On vector and pseudovector Lagrangians for electromagnetic field

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A Lagrange function in terms of electromagnetic field strengths is constructed which is a 4-vector with respect to the total Poincaré group $\tilde{P}(1,3)$ and whose Euler–Lagrange equations are equivalent to the Maxwell equations. The advantages of the Lagrange function proposed in comparison with the known pseudovector with respect to the $\tilde{P}(1,3)$ group Lagrange function are shown. The conserved quantitites on the basis of corresponding generalization of Noether theorem are found.

A development of Lagrange approach (L-approach) in electro-dynamics in terms of field-strength tensor $F=(F^{\mu\nu})=(\vec{E},\vec{H})$ of electromagnetic field, without using the potentials A_μ, was discussed in [1–4]. It is easy to show, that in terms of $(\vec{E},\vec{H})$ there is no scalar, with respect to the Poincaré group $P(1,3)$, Lagrange function, for which the Euler-Lagrange (EL) equations coincide with the Maxwell equations.

The purpose of this work is to construct a vector (with respect to the total Poincaré group $\tilde{P}(1,3)$ i.e., $P(1,3)$ group including the space and time reflections) Lagrange function in terms of $(\vec{E},\vec{H})$, with the help of which the system of equations equivalent to the Maxwell equations can be received from the EL equations. The conserved quantities are constructed on the basis of corresponding generalization of Noether theorem. Further we will call such Lagrange vector-function a Lagrange vector.

Let us represent the Maxwell equations

$$\partial_0\vec{E}=\mathrm{rot}\,\vec{H}-\vec{j},\quad \mathrm{div}\,\vec{E}=\rho,\quad \partial_0\vec{H}=-\mathrm{rot}\,\vec{E},\quad \mathrm{div}\,\vec{H}=0 \tag{1}$$

in a manifestly covariant form

$$Q^\mu=j^\mu,\quad R^\mu=0,\quad \mu=0,1,2,3, \tag{2}$$

where

$$Q^\mu\equiv F^{\mu\nu}_{,\nu},\quad R^\mu\equiv\varepsilon F^{\mu\nu}_{,\nu},\quad \varepsilon F^{\mu\nu}\equiv\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}, \tag{3}$$

$F=(F^{\mu\nu})$ is a tensor of electromagnetic field:

$$F=(F^{\mu\nu})=(\vec{E},\vec{H}):\quad F^{0i}=E^i,\quad F^{ij}=\varepsilon^{ijk}H^k,\quad F^{\mu\nu}=-F^{\nu\mu}, \tag{4}$$

j is a 4-vector of current:

$$j\equiv(j^\mu)=(\rho,\vec{j}),\quad j^0=\rho,\quad \vec{j}=(j^i),\quad i=1,2,3, \tag{5}$$

and $\varepsilon^{\mu\nu\rho\sigma}$ is a completely antisymmetric unit tensor, $\varepsilon^{0123}=1$:

$$x=(x^\mu)\in R(1,3),\quad \partial_\mu\equiv\partial/\partial x^\mu. \tag{6}$$

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The explicit form of the components Q^μ, R^μ is the following

$$Q^0 = \mathrm{div}\,\vec{E}, \quad Q^i = (-\partial_0\vec{E} + \mathrm{rot}\,\vec{H})^i \equiv -\partial_0 E^i + \varepsilon^{ijk}\partial_j H^k, \tag{7}$$

$$R^0 = \mathrm{div}\,\vec{H}, \quad R^i = (-\partial_0\vec{H} + \mathrm{rot}\,\vec{E})^i \equiv -\partial_0 H^i + \varepsilon^{ijk}\partial_j E^k. \tag{8}$$

Now consider the tensor $T_{\mu\rho\sigma}$ and pseudotensor $T'_{\mu\rho\sigma}$ of 3-rd rank (with respect to $\tilde{P}(1,3)$ group), which are constructed from 4-vectors Q^μ, R^μ (3):

$$T_{\mu\rho\sigma} \equiv a[g_{\mu\rho}(Q_\sigma - j_\sigma) - g_{\mu\sigma}(Q_\rho - j_\rho)] + b\varepsilon_{\mu\nu\rho\sigma}R^\nu, \tag{9}$$

$$T'_{\mu\rho\sigma} \equiv a'(g_{\mu\rho}R_\sigma - g_{\mu\sigma}R_\rho) + b'\varepsilon_{\mu\nu\rho\sigma}(Q^\nu - j^\nu), \tag{10}$$

a, b, a', b' are constant coefficients.

Theorem 1. *For any $ab \neq 0 \neq a'b'$ each of the set of equations*

$$T_{\mu\rho\sigma} = 0, \tag{11}$$

$$T'_{\mu\rho\sigma} = 0, \tag{12}$$

is equivalent to the initial Maxwell equations (2).

One can easily varify the validity of this statement by rewriting the components of tensors T, T' (11), (12) in the evident form.

Only the $\tilde{P}$-tensor set of equations (11) and $\tilde{P}$-pseudotensor set of equations (12) will be used in this work for the construction of $\tilde{P}$-vector L-approach for the electromagnetic field $F = (\vec{E}, \vec{H})$.

Let us introduce in addition to the Lagrange variables for tensor electromagnetic field new Lagrange variables $\bar{F}$, $\bar{F}_{,\mu}$ which are dually conjugated to F, $F_{,\mu}$ (on the manifold ϕ_0 of the solutions of Maxwell's equations $\bar{F} = \varepsilon F$ see (3)). The general form of $\tilde{P}$-vector Lagrange function

$$\mathcal{L}_\mu = \mathcal{L}_\mu(F, F_{,\nu}, \bar{F}, \bar{F}_{,\nu}), \quad \mathcal{L}_\mu : R^{60} \to R^1 \tag{13}$$

up to a total 4-divergence terms is the following:

$$\begin{aligned}\mathcal{L}_\mu = {} & a_1 F_{\mu\nu}Q^\nu + a_2 F_{\mu\nu}\bar{R}^\nu + a_3\varepsilon F_{\mu\nu}R^\nu + a_4\varepsilon F_{\mu\nu}\bar{Q}^\nu + a_5\bar{F}_{\mu\nu}\bar{Q}^\nu + \\ & + a_6\bar{F}_{\mu\nu}R^\nu + a_7\varepsilon\bar{F}_{\mu\nu}\bar{R}^\nu + a_8\varepsilon\bar{F}_{\mu\nu}Q^\nu + (q_1 F_{\mu\nu} + q_2\varepsilon\bar{F}_{\mu\nu})j^\nu.\end{aligned} \tag{14}$$

Here we are using also notations

$$\bar{Q}^\mu \equiv \bar{F}^{\mu\nu}_{,\nu}, \quad \bar{R}^\mu \equiv \bar{F}^{\mu\nu}_{,\nu}, \quad \varepsilon\bar{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\bar{F}_{\rho\sigma}. \tag{15}$$

Theorem 2. *The EL equations for $\tilde{P}$-vector $\mathcal{L} = (\mathcal{L}^\mu)$ are equivalent to the Maxwell equations if and only if the following conditions on the coefficients in (14) are fulfilled*

$$\begin{aligned}& a_8 - a_2 = a = -b' = -q_1 \equiv -q = 0, \quad a_6 - a_4 = a' = -b \neq 0, \\ & a_1 - a_3 - a_6 - a_8 = a_2 + a_4 + a_5 - a_7 = 0.\end{aligned} \tag{16}$$

Proof. The calculation of Lagrange derivatives $\delta\mathcal{L}_\mu/\delta F_{\rho\sigma}$ and $\delta\mathcal{L}_\mu/\delta\bar{F}_{\rho\sigma}$ from $\mathcal{L}_\mu$ (14) leads to the result that the EL equations for the Lagrange vector (14) may coincide only with the equations (11), (12), and only in the following form

$$\delta\mathcal{L}_\mu/\delta F_{\rho\sigma} = T_{\mu\rho\sigma} = 0, \quad \delta\mathcal{L}_\mu/\delta\bar{F}_{\rho\sigma} = T'_{\mu\rho\sigma} = 0, \tag{17}$$

and it is possible only if the conditions (16) are fulfilled. ■

The four component of the Lagrange vector (14) generate four actions

$$W^{\mu}(F,\bar{F}) = \int d^3x\mathcal{L}^{\mu}\left(F(x),\bar{F}(x),\partial vF(x),\partial v\bar{F}(x)\right), \tag{18}$$

where F, $\bar{F}$ belong to the set Φ of twice differentiable functions, and Φ_0^{μ} defines the set of extremals of the action (18) with a fixed μ.

Theorem 3. *The intersection $\Phi_0 = \cap_{\mu}\Phi_0^{\mu}$ of the sets Φ_0^{μ} of extremals of four actions (18) given by the Lagrange function $\mathcal{L}_{\mu}$ (14) whose coefficients obey the equations (16), coincides with the set of solutions of Maxwell's equations (1).*

Proof. The validity of this theorem follows from the derivation of the evident form of EL equations for (14), i.e. from (17) and the theorem I about the equivalence of the sytems of equations (11), (12) and the Maxwell equations (2), i.e. (1). ■

The $\tilde{P}$-vector Lagrangian (14), proposed here, has several advantages in comparison with the $\tilde{P}$-pseudovector Lagrangian from [3], which in our notation has the form

$$\mathcal{L}^{\mu} = \mathcal{L}^{\mu}(F,F_{,\nu}) = F^{\mu\nu}R_{\nu} - \varepsilon F^{\mu\nu}(Q_{\nu} - j_{\nu}). \tag{19}$$

Firstly, Lagrangian (19) leads only to the pseudotensor system of equations (12), i.e. it unreasonably separates the pseudo-tensor system of equations (12) in comparison with the tensor system of equations (11). That is a direct consequence of a pseudovector character of Lagrangian (19). Let us note, that without appealing to the additional Lagrange variable $\bar{F}$ it is impossible to construct a $\tilde{P}$-vector Lagrange function: the demand of function $\mathcal{L}^{\mu}(F,F_{,\nu})$ being a $\tilde{P}$-vector leads to the expression

$$\mathcal{L}^{\mu} = \mathcal{L}^{\mu}(F,F_{,\nu}) = F^{\mu\nu}Q_{\nu} + \varepsilon F^{\mu\nu}R_{\nu}, \tag{20}$$

for which the EL equations are the identities.

Secondly, as it is seen from the terms with the current in (19) the interaction Lagrangian in [3] also is a $\tilde{P}$-pseudovector one:

$$\mathcal{L}^{I} = \varepsilon F^{\mu\nu}j_{\nu}, \quad \mathcal{L}^{I_0} = \vec{j}\cdot\vec{H}, \quad \mathcal{L}^{I_i} = (\vec{j}\times\vec{E} - \rho\vec{H})^i. \tag{21}$$

A physically unsatisfactority of such an interaction is evident already from the fact, that density of electric charge in (21) is connected not with the electric field strengths but with the magnetic field strength $\vec{H}$.

Finally, thirdly, during the derivation of conserved quantities the Lagrange function (19) put into correspondence for $\tilde{P}$-tensor generator of the Poincaré group a pseudotensor conserved currents. This shortcoming together with the above mentioned ones, is overcome using the $\tilde{P}$-vector Lagrange function (14).

Derivation of conservation quantities in the framework of L-approach formulating here inquires a generalization of Noether theorem for the case of vector Lagrangians.

Theorem 4. *Let*

$$\hat{q}: \ F(x) \to F'(x) = \hat{g}F(x) \tag{22}$$

be the arbitrary transformation of invariance of equations (2) with $j = 0$. Then the tensor of current θ^{μ}_{ν}, constructed on the basis of $\mathcal{L}_{\mu}$ (14) (of course with $j = 0$) according to the formula

$$\begin{aligned} &\hat{q} \to \theta^{\mu}_{\nu} \stackrel{\mathrm{df}}{=} \frac{1}{2}\left(\frac{\partial \mathcal{L}_{\nu}}{\partial F^{\rho\sigma}_{,\mu}} F'^{\rho\sigma} + \frac{\partial \mathcal{L}_{\nu}}{\partial \bar{F}^{\rho\sigma}_{,\mu}} \bar{F}'^{\rho\sigma}\right), \\ &F' \equiv \hat{q}F, \quad \bar{F}' \equiv \overline{\hat{q}F} = \varepsilon \hat{q} F. \end{aligned} \tag{23}$$

is symmetric and its divergence vanishes for any solution of the equation (2) with

$$\partial_{\mu}\theta^{\mu}_{\nu} = 0. \tag{24}$$

Proof. Derivation of currents (23) for $\mathcal{L}_{\mu}$ (14) with $j = 0$ leads to the result

$$\begin{aligned} &\hat{q} \to \theta^{\mu}_{\nu} = A\left(F^{\mu\alpha}F'_{\alpha\nu} + F'^{\mu\alpha}F_{\alpha\nu} + \frac{1}{2}\delta^{\mu}_{\nu}F^{\alpha\beta}F'_{\alpha\beta}\right), \\ &A = a_1 - a_2 + a_7 - a_8 = a_3 + a_4 + a_5 + a_6. \end{aligned} \tag{25}$$

Symmetry of the tensor (25) is evident and the equation (24) is a consequency of the Maxwell equations (2) with $j = 0$. ■

Note that in the vector L-approach the four conservation quantities correspond (according to the Noether theorem) to one generator of invariance transformation.

Let us give the analysis of conserved quantities which are the consequences of (25). We receive, taking $A = 1$, that generators of 4-translations ∂_{ρ} according to the formula (25) give the trivial current

$$\partial_{\rho} \to \theta^{\mu\nu}(\hat{q} = \partial_{\rho}) = (\partial_{\rho})^{\mu\nu} \equiv \partial_{\rho}T^{\mu\nu}, \tag{26}$$

where $T^{\mu\nu}$ is standard energy-moment tensor for the field

$$T^{\mu}_{\nu} = F^{\mu\alpha}F_{\alpha\nu} + \frac{1}{4}\delta^{\mu}_{\nu}F^{\alpha\beta}F_{\alpha\beta}, \quad T^{0}_{\mu} = \mathcal{P}_{\mu}, \tag{27}$$

$$\mathcal{P}_0 \equiv \frac{1}{2}(\vec{E}^2 + \vec{H}^2), \quad \mathcal{P} \equiv (\vec{E} \times \vec{H})_j. \tag{28}$$

For the analysis of integral conserved quantities

$$\bar{\theta}^{\mu} = \int d^3x\theta^{0\mu}(x) = \mathrm{const}, \quad \theta^{0\mu}(x) = \theta^{0\mu}(\hat{q}) \equiv (\hat{q}^{0\mu}) \tag{29}$$

it is sufficient to represent the densities $\theta^{0\mu}$, ommiting the terms with spacelike derviatives, which do not contribute to the integral $\bar{\theta}^{\mu}$ (29). We obtain from the formula (25) for the densities $\theta^{0\mu}$, corresponding to the rest of the generators of conformal algebra $C(1,3)$ (the definition of algebra $C(1,3)$ see, for example in [5]) the following expressions:

$$\hat{j}_{\rho\sigma} \to J^{0\mu}_{\rho\delta} = \delta^{\mu}_{\rho}\mathcal{P}_{\sigma} - \delta^{\mu}_{\sigma}\mathcal{P}_{\rho}, \quad d \to D^{0\mu} = \mathcal{P}^{\mu}, \tag{30}$$

$$\hat{K}_{\rho} \to K^{0\mu}_{\rho} = 2(\delta^{\mu}_{\rho}\mathcal{D} + \mathcal{J}_{\rho\sigma}\mathcal{G}^{\delta\mu}), \tag{31}$$

where

$$\mathcal{D} \equiv x^{\mu}\mathcal{P}_{\mu}, \quad \mathcal{J}_{\rho\sigma} \equiv x_{\rho}\mathcal{P}_{\sigma} - x_{\sigma}\mathcal{P}_{\rho}. \tag{32}$$

As one can see, $C(1,3)$-generators $\hat{q} = (\hat{\partial}, \hat{j}, \hat{d}, \hat{k})$ lead here to the conserved quantities, which are expressed in terms of well-known series of main conservation quantities for the electromagnetic field $F = (\vec{E}, \vec{H})$, found by Bessel-Hagen [6] on the basis of L-approach for vector field $A = (A^\mu)$ of potentials, namely:

$$\begin{aligned} &P_\rho = \int d^3x P_\rho(x), \quad J_{\rho\sigma} = \int d^3x (x_\rho P_\sigma(x) - x_\sigma P_\rho(x)), \\ &D = \int d^3x \mathcal{D}(x), \quad K_\rho = \int d^3x (2x_\rho \mathcal{D}(x) - x^2 P_\rho(x)). \end{aligned} \tag{33}$$

It is interesting to note, that according to the formula (25) the duality transformation ε gives identically zero. Nontrivial conservation laws are given here by the generators of the algebra $A_{32} \supset C(1,3)$ of invariance of free Maxwell's equations (1) found in [1], which has the form of composition $\hat{q}' = \varepsilon\hat{q}$ of $C(1,3)$ generators $\hat{q}$ and the generator ε. Integral conserved quantities, which are found on the basis of formulae (23) or (25) and (29) for $\varepsilon C(1,3)$-generators $\hat{q}' = (\varepsilon\hat{\partial}, \varepsilon\hat{j}, \varepsilon\hat{d}, \varepsilon\hat{K})$ are expressed in terms of series

$$\begin{aligned} &Z^\mu_\rho = \int d^3x Z^\mu_\rho(x), \quad Z^\mu_{\rho\sigma} = \int d^3x (x_\rho Z^\mu_\sigma - x_\sigma Z^\mu_\rho), \\ &Z^\mu = \int d^3x x^\nu Z^\mu_\nu(x), \quad Z^\mu_\rho = \int d^3x (2x_\rho x^\sigma Z^\mu_\sigma - x^2 Z^\mu_\rho), \end{aligned} \tag{34}$$

of conserved quantities having polarization nature, of Lipkin [7] and others [8–10] (in [7–10] the conservation laws (34) were found without using the L-approach and Noether theorem). In (34) the densities Z of conserved quantities are expressed in the terms of Lipkin's Zilch tensor

$$Z^\mu_\rho \equiv Z^{0|\mu}_\rho, \quad Z^{\nu|\mu}_\rho = F^{\nu\alpha}\varepsilon F^{,\mu}_{\alpha\rho} - \varepsilon F^{\nu\alpha} F^{,\mu}_{\alpha\rho}. \tag{35}$$

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