On some new exact solutions of the nonlinear d'Alembert–Hamilton system

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Some new exact solutions of the d'Alembert–Hamilton system of partial differential equations are obtained. The necessary conditions of integrability of an over-determined d'Alembert–Hamilton system are established.

Since Euler (1734–1740) the method of reduction of partial differential equations (PDEs) to ordinary differential equations (ODEs) is one of the most effective ways to construct partial solutions of PDEs.

In refs. [1–4] the symmetry reduction of the d'Alembert equation,

$$\Box u = F_1(u), \quad \Box = \partial^2_{x_0} - \partial^2_{x_1} - \partial^2_{x_2} - \partial^2_{x_3} \tag{1}$$

(where $F_1(u)$ is an arbitrary smooth function), to an ODE has been carried out. So the four-dimensional PDE (1) with the ansatz

$$u(x) = \varphi(\omega), \tag{2}$$

where $\varphi \in C^2(\mathbb{R}^1, \mathbb{R}^1)$, and $\omega = \omega(x) \in C^2(\mathbb{R}^4, \mathbb{R}^1)$ being the new variable, is reduced to an ODE having variable coefficients,

$$(\omega_\mu \omega_\mu)\ddot{\varphi} + (\Box\omega)\dot{\varphi} = F_1(\varphi), \tag{3}$$

where $\omega_\mu \equiv \partial\omega/\partial x_\mu$, $\mu = 0, \ldots, 3$, $\dot{\varphi} \equiv d\varphi/d\omega$. Hereafter the summation over repeated indices in Minkowski space $\mathbb{R}(1,3)$ having the metric $g_{\mu\nu} = \operatorname{diag}(1, -1, -1, -1)$ is supposed, i.e.

$$\omega_\mu \omega_\mu = g_{\mu\nu}\omega_\mu\omega_\nu = \omega_1^2 - \omega_1^2 - \omega_2^2 - \omega_3^2.$$

In refs. [3, 4] using the symmetry properties of eq. (1) and the subgroup structure of the Poincaré group $P(1,3)$ new variables have been constructed for eq. (3) depending on ω only.

In the present paper we suggest a more general approach to the problem of reduction of the PDE (1) to an ODE than the approach based on the employment of its symmetry properties [1–4].

We say that the ansatz (2) reduces the PDE (1) to an ODE if the new variable $\omega(x)$ satisfies both the d'Alembert and the Hamilton equation,

$$\Box\omega = F_2(\omega), \tag{4}$$

$$\omega_\mu \omega_\mu = F_3(\omega), \tag{5}$$

where F_2, F_3 are arbitrary smooth functions.

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Evidently for every $\omega(x)$ satisfying the system (4), (5) the ODE (3) depends on ω only

$$F_3(\omega)\ddot{\varphi} + F_2(\omega)\dot{\varphi} = F_1(\varphi) \tag{6}$$

(one can be easily convinced that the invariants obtained by Winternitz et al. [4] satisfy this system). Thus the problem of finding the ansatz (2) reducing the PDE (1) to an ODE leads to the construction of solutions of the d'Alembert–Hamilton system (4), (5).

In the present paper the compatibility of the overdetermined system (4), (5) is investigated, i.e. all smooth functions ensuring the compatibility of the d'Alembert–Hamilton system are described. Besides wide classes of exact solutions of the system (4), (5) are presented.

System (4), (5) via the change of the dependent variable $Z = Z(\omega)$ can be reduced to the allowing system:

$$\Box\omega = F(\omega), \tag{7}$$

$$\omega_\mu\omega_\mu = \lambda, \quad \lambda = \mathrm{const}. \tag{8}$$

The ODE (6) then takes the form

$$\lambda\ddot{\varphi} + F(\omega)\dot{\varphi} = F_1(\varphi). \tag{9}$$

Before formulating the principal result of the paper we adduce without proof some auxiliary statements.

Lemma 1. *Solutions of the system (7), (8) satisfy the identities*

$$\begin{aligned}
&\omega_{\mu\nu_1}\omega_{\nu_1\mu} = -\lambda\dot{F}(\omega),\\
&\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\omega_{\nu_2\mu} = \frac{1}{2!}\lambda^2\ddot{F}(\omega), \quad \ldots,\\
&\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\cdots\omega_{\nu_n\mu} = \frac{1}{n!}\lambda^n\frac{d^nF(\omega)}{d\omega^n},
\end{aligned} \tag{10}$$

where $\omega_{\alpha\beta} \equiv \partial^2\omega/\partial x_\alpha\partial x_\beta$, $\alpha, \beta = 0, \ldots, 3$, $n \geq 1$.

Lemma 2. *Solutions of the system (7), (8) satisfy the following equality:*

$$\det(\omega_{\mu\nu}) = 0. \tag{10$'$}$$

Let us now formulate the principal statement.

Theorem 1. *The necessary condition of compatibility of the overdetermined system (7), (8) is*

$$F(\omega) = \begin{cases} 0, \\ \lambda(\omega + C_1)^{-1}, \\ 2\lambda(\omega + C_1)[(\omega + C_1)^2 + C_2]^{-1}, \\ 3\lambda[(\omega + C_1)^2 + C_2][(\omega + C_1)^3 + 3C_2(\omega + C_1) + C_3]^{-1}. \end{cases} \tag{11}$$

where C_1, C_2, C_3 are arbitrary constants.

Proof. By direct (and rather tiresome) verification one can be convinced that the following identity holds,

$$6(\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\omega_{\nu_2\nu_3}\omega_{\nu_3\mu}) - 8(\Box\omega)(\omega_{\mu\nu_1}\omega_{\nu_1\nu_2}\omega_{\nu_2\mu}) - 3(\omega_{\mu\nu_1}\omega_{\nu_1\mu})^2 + \\ + 6(\Box\omega)^2(\omega_{\mu\nu_1}\omega_{\nu_1\mu}) - (\Box\omega)^4 = 24\det(\omega_{\alpha\beta}). \tag{12}$$

Substituting (10), (10′) into (12) one obtains a nonlinear ODE for $F(\omega)$

$$\lambda^3\dddot{F} + 4\lambda^2 F\ddot{F} + 3\lambda^2\dot{F}^2 + 6\lambda\dot{F}F^2 + F^4 = 0, \tag{13}$$

where $\dot{F} = dF/d\omega$.

The general solution of eq. (13) is given by formulae (11). The theorem is proved.

Note 1. Compatibility of the three-dimensional d'Alembert–Hamilton system has been investigated in detail by Collins [5]. Collins essentially used geometrical methods which could not be generalized to higher dimensions.

Using Lie's method (see e.g. ref. [6]) one can prove the following statement.

Theorem 2. *System (7), (8) is invariant under the 15-parameter conformal group $C(1,3)$ iff*

$$F(\omega) = 3\lambda(\omega + C)^{-1}, \quad \lambda > 0, \quad C = \text{const}. \tag{14}$$

Note 2. Formula (14) can be obtained from (11) by putting $C_2 = C_3 = 0$. So Theorem 2 demonstrates the close connection between compatibility of a system of PDEs and its symmetry.

Note 3. It is common knowledge that the PDE (7) is invariant under the group $C(1,3)$ iff $F(\omega) = \lambda\omega^3$ [3]. Consequently, an additional constraint (8) changes essentially the symmetry properties of the d'Alembert equation (choosing $F_3(\omega)$ in a proper way one can obtain a conformally-invariant system of the form (4), (5) under arbitrary $F_2(\omega)$).

In Table 1 we list the explicit form of some exact solutions of the d'Alembert–Hamilton system (7), (8) and the reduced ODEs for the function $\varphi(\omega)$. h_1, g_1 are arbitrary smooth functions on $a_\mu x^\mu + d_\mu x^\mu$; h_2, g_2 on $\omega + d_\mu x^\mu$; and a_μ, b_μ, c_μ, d^μ are arbitrary real parameters satisfying conditions of the form

$$-a_\mu a^\mu = b_\mu b^\mu = c_\mu c^\mu = d_\mu d^\mu = -1, \\ a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.$$

Table 1

No.	λ	$F(\omega)$	$\omega = \omega(x)$	QDE for $\varphi(\omega)$
1	1	0	$a_\mu x^\mu$	$\ddot{\varphi} = F_1(\varphi)$
2	1	ω^{-1}	$[(a_\mu x^\mu)^2 - (b_\mu x^\mu)^2]^{1/2}$	$\ddot{\varphi} + \omega^{-1}\dot{\varphi} = F_1(\varphi)$
3	1	$2\omega^{-1}$	$[(a_\mu x^\mu)^2 - (b_\mu x^\mu)^2 - (c_\mu x^\mu)^2]^{1/2}$	$\ddot{\varphi} + 2\omega^{-1}\dot{\varphi} = F_1(\varphi)$
4	1	$3\omega^{-1}$	$(x_\mu x^\mu)^{1/2}$	$\ddot{\varphi} + 3\omega^{-1}\dot{\varphi} = F_1(\varphi)$
5	-1	0	$b_\mu x^\mu \cos h_1 + c_\mu x^\mu \sin h_1 + g_1$	$\ddot{\varphi} = -F_1(\varphi)$
			$a_\mu x^\mu - b_\mu x^\mu \cos h_2 - c_\mu x^\mu \sin h_2 - g_2 = 0$	
6	-1	$-\omega^{-1}$	$[(b_\mu x^\mu + h_1)^2 + (c_\mu x^\mu + h_2)^2]^{1/2}$	$\ddot{\varphi} + \omega^{-1}\dot{\varphi} = F_1(\varphi)$
7	-1	$-2\omega^{-1}$	$[(b_\mu x^\mu)^2 + (c_\mu x^\mu)^2 + (d_\mu x^\mu)^2]^{1/2}$	$\ddot{\varphi} + 2\omega^{-1}\dot{\varphi} = F_1(\varphi)$
8	0	0	h_1	$0 = F_1(\varphi)$

Choosing in a proper way constants a_μ, b_μ, c_μ, d^μ and functions f_i, g_i one can obtain from Table 1 symmetry ansatze constructed by Winternitz et al. [4]. Such an approach based on the d'Alembert–Hamilton system makes it possible to obtain a wider family of ansatze for the nonlinear d'Alembert equation (1) (see also ref. [7]).

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