left(-\dfrac{1}{2}\gamma_1\gamma_2\operatorname{arctg}\dfrac{x_1+\phi_1}{x_2+\phi_2}\right)\varphi_2([(x_1+\phi_1)^2+(x_2+\phi_2)^2]/\phi_0^2),\end{cases}
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
\psi(x)=\;&\frac{\gamma_0x_0-\gamma_1x_1-\gamma_2x_2}{(x_0^2-x_1^2-x_2^2)^{3/2}}\times\\
&\times\begin{cases}\varphi_3(x_0/(x_0^2-x_1^2-x_2^2)),\\ \varphi_4(x_1/(x_0^2-x_1^2-x_2^2)),\\ \exp\left[-\dfrac{1}{2}\gamma_1\gamma_2\operatorname{arctg}\dfrac{x_1}{x_2}\right]\varphi_5((x_1^2+x_2^2)/(x_0^2-x_1^2-x_2^2)^2),\end{cases}
\end{aligned}
\tag{3.19}
$$

$$\psi(x) = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{(\boldsymbol{x}^2)^{3/2}} \times \begin{cases} \varphi_6(x_1(\boldsymbol{x}^2)^{-1}), \\ \exp\left[-\frac{1}{2}\gamma_1\gamma_2 \operatorname{arctg}\frac{x_1}{x_2}\right]\varphi_7((x_1^2+x_2^2)/(\boldsymbol{x}^2)^2), \end{cases} \tag{3.20}$$

$k \neq 1$:

$$\begin{gathered} \psi(x) = \exp\left[\phi_3\gamma_4(\gamma_0+\gamma_3) - \frac{1}{2}(\dot{\phi}_1\gamma_1 + \dot{\phi}_2\gamma_2)(\gamma_0+\gamma_3)\right] \times \\ \times \begin{cases} \varphi_8(x_1+\phi_1), \\ \exp\left(-\frac{1}{2}\gamma_1\gamma_2 \operatorname{arctg}\frac{x_1+\phi_1}{x_2+\phi_2}\right)\varphi_9((x_1+\phi_1)^2+(x_2+\phi_2)^2), \end{cases} \end{gathered} \tag{3.21}$$

In eqs. (3.18)–(3.21) $\phi_0, \ldots, \phi_3$ are arbitrary smooth functions of $x_0 + x_3$, $\varphi_1, \ldots, \varphi_9$ are new unknown spinors. While obtaining formulae (3.19)–(3.21) we essentially used the conformally invariant ansatz suggested in refs. [20, 21] and the results of refs. [24, 29].

Let us turn now to eq. (3) of (3.16). If one chooses $\varphi = \varphi(\omega_1, \omega_2)$ and introduces the notations

$$\Gamma_1 = \gamma_3 + \beta(\gamma_0 - \gamma_2), \quad \Gamma_2 = \gamma_1, \quad z_1 = \omega_1, \quad z_2 = \frac{1}{2}\omega_2,$$

then one obtains the following PDE:

$$\Gamma_1\varphi_{z_1} + \Gamma_2\varphi_{z_2} = -\Phi_2(\bar{\varphi}, \varphi), \tag{3.22}$$

where $\Gamma_1^2 = \Gamma_2^2 = -1$, $\Gamma_1\Gamma_2 + \Gamma_2\Gamma_1 = 0$.

With the aid of the Lie method [32–34] it is possible to prove that eq. (3.22) is invariant under the conformal group $C(2)$ if $k = 1/2$ [consequently, for $k = 1/2$ the initial PDE (3.14) is conditionally invariant under the conformal group $C(2)$]. Using this fact we have constructed the ansatz that reduces (3.14) to a system of ODE [24]:

$$\begin{gathered} \psi(x) = \rho^{-1}\exp\left[\frac{1}{2}\gamma_1(\gamma_0-\gamma_2)(x_0-x_2)\right] \times \\ \times \left\{[\gamma_3+\beta(\gamma_0-\gamma_2)][x_3+\beta(x_0-x_2)] + \frac{1}{2}\gamma_1(2x_1+(x_0-x_2)^2)\right\}\varphi(\omega), \end{gathered} \tag{3.23}$$

where

$$\omega = \left\{\beta_1[x_3+\beta(x_0-x_2)] + \frac{1}{2}\beta_2[2x_1+(x_0-x_2)^2]\right\}\rho^{-1},$$

$$\rho = [x_3+\beta(x_0-x_2)]^2 + \frac{1}{4}[2x_1+(x_0-x_2)^2]^2,$$

β, β_1, β_2 are constants.

3.2. Reduction of the nonlinear Dirac equation to systems of ODE. To reduce the nonlinear Dirac equation (3.10) via ansatze from table 1 one has to make rather cumbersome calculations. Therefore we give the final result, systems of ODE for $\varphi(\omega)$, omitting intermediate calculations.

$$
\begin{aligned}
&(1) \quad i\gamma_2\dot{\varphi} = \Phi_1,\\
&(2) \quad i\gamma_0\dot{\varphi} = \Phi_1,\\
&(3) \quad i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(4) \quad \frac{1}{2}i(\gamma_0+\gamma_3)\varphi + i[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3]\dot{\varphi} = \Phi_1,\\
&(5) \quad \frac{1}{2}i(\gamma_0+\gamma_3)\varphi + i\gamma_2\dot{\varphi} = \Phi_1,\\
&(6) \quad -\frac{i}{2\alpha}\gamma_1\gamma_4\varphi + i\gamma_1\dot{\varphi} = \Phi_1,\\
&(7) \quad -\frac{i}{2\alpha}\gamma_1\gamma_4\varphi + i[\alpha(\gamma_0+\gamma_3)e^{-\omega/\alpha}-\gamma_2]\dot{\varphi} = \Phi_1,\\
&(8) \quad \frac{1}{2}i\omega^{-1/2}\gamma_2\varphi + 2i\omega^{1/2}\gamma_2\dot{\varphi} = \Phi_1,\\
&(9) \quad -\frac{i}{2\alpha}\gamma_3\gamma_4\varphi + i\gamma_3\dot{\varphi} = \Phi_1,\\
&(10) \quad \frac{i}{2\alpha}\gamma_0\gamma_4\varphi + i\gamma_0\dot{\varphi} = \Phi_1,\\
&(11) \quad \frac{1}{4}i(\gamma_0-\gamma_3)\gamma_4\varphi + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(12) \quad \frac{1}{2}i\omega^{-1}(\gamma_0+\gamma_3)\varphi + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(13) \quad \frac{i}{2\alpha}\omega^{-1}(\alpha+\gamma_4)(\gamma_0+\gamma_3)\varphi + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(14) \quad \frac{1}{2}i(\gamma_0+\gamma_3)\gamma_4\varphi + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(15) \quad 2i\gamma_1\dot{\varphi} = \Phi_1,\\
&(16) \quad 2i(\gamma_2-\alpha\gamma_1)\dot{\varphi} = \Phi_1,\\
&(17) \quad \frac{i}{2\alpha}\omega^{-1/2}\gamma_2(\alpha-\gamma_4)\varphi + 2i\omega^{1/2}\gamma_2\dot{\varphi} = \Phi_1,\\
&(18) \quad \frac{1}{2}i(\gamma_0+\gamma_3)(1+\alpha\gamma_4)\varphi + i[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3]\dot{\varphi} = \Phi_1,\\
&(19) \quad i\omega^{-1}(\gamma_0+\gamma_3)\varphi + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(20) \quad \frac{1}{2}i[\omega(\omega+\beta)-\alpha]^{-1}(\gamma_0+\gamma_3)\{[2\omega(\omega+\beta)-\alpha-1]\gamma_4-2\omega-\beta\}\varphi +\\
&\qquad\qquad + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(21) \quad \frac{1}{2}i[\omega(\omega+\beta)]^{-1}(\gamma_0+\gamma_3)(2\omega+\beta-\gamma_4)\varphi + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(22) \quad \frac{1}{2}i[\omega(\omega+1)]^{-1}(2\omega+1)(\gamma_0+\gamma_3)\varphi + i(\gamma_0+\gamma_3)\dot{\varphi} = \Phi_1,\\
&(23) \quad i(\gamma_0+\gamma_3)\varphi + i[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3]\dot{\varphi} = \Phi_1,\\
&(24) \quad i(\gamma_0+\gamma_3)\varphi + i[\gamma_2-\beta(\gamma_0+\gamma_3)]\dot{\varphi} = \Phi_1,\\
&(25) \quad i(\gamma_0+\gamma_3)\dot{\varphi} + i\left[(\gamma_0+\gamma_3)\omega^{-1} + \frac{1}{4}(\gamma_0-\gamma_3)\gamma_4\right]\varphi = \Phi_1,\\
&(26) \quad \frac{1}{2}i(\gamma_0+\gamma_3)(3+\alpha\gamma_4)\varphi + i[(\gamma_0+\gamma_3)\omega+\gamma_0-\gamma_3]\dot{\varphi} = \Phi_1,
\end{aligned}
\tag{3.24}
$$

where

$$\Phi_1 \equiv (F_1 + F_2\gamma_4)\varphi, \quad F_i = F_i(\bar{\varphi}\varphi, \bar{\varphi}\gamma_4\varphi), \quad \dot{\varphi} \equiv d\varphi/d\omega.$$

To integrate eqs. (3.24) one can again apply group-theoretical methods. In ref. [29] it was pointed out how to obtain some information about the symmetry of the reduced PDE by purely algebraic methods (without application of the infinitesimal Lie method). It is based on the following statement:

Let G be a Lie group of transformations, H be a normal divisor in G. And let there be a PDE invariant under the group G.

Theorem 5. *The equation obtained via reduction with the help of H-invariant solutions admits the factor group G/H.*

A proof can be found in ref. [33].

We use the equivalent formulation of this theorem in terms of Lie algebras: If there is a PDE with the symmetry algebra AG and subalgebra Q which is an ideal in AG, then the equation obtained via reduction with the help of Q-invariant solutions admits the Lie algebra AG/Q.

Straightforward application of the above theorem to the three-dimensional algebras listed in the table 1 is impossible because these algebras are not, in general, ideals in $AP(1,3)$. Therefore there arises the intermediate problem of constructing the maximal subalgebras $A_1, \ldots, A_{26}$ of the algebra $AP(1,3)$ having the algebras of the table 1 as ideals.

It is known from the theory of Lie algebras [33] that the algebra $\langle Q_1, Q_2, Q_3\rangle$ is the ideal in the Lie algebra $\langle \Sigma_1, \Sigma_2, \ldots, \Sigma_s\rangle$ iff

$$[Q_i, \Sigma_j] = \lambda_{ij}^k Q_k, \quad \lambda_{ij}^k = \text{const},$$

where $[Q_i, \Sigma_j]$ is the commutator, and summation over repeated indices is understood.

Consequently, the operator $\theta_k^{\mu\nu} J_{\mu\nu} + \theta_k^{\mu} P_\mu$ belongs to the algebra A_k iff

$$[Q_i, \theta_k^{\mu\nu} J_{\mu\nu} + \theta_k^{\mu} P_\mu] = \lambda_{ik}^j Q_j, \quad i = 1,2,3, \quad k = 1, \ldots, 26. \tag{3.25}$$

Here $\theta_k^{\mu\nu}$, θ_k^{μ} and λ_{ik}^j are constants; Q_1, Q_2, Q_3 is the triplet of operators in table 1 under number k.

When one calculates the commutators on the left-hand sides of equalities (3.25) and equates the coefficients to zero at linearly independent operators $J_{\mu\nu}$ and P_μ, one obtains a system of algebraic equations for $\theta_i^{\mu\nu}$ and θ_i^{μ}. The solution of these equations gives the explicit expression for the basis operators of the algebras A_1 to A_{26}.

The next step is the calculation of the factor algebras $\{A_i/Q_i,\ i = 1, \ldots, 26\}$, which generate the invariance groups of the reduced equations (3.24). We shall realize the above scheme for the algebra $\langle P_0, P_1, P_2\rangle$, the remaining algebras being treated in the same way. To do this one needs the commutation relations of the algebra $AP(1,3)$ [31],

$$\begin{aligned} &[J_{\mu\nu}, J_{\alpha\beta}] = i(g_{\mu\beta}J_{\nu\alpha} + g_{\nu\alpha}J_{\mu\beta} - g_{\mu\alpha}J_{\nu\beta} - g_{\nu\beta}J_{\mu\alpha}), \\ &[P_\mu, J_{\alpha\beta}] = i(g_{\mu\alpha}P_\beta - g_{\mu\beta}P_\alpha), \quad [P_\mu, P_\nu] = 0. \end{aligned} \tag{3.26}$$

Relations (3.25) are rewritten for $Q_1 = P_0$, $Q_2 = P_1$, $Q_3 = P_2$ in the following way:

$$\begin{aligned} &[P_0, \theta_1^{\mu\nu} J_{\mu\nu} + \theta_1^{\mu} P_\mu] = \lambda_{11}^1 P_0 + \lambda_{11}^2 P_1 + \lambda_{11}^3 P_2, \\ &[P_1, \theta_1^{\mu\nu} J_{\mu\nu} + \theta_1^{\mu} P_\mu] = \lambda_{21}^1 P_0 + \lambda_{21}^2 P_1 + \lambda_{21}^3 P_2, \\ &[P_2, \theta_1^{\mu\nu} J_{\mu\nu} + \theta_1^{\mu} P_\mu] = \lambda_{31}^1 P_0 + \lambda_{31}^2 P_1 + \lambda_{31}^3 P_2. \end{aligned} \tag{3.27}$$

Taking into account relations (3.26) one obtains the following equalities:

$$
\begin{aligned}
&2i\theta_1^{0\mu} P_\mu = \lambda_{11}^1 P_0 + \lambda_{11}^2 P_1 + \lambda_{11}^3 P_2,\\
&-2i\theta_1^{1\mu} P_\mu = \lambda_{21}^1 P_0 + \lambda_{21}^2 P_1 + \lambda_{21}^3 P_2,\\
&-2i\theta_1^{2\mu} P_\mu = \lambda_{31}^1 P_0 + \lambda_{31}^2 P_1 + \lambda_{31}^3 P_2,
\end{aligned}
\tag{3.28}
$$

whence it follows that $\theta_1^{03} = \theta_1^{13} = \theta_1^{23} = 0$, θ_1^{01}, θ_1^{02}, θ_1^{12}, θ_1^{μ} are arbitrary real parameters. Consequently, the set of linearly independent solutions of the system (3.27) is exhausted by the operators

$$\langle J_{01}, J_{02}, J_{12}, P_0, P_1, P_2, P_3\rangle = A_1,$$

whence one obtains

$$A_1/\langle P_0, P_1, P_2\rangle = \langle J_{01}, J_{02}, J_{12}, P_3\rangle. \tag{3.29}$$

To construct the invariance algebra of eq. (1) of (3.24) it is necessary to rewrite the operators (3.29) in the new variables ω, φ. As a result one has

(1) $\langle \gamma_0\gamma_1, \gamma_0\gamma_2, \gamma_1\gamma_2, \partial_\omega\rangle$.

The invariance algebras of the other equations of (3.24) are as follows:

(2) $\langle \gamma_1\gamma_2, \gamma_2\gamma_3, \gamma_1\gamma_3, \partial_\omega\rangle$;
(3) $\langle \gamma_1(\gamma_0+\gamma_3), \gamma_2(\gamma_0+\gamma_3), \gamma_1\gamma_2, \omega\partial_\omega - \frac{1}{2}\gamma_0\gamma_3, \partial_\omega\rangle$;
(4) $\langle \gamma_1\gamma_2\rangle$;
(5) $\langle \partial_\omega\rangle$;
(6) $\langle \gamma_0\gamma_3, \partial_\omega\rangle$;
(7) $\langle 2\alpha\partial_\omega - \gamma_0\gamma_3\rangle$;
(8) $\langle \gamma_0\gamma_3\rangle$;
(9) $\langle \partial_\omega, \gamma_1\gamma_2\rangle$;
(10) $\langle \partial_\omega, \gamma_1\gamma_2\rangle$;
(11) $\langle \partial_\omega, \gamma_1\gamma_2\rangle$;
(12) $\langle \gamma_2(\gamma_0+\gamma_3), \omega^{-1}\gamma_1(\gamma_0+\gamma_3), \omega\partial_\omega - \frac{1}{2}\gamma_0\gamma_3\rangle$;
(13) $\langle (\gamma_1+\alpha\gamma_2)(\gamma_0+\gamma_3), \omega^{-1}\gamma_1(\gamma_0+\gamma_3), \omega\partial_\omega - \frac{1}{2}\gamma_0\gamma_3\rangle$;
(14) $\langle \gamma_1(\gamma_0+\gamma_3), \gamma_2(\gamma_0+\gamma_3), \partial_\omega\rangle$;
(15) $\langle \partial_\omega, \gamma_2(\gamma_0+\gamma_3)\rangle$;
(16) $\langle \partial_\omega\rangle$;
(17) $\langle \gamma_0\gamma_3\rangle$;

$$
\begin{aligned}
&(18)\quad \langle\gamma_1\gamma_2\rangle;\\
&(19)\quad \langle\omega\partial_\omega - \frac{1}{2}\gamma_0\gamma_3, \omega^{-1}\gamma_1(\gamma_0+\gamma_3), \omega^{-2}\gamma_2(\gamma_0+\gamma_3), \gamma_1\gamma_2\rangle;\\
&(20)\quad \langle[\omega(\omega+\beta)-\alpha]^{-1}(\gamma_0+\gamma_3)[(\omega+\beta)\gamma_1-\gamma_2],\\
&\qquad\qquad [\omega(\omega+\beta)-\alpha]^{-1}(\gamma_0+\gamma_3)(\omega\gamma_2-\alpha\gamma_1), \gamma_1\gamma_2\rangle;\\
&(21)\quad \langle\omega^{-1}\gamma_1(\gamma_0+\gamma_3), [\omega(\omega+\beta)](\gamma_0+\gamma_3)(\omega\gamma_2-\gamma_1)\rangle;\\
&(22)\quad \langle\omega^{-1}\gamma_1(\gamma_0+\gamma_3), (\omega+1)^{-1}(\gamma_0+\gamma_3)\gamma_2\rangle;\\
&(23)\quad \langle\gamma_1\gamma_2\rangle;\\
&(24)\quad \langle\partial_\omega, \gamma_1(\gamma_0+\gamma_3)\rangle;\\
&(25)\quad \langle\gamma_1\gamma_2\rangle;\\
&(26)\quad \langle\gamma_1\gamma_2\rangle.
\end{aligned}
\tag{3.30}
$$

Here $\langle Q_1,\ldots,Q_s\rangle$ denotes the set of all linear combinations of the operators $Q_1,\ldots,Q_s$.

Let us note that the Lie algebras (3.30) are not, in general, the maximal invariance algebras of the equations of (3.24). As an example we shall consider eq. (3). By direct verification one can check that this equation is invariant under the infinite-parameter group of the form

$$
\omega' = \omega, \quad \varphi' = \exp\{[f_1(\omega)\gamma_1 + f_2(\omega)\gamma_2](\gamma_0+\gamma_3)\}\varphi,
\tag{3.31}
$$

where $f_i(\omega)$ are arbitrary smooth functions; the Lie group generated by the operators $\gamma_1(\gamma_0+\gamma_3)$, $\gamma_2(\gamma_0+\gamma_3)$ in line (3) of (3.30) is a two-parameter subgroup of the group (3.31).

Nevertheless, the information obtained about the symmetry of the ODE (3.24) proves to be very useful while constructing their particular solutions. Besides if an ODE has a lagrangian then one can construct its first integrals using Noether's theorem.

Let us also stress that an arbitrary Poincaré-invariant equation for a spinor field, after being reduced to systems of ODE with the help of the ansatze of table 1, possesses the symmetry (3.30).

Let us turn to the system (3.14). Substitution of the ansätze (3.18)–(3.21) into (3.14) gives rise to the following systems of equations for the spinors $\varphi_1,\ldots,\varphi_9$:

$k=1$:

$$
\begin{aligned}
&(1)\quad i\gamma_1\dot\varphi_1 = \Phi_2(\bar\varphi_1,\varphi_1);\\
&(2)\quad \frac{1}{2}iz_2^{-1/2}\gamma_2\varphi_2 + 2i\gamma_2 z_2^{1/2}\dot\varphi_2 = \Phi_2(\bar\varphi_2,\varphi_2);\\
&(3)\quad -i\gamma_0\dot\varphi_3 = \Phi_2(\bar\varphi_3,\varphi_3);\\
&(4)\quad -i\gamma_1\dot\varphi_4 = \Phi_2(\bar\varphi_4,\varphi_4);\\
&(5)\quad \frac{1}{2}iz_5^{-1/2}\gamma_2\varphi_5 + 2iz_5^{1/2}\gamma_2\dot\varphi_5 = -\Phi_2(\bar\varphi_5,\varphi_5);\\
&(6)\quad -i\gamma_1\dot\varphi_6 = \Phi_2(\bar\varphi_6,\varphi_6);\\
&(7)\quad \frac{1}{2}iz_7^{-1/2}\gamma_2\varphi_7 + 2iz_7^{1/2}\gamma_2\dot\varphi_7 = -\Phi_2(\bar\varphi_7,\varphi_7);
\end{aligned}
\tag{3.32}
$$

$k\in\mathbb{R}^1$:

$$
(8)\quad i\gamma_1\dot\varphi_8 = \Phi_2(\bar\varphi_8,\varphi_8);
$$

$$(9)\quad \frac{1}{2}iz_9^{-1/2}\gamma_2\varphi_9 + 2iz_9^{1/2}\gamma_2\dot{\varphi}_9 = \Phi_2(\bar{\varphi}_9, \varphi_9),$$

where $\varphi_l = \varphi_l(z_l)$, z_l being determined by formulae (3.18)–(3.21).

Following ref. [24] we obtain via direct reduction of (3.16) to ODE equations of the form (the number of PDE from which the ODE is obtained is given in parentheses)

$$\begin{aligned}
&(1)\quad \frac{1}{2}(1-2k)\gamma_3\varphi + 2(\gamma_3 + a\gamma_2)\varphi_{\omega_2} = -i\Phi_2(\bar{\varphi}, \varphi) \qquad (5);\\
&(2)\quad \frac{1}{2}(\gamma_0 + \gamma_1 + \gamma_3\omega_2^{-1/2})\varphi + 2\gamma_3\omega_2^{1/2}\varphi_{\omega_2} = -i\Phi_2(\bar{\varphi}, \varphi) \qquad (8);\\
&(3)\quad \frac{1}{2}[(1-2k)(\gamma_0+\gamma_1) + \gamma_3\omega_2]\varphi + \\
&\qquad\qquad + 2\omega_2^{1/2}(\gamma_0 + \gamma_1 - \gamma_3\omega_2^{1/2})\varphi_{\omega_2} = -i\Phi_2(\bar{\varphi}, \varphi) \qquad (9);\\
&(4)\quad -k\gamma_0\varphi + (\gamma_1 - \omega_1\gamma_0)\varphi_{\omega_1} = -i\Phi_2(\bar{\varphi}, \varphi) \qquad (12).
\end{aligned} \tag{3.33}$$

So we have constructed systems of ODE whose solutions, when substituted into corresponding ansatze, give rise to exact solutions of the initial nonlinear Dirac equation.

3.3. Exact solutions of the nonlinear Dirac–Heisenberg equation. To integrate eqs. (3.24), (3.32) and (3.33) one can apply various methods. We restrict ourselves to those ODE which can be integrated in quadrature. Let us put $\Phi_1 \equiv \Phi_2 \equiv \lambda(\bar{\psi}\psi)^{1/2k}\psi$, λ and k const. Then the PDE (3.10) and (3.14) take the form

$$[\gamma_\mu p^\mu - \lambda(\bar{\psi}\psi)^{1/2k}]\psi(x) = 0. \tag{3.34}$$

The PDE (3.34) was suggested by W. Heisenberg [4, 36] as a possible basic equation for the unified field theory. According to theorems 2 and 3 it is invariant under the extended Poincaré group $P(1,3)$. In the case $k = 3/2$, eq. (3.34) admits the conformal group $C(1,3)$. Therefore, to reduce the PDE (3.34) one can apply both the ansätze of table 1 and of table 2. As a result we obtain eqs. (3.24), (3.22) and (3.33), where

$$\Phi_1(\bar{\varphi}, \varphi) \equiv \Phi_2(\bar{\varphi}, \varphi) \equiv \lambda(\bar{\varphi}, \varphi)^{1/2k}\varphi.$$

If one multiplies ODE (3) of (3.24) by $\gamma_0 + \gamma_3$ and uses the identity $(\gamma_0 + \gamma_3)^2 = 0$, then the following compatibility condition of eq. (3) of (3.24) appears:

$$(\gamma_0 + \gamma_3)\varphi = 0,$$

whence it easily follows that $\bar{\varphi}\varphi = 0$. So $\bar{\psi}\psi = \bar{\varphi}\varphi = 0$, i.e., the factor $(\bar{\psi}\psi)^{1/2k}$ determining the nonlinear character of the PDE (3.34) vanishes. Analogous results hold for eqs. (12)–(14) and (19)–(22) of (3.24). Such solutions are not considered.

ODE (1), (2), (15), (16) and (24) of (3.24) are trivially integrated if one notes that the condition $\bar{\varphi}\varphi = \text{const}$ holds. Let us consider, for example, eq. (1). After multiplying it by $i\gamma_2$ one obtains

$$\dot{\varphi} = i\lambda(\bar{\varphi}\varphi)^{1/2k}\gamma_2\varphi. \tag{3.35}$$

The conjugate spinor satisfies the following equation:

$$\dot{\bar{\varphi}} = -i\lambda(\bar{\varphi}\varphi)^{1/2k}\bar{\varphi}\gamma_2. \tag{3.36}$$

Multiplying (3.35) by $\bar\varphi$ and (3.36) by φ we come to the equality

$$(d/d\omega)(\bar\varphi\varphi) \equiv \dot{\bar\varphi}\varphi + \bar\varphi\dot\varphi = 0,$$

whence it follows that $\bar\varphi\varphi = \mathrm{const}$. Consequently, the ODE (3.35) is equivalent to a linear one,

$$\dot\varphi = i\lambda C^{1/2k}\gamma_2\varphi, \quad \bar\varphi\varphi = C,$$

whose general solution has the form $\varphi(\omega) = \exp(i\lambda C^{1/2k}\gamma_2\omega)\chi$. Since $\bar\varphi(\omega) = \bar\chi\exp(-i\lambda C^{1/2k}\gamma_2\omega)$, then $\bar\varphi\varphi = \bar\chi\chi$ or $\bar\chi\chi = C$. Finally, the general solution of (3.35) takes the form

$$\varphi(\omega) = \exp[i\lambda(\bar\chi\chi)^{1/2k}\gamma_2\omega]\chi. \tag{3.37}$$

Hereafter χ is an arbitrary constant spinor. Let us note that, taking into account the identity $(i\gamma_2)^2 = 1$, expression (3.37) can be rewritten in the following way:

$$\varphi(\omega) = \{\cosh[\lambda(\bar\chi\chi)^{1/2k}\omega] + i\gamma_2\sinh[\lambda(\bar\chi\chi)^{1/2k}\omega]\}\chi.$$

The general solutions of eqs. (2), (15), (16) and (24) of (3.24) are constructed in the same way. Omitting intermediate calculations we write down the final result,

$$\begin{aligned}
&\varphi(\omega) = \exp[-i\lambda(\bar\chi\chi)^{1/2k}\gamma_0\omega]\chi,\\
&\varphi(\omega) = \exp\left[\frac{1}{2}i\lambda(\bar\chi\chi)^{1/2k}\gamma_1\omega\right]\chi,\\
&\varphi(\omega) = \exp\left(\frac{i\lambda}{2(1+\alpha^2)}(\bar\chi\chi)^{1/2k}(\gamma_2 - \alpha\gamma_1)\omega\right)\chi,\\
&\varphi(\omega) = \exp\{[\gamma_2(\gamma_0+\gamma_3) + i\lambda(\bar\chi\chi)^{1/2k}(\gamma_2 - \beta(\gamma_0+\gamma_3))]\omega\}\chi.
\end{aligned} \tag{3.38}$$

To construct the solution of ODE (6) of (3.24) we use its symmetry properties. Above it was established that this equation is invariant under the Lie algebra $\langle\partial_\omega, \gamma_0\gamma_3\rangle$. We look for the solution which is invariant under the group generated by the operator $Q = \partial_\omega - \theta\gamma_0\gamma_3$, $\theta = \mathrm{const}$, i.e., $\varphi(\omega)$ has to satisfy the additional constraint

$$Q\varphi \equiv (\partial_\omega - \theta\gamma_0\gamma_3)\varphi = 0.$$

The general solution of the above equation is given by the formula

$$\varphi(\omega) = \exp(\theta\gamma_0\gamma_3\omega)\chi_1,$$

where χ_1 is an arbitrary constant spinor. Substituting this expression into the initial ODE one has

$$\left(\theta\gamma_1\gamma_0\gamma_3 - \frac{1}{2\alpha}\gamma_1\gamma_4\right)\exp(\theta\gamma_0\gamma_3\omega)\chi_1 = -i\lambda\tau\exp(\theta\gamma_0\gamma_3\omega)\chi_1,$$

where $\tau = (\bar\chi_1\chi_1)^{1/2k}$. Multiplying this equality by $\exp(-\theta\gamma_0\gamma_3)$ we come to the system of linear algebraic equations for χ_1,

$$\left(\theta\gamma_2 - \frac{1}{2\alpha}\gamma_1\right)\gamma_4\chi_1 = -i\lambda\tau\chi_1. \tag{3.39}$$

The system (3.39) is diagonalized by the following substitution:

$$\chi_1 = \left[\left(\theta\gamma_2 - \frac{1}{2\alpha}\gamma_1\right)\gamma_4 - i\lambda\tau\right]\chi,$$

whence it follows that

$$\left(-\theta^2 - \frac{1}{4\alpha^2} + \lambda^2\tau^2\right)\chi = 0.$$

Consequently

$$\theta = \varepsilon(4\alpha^2\tau^2\lambda^2 - 1)^{1/2}/2\alpha, \quad \varepsilon = \pm 1. \tag{3.40}$$

Imposing on τ the condition $\tau = (\bar{\chi}_1\chi_1)^{1/2k}$ one obtains a nonlinear algebraic equation for τ,

$$\tau^{2k} = 2\lambda^2\tau^2(\bar{\chi}\chi) + 2i\lambda\tau\theta(\bar{\chi}\gamma_2\gamma_4\chi) - i\lambda\tau\alpha^{-1}(\bar{\chi}\gamma_1\gamma_4\chi). \tag{3.41}$$

Finally, the particular solution of ODE (6) of (3.34) takes the form

$$\varphi(\omega) = \exp(\theta\gamma_0\gamma_3\omega)\left[\left(\theta\gamma_2 - \frac{1}{2\alpha}\gamma_1\right)\gamma_4 - i\lambda\tau\right]\chi, \tag{3.42}$$

θ and τ being determined by formulae (3.40) and (3.41).

An analogous method can be applied to construct solutions of equations (9–11), the result being

$$\varphi(\omega) = \exp(\theta\gamma_1\gamma_2\omega)\left[\left(\theta\gamma_0 - \frac{1}{2\alpha}\gamma_3\right)\gamma_4 - i\lambda\tau\right]\chi, \tag{3.43}$$

θ and τ being determined by the formulae

$$\begin{aligned}&\theta = \varepsilon(1 - 4\alpha^2\lambda^2\tau^2)^{1/2}/2\alpha,\\ &\tau^{2k} = 2\lambda^2\tau^2(\bar{\chi}\chi) - i\lambda\tau\alpha^{-1}(\bar{\chi}\gamma_3\gamma_4\chi) + 2i\lambda\tau\theta(\bar{\chi}\gamma_0\gamma_4\chi);\end{aligned} \tag{3.44}$$

$$\varphi(\omega) = \exp(\theta\gamma_1\gamma_2\omega)\left[\left(\theta\gamma_3 + \frac{1}{2\alpha}\gamma_0\right)\gamma_4 - i\lambda\tau\right]\chi, \tag{3.45}$$

θ and τ being determined by the formulae

$$\begin{aligned}&\theta = \varepsilon(4\alpha^2\lambda^2\tau^2 + 1)^{1/2}/2\alpha,\\ &\tau^{2k} = 2\lambda^2\tau^2(\bar{\chi}\chi) + 2i\lambda\tau\theta(\bar{\chi}\gamma_3\gamma_4\chi) + i\lambda\tau\alpha^{-1}(\bar{\chi}\gamma_0\gamma_4\chi);\end{aligned} \tag{3.46}$$

$$\varphi(\omega) = \exp(\theta\gamma_1\gamma_2\omega)[4\theta(\gamma_0 + \gamma_3)\gamma_4 + (\gamma_0 - \gamma_3)\gamma_4 - 4i\lambda\tau]\chi, \tag{3.47}$$

θ and τ being determined by the formulae

$$\begin{aligned}&\theta = -\lambda^2\tau^2,\\ &\tau^{2k} = 32\lambda^2\tau^2(\bar{\chi}\chi) + 8i\lambda\tau[\bar{\chi}(\gamma_0 - \gamma_3)\gamma_4\chi] - 32i\lambda^3\tau^3[\bar{\chi}(\gamma_0 + \gamma_3)\gamma_4\chi].\end{aligned} \tag{3.48}$$

Eq. (8) of (3.24) is, via the change of variables

$$\varphi(\omega) = \omega^{-1/4}\phi(\omega),$$

reduced to the following ODE:

$$2i\omega^{1/2}\gamma_2\dot{\phi} = \lambda\omega^{-1/4k}(\bar{\phi}\phi)^{1/2k}\phi.$$

Multiplying it by $\frac{1}{2}i\gamma_2\omega^{-1/2}$ we come to an equation of the form

$$\dot{\phi} = \frac{1}{2}i\lambda\omega^{-(1+2k)/4k}(\bar{\phi}\phi)^{1/2k}\phi,$$

whose general solution is given by the formulae

$$k \neq 1/2: \quad \phi(\omega) = \exp\left(\frac{2i\lambda k}{1-2k}(\bar{\chi}\chi)^{1/2k}\gamma_2\omega^{(2k-1)/4k}\right)\chi,$$
$$k = 1/2: \quad \phi(\omega) = \exp\left[\frac{1}{2}i\lambda(\bar{\chi}\chi)\gamma_2\ln\omega\right]\chi.$$

So the general solution of ODE (8) has the form

$$\begin{aligned} &k \neq 1/2: \quad \varphi(\omega) = \omega^{-1/4}\exp\left(\frac{2i\lambda k}{1-2k}(\bar{\chi}\chi)^{1/2k}\gamma_2\omega^{(2k-1)/4k}\right)\chi,\\ &k = 1/2: \quad \varphi(\omega) = \omega^{-1/4}\exp\left[\tfrac{1}{2}i\lambda(\bar{\chi}\chi)\gamma_2\ln\omega\right]\chi. \end{aligned} \tag{3.49}$$

Besides we have succeeded in integrating eqs. (4), (23) and (26) of (3.24) (for $\alpha = 0$). These ODE can be written in the following way:

$$\frac{1}{2}m(\gamma_0+\gamma_3)\varphi + [\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3]\dot{\varphi} = -i\lambda(\bar{\varphi}\varphi)^{1/2k}\varphi,$$

where for $m = 1, 2, 3$ eqs. (4), (23), (26) of (3.24) are obtained. Multiplying both parts of the equality by $\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3$, comes to the ODE

$$4\omega\dot{\varphi} = -\{m(1+\gamma_0\gamma_3) + i\lambda(\bar{\varphi}\varphi)^{1/2k}[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3]\}\varphi, \tag{3.50}$$

and the equation for the conjugate spinor has the form

$$4\omega\dot{\bar{\varphi}} = -\bar{\varphi}\{m(1-\gamma_0\gamma_3) - i\lambda(\bar{\varphi}\varphi)^{1/2k}[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3]\}.$$

Multiplying the first equation by $\bar{\varphi}$ and the second by φ one obtains the following relation:

$$4(\dot{\bar{\varphi}}\varphi + \bar{\varphi}\dot{\varphi}) = -2m\bar{\varphi}\varphi,$$

whence it follows that $\bar{\varphi}\varphi = c\omega^{-m/2}$, $C = \text{const}$. Substitution of the above result into (3.50) gives rise to a linear equation for $\varphi(\omega)$,

$$4\dot{\varphi} = -\{m(\gamma_0\gamma_3+1) + i\tau\omega^{-m/4k}[\omega(\gamma_0+\gamma_3)+\gamma_0-\gamma_3]\}\varphi,$$

where $\tau = -\lambda C^{1/2k}$. Writing this equality in components we obtain a system of ODE of the form

$$\begin{aligned} &2\omega\dot{\varphi}^0 = i\tau\omega^{\alpha+1}\varphi^2, &&2\omega\dot{\varphi}^1 = -m\varphi^1 + i\tau\omega^{\alpha}\varphi^3,\\ &2\omega\dot{\varphi}^2 = -m\varphi^2 + i\tau\omega^{\alpha}\varphi^0, &&2\omega\dot{\varphi}^3 = i\tau\omega^{\alpha+1}\varphi^1, \quad \alpha = -m/4k. \end{aligned} \tag{3.51}$$

It is not difficult to convince oneself that (3.51) is equivalent to the following system of ODE:

$$\omega^2\ddot{\varphi}^0 + \frac{1}{2}(m-2\alpha)\omega\dot{\varphi}^0 + \frac{1}{4}\tau^2\omega^{2\alpha+1}\varphi^0 = 0,$$
$$\omega^2\ddot{\varphi}^3 + \frac{1}{2}(m-2\alpha)\omega\dot{\varphi}^3 + \frac{1}{4}\tau^2\omega^{2\alpha+1}\varphi^3 = 0,$$
$$\varphi^2 = -\frac{2i}{\tau}\omega^{-\alpha}\dot{\varphi}^0, \quad \varphi^1 = -\frac{2i}{\tau}\omega^{-\alpha}\dot{\varphi}^3.$$

The first and the second equations of this system are Bessel-type equations.

For $\alpha \neq -1/2$ their general solutions are determined by the formulae

$$\begin{aligned}\varphi^0 &= \omega^{(2+\alpha-m)/4}[\chi^0 J_\nu(z) + \chi^2 Y_\nu(z)],\\ \varphi^3 &= \omega^{(2+\alpha-m)/4}[\chi^3 J_\nu(z) + \chi^1 Y_\nu(z)],\end{aligned} \tag{3.52}$$

where J_ν, Y_ν are Bessel functions, $z = \tau\omega^{(2\alpha+1)/2}/(\alpha+1)$, $\nu = (2\alpha-m+2)/2(1+2\alpha)$.

Consequently

$$\begin{aligned}\varphi^2 &= \omega^{(2+2\alpha-m)/4}\left[\frac{i(m-2\alpha-2)}{2\tau}\omega^{-\alpha-1}[\chi^0 J_\nu(z) + \chi^2 Y_\nu(z)] - \right.\\ &\qquad \left. - i\omega^{-1/2}\left(\chi^0\frac{dJ_\nu(z)}{dz} + \chi^2\frac{dY_\nu(z)}{dz}\right)\right],\\ \varphi^1 &= \omega^{(2+2\alpha-m)/4}\left[\frac{i(m-2\alpha-2)}{2\tau}\omega^{-\alpha-1}[\chi^3 J_\nu(z) + \chi^1 Y_\nu(z)] - \right.\\ &\qquad \left. - i\omega^{-1/2}\left(\chi^3\frac{dJ_\nu(z)}{dz} + \chi^1\frac{dY_\nu(z)}{dz}\right)\right],\end{aligned} \tag{3.53}$$

where $\chi^\mu = \mathrm{const}$, $\mu = 0,1,2,3$. Formulae (3.52), (3.53) determine the general solution of the initial nonlinear system (3.50) if the following condition holds:

$$\bar{\varphi}\varphi \equiv \varphi^{0*}\varphi^2 + \varphi^{2*}\varphi^0 + \varphi^{3*}\varphi^1 + \varphi^{1*}\varphi^3 = C\omega^{-m/2}.$$

Substitution of (3.52), (3.53) into this formula yields the following equality:

$$\frac{2i(2\alpha+1)}{\tau\pi}(\chi^0\chi^{2*} - \chi^2\chi^{0*} + \chi^3\chi^{1*} - \chi^1\chi^{3*})\omega^{-m/2} = C\omega^{-m/2},$$

where we used the well-known identity for Bessel functions

$$W[J_\nu, Y_\nu] \equiv J_\nu\frac{dY_\nu}{dz} - Y_\nu\frac{dJ_\nu}{dz} = 2/\pi z.$$

Comparing both sides of the equality one obtains

$$C = \frac{2i(2\alpha+1)}{\tau\pi}(\chi^{0*}\chi^2 - \chi^{2*}\chi^0 + \chi^3\chi^{1*} - \chi^1\chi^{3*}),$$

whence it follows that

$$C = \left(-\frac{i(m-2k)}{\pi k\lambda}(\chi^{0*}\chi^2 - \chi^0\chi^{2*} + \chi^{3*}\chi^1 - \chi^{1*}\chi^3)\right)^{2k/(2k+1)}. \tag{3.54}$$

For $\alpha=-1/2$ ($k=m/2$) one has to consider three cases,

(1) $(m-1)^2-4\tau^2\neq 0, \quad m=2,3;$
(2) $\tau\neq 0, \quad m=1;$
(3) $\tau=\varepsilon(m-1)/2, \quad \varepsilon=\pm 1.$

The general solution of the system (3.50) is given by the following formulae:

$$(1)\quad \begin{aligned}&\varphi^0=\chi^0\omega^{\theta_+}+\chi^2\omega^{\theta_-}, \quad \varphi^1=-\frac{2i}{\tau}(\theta_+\chi^3\omega^{\theta_+}+\theta_-\chi^1\omega^{\theta_-})\omega^{-1/2},\\ &\varphi^2=-\frac{2i}{\tau}(\theta_+\chi^0\omega^{\theta_+}+\theta_-\chi^2\omega^{\theta_-})\omega^{-1/2}, \quad \varphi^3=\chi^3\omega^{\theta_+}+\chi^1\omega^{\theta_-},\end{aligned} \tag{3.55}$$

where

$$\theta_\pm=\frac{1}{4}\left(1-m\pm\sqrt{(m-1)^2-4\tau^2}\right),$$

$\chi^0,\ldots,\chi^3$ are arbitrary complex constants; τ satisfies the equality $(-1)^m i(\chi^{0*}\chi^2-\chi^0\chi^{2*}+\chi^{3*}\chi^1-\chi^3\chi^{1*})[(m-1)^2-4\tau^2]^{1/2}=\tau^{m+1}\lambda^{-m}$;

$$(2)\quad \begin{aligned}&\varphi^0=\chi^0\cos\left(\frac{1}{2}\tau\ln\omega\right)+\chi^2\sin\left(\frac{1}{2}\tau\ln\omega\right),\\ &\varphi^1=-i\omega^{-1/2}\left[\chi^1\cos\left(\frac{1}{2}\tau\ln\omega\right)-\chi^3\sin\left(\frac{1}{2}\tau\ln\omega\right)\right],\\ &\varphi^2=-i\omega^{-1/2}\left[\chi^2\cos\left(\frac{1}{2}\tau\ln\omega\right)-\chi^0\sin\left(\frac{1}{2}\tau\ln\omega\right)\right],\\ &\varphi^3=\chi^3\cos\left(\frac{1}{2}\tau\ln\omega\right)+\chi^1\sin\left(\frac{1}{2}\tau\ln\omega\right),\end{aligned} \tag{3.56}$$

where $\chi^0,\ldots,\chi^3$ are constants; τ satisfies the equality $\tau=i\lambda(\chi^0\chi^{2*}-\chi^2\chi^{0*}+\chi^3\chi^{1*}-\chi^1\chi^{3*})$;

$$(3)\quad \begin{aligned}&\varphi^0=\omega^{(1-m)/4}(\chi^0+\chi^2\ln\omega),\\ &\varphi^1=\frac{1}{2\tau}i(m-1)\omega^{-1/2}\varphi^3+\frac{4i\varepsilon}{1-m}\omega^{-(m+1)/4}\chi^1,\\ &\varphi^2=\frac{1}{2\tau}i(m-1)\omega^{-1/2}\varphi^0+\frac{4i\varepsilon}{1-m}\omega^{-(m+1)/4}\chi^2,\\ &\varphi^3=\omega^{(1-m)/4}(\chi^3+\chi^1\ln\omega), \quad \varepsilon=\pm 1,\end{aligned} \tag{3.57}$$

while the following equality holds:

$$2i(\chi^0\chi^{2*}-\chi^{0*}\chi^2+\chi^3\chi^{1*}-\chi^{3*}\chi^1)=\lambda\left(\frac{m-1}{2\varepsilon\lambda}\right)^{m+1}(-1)^m.$$

So the general solution of the system (3.50) [and consequently, of the systems (4), (23) and (26) of (3.24) ($\alpha=0$)] is given by formulae (3.52), (3.53), for $k\neq m/2$ and by formulae (3.55)–(3.57) for $k=m/2$.

Let us turn now to eqs. (3.32). The systems of ODE (1), (3), (4), (6) and (8) are integrated in the same way as eqs. (1) and (2) of (3.24). As a result one has

$$\begin{aligned}&\varphi_1 = \exp[i\lambda\gamma_1(\bar{\chi}\chi)^{1/2}z_1]\chi; \quad \varphi_2 = \exp[i\lambda(\bar{\chi}\chi)^{1/2}\gamma_0 z_3]\chi;\\ &\varphi_j = \exp[-i\lambda(\bar{\chi}\chi)^{1/2}\gamma_1 z_j]\chi, \quad j=4,6; \quad \varphi_8 = \exp[i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_1 z_8]\chi.\end{aligned} \tag{3.58}$$

Equations (2), (5), (7) and (9) of (3.32) coincide with ODE (8) of (3.24) up to the sign of the nonlinear term $\lambda(\bar{\varphi}\varphi)^{1/2k}\varphi$. Using this fact one easily obtains their general solutions,

$$\begin{aligned}&\varphi_2(z_2) = z_2^{-1/4}\exp[-2i\lambda(\bar{\chi}\chi)^{1/2}\gamma_2 z_2^{1/4}]\chi;\\ &\varphi_j(z_j) = z_j^{-1/4}\exp[2i\lambda(\bar{\chi}\chi)^{1/2}\gamma_2 z_j^{1/4}]\chi, \quad j=5,7;\\ &\varphi_9(z_9) = z_9^{-1/4}\exp\left(\frac{2i\lambda k}{1-2k}(\bar{\chi}\chi)^{1/2k}\gamma_2 z_9^{(2k-1)/2k}\right)\chi.\end{aligned} \tag{3.59}$$

Besides we have succeeded in integrating ODE (1) of (3.33) (for $k=1/2$) and (2) of (3.33). The final result has the form

$$\begin{aligned}&\varphi(\omega_2) = \exp\left(\frac{i\lambda}{2(1+a^2)}(\bar{\chi}\chi)(\gamma_3 + a\gamma_2)\omega_2\right)\chi,\\ &\varphi(\omega_2) = \omega_2^{-1/4}[f_1 + \gamma_3 f_2 + (\gamma_0+\gamma_1)f_3 + \gamma_3(\gamma_0+\gamma_1)f_4]\chi,\end{aligned}$$

where the functions $f_i(\omega)$ are determined by the following equalities:

$k \neq 1/2$:

$$\begin{aligned}&f_1 = \cosh(\tau\omega_2^\alpha), \quad f_2 = i\sinh(\tau\omega_2^\alpha),\\ &f_3 = \frac{1}{4}i\left(\cosh(\tau\omega_2^\alpha)\int^{\omega_2}\sinh(2\tau z^\alpha)dz - \sinh(\tau\omega_2^\alpha)\int^{\omega_2}\cosh(2\tau z^\alpha)dz\right),\\ &f_4 = \frac{1}{4}i\left(-\sinh(\tau\omega_2^\alpha)\int^{\omega_2}\sinh(2\tau z^\alpha)dz + \cosh(\tau\omega_2^\alpha)\int^{\omega_2}\cosh(2\tau z^\alpha)dz\right),\\ &\tau = \frac{2\lambda k(\bar{\chi}\chi)^{1/2k}}{2k-1}, \quad \alpha = \frac{2k-1}{4k};\end{aligned} \tag{3.60}$$

$k = 1/2$:

$$\begin{aligned}&f_1 = \frac{1}{2}(2^\tau\omega_2^{\tau/2} + 2^{-\tau}\omega_2^{-\tau/2}), \quad f_2 = \frac{1}{2}i(2^\tau\omega_2^{\tau/2} - 2^{-\tau}\omega_2^{-\tau/2}),\\ &f_3 = \frac{1}{4}i\omega_2^{1/2}\left(\frac{2^\tau\omega_2^{\tau/2}}{2\tau+1} - \frac{2^{-\tau}\omega_2^{-\tau/2}}{1-2\tau}\right),\\ &f_4 = \frac{1}{4}\omega_2^{1/2}\left(\frac{2^\tau\omega_2^{\tau/2}}{2\tau+1} + \frac{2^{-\tau}\omega_2^{-\tau/2}}{1-2\tau}\right), \quad \tau = \lambda(\bar{\chi}\chi).\end{aligned} \tag{3.61}$$

The possibility of integrating the nonlinear systems of ODE (3.24), (3.22) and (3.33) in quadratures is closely connected with the nontrivial symmetry admitted by these equations. And this property, in its turn, is connected with the large invariance group admitted by the initial equation [in the present case the group $\tilde{P}(1,3)$]. That is why, when the symmetry properties of the equations are better, the group-theoretical methods of constructing exact solutions are more effective. It is worth noting that other classical methods of constructing particular solutions (separation of variables, d'Alembert method and so on) use explicitly or implicitly the symmetry properties of PDE [30].

Substitution of the above results into the corresponding ansätze in tables 1 and 2 or into the ansätze (3.18)–(3.21) yields the exact solutions of the nonlinear Dirac–Heisenberg equation (3.34):

$k \in \mathbb{R}^1$:

$$\psi_1(x) = \exp[i\lambda(\bar\chi\chi)^{1/2k}\gamma_3 x_3]\chi;$$
$$\psi_2(x) = \exp[-i\lambda(\bar\chi\chi)^{1/2k}\gamma_0 x_0]\chi;$$
$$\psi_3(x) = \exp\left[-\frac{1}{2}(\gamma_0+\gamma_3)\gamma_1(x_0+x_3)\right]\times$$
$$\times\exp\left\{\frac{1}{2}i\lambda(\bar\chi\chi)^{1/2k}\gamma_1[2x_1+(x_0+x_3)^2]\right\}\chi;$$
$$\psi_4(x) = \exp\left[-\frac{1}{2}(\gamma_0+\gamma_3)\gamma_1(x_0+x_3)\right]\times$$
$$\times\exp\left(\frac{i\lambda}{2(1+\alpha^2)}(\bar\chi\chi)^{1/2k}(\gamma_2-\alpha\gamma_1)[2(x_2-\alpha x_1)-\alpha(x_0+x_3)^2]\right\}\chi;$$
$$\psi_5(x) = \exp\left(\frac{x_1-\alpha\ln(x_0+x_3)}{2(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1\right)\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right]\times$$
$$\times\exp\left\{[\gamma_2(\gamma_0+\gamma_3)+i\lambda(\bar\chi\chi)^{1/2k}(\gamma_2-\beta(\gamma_0+\gamma_3))][x_2-\beta\ln(x_0+x_3)]\right\}\chi;$$
$$\psi_6(x) = \exp\left(\frac{x_2+2\alpha\theta x_1}{2\alpha}\gamma_0\gamma_3\right)\left[\left(\theta\gamma_0-\frac{1}{2\alpha}\gamma_1\right)\gamma_4-i\lambda\tau\right]\chi,$$

where $\alpha\in\mathbb{R}^1$; θ and τ being determined by formulae (3.40) and (3.41);

$$\psi_7(x) = \exp\left(\frac{2\alpha\theta x_3-x_0}{2\alpha}\gamma_1\gamma_2\right)\left[\left(\theta\gamma_0-\frac{1}{2\alpha}\gamma_3\right)\gamma_4-i\lambda\tau\right]\chi,$$

where $\alpha\in\mathbb{R}^1$; θ and τ being determined by formulae (3.44);

$$\psi_8(x) = \exp\left(\frac{x_3+2\alpha\theta x_0}{2\alpha}\gamma_1\gamma_2\right)\left[\left(\theta\gamma_3+\frac{1}{2\alpha}\gamma_0\right)\gamma_4-i\lambda\tau\right]\chi,$$

where $\alpha\in\mathbb{R}^1$; θ and τ being determined by formulae (3.46);

$$\psi_9(x) = \exp\left\{\frac{1}{4}[x_3-x_0+4\theta(x_0+x_3)]\gamma_1\gamma_2\right\}\times$$
$$\times[4\theta(\gamma_0+\gamma_3)\gamma_4+(\gamma_0-\gamma_3)\gamma_4-4i\lambda\tau]\chi,$$

θ and τ being determined by formulae (3.48);

$$\psi_{10}(x) = \exp\left\{\left[-\frac{1}{2}(\dot\phi_1\gamma_1+\dot\phi_2\gamma_2)+\phi_3\gamma_4\right](\gamma_0+\gamma_3)\right\}\times$$
$$\times\exp[i\lambda(\bar\chi\chi)^{1/2k}\gamma_1(x_1+\phi_1)]\chi,$$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary smooth functions of x_0+x_3.

$k \in \mathbb{R}^1$, $k \neq 1/2$:

$$\psi_{11}(x) = \exp\left[\frac{1}{2}\gamma_0\gamma_3 \ln(x_0 + x_3)\right] \varphi(x_0^2 - x_3^2),$$

$\varphi(\omega)$ being determined by formulae (3.52)–(3.54) with $m = 1$;

$$\begin{aligned}\psi_{12}(x) = {} & [(x_1 + \phi_1)^2 + (x_2 + \phi_2)^2]^{-1/4} \times \\ & \times \exp\left\{\left[-\frac{1}{2}(\dot{\phi}_1\gamma_1 + \dot{\phi}_2\gamma_2) + \phi_3\gamma_4\right](\gamma_0 + \gamma_3)\right\} \times \\ & \times \exp\left(-\frac{1}{2}\gamma_1\gamma_2 \operatorname{arctg}\frac{x_1 + \phi_1}{x_2 + \phi_2}\right) \times \\ & \times \exp\left(\frac{2i\lambda k}{1 - 2k}(\bar{\chi}\chi)^{1/2k}\gamma_2[(x_1 + \phi_1)^2 + (x_2 + \phi_2)^2]^{(2k-1)/4k}\right)\chi,\end{aligned}$$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary smooth functions of $x_0 + x_3$;

$$\begin{aligned}\psi_{13}(x) = {} & (x_2^2 + x_3^2)^{-1/4} \exp\left[-\frac{1}{4}\gamma_0\gamma_1 \ln(x_0 - x_1) - \frac{1}{2}\gamma_2\gamma_3 \operatorname{arctg}\frac{x_2}{x_3}\right] \times \\ & \times [f_1 + \gamma_3 f_2 + (\gamma_0 + \gamma_1) f_3 + \gamma_3(\gamma_0 + \gamma_1) f_4]\chi,\end{aligned}$$

where $f_i = f_i(x_2^2 + x_3^2)$ are determined by formulae (3.60);
$k \in \mathbb{R}^1$, $k \neq 1$:

$$\psi_{14}(x) = \exp\left(\frac{x_1}{2(x_0 + x_3)}(\gamma_0 + \gamma_3)\gamma_1\right)\exp\left[\frac{1}{2}\gamma_0\gamma_3 \ln(x_0 + x_3)\right] \varphi(x_0^2 - x_1^2 - x_3^2),$$

$\varphi(\omega)$ being given by formulae (3.52)–(3.54) with $m = 2$;
$k \in \mathbb{R}^1$, $k \neq 3/2$:

$$\psi_{15}(x) = \exp\left(\frac{\gamma_0 + \gamma_3}{2(x_0 + x_3)}(\gamma_1 x_1 + \gamma_2 x_2)\right) \exp\left[\frac{1}{2}\gamma_0\gamma_3 \ln(x_0 + x_3)\right] \varphi(x \cdot x),$$

$\varphi(\omega)$ being given by formulae (3.52)–(3.54) with $m = 3$;
$k = 1/2$:

$$\varphi_{16}(x) = \exp\left[\frac{1}{2}\gamma_0\gamma_3 \ln(x_0 + x_3)\right] \varphi(x_0^2 - x_3^2),$$

$\varphi(\omega)$ being given by formulae (3.56);

$$\begin{aligned}\psi_{17}(x) = {} & (x_2^2 + x_3^2)^{-1/4} \exp\left[-\frac{1}{2}\gamma_2\gamma_3 \operatorname{arctg}\frac{x_2}{x_3}\right] \times \\ & \times \exp\left(\frac{i\lambda}{2(1 + a^2)}(\bar{\chi}\chi)(\gamma_3 + a\gamma_2)\left[\ln(x_2^2 + x_3^2) + 2a \operatorname{arctg}\frac{x_2}{x_3}\right]\right\}\chi;\end{aligned}$$

$k = 1$:

$$\begin{aligned}\psi_{18}(x) = {} & \phi_0^{-1} \exp\left\{\left[-\frac{1}{2}(\dot{\phi}_1\gamma_1 + \dot{\phi}_2\gamma_2) + \phi_3\gamma_4 - \right.\right. \\ & \left.\left. - \frac{1}{2}\dot{\phi}_0\phi_0^{-1}(\gamma_1(x_1 + \phi_1) + \gamma_2(x_2 + \phi_2)\right](\gamma_0 + \gamma_3)\right\} \times \\ & \times \exp\left(\frac{i\lambda}{\phi_0}(\bar{\chi}\chi)^{1/2}\gamma_1(x_1 + \phi_1)\right)\chi;\end{aligned}$$

$$\psi_{19}(x) = \phi_0^{-1/2}[(x_1+\phi_1)^2+(x_2+\phi_2)^2]^{-1/4}\exp\left\{\left[-\frac{1}{2}(\dot\phi_1\gamma_1+\dot\phi_2\gamma_2)+\phi_3\gamma_4-\right.\right.$$
$$\left.\left.-\frac{1}{2}\dot\phi_0\phi_0^{-1}(\gamma_1(x_1+\phi_1)+\gamma_2(x_2+\phi_2)\right](\gamma_0+\gamma_3)\right\}\times$$
$$\times\exp\left(-\frac{1}{2}\gamma_1\gamma_2\operatorname{arctg}\frac{x_1+\phi_1}{x_2+\phi_2}\right)\times$$
$$\times\exp\left\{-2i\lambda(\bar\chi\chi)^{1/2}\gamma_2[(x_1+\phi_1)^2+(x_2+\phi_2)^2]^{1/4}\phi_0^{-1/2}\right\}\chi;$$

where ϕ_0, ϕ_2, ϕ_3 are arbitrary smooth functions x_0+x_3;

$$\psi_{20}(x) = \frac{\gamma_0x_0-\gamma_1x_1-\gamma_2x_2}{(x_0^2-x_1^2-x_2^2)^{3/2}}\exp[i\lambda(\bar\chi\chi)^{1/2}\gamma_0x_0(x_0^2-x_1^2-x_2^2)^{-1}]\chi;$$
$$\psi_{21}(x) = \frac{\gamma_0x_0-\gamma_1x_1-\gamma_2x_2}{(x_0^2-x_1^2-x_2^2)^{3/2}}\exp[-i\lambda(\bar\chi\chi)^{1/2}\gamma_1x_1(x_0^2-x_1^2-x_2^2)^{-1}]\chi;$$
$$\psi_{22}(x) = \frac{\gamma_0x_0-\gamma_1x_1-\gamma_2x_2}{x_0^2-x_1^2-x_2^2}(x_1^2+x_2^2)^{-1/4}\exp\left[-\frac{1}{2}\gamma_1\gamma_2\operatorname{arctg}\frac{x_1}{x_2}\right]\times$$
$$\times\exp[2i\lambda(\bar\chi\chi)^{1/2}\gamma_2(x_1^2+x_2^2)^{1/4}(x_0^2-x_1^2-x_2^2)^{-1/2}]\chi;$$
$$\psi_{23}(x) = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{(\boldsymbol{x}^2)^{3/2}}\exp[-i\lambda(\bar\chi\chi)^{1/2}\gamma_1x_1(\boldsymbol{x}^2)^{-1}]\chi;$$
$$\psi_{24}(x) = \frac{\boldsymbol{\gamma}\cdot\boldsymbol{x}}{\boldsymbol{x}^2}(x_1^2+x_2^2)^{-1/4}\exp\left[-\frac{1}{2}\gamma_1\gamma_2\operatorname{arctg}\frac{x_1}{x_2}\right]\times$$
$$\times\exp[2i\lambda(\bar\chi\chi)^{1/2}\gamma_2(x_1^2+x_2^2)^{1/4}(\boldsymbol{x}^2)^{-1/2}]\chi;$$
$$\psi_{25}(x) = \exp\left(\frac{x_1}{2(x_0+x_3)}(\gamma_0+\gamma_3)\gamma_1\right)\times$$
$$\times\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right]\varphi(x_0^2-x_1^2-x_3^2),$$

$\varphi(\omega)$ being determined by formulae (3.55) or (3.57) with $m=2$;

$$\psi_{26}(x) = \exp\left[\frac{1}{2}\gamma_0\gamma_1\ln(x_0+x_1)-\frac{1}{2}\gamma_2\gamma_3\operatorname{arctg}\frac{x_2}{x_3}\right]\times$$
$$\times(x_2^2+x_3^2)^{-1/4}[f_1+\gamma_3f_2+(\gamma_0+\gamma_1)f_3+\gamma_3(\gamma_0+\gamma_1)f_4]\chi,$$

$f_i=f_i(x_2^2+x_3^2)$ being given by formulae (3.61);

$k=3/2$:

$$\psi_{27}(x) = \exp\left(\frac{1}{2(x_0+x_3)}(\gamma_0+\gamma_3)(\gamma_1x_1+\gamma_2x_2)\right)\times$$
$$\times\exp\left[\frac{1}{2}\gamma_0\gamma_3\ln(x_0+x_3)\right]\varphi(x\cdot x),$$

where $\varphi(\omega)$ is determined by formulae (3.55) or (3.57) with $m=3$.

Besides, in ref. [24] two other classes of exact solutions were obtained, essentially using ansatz (3.23) and the Heisenberg ansatz [14],

$k<0$:

$$\psi_{28}(x) = \exp\left[\frac{1}{2}\gamma_1(\gamma_0-\gamma_2)(x_0-x_2)\right]\left\{\left[(\gamma_3+\beta(\gamma_0-\gamma_2))(x_3+\beta(x_0-x_2))+\right.\right.$$
$$\left.\left.+\frac{1}{2}\gamma_1(2x_1+(x_0-x_2)^2)\right]f(\omega)+ig(\omega)\right\}\chi;$$

$k=1/2$:

$$\psi_{29}(x)=\exp\left[\frac{1}{2}\gamma_1(\gamma_0-\gamma_2)(x_0-x_2)\right]\times$$
$$\times\left[(\gamma_3+\beta(\gamma_0-\gamma_2))(x_3+\beta(x_0-x_2))+\frac{1}{2}\gamma_1(2x_1+(x_0-x_2)^2)\right]\omega^{-1}\times$$
$$\times\exp\Big(\frac{i\lambda(\bar\chi\chi)}{\beta_1^2+\beta_2^2}\{\beta_1[\gamma_3+\beta(\gamma_0-\gamma_2)]+\beta_2\gamma_1\}\times$$
$$\times\left\{\beta_1[x_3+\beta(x_0-x_2)]+\frac{1}{2}\beta_2[2x_1+(x_0-x_2)^2]\right\}\omega^{-1}\Big)\chi,$$

where

$$\omega=[x_3+\beta(x_0-x_2)]^2+\frac{1}{4}[2x_1+(x_0-x_2)^2]^2,$$
$$f(\omega)=|k|^{-1/2}\left(\mp\frac{(1-k)^{1/2}}{\lambda(\bar\chi\chi)^{1/2k}}\right)^k\omega^{-(k+1)/2},$$
$$g(\omega)=\pm(1-k)^{1/2}\left(\mp\frac{(1-k)^{1/2}}{\lambda(\bar\chi\chi)^{1/2k}}\right)^k\omega^{-k/2},$$

β, β_1 and β_2 are arbitrary constants.

The existence of exact solutions depending on arbitrary functions is connected with the fact that the additional constraint

$$(p_0+p_3)\psi(x)=0$$

selects the subset of solutions of the Dirac–Heisenberg equation admitting the infinite-dimensional algebra (3.17). As established in ref. [29], the large class of Poincaré-invariant equations (Bhabha-type equations)

$$[\beta_\mu p^\mu+m]\Psi(x)=0,\quad m=\mathrm{const},\tag{3.62}$$

possess such a property. In (3.62) $\Psi=\{\Psi^1,\ldots,\Psi^n\}$, $x=(x_0,x_1,x_2,\ldots,x_l\}$, $l\geq 2$, β_μ are $n\times n$ matrices satisfying the conditions

$$\begin{gathered}[\beta_\alpha,S_{\mu\nu}]=i(g_{\mu\alpha}\beta_\nu-g_{\nu\alpha}\beta_\mu),\quad S_{\mu\nu}=i(\beta_\mu\beta_\nu-\beta_\nu\beta_\mu),\\ g_{\mu\nu}=\mathrm{diag}\,(1,-1,\ldots,-1,-1),\quad \alpha,\mu,\nu=0,\ldots,l.\end{gathered}\tag{3.63}$$

It is well known that eq. (3.62) is invariant under the Poincaré algebra $P(1,l)$ having basis operators of the form [46]

$$P_\mu=ig^{\mu\nu}\partial/\partial x_\nu,\quad J_{\mu\nu}=x_\mu P_\nu-x_\nu P_\mu+S_{\mu\nu}.$$

We impose on $\Psi(x)$ the additional constraint

$$(P_0+P_l)\Psi(x)=0,$$

from which an equation for $\Psi(\omega)=\Psi(x_0+x_l,x_1,\ldots,x_{l-1})$ follows,

$$\left(i(\beta_0+\beta_l)\partial_{\omega_0}+\sum_{j=1}^{l-1}\beta_j\partial_{\omega_j}+m\right)\Psi(\omega)=0.\tag{3.64}$$

Proposition 3. *Equation (3.64) is invariant under the infinite-dimensional Lie algebra with the following basis operators:*

$$Q_1 = \partial_{\omega_0}, \quad Q_2 = \sum_{k=1}^{l-1} \left[\phi^k(\omega_0)\partial_{\omega_k} + \frac{1}{2} i\dot{\phi}_k(\omega_0)(S_{kl} - S_{0k}) \right], \tag{3.65}$$
$$Q_{ab} = \omega_a\partial_{\omega_b} - \omega_b\partial_{\omega_a} + iS_{ab}, \quad a, b = 1, \dots, l-1,$$

where $\partial_{\omega_\mu} = \partial/\partial\omega_\mu$, $\mu = 0, \dots, l-1$, $\dot{\phi}^k = d\phi^k/d\omega_0$, $\phi^k(\omega)$ *are arbitrary functions.*

Proof. For linear equations the following statement holds [31]: An operator Q is the symmetry operator of the linear equation

$$L(x)\Psi = 0$$

iff there exists a matrix $R(x)$ such that

$$[Q, L] = R(x)L.$$

We shall prove that

$$[Q, L] = 0. \tag{3.66}$$

If $Q \in \langle Q_1, Q_{ab}\rangle$, then the statement is quite evident. Let us consider the case $Q = Q_2$. If we shall show that $(\beta_0 + \beta_l)(S_{0k} - S_{kl}) = 0$, then proposition 3 will be proved. Choosing $k = 1$ one has

$$\begin{aligned}(\beta_0 + \beta_l)(S_{01} - S_{1l}) &= i(\beta_0 + \beta_l)(\beta_0\beta_1 - \beta_1\beta_0 - \beta_1\beta_l + \beta_l\beta_1) = \\ &= i(\beta_0\beta_0\beta_1 - \beta_0\beta_1\beta_0) + i(\beta_l\beta_l\beta_1 - \beta_l\beta_1\beta_l) + \\ &\quad + i(\beta_l\beta_0\beta_1 - \beta_0\beta_1\beta_l) + i(\beta_0\beta_l\beta_1 - \beta_l\beta_1\beta_0) = \\ &= i\beta_1 - i\beta_1 = 0.\end{aligned}$$

The cases $k = 2, 3, \dots, l-1$ are treated in the same way.

Consequence. *On the set of solutions of eq. (3.64) the following representation of the Galilei algebra* $AG(1, l-1)$ *is realized:*

$$P_0 = i\partial_{\omega_0}, \quad P_a = -i\partial_{\omega_a}, \quad J_{ab} = \omega_a P_b - \omega_b P_a + S_{ab},$$
$$G_a = \omega_0 P_a + \frac{1}{2}(S_{al} - S_{0a}), \quad a, b = 1, \dots, l-1.$$

Note 1. In general, the algebra (3.65) is not a maximal invariance algebra of (3.64). As an example one can take eq. (13) of (3.16), whose symmetry is described by proposition 1. Other examples are given in refs. [27, 29].

Note 2. Proposition 3 holds true for Poincaré-invariant generalizations of the Bhabha equation of the form

$$[\beta_\mu p^\mu + F(\Psi^*, \Psi)]\Psi(x) = 0.$$

This makes it possible to construct exact solutions of the above nonlinear equations including arbitrary functions with the help of the procedure of generating solutions [27, 29]. By a special choice of the arbitrary functions one can pick out classes of solutions possessing some additional properties.

Choosing in $\psi_{18}(x)$

$$\phi_0 \equiv \exp[\theta^2(x_0+x_3)^2], \quad \theta = \text{const}, \quad \phi_1 \equiv \phi_2 \equiv \phi_3 \equiv 0,$$

one obtains the following solution of the Dirac–Heisenberg equation:

$$\begin{aligned}\psi(x) = {} & \exp[-\theta^2(x_0+x_3)^2][I + \theta^2(x_0+x_3)(\gamma_1 x_1 + \gamma_2 x_2)(\gamma_0+\gamma_3)] \times \\ & \times \exp\{i\lambda(\bar\chi\chi)^{1/2}\gamma_1 x_1 \exp[-\theta^2(x_0+x_3)^2]\}\chi.\end{aligned} \tag{3.67}$$

This solution is not localized in $\mathbb{R}^3$ but it is localized inside an infinite cylinder with its axis parallel to the Ox_3 axis. Moreover (3.67) decreases exponentially in all points of $\mathbb{R}^3$ as $x_0 \to +\infty$.

Let us mention that for $\theta = 0$ takes the form

$$\psi(x) = \exp[i\lambda(\bar\chi\chi)^{1/2}\gamma_1 x_1]\chi. \tag{3.68}$$

Consequently, (3.67) can be considered as a perturbation of the stationary state (3.68).

3.4. Nongenerable families of solutions of the nonlinear Dirac equation. The solutions $\psi_1(x)$–$\psi_{29}(x)$ depend on the variables x_μ in an asymmetrical way, while in the Dirac–Heisenberg equation all independent variables have equal rights. Using physical language one can say that the system (3.34) is solved in some fixed reference system. To obtain solutions (more precisely families of solutions) which do not depend on the chosen reference system it is necessary to apply a procedure of generating solutions by a group of transformations [21, 47]. This procedure is based on the following statement.

Let eq. (3.34) be invariant under the group of transformations

$$\psi'(x') = A(x,\theta)\psi(x), \quad x'_\mu = f_\mu(x,\theta), \tag{3.69}$$

where $A(x,\theta)$ is an invertible 4×4 matrix, $\theta = (\theta_1,\ldots,\theta_r)$ are group parameters. Besides there is some solution $\psi = \psi_I(x)$ of eq. (3.34).

Proposition 4. *The spinor $\psi_{II}(x)$,*

$$\psi_{II}(x) = A^{-1}(x,\theta)\psi_I(f(x,\theta)), \tag{3.70}$$

satisfies eq. (3.34) too.

The proof can be found in refs. [21, 32].

We call formula (3.70) the solutions generating formula. Let us mention the solution generating formulae with transformations of the conformal group $C(1,3)$.

(1) The group of translations,

$$\psi_{II}(x) = \Psi_I(x'), \quad x'_\mu = x_\mu + \theta_\mu, \quad \theta_\mu = \text{const}, \tag{3.71}$$

(2) the Lorentz group $O(1,3)$,

(a) the group of rotations

$$\begin{aligned}&\psi_{II}(x) = \exp\left(-\frac{i}{2}\varepsilon_{abc}\theta_a S_{bc}\right)\psi_I(x'),\\ &x'_0 = x_0, \quad \boldsymbol{x}' = \boldsymbol{x}\cos\theta - \frac{\boldsymbol{\theta}\times\boldsymbol{x}}{\theta}\sin\theta + \frac{\boldsymbol{\theta}(\boldsymbol{\theta}\cdot\boldsymbol{x})}{\theta^2}(1-\cos\theta),\\ &\theta_k = \text{const}, \quad \theta = (\boldsymbol{\theta}\cdot\boldsymbol{\theta})^{1/2}, \quad S_{ab} = \frac{1}{4}i(\gamma_a\gamma_b - \gamma_b\gamma_a),\end{aligned} \tag{3.72a}$$

(b) the Lorentz transformations,

$$\begin{aligned}
&\psi_{II}(x) = \exp\left(-\frac{1}{2}\theta\gamma_0\gamma_a\right)\psi_I(x'),\\
&x'_0 = x_0\cosh\theta + x_a\sinh\theta, \quad x'_a = x_a\cosh\theta + x_0\sinh\theta,\\
&x'_b = x_b, \quad b \neq a, \quad a,b = 1,2,3, \quad \theta = \text{const};
\end{aligned} \tag{3.72b}$$

(3) the group of scale transformations,

$$\psi_{II}(x) = e^{k\theta}\psi_I(x'), \quad x'_\mu = e^\theta x_\mu, \quad \theta = \text{const}; \tag{3.73}$$

(4) the group of special conformal transformations,

$$\begin{aligned}
&\psi_{II}(x) = \sigma^{-2}(x)[1 - (\gamma\cdot x)(\gamma\cdot\theta)]\psi_I(x'),\\
&x'_\mu = [x_\mu - \theta_\mu(x\cdot x)]\sigma^{-1}(x), \quad \mu = 0,1,2,3,
\end{aligned} \tag{3.74}$$

where $\sigma(x) = 1 - 2\theta\cdot x + (\theta\cdot\theta)(x\cdot x)$, $\theta_\mu = \text{const}$.

As an example we shall consider the procedure of generating the solution $\psi_1(x)$, the remaining cases being treated in an analogous way. Let us apply to $\psi_1(x)$ formula (3.72b) with $a = 3$,

$$\psi(x) = \exp\left(-\frac{1}{2}\theta\gamma_0\gamma_3\right)\exp[i\lambda(\bar\chi\chi)^{1/2k}(x_3\cosh\theta + x_0\sinh\theta)\gamma_3]\chi.$$

Rewriting the above formula in the equivalent form one obtains

$$\begin{aligned}
\psi(x) = {}&\exp\left(-\frac{1}{2}\theta\gamma_0\gamma_3\right)\exp[i\lambda(\bar\chi\chi)^{1/2k}(x_3\cosh\theta + x_0\sinh\theta)\gamma_3]\times\\
&\times\exp\left(\frac{1}{2}\theta\gamma_0\gamma_3\right)\exp\left(-\frac{1}{2}\theta\gamma_0\gamma_3\right)\chi.
\end{aligned}$$

Taking into account the identities

$$\exp\left(-\frac{1}{2}\theta\gamma_0\gamma_3\right)\gamma_\mu\exp\left(\frac{1}{2}\theta\gamma_0\gamma_3\right) = \begin{cases}\gamma_0\cosh\theta + \gamma_3\sinh\theta, & \mu = 0,\\ \gamma_3\cosh\theta + \gamma_0\sinh\theta, & \mu = 3,\\ \gamma_\mu, & \mu = 1,2,\end{cases}$$

one has

$$\psi_{II}(x) = \exp[i\lambda(\bar\chi\chi)^{1/2k}(\gamma_3\cosh\theta + \gamma_0\sinh\theta)(x_3\cosh\theta + x_0\sinh\theta)]\tilde\chi,$$

where $\tilde\chi = \exp\left(-\frac{1}{2}\theta\gamma_0\gamma_3\right)\chi$. Using formula (3.72a) one comes to the following family of solutions:

$$\psi_{II}(x) = \exp[i\lambda(\bar\chi\chi)^{1/2k}(\gamma\cdot d)(d\cdot x)]\chi. \tag{3.75}$$

Hereafter a_μ, b_μ, c_μ and d_μ are arbitrary real parameters satisfying the relations

$$-a\cdot a = b\cdot b = c\cdot c = d\cdot d = -1, \quad a\cdot b = a\cdot c = a\cdot d = b\cdot c = b\cdot d = c\cdot d = 0 \tag{3.76}$$

[in other words, the four-vectors a, b, c, d create an orthonormal basis in the Minkowski space $\mathbb{R}(1,3)$]. It is not difficult to verify that the family (3.74) is invariant under the transformations (3.71), (3.73).

Solution (3.75) depends on the variables x_μ in a symmetrical way and its form is not changed both under a transition from one inertial reference system to another and under a change of the scale according to formula (3.73). In other words, we have constructed a $\tilde{P}(1,3)$-nongenerable family of solutions of the nonlinear Dirac–Heisenberg equation (the corresponding definition is given in ref. [48]). The transition from the solution $\psi_1(x)$ to the family of solutions (3.75) seems to be very important because one obtains a class of exact solutions having the same symmetry as the equation of motion (3.34).

Generating $\psi_2(x)$–$\psi_5(x)$ we obtain the following $\tilde{P}(1,3)$-nongenerable families of solutions of eq. (3.34):

$$
\begin{aligned}
\psi_2(x) &= \exp[-i\lambda(\bar{\chi}\chi)^{1/2k}(\gamma\cdot a)(a\cdot x)]\chi;\\
\psi_3(x) &= \exp\left[-\frac{1}{2}\theta(\gamma\cdot a+\gamma\cdot d)(\gamma\cdot b)(a\cdot z+d\cdot z)\right]\times\\
&\quad\times\exp\left\{\frac{1}{2}i\lambda(\bar{\chi}\chi)^{1/2k}(\gamma\cdot b)[2b\cdot z+\theta(a\cdot z+d\cdot z)^2]\right\}\chi;\\
\psi_4(x) &= \exp\left[-\frac{1}{2}\theta(\gamma\cdot a+\gamma\cdot d)(\gamma\cdot b)(a\cdot z+d\cdot z)\right]\exp\Big(\frac{i\lambda}{2(1+\alpha^2)}(\bar{\chi}\chi)^{1/2k}\times\\
&\quad\times(\gamma\cdot c-\alpha\gamma\cdot b)[2(c\cdot z-\alpha b\cdot z)-\alpha\theta(a\cdot z+d\cdot z)^2]\Big)\chi;\\
\psi_5(x) &= \exp\left(\frac{\theta b\cdot z-\alpha\ln[\theta(a\cdot z+d\cdot z)]}{2\theta(a\cdot z+d\cdot z)}(\gamma\cdot a+\gamma\cdot d)\gamma\cdot b\right)\times\\
&\quad\times\exp\left\{\frac{1}{2}(\gamma\cdot a)(\gamma\cdot d)\ln[\theta(a\cdot z+d\cdot z)]\right\}\times\\
&\quad\times\exp\{[(\gamma\cdot c)(\gamma\cdot a+\gamma\cdot d)+i\lambda(\bar{\chi}\chi)^{1/2k}][\gamma\cdot c-\beta(\gamma\cdot a+\gamma\cdot d)]\times\\
&\quad\times[c\cdot z-(\beta/\theta)\ln[\theta(a\cdot z+d\cdot z)]]\}\chi,
\end{aligned}
$$

where $z_\mu = x_\mu+\theta_\mu$; $\alpha,\beta,\theta,\theta_\mu = \text{const}$.

If in (3.34) $k=3/2$, then the equation is invariant under the conformal group $C(1,3)$. Therefore one can generate solutions by the transformations (3.74). Generating solutions $\psi_1(x)$–$\psi_{14}(x)$ (for $k=3/2$) one comes to $C(1,3)$-nongenerable families of solutions. The corresponding formulae are omitted because of their cumbersome character.

3.5. Conditionally invariant solutions of the Dirac–Heisenberg equation. As emphasized in refs. [37, 42] additional constraints enlarging the symmetry of the equation are not necessarily differential ones. Let us impose on the solutions of PDE (3.34) an algebraic condition $\bar{\psi}\psi=1$, i.e., we consider the over-determined system

$$
[\gamma_\mu p^\mu-\lambda(\bar{\psi}\psi)^{1/2k}]\psi(x)=0,\quad \bar{\psi}\psi=1,
$$

or

$$
(\gamma_\mu p^\mu-\lambda)\psi(x)=0,\quad \bar{\psi}\psi=1. \tag{3.77}
$$

Proposition 5. *The system (3.77) is conditionally invariant under the operators* $Q_1=p_0-\lambda\gamma_0$, $Q_2=p_3-\lambda\gamma_3$.

Proof. According to the definition of conditional invariance it is to be proved that the system

$$(\gamma_\mu p^\mu - \lambda)\psi(x) = 0, \quad \bar{\psi}\psi = 1, \quad Q_1\psi = 0 \tag{3.78}$$

is invariant in the Lie sense [32] under the group of transformations generated by Q_1. Acting on the system (3.78) with the extended operator $\tilde{Q}_1$ [32] one obtains

$$\tilde{Q}_1\bar{\psi}\psi = 0, \quad \tilde{Q}_1(p_0\psi - \lambda\gamma_0\psi) = 0,$$
$$\tilde{Q}_1(\gamma_\mu p^\mu\psi - \lambda\psi) = i\lambda\gamma_0(\gamma_\mu p^\mu\psi - \lambda\psi) - 2i\lambda(p_0\psi - \lambda\gamma_0\psi),$$

whence it follows that the statement holds true. The case of the operator Q_2 is treated in the same way.

Let us perform a reduction of the system (3.37) using the above statement. Integration of the equation $Q_1\psi = 0$ yields the following ansatz:

$$\psi(x) = \exp(-i\lambda\gamma_0 x_0)\varphi(\boldsymbol{x}). \tag{3.79}$$

Substituting (3.79) into (3.77) one obtains

$$\gamma_1\varphi_{x_1} + \gamma_2\varphi_{x_2} + \gamma_3\varphi_{x_3} = 0, \quad \bar{\varphi}\varphi = 1. \tag{3.80}$$

Analogously integration of the equation $Q_2\psi = 0$ yields the ansatz

$$\psi(x) = \exp(i\lambda\gamma_3 x_3)\varphi(x_0, x_1, x_2), \tag{3.81}$$

$\varphi(x_0, x_1, x_2)$ satisfying a PDE of the form

$$\gamma_0\varphi_{x_0} + \gamma_1\varphi_{x_1} + \gamma_2\varphi_{x_2} = 0, \quad \bar{\varphi}\varphi = 1. \tag{3.82}$$

If one chooses in (3.80) $\varphi = \varphi(x_1, x_2)$, then the obtained two-dimensional PDE can be integrated. Its general solution is given by

$$\varphi = (\varphi^0(z^*), \varphi^1(z), \varphi^2(z^*), \varphi^3(z))^T,$$

where φ^1, φ^3 (φ^0, φ^2) are arbitrary analytical (anti-analytical) functions.

Imposing on φ the condition $\bar{\varphi}\varphi = 1$ [we use the form (1.2b) of the γ-matrices] one comes to the following relation for φ^μ:

$$|\varphi^)|^2 + |\varphi^1|^2 - |\varphi^2|^2 - |\varphi^3|^2 = 1, \quad |\varphi^\mu|^2 = \varphi^{\mu *}\varphi^\mu. \tag{3.83}$$

Analogously choosing in (3.82) $\varphi = \varphi(x_0, x_1)$ and integrating the obtained equation one has

$$\varphi = (h_0 + g_0, h_0 - g_0, h_1 + g_1, -h_1 + g_1)^T,$$

where

$$h_\mu = h_\mu^1(x_0 + x_3) + ih_\mu^2(x_0 + x_3),$$
$$g_\mu = g_\mu^1(x_0 - x_3) + ig_\mu^2(x_0 - x_3), \quad \mu = 0, 1,$$

h_μ^i, g_μ^i are arbitrary smooth functions. From $\bar{\varphi}\varphi = 1$ it follows that h_μ^i, g_μ^i satisfy the equality

$$h_1^1 g_0^1 + h_1^2 g_0^2 + h_0^1 g_1^1 + h_0^2 g_1^2 = \frac{1}{4}. \tag{3.84}$$

It is easy to convince oneself that (3.83), (3.84) can be written in the form

$$A_l(\xi)B_l(\eta) = C \tag{3.85}$$

(summation over repeated indices from 1 to 4 is implied).

Lemma. *The general solution of the algebraic equation (3.85) is given by formulae*

$$(a) \quad A_k = \theta_k(\phi_l\theta_l) - \phi_k(\theta_l\theta_l) + C\theta_k/(\theta_l\theta_l), \quad B_k = \theta_k, \quad k = 1,2,3,4, \tag{3.86}$$

where $\theta_k = \text{const}$, $\theta_k = \phi_k(\xi)$ *are arbitrary functions;*

$$\begin{aligned} (b) \quad & A_1 = C_1\phi + C_4, \quad A_2 = C_2\phi + C_5, \quad A_3 = C_3\phi + C_6, \quad A_4 = \phi, \\ & B_1 = \rho_1, \quad B_2 = \rho_2, \quad B_3 = C_6^{-1}(C - C_4\rho_1 - C_5\rho_2), \\ & B_4 = C_6^{-1}[(C_3C_4 - C_1C_6)\rho_1 + (C_3C_5 - C_2C_6)\rho_2 - CC_3], \end{aligned} \tag{3.87}$$

where $C_1, \ldots, C_6$ *are constants,* $\phi = \phi(\xi)$, $\rho_i = \rho_i(\eta)$ *are arbitrary functions;*

(c) two other classes of solutions are obtained via the transposition $A_k \to B_k$, $B_k \to A_k$ *in formulae (3.86), (3.87).*

The proof is rather formal; therefore it is omitted.

Using formulae (3.79), (3.81), (3.86) and (3.87) we constructed the following classes of exact solutions of the initial PDE:

$$\psi(x) = \exp(-i\lambda\gamma_0 x_0)\begin{pmatrix} e^{iC_1} \\ e^{iC_2}\phi(z) \\ e^{iC_4}\phi(z^*)\cos C_3 \\ e^{iC_5}\phi(z)\sin C_3 \end{pmatrix}, \tag{3.88}$$

where $\{C_1, \ldots, C_5\} \subset \mathbb{R}^1$, ϕ is an arbitrary analytical function,

$$\psi(x) = \exp(i\lambda\gamma_3 x_3)\begin{pmatrix} A_3 + B_1 + i(A_4 + B_2) \\ A_3 - B_1 + i(A_4 - B_2) \\ A_1 + B_3 + i(A_2 + B_4) \\ -A_1 + B_3 + i(-A_2 + B_4) \end{pmatrix}, \tag{3.89}$$

where the real functions $A^l(x_0 + x_1)$, $B^l(x_0 - x_1)$ are determined by formulae (3.86), (3.87) with $C = 1/4$.

It is worth noting that solutions (3.88), (3.89) are essentially different from $\psi_1(x)$–$\psi_{29}(x)$. They cannot be obtained with the help of the ansatze in tables 1 and 2.

3.6. On scalar fields generated by the solutions of the nonlinear Dirac–Heisenberg equation. In this subsection we construct a scalar field with spin $s = 0$ using the exact solutions of the nonlinear Dirac–Heisenberg equation for a spinor field. The solutions obtained in this way prove to satisfy the nonlinear d'Alembert equation.

The scalar field generated by the solutions of PDE (3.34) is looked for in the form

$$u(x) = \bar\psi\psi e^{i\theta(x)}, \tag{3.90}$$

where $\theta(x)$ is the phase of the field $u(x)$. For ψ_1–ψ_{10} we have the equality

$$\bar\psi\psi = \text{const},$$

whence it follows that $u(x) = Ce^{i\theta(x)}$. Choosing $\theta(x) = \tau a_\mu x^\mu$, $\tau = \text{const}$, one obtains the plane-wave solution

$$u(x) = Ce^{i\tau a_\mu x^\mu}. \tag{3.91}$$

So the spinors ψ_1–ψ_{10} generate plane-wave solutions of the form (3.91) sastisfying the following equation:

$$p_\mu p^\mu u(x) = F(|u|)u(x), \quad |u|^2 = u^* u. \tag{3.92}$$

We did not succeed in establishing a correspondence between the spinor fields ψ_{22}, ψ_{24} and a scalar field $u(x)$. Spinor ψ_{19} generates a scalar field of the form

$$u(x) = C[\phi_0(x_0 + x_3)]^{-1}[(x_1 + \phi_1)^2 + (x_2 + \phi_2)^2]^{-1/2} \exp[i\phi_3(x_0 + x_3)],$$

where ϕ_μ are arbitrary smooth functions of $x_0 + x_3$. It is easy to check that the above function satisfies the nonlinear wave equation with variable coupling constant $\kappa(x) = \tilde{\kappa}[\phi_0(x_0 + x_3)]^2$, $\kappa = \text{const}$, i.e.,

$$p_\mu p^\mu u(x) = \tilde{\kappa}[\phi_0(x_0 + x_3)]^2 |u|^2 u(x). \tag{3.93}$$

The remaining solutions of the nonlinear Dirac–Heisenberg equation (3.34) generate scalar fields satisfying the nonlinear d'Alembert equation

$$p_\mu p^\mu = \kappa |u|^\alpha u, \quad \kappa = \text{const}. \tag{3.94}$$

The corresponding results are given in table 3.

Table 3

No.	$u(x)$	α
11	$C(x_1^2 + x_2^2)^{-1/2} \exp[i\phi_0(x_0 + x_3)]$	2
12	$C[(x_1 + \phi_1)^2 + (x_2 + \phi_2)^2]^{-1/2} \exp[i\phi_0(x_0 + x_3)]$	2
13	$C(x_2^2 + x_3^2)^{-1/2} \exp[i\rho(x_0 + x_1)]$	2
14	$C(x_0^2 - x_1^2 - x_3^2)^{-1}$	1
15	$C(x \cdot x)^{-3/2}$	2/3
16	$C(x_0^2 - x_3^2)^{-1/2}$	2
17	$C(x_2^2 + x_3^2)^{-1/2} \exp[i\rho(x_0 + x_1)]$	2
18	$C\phi_0^{-2}(x_0 + x_3) \exp[i(x_1 + \phi_1)]$	0
20	$C(x_0^2 - x_1^2 - x_2^2)^{-2}$	1/2
21	$C(x_0^2 - x_1^2 - x_2^2)^{-2}$	1/2
23	$C(x_1^2 + x_2^2 + x_3^2)^{-2}$	1/2
25	$C(x_0^2 - x_1^2 - x_3^2)^{-1}$	1
26	$C(x_2^2 + x_3^2)^{-1/2} \exp[i\rho(x_0 + x_1)]$	2
27	$C(x \cdot x)^{-3/2}$	2/3
28	$C\left\{[x_3 + \beta(x_0 - x_2)]^2 + \left[x_1 + \frac{1}{2}(x_0 - x_2)^2\right]^2\right\}^{-k}$	$1/k$, $k < 0$
29	$C\left\{[x_3 + \beta(x_0 - x_2)]^2 + \left[x_1 + \frac{1}{2}(x_0 - x_2)^2\right]^2\right\}^{-1}$	1

ϕ_0, ϕ_1, ϕ_2 are arbitrary smooth functions of $x_0 + x_3$, ρ of $x_0 + x_1$;
C and β are constants.

Thus the spinors ψ_1–ψ_{29} generate complex scalar fields satisfying the nonlinear d'Alembert equation (3.94). Let us note that (3.94) with $\alpha = 2$ admits the conformal group $C(1,3)$. Consequently the fields $u(x)$ generated by the spinors ψ_{11}–ψ_{13}, ψ_{16}, ψ_{17} and ψ_{26} satisfy the conformally invariant d'Alembert equation [though the Dirac–Heisenberg equation may not be invariant under the group $C(1,3)$].

Another interesting feature inherent to the fields $u(x)$ is that $u(x) \to 0$ as $x_0 =$ const, $|\boldsymbol{x}| \to +\infty$ (the only exception is ψ_{28}). What is more, all the functions $u(x)$ have a nonintegrable singularity.

4. Exact solutions of the system of nonlinear Klein–Gordon–Dirac equations

In this section we construct multi-parameter families of exact solutions of the system of PDE describing the interaction of the spinor field $\psi(x)$ and the complex scalar field $u(x)$,

$$\gamma_\mu p^\mu \psi = [\lambda_1 |u|^{k_1} + \lambda_2 (\bar{\psi}\psi)^{k_2}]\psi, \quad p_\mu p^\mu u = [\mu_1 |u|^{k_1} + \mu_2 (\bar{\psi}\psi)^{k_2}]^2 u, \tag{4.1}$$

where $x = (x_0, x_1, x_2, x_3)$, $|u| = (uu^*)^{1/2}$, λ_1, λ_2, μ_1, μ_2, k_1, k_2 are constants.

Let us note that for $\lambda_1 = \mu_2 = 0$, $k_1 = k_2 = 0$ the system of equations (4.1) decomposes into the Dirac equation with mass λ_2 and the Klein–Gordon equation with mass μ_1. For $\lambda_1 = \mu_2 = 0$, $k_1 = 1$, $k_2 = 1/3$ one obtains the nonlinear conformally invariant Dirac–Gürsey [36] and d'Alembert [49] equations.

With the help of the Lie method one can prove that the system of equations (4.1) for arbitrary, non-null k_1, k_2 is invariant under the extended Poincaré group. For $k_1 = 1$, $k_2 = 1/3$ then (4.1) is invariant under the conformal group $C(1,3)$. The above facts make it possible to apply the technique of group-theoretical reduction (as was done in the previous section). But we use another approach which essentially uses the connection between spinor and scalar fields established earlier and the ansatz

$$\psi(x) = \{ig_1(\omega) + g_2(\omega)\gamma_4 - [if_1(\omega) + f_2(\omega)\gamma_4]\gamma_\mu p^\mu \omega\}\chi, \tag{4.2}$$

where g_1, g_2, f_1, f_2 are unknown real functions, $\omega = \omega(x)$ is a scalar function satisfying the system of PDE

$$p_\mu p^\mu \omega + A(\omega) = 0, \quad (p_\mu \omega)(p^\mu \omega) + B(\omega) = 0, \quad A, B : \mathbb{R}^1 \to \mathbb{R}^1. \tag{4.3}$$

Ansatz (4.2) was suggested in refs. [23, 24] for the purpose of constructing exact solutions of the nonlinear Dirac equation. As shown in ref. [28] it can be used to obtain solutions of the system (4.1). The scalar field $u(x)$ is looked for in the form

$$u(x) \sim C(\bar{\psi}\psi), \quad C = \text{const} \quad \text{or} \quad u(x) = \phi(x), \quad \phi \in C^2(\mathbb{R}^1, \mathbb{C}^2). \tag{4.4}$$

Substitution of expressions (4.2), (4.4) into (4.1), $\omega = \omega(x)$ satisfying (4.3), gives rise to the following system of ODE for g_i, f_i and ϕ:

$$\begin{aligned}
&B\ddot{\phi} + A\dot{\phi} = -\{\mu_1 |\phi|^{k_1} + \tilde{\mu}_2 [g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^{k_2}\}^2 \phi, \\
&B\dot{f}_1 + Af_1 = \{\lambda_1 |\phi|^{k_1} + \tilde{\lambda}_2 [g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^{k_2}\} g_1, \\
&\dot{g}_1 = -\{\lambda_1 |\phi|^{k_1} + \tilde{\lambda}_2 [g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^{k_2}\} f_1, \\
&\dot{g}_2 = \{\lambda_1 |\phi|^{k_1} + \tilde{\lambda}_2 [g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^{k_2}\} f_2, \\
&B\dot{f}_2 + Af_2 = -\{\lambda_1 |\phi|^{k_1} + \tilde{\lambda}_2 [g_1^2 - g_2^2 + B(f_1^2 - f_2^2)]^{k_2}\} g_2,
\end{aligned} \tag{4.5}$$

where $\tilde{\lambda}_2 = \lambda_2(\bar{\chi}\chi)^{k_2}$, $\tilde{\mu}_2 = \mu_2(\bar{\chi}\chi)^{k_2}$, dot means differentiation with respect to ω. The system of equations (4.3) is over-determined. Therefore one has to investigate its compatibility. The compatibility of three-dimensional systems of the form (4.3) was investigated in detail by C. Collins [50]. He has proved that the system (4.3) is compatible iff

(1) $B(\omega) \equiv 0, \quad A(\omega) \equiv 0;$
(2) $B(\omega) = \pm 1, \quad A(\omega) = N(\omega+\theta)^{-1}, \quad N = -1, 0, 1, 2.$

In each case the general solution was constructed.

Generalizing Collins' results to the four-dimensional case we obtain the following classes of particular solutions of the system of equations (4.3):

(1) $A(\omega) = -m\omega^{-1}$, $B(\omega) = -1$, $m = 1, 2$:

$$\omega = [(b\cdot y)^2 + (c\cdot y)^2 + (d\cdot y)^2]^{1/2}, \quad m = 2, \tag{4.6}$$

$$\omega = [(b\cdot y + \rho_1)^2 + (c\cdot y + \rho_2)^2]^{1/2}, \quad m = 1; \tag{4.7}$$

(2) $A(\omega) = 0$, $B(\omega) = -1$:

$$\omega = (b\cdot y)\cos\rho_1 + (c\cdot y)\sin\rho_1 + \rho_2, \quad a\cdot y = (b\cdot y)\cos\rho_3 + (c\cdot y)\sin\rho_3 + \rho_4; \tag{4.8}$$

(3) $A(\omega) = 0$, $B(\omega) = 1$:

$$\omega = a\cdot y; \tag{4.9}$$

(4) $A(\omega) = m\omega^{-1}$, $B(\omega) = 1$, $m = 1, 3$:

$$\begin{aligned} &\omega = [(a\cdot y)^2 - (b\cdot y)^2]^{1/2}, \quad m = 1,\\ &\omega = [(a\cdot y)^2 - (b\cdot y)^2 - (c\cdot y)^2]^{1/2}, \quad m = 2,\\ &\omega = (y\cdot y)^{1/2}, \quad m = 3, \end{aligned} \tag{4.10}$$

In (4.6)–(4.10) $y_\mu = x_\mu + \theta_\mu$, $\theta_\mu = \text{const}$; ρ_1, ρ_2 are arbitrary smooth functions of $a\cdot y + d\cdot y$, ρ_3, ρ_4 of $\omega + d\cdot y$; a_μ, b_μ, c_μ, d_μ are arbitrary real parameters satisfying (3.76).

We have succeeded in obtaining the general solution of the system of ODE (4.5) for $A(\omega) = 0$, while in the remaining cases partial solutions are obtained. Let us give the final result:

(1) $A(\omega) = -m\omega^{-1}$, $B(\omega) = -1$, $m = 1, 2$:

$$\begin{aligned} &f_n(\omega) = C_n\omega^{-1/2k_2}, \quad g_n(\omega) = \mp(-1)^n(1-2k_2m)^{1/2}C_n\omega^{-1/2k_2},\\ &n = 1, 2, \quad \phi(\omega) = E\omega^{-1/k_1}, \end{aligned} \tag{4.11}$$

the constants k_1, k_2, C_1, C_2 and E satisfying the conditions

$$\begin{aligned} &[(m-1)k_1 - 1]k_1^{-2} + \{\mu_1|E|^{k_1} + \tilde{\mu}_2[2mk_2(C_1^2 - C_2^2)]^{k_2}\}^2 = 0,\\ &\pm(1-2k_2m)^{1/2} - 2k_2\{\lambda_1|E|^{k_1} + \tilde{\lambda}_2[2mk_2(C_1^2 - C_2^2)]^{k_2}\} = 0,\\ &k_2 < 1/2m, \quad k_1 < 1/(m-1); \end{aligned} \tag{4.12}$$

(2) $A(\omega) = 0,\ B(\omega) = -1$:

$$
\begin{aligned}
&f_1 = C_1 \cosh\left(-\lambda_1 \int [\rho(\omega)]^{k_1} d\omega - \tilde{\lambda}_2 (C_3^2 - C_1^2)^{k_2}\omega + C_2\right),\\
&f_2 = C_3 \cosh\left(\lambda_1 \int [\rho(\omega)]^{k_1} d\omega + \tilde{\lambda}_2 (C_3^2 - C_1^2)^{k_2}\omega + C_4\right),\\
&g_1 = C_1 \sinh\left(-\lambda_1 \int [\rho(\omega)]^{k_1} d\omega - \tilde{\lambda}_2 (C_3^2 - C_1^2)^{k_2}\omega + C_2\right),\\
&g_2 = C_3 \sinh\left(\lambda_1 \int [\rho(\omega)]^{k_1} d\omega + \tilde{\lambda}_2 (C_3^2 - C_1^2)^{k_2}\omega + C_4\right),\\
&\phi = \rho(\omega)\exp[i\theta(\omega)],\\
&\int^{\rho(\omega)} [a_-(z) + C_6]^{-1/2} dz = \omega + C_7, \quad \theta(\omega) = C_5 \int [\rho(\omega)]^{-1/2} d\omega + C_8,
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
a_-(z) = {}& \frac{\mu_1^2}{k_1+1} z^{2(k_1+1)} + \tilde{\mu}_2^2 (C_3^2 - C_1^2)^{2k_2} z^2 + \\
& + 4\frac{\mu_1\tilde{\mu}_2}{k_1+2}(C_3^2 - C_1^2)^{k_2} z^{k_1+2} + 2C_5^2 z;
\end{aligned}
$$

(3) $A(\omega) = 0,\ B(\omega) = 1$:

$$
\begin{aligned}
&f_1 = C_1 \sin\left(\lambda_1 \int [\rho(\omega)]^{k_1} d\omega + \tilde{\lambda}_2 (C_1^2 - C_3^2)^{k_2}\omega + C_2\right),\\
&f_2 = C_3 \cos\left(\lambda_1 \int [\rho(\omega)]^{k_1} d\omega + \tilde{\lambda}_2 (C_1^2 - C_3^2)^{k_2}\omega + C_4\right),\\
&g_1 = C_1 \cos\left(\lambda_1 \int [\rho(\omega)]^{k_1} d\omega + \tilde{\lambda}_2 (C_1^2 - C_3^2)^{k_2}\omega + C_2\right),\\
&g_2 = C_3 \sin\left(\lambda_1 \int [\rho(\omega)]^{k_1} d\omega + \tilde{\lambda}_2 (C_1^2 - C_3^2)^{k_2}\omega + C_4\right),\\
&\phi = \rho(\omega)\exp[i\theta(\omega)],\\
&\int^{\rho(\omega)} [a_+(z) + C_6]^{-1/2} dz = \omega + C_7, \quad \theta(\omega) = C_5 \int [\rho(\omega)]^{-1/2} d\omega + C_8,
\end{aligned}
\tag{4.14}
$$

where

$$
\begin{aligned}
a_+(z) = {}& -\frac{\mu_1^2}{k_1+1} z^{2(k_1+1)} - \tilde{\mu}_2^2 (C_1^2 - C_3^2)^{2k_2} z^2 - \\
& - 4\frac{\mu_1\tilde{\mu}_2}{k_1+2}(C_1^2 - C_3^2)^{k_2} z^{k_1+2} + 2C_5^2 z
\end{aligned}
$$

(in the above formulae $C_1, \dots, C_8$ are arbitrary constants);

(4) $A(\omega) = m\omega^{-1},\ B(\omega) = 1,\ m = 2, 3$:

(a) $k_1 > 1/(m-1)$, $k_2 > 1/2m$:

$$\begin{aligned} &f_n(\omega) = C_n\omega^{-1/2k_2}, \quad g_n(\omega) = \mp(-1)^n(2k_2m-1)^{1/2}C_n\omega^{-1/2k_2}, \\ &n = 1, 2, \quad \phi(\omega) = E\omega^{-1/k_1}, \end{aligned} \tag{4.15}$$

where C_1, C_2 and E are constants satisfying the following conditions:

$$\begin{aligned} &[(1-m)k_1+1]k_1^{-2} + \{\mu_1|E|^{k_1} + \tilde{\mu}_2[2mk_2(C_1^2-C_2^2)]^{k_2}\}^2 = 0, \\ &\pm(2k_2m-1)^{1/2} - 2k_2\{\lambda_1|E|^{k_1} + \tilde{\lambda}_2[2mk_2(C_1^2-C_2^2)]^{k_2}\} = 0; \end{aligned} \tag{4.16}$$

(b) $k_1 = 2(m-1)^{-1}$, $k_2 > m^{-1}$:

$$\begin{aligned} &f_n(\omega) = (-1)^n\theta\omega g_n(\omega), \quad g_n(\omega) = C_n(1+\theta^2\omega^2)^{-(m+1)/2}, \\ &n = 1, 2, \quad \phi(\omega) = E(1+\theta^2\omega^2)^{(1-m)/2}, \end{aligned} \tag{4.17}$$

where the constants C_1, C_2 and E satisfy the conditions

$$\begin{aligned} &\theta^2(m^2-1) = [\mu_1|E|^{2/(m-1)} + \tilde{\mu}_2(C_1^2-C_2^2)^{1/m}]^2, \\ &(m+1)\theta = [\lambda_1|E|^{2/(m-1)} + \tilde{\lambda}_2(C_1^2-C_2^2)^{1/m}]. \end{aligned} \tag{4.18}$$

To obtain the exact solutions of the initial system (4.1) one has to substitute formulae (4.6)–(4.10), (4.11)–(4.17) into the ansatz (4.2), (4.4). The obtained expressions are very cumbersome and will not be given here.

Let us make some remarks.

Note 1. If one interprets the nonlinearities $\lambda_1|u|^{k_1} + \lambda_2(\bar{\psi}\psi)^{k_2}$, $\mu_1|u|^{k_1} + \mu_2(\bar{\psi}\psi)^{k_2}$, as the masses of a spinor field (M_ψ) and of a scalar field (M_u) created because of the nonlinear interaction of these fields, then for solutions (4.11), (4.15) and (4.17) the following remarkable relations hold:

$$\begin{aligned} &\left(\frac{M_u}{M_\psi}\right)^2 = \frac{4k_2^2[1+(1-m)k_1]}{k_1^2(1-2mk_2)}, \quad m = 1, 2, \\ &\left(\frac{M_u}{M_\psi}\right)^2 = \frac{4k_2^2[(m-1)k_1-1]}{k_1^2(2mk_2-1)}, \quad m = 2, 3, \\ &\left(\frac{M_u}{M_\psi}\right)^2 = \frac{m-1}{m+1}, \quad m = 2, 3. \end{aligned} \tag{4.19}$$

These relations can be interpreted as formulae for the mass spectrum of spinor and scalar particles. What is more, the discrete variable m arises as the compatibility condition of the over-determined system (4.3) (compare ref. [50]). So the mass spectrum is determined by the geometry of the solutions of the form (4.2), (4.4).

Note 2. If one puts in (4.2) $g_2 \equiv f_2 \equiv 0$, $\omega(x) = x\cdot x$, then the ansatz suggested by Heisenberg [2, 14] is obtained,

$$\psi(x) = [ig_1(x\cdot x) + \gamma\cdot xf_1(x\cdot x)]\chi.$$

Note 3. If one chooses $\lambda_1 = \mu_2 = 0$ then formulae (4.2), (4.4), (4.6)–(4.18) give exact solutions of the nonlinear Dirac–Heisenberg equation and of the d'Alembert equations.

Note 4. Ansätze (4.2), (4.4) can be used to reduce nonlinear systems of PDE of more general form than (4.1), namely

$$\gamma_\mu p^\mu = F_1(\bar{\psi}\psi, |u|)\psi, \quad p_\mu p^\mu = F_2(\bar{\psi}\psi, u, u^*), \tag{4.20}$$

where F_1, F_2 are arbitrary continuous functions. In particular, the solutions of a system of equations of the form (4.20) constructed in refs. [12, 13, 51] can be obtained via ansätze (4.2), (4.4).

5. Exact solutions of the nonlinear Maxwell–Dirac equations

There is a vast literature devoted to the system of equations of classical electrodynamics (Maxwell–Dirac equations)

$$\begin{aligned} &[\gamma_\mu(p^\mu + eA^\mu) + m]\psi(x) = 0, \\ &p_\nu p^\nu A_\mu - p_\mu p_\nu A^\nu = e\bar{\psi}\gamma_\mu\psi, \quad \mu,\nu = 0,1,2,3, \end{aligned} \tag{5.1}$$

where $A_\mu = A_\mu(x)$ is the vector potential of the electromagnetic field; m and e are the mass and the charge of the electron. A number of existence theorems have been proved (in particular, in ref. [52] the solubility of the Cauchy problem has been investigated). However, as far as we know there are no publications containing exact solutions of this system in explicit form.

We look for solutions of eqs. (5.1) in the form

$$\psi(x) = (\gamma\cdot\theta)\varphi(\boldsymbol{\omega}), \quad A_\mu(x) = \theta_\mu\phi(\boldsymbol{\omega}), \quad \mu = 0,1,2,3, \tag{5.2}$$

where $\boldsymbol{\omega} = \{\omega_0, \omega_1, \omega_2\} \equiv \{\theta\cdot x, b\cdot x, c\cdot x\}$, $\theta_\mu = a_\mu + d_\mu$, $\varphi(\boldsymbol{\omega})$ and $\phi(\boldsymbol{\omega})$ are unknown functions. Substitution of (5.2) into (5.1) gives rise to the following system of two-dimensional PDE for $\varphi(\boldsymbol{\omega})$ and $\phi(\boldsymbol{\omega})$

$$(\gamma\cdot b)\varphi_{\omega_1} + (\gamma\cdot c)\varphi_{\omega_2} + im\varphi = 0, \tag{5.3a}$$

$$\phi_{\omega_1\omega_1} + \phi_{\omega_2\omega_2} = 2e\bar{\varphi}(\gamma\cdot\theta)\varphi. \tag{5.3b}$$

Let us note that in (5.3) there is no differentiation with respect to ω_0, therefore φ and ϕ contain ω_0 as a parameter.

The general solution of eq. (5.3a) is given by the elliptic analogue of the d'Alembert formula for the wave equation [38]

$$\phi(\boldsymbol{\omega}) = F(z,\omega_0) + F(z^*,\omega_0) - ie\int_0^{\omega_2}\int_{\omega_1 - i(\omega_2-\tau)}^{\omega_1 + i(\omega_2-\tau)} \bar{\varphi}(\gamma\cdot\theta)\varphi(\xi,\tau)d\xi d\tau, \tag{5.4}$$

where F is an arbitrary analytical function of $z = \omega_1 + i\omega_2$. So the problem of constructing particular solutions of the initial system of equations (5.1) is reduced to that of integrating the linear two-dimensional Dirac equation (5.3a).

Choosing the eigenfunction of the Hermitian operator $-i\partial_{\omega_1}$ as a partial solution of eq. (5.3a) one obtains

$$\varphi = \exp[i\lambda\omega_1 + i\gamma\cdot c(m + \lambda\gamma\cdot b)\omega_2]\varphi_0(\omega_0), \tag{5.5}$$

where φ_0 is a four-component spinor depending on ω_0 in an arbitrary way. Imposing on (5.5) the additional condition of being periodical with respect to the variable ω_1, we come to the following relation:

$$\lambda = \lambda_n = 2\pi n, \quad n \in \mathbb{Z}. \tag{5.6}$$

Substitution of (5.5) into formula (5.4) gives the explicit form of $\phi(\boldsymbol{\omega})$,

$$\begin{aligned}\phi^{(n)}(\boldsymbol{\omega}) &= F(z,\omega_0) + F(z^*,\omega_0) + \frac{1}{4}(m^2+\lambda_n^2)^{-1}\times\\ &\times[\tau_1\cosh 2(m^2+\lambda_n^2)^{1/2}\omega_2 + \tau_2\sinh 2(m^2+\lambda_n^2)^{1/2}\omega_2],\quad n\in\mathbb{Z},\end{aligned}\tag{5.7}$$

where

$$z=\omega_1+i\omega_2,\quad \tau_1 = 2e\bar\varphi_0(\gamma\cdot\theta)\varphi_0,$$
$$\tau_2 = 2ie(m^2+\lambda_n^2)^{-1/2}\bar\varphi_0(\gamma\cdot\theta)(m+\lambda_n\gamma\cdot b)\varphi_0.$$

Substituting (5.5), (5.7) into the ansatz (5.2) one obtains a multi-parameter family of exact solutions of the Maxwell–Dirac equations depending on three arbitrary complex functions,

$$\begin{aligned}&\psi^{(n)}(x) = (\gamma\cdot a+\gamma\cdot d)\exp[i\lambda_n b\cdot x + i\gamma\cdot c(m+\lambda_n\gamma\cdot b)c\cdot x]\varphi_0(a\cdot x+d\cdot x),\\ &A_\mu^{(n)}(x) = (a_\mu+d_\mu)\Big\{F(z,a\cdot x+d\cdot x)+F(z^*,a\cdot x+d\cdot x)+\\ &+\frac{1}{4}(m^2+\lambda_n^2)^{-1}[\tau_1\cosh(2(m^2+\lambda_n^2)^{1/2}c\cdot x)+\tau_2\sinh(2(m^2+\lambda_n^2)^{1/2}c\cdot x)]\Big\}.\end{aligned}\tag{5.8}$$

Analogously if one chooses the following solution of eq. (5.3a):

$$\begin{aligned}\varphi(\boldsymbol{\omega}) &= (\omega_1^2+\omega_2^2)^{-1/4}\exp\left[-\frac{1}{2}(\gamma\cdot b)(\gamma\cdot c)\,\mathrm{arctg}\,\frac{\omega_1}{\omega_2}\right]\times\\ &\times\exp[im(\gamma\cdot c)(\omega_1^2+\omega_2^2)^{1/2}]\varphi_0(\omega_0)\end{aligned}$$

as $\varphi(\boldsymbol{\omega})$, then formulae (5.2), (5.4) give rise to the following family of exact solutions:

$$\begin{aligned}\psi(x) &= (\gamma\cdot a+\gamma\cdot d)|z|^{-1/2}\exp\left(-\frac{1}{2}(\gamma\cdot b)(\gamma\cdot c)\,\mathrm{arctg}\,\frac{b\cdot x}{c\cdot x}\right)\times\\ &\times\exp[im\gamma\cdot c|z|]\varphi_0(a\cdot x+d\cdot x),\\ A_\mu(x) &= (a_\mu+d_\mu)\Big(F(z,a\cdot x+d\cdot x)+F(z^*,a\cdot x+d\cdot x)+\\ &+\int^{|z|}(\tau_1\sinh 2m\rho+\tau_2\cosh 2m\rho)\rho^{-1}d\rho\Big),\end{aligned}\tag{5.9}$$

where F is an arbitrary analytical function of $z=b\cdot x+ic\cdot x$,

$$|z| = (z^*z)^{1/2} = [(b\cdot x)^2+(c\cdot x)^2]^{1/2},\quad \tau_1 = 2e\bar\varphi_0[\gamma\cdot a+\gamma\cdot d)\varphi_0,$$
$$\tau_2 = 2ie\bar\varphi_0(\gamma\cdot a+\gamma\cdot d)(\gamma\cdot c)\varphi_0.$$

Let us consider in more detail the solution of the Maxwell–Dirac equations (5.8) putting

$$F\equiv 0,\quad \varphi_0 = \exp[-\kappa^2(a\cdot x+d\cdot x)^2]\chi,$$

where χ is an arbitrary constant spinor, $\kappa=\mathrm{const}$. By direct verification one can convince oneself that the following equalities hold:

$$\begin{aligned}&p_\mu p^\mu A_\nu^{(n)} = 4(m^2+\lambda_n^2)A_\nu^{(n)},\quad n\in\mathbb{Z},\\ &p^\nu A_\nu^{(n)} = 0,\quad p_\mu p^\mu\psi^{(n)} = m^2\psi^{(n)}.\end{aligned}\tag{5.10}$$

The above relations seem to admit the following interpretation: the interaction of a spinor and a massless electromagnetic field according to the nonlinear eqs. (5.1) generates massive electromagnetic fields $A_\mu^{(n)}(x)$ with masses $M_n = 2(m^2 + \lambda_n^2)^{1/2}$ (in other words, the nonlinear interaction of the fields $A_\mu(x)$ and $\psi(x)$ generates the mass spectrum). If one puts $n = 0$ then $M_0 = 2m$, m being the mass of the electron.

As solutions (5.8), (5.9) have an analytical dependence on m, then the solutions of the massless Maxwell–Dirac equations can be obtained by putting $m = 0$. The case $m = 0$ deserves special consideration because the massless Maxwell–Dirac equations are conformally invariant (see, e.g., ref. [53]).

It is not difficult to obtain the general solution of the two-dimensional massless Dirac equation

$$\varphi = (\gamma \cdot b + i\gamma \cdot c)\varphi_1(z, \omega_0) + (\gamma \cdot b - i\gamma \cdot c)\varphi_2(z^*, \omega_0), \tag{5.11}$$

where φ_1, φ_2 are arbitrary spinors depending analytically on z, z^*; $z = b \cdot x + ic \cdot x$. Substituting (5.11) into (5.4) one obtains the following expression for $\phi(\boldsymbol{\omega})$:

$$\begin{gathered}\phi(\boldsymbol{\omega}) = F(z, \omega_0) + F(z^*, \omega_0) + \\ + e\left(z^* \int_0^z f_1(z, \omega_0) dz + z \int_0^{z^*} f_2(z^*, \omega_0) dz^*\right), \\ f_1 = \bar{\varphi}_1(\gamma \cdot \theta)[1 - i(\gamma \cdot b)(\gamma \cdot c)]\varphi_2, \quad f_2 = \bar{\varphi}_2(\gamma \cdot \theta)[1 + i(\gamma \cdot b)(\gamma \cdot c)]\varphi_1.\end{gathered} \tag{5.12}$$

Substitution of the above formulae into (5.2) gives rise to a multi-parameter family of exact solutions including three arbitrary complex functions,

$$\begin{gathered}\psi(x) = (\gamma \cdot a + \gamma \cdot d)[(\gamma \cdot b + i\gamma \cdot c)\varphi_1(z, a \cdot x + d \cdot x) + \\ + (\gamma \cdot b - i\gamma \cdot c)\varphi_2(z^*, a \cdot x + d \cdot x)], \\ A_\mu(x) = (a_\mu + d_\mu)\Big[F(z, a \cdot x + d \cdot x) + F(z^*, a \cdot x + d \cdot x) + \\ + e\Big(z^* \int_0^z f_1(z, a \cdot x + d \cdot x) dz + z \int_0^{z^*} f_2(z^*, a \cdot x + d \cdot x) dz^*\Big)\Big], \\ z = b \cdot x + ic \cdot x.\end{gathered} \tag{5.13}$$

Using the solution generating formula with the group of special conformal transformations [24, 47]

$$\begin{gathered}\psi_{II}(x) = \sigma^{-2}(x)[1 - (\gamma \cdot x)(\gamma \cdot \theta)]\psi_I(x'), \\ A_\mu^{II}(x) = \sigma^{-2}(x)[g_{\mu\nu}\sigma(x) + 2(\theta_\mu x_\nu - \theta_\nu x_\mu + \\ + 2\theta \cdot x x_\mu \theta_\nu - x \cdot x \theta_\mu \theta_\nu - \theta \cdot \theta x_\mu x_\nu)]A_I^\nu(x'), \\ x'_\mu = (x_\mu - \theta_\mu x \cdot x)\sigma^{-1}(x), \quad \sigma(x) = 1 - 2\theta \cdot x + (\theta \cdot \theta)(x \cdot x),\end{gathered}$$

it is possible to obtain a larger family of solutions of the system of equations (5.1). We omit the corresponding formulae because of their cumbersome character.

6. Conclusions

In this review we described Poincaré-invariant nonlinear systems of first-order differential equations for spinor fields which are nonlinear generalizations of the classical Dirac equation without using variational principles. The large class of nonlinear spinor equations invariant under the extended Poincaré group $\tilde{P}(1,3)$ and the conformal group is constructed. It contains, in particular, the well-known nonlinear Dirac–Ivanenko, Dirac–Heisenberg and Dirac–Gürsey equations. Besides there are many equations which so far have not been considered in the literature.

The main aim of this review is to suggest a constructive method of solution of nonlinear Dirac-type spinor equations, that is, to construct in explicit form families of exact solutions of these equations without applying methods of perturbation theory. The key idea of our method is a symmetry reduction of the many-dimensional spinor equation to systems of ordinary differential equations. Many of them can be integrated in quadratures. Such a reduction is carried out with the help of special ansatze constructed using the symmetry properties of the equation in question.

To our mind the important result of the present paper is that we have obtained nongenerable families of exact solutions of nonlinear spinor equations. These solutions possess the same symmetry as the equation of motion. So nongenerable families of solutions can be quantized in a standard way without losing the invariance under the Poincaré group.

It is worth noting that some solutions depend on the coupling constant λ in a singular way.

It is shown how to construct the simplest fields with spin $s = 0$ using solutions of the fundamental spinor equation. Such bosonic fields satisfy the nonlinear d'Alembert equations.

A new approach to the problem of the mass spectrum is suggested (section 4). It is established that exact solutions of the system of nonlinear equations for spinor and scalar fields make it possible to calculate the ratio of the masses of spinor and scalar fields. It occurs that this ratio is determined by the non-linearity degrees of the spinor and scalar fields.

We hope that the results presented in our paper will make it possible to understand more deeply the role played by nonlinear spinor equations in the unified theory of bosonic and fermionic fields with spins $s = 0,\ 1/2,\ 1,\ 3/2,\ 2,\ \ldots$.

Suggested methods can be applied to equations of motion in $\mathbb{R}(1,n)$ [54, 55]. The problem of subgroup classification of generalized Poincaré groups $P(1,n)$, $\tilde{P}(1,n)$, $P(2,n)$ and Galilei groups $G(1,n)$ was solved in refs. [55–60].

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