

Exact solutions of multidimensional nonlinear Dirac's and Schrödinger's equations

W.I. FUSHCHYCH

A class of nonlinear spinor equations invariant under the extended Poincaré group and conformal group is described. New Ansätze for spinor fields are suggested. Multiparameter families of exact solutions for the multidimensional families of exact solutions for the multidimensional nonlinear Dirac and Schrödinger equations are obtained.

In this paper I present some new results, obtained in Institute of Mathematics, Academy of Sciences of the Ukrainian SSR in Kiev by R. Zhdanov, W. Shtelen, N. Serov and me on multiparameter families of exact solutions of nonlinear Dirac and Schrödinger equations

$$\gamma_\mu p^\mu \Psi + F_1(x, \Psi^*, \Psi) \Psi = 0, \quad (1)$$

$$\left(p_0 + \frac{1}{2m} p_a p_a \right) u + F_2(x, u, u^*) = 0, \quad (2)$$

where $\Psi \equiv \Psi(x) = (\Psi_0, \Psi_1, \Psi_2, \Psi_3)$ is 4-component spinor, $x = (x_0, x_1, x_2, x_3)$, Ψ^* is complex conjugated spinor, γ_μ are 4×4 Dirac matrices, $u \equiv u(x_0, x_1, x_2, x_3)$, $x_0 \equiv t$, u^* is complex conjugated wave function,

$$p_0 = i \frac{\partial}{\partial x_0}, \quad p_j = -i \frac{\partial}{\partial x_j}, \quad \mu, \nu = \overline{0, 3}, \quad j = 1, 2, 3,$$

F_1, F_2 are arbitrary smooth function, m is the particle mass.

Fifty years ago D. Ivanenko (1938) considered the simplest equation of the type (1), the case in which

$$F_1 = \lambda(\bar{\Psi}\Psi), \quad (3)$$

where $\bar{\Psi} \equiv \Psi^\dagger \gamma_0$ is Dirac-conjugated spinor, λ is arbitrary parameter.

W. Heisenberg and his collaborators (1954–1959) have analysed the equation (1) from a different point of view with the nonlinearity

$$F_1 = \lambda \bar{\Psi} \gamma_\mu \gamma_4 \Psi \gamma^\mu \gamma_4, \quad \gamma_4 = \gamma_0 \gamma_1 \gamma_2 \gamma_3. \quad (4)$$

The main efforts of W. Heisenberg directed into the construction of unified quantum field theory based on eq. (1) with the nonlinearities (3), (4). In the works by R. Finkelstein and his collaborators (1951–1956) eq. (1) has been studied from the classical point of view, i.e. they studied the exact and approximate solutions of spinor systems of the type (1).

Some exact solutions of the Dirac equation were obtained by F. Kortel (1956), D. Kurgelidze (1957), K.G. Akdezin., A. Smailovic (1984), A.O. Barut, B.W. Xu (1982), K. Takahashi (1979).

The classical Lie's method for finding exact solutions of multidimensional nonlinear Schrödinger equation and d'Allembert equation was applied by Fushchych (1981, 1983), Fushchych and Serov (1983, 1987), Gagnon and Winternitz (1988), Grundland, Harmad and Winternitz (1984), Tajiri (1983).

Evidently if we do not specify the functions F_1, F_2 in eqs. (1), (2) there is no hope to get any profound information about exact solutions of these equations. To specify the functions F_1 and F_2 we shall study the symmetry properties of equations (1), and (2). In what follows, I shall essentially use the classical ideas of S. Lie in application to nonlinear wave equations.

The wide symmetry of equations (1), (2) makes it possible to reduce the multidimensional partial differential equation (PDE) a set of systems of ordinary differential equations (ODE). Many of these ODEs can be solved exactly. In this way we are able to construct many parameter families of exact solutions of the multidimensional wave equations (1), (2).

1. The symmetry of the nonlinear spinor equation. In this section we will present the theorems concerning the symmetry properties of equation (1).

Theorem 1. *Equation (1) is invariant under the Poincaré group $P(1, 3)$ iff*

$$F_{11}(x, \Psi^*, \Psi) = F_{11}(s) + F_{12}(s)\gamma_4 + F_{13}(s)\gamma^\mu(\bar{\Psi}\gamma_4\gamma_\mu\Psi) + F_{14}(s)S^{\mu\nu}(\bar{\Psi}\gamma_4S_{\mu\nu}\Psi),$$

$$s = (\bar{\Psi}\gamma_4\Psi, \bar{\Psi}\Psi), \quad S_{\mu\nu} = [\gamma_\mu, \gamma_\nu] = \frac{1}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad (1.1)$$

where $F_{11}, F_{12}, F_{13}, F_{14}$ are arbitrary smooth scalar functions of the invariant variable s .

Theorem 2. *Equation (1) is invariant under the extend Poincaré group $\tilde{P}(1, 3)$, i.e. the $P(1, 3)$ group expanded by the one-parameter group of scale transformations of the type*

$$x'_\mu = x_\mu \exp(\theta), \quad \Psi'_{x'} = \Psi(x) \exp(k\theta),$$

iff

$$F_{1i} = (\bar{\Psi}\Psi)^{-1/2k} \tilde{F}_{1i}, \quad i = 1, 2, \quad (1.2)$$

$$F_{1j} = (\bar{\Psi}\Psi)^{-\frac{(1+2k)}{2k}} \tilde{F}_{1j}, \quad j = 3, 4, \quad (1.3)$$

where $\tilde{F}_{1i}, \tilde{F}_{1j}$ are arbitrary functions of $\frac{(\bar{\Psi}\Psi)}{(\bar{\Psi}\gamma_4\Psi)}$.

Theorem 3. *Equation (1) is invariant under the conformal group $C(1, 3) = \langle P(1, 3), D, K_\mu \rangle$*

$$x'_\mu = \{x_\mu - c_\mu(x \cdot x)\}\sigma^{-1}(x), \quad \sigma(x) = 1 - 2c_\mu x^\mu + c^2 x^2,$$

$$\Psi'(x) = \sigma(x)\{1 - (\gamma \cdot c)(\gamma \cdot x)\}\Psi$$

iff

$$F_{1i} = (\bar{\Psi}\Psi)^{1/3} \tilde{F}_{1i}, \quad i = 1, 2, \quad k = -3/2, \quad (1.4)$$

$$F_{1j} = (\bar{\Psi}\Psi)^{-2/3} \tilde{F}_{1j}, \quad j = 3, 4. \quad (1.5)$$

Note 1. The conformally-invariant Dirac–Gürsey equation (1956)

$$\{\gamma_\mu p^\mu + D(\bar{\Psi}\Psi)^{1/3}\}\Psi = 0 \quad (1.6)$$

belongs to the class (1), (1.4).

Note 2. The equation of the type

$$\{\gamma_\mu p^\mu + D\gamma_\mu(\bar{\Psi}\gamma^\mu\Psi) \cdot [(\bar{\Psi}\gamma_\alpha\Psi)(\bar{\Psi}\gamma^\alpha\Psi)]^{-1/3}\}\Psi = 0 \quad (1.7)$$

is invariant under the conformal group $C(1, 3)$.

2. The Ansatzes for $\tilde{P}(1, 3)$ -invariant equation. To be specific let us consider the nonlinear spinor equation of the type

$$\{\gamma_\mu p^\mu + \lambda(\bar{\Psi}\Psi)^{1/2k}\}\Psi = 0, \quad (2.1)$$

where λ, k are arbitrary constants, $k \neq 0$.

We look for solutions of (2.1) in the form Fushchych (1981)

$$\Psi = A(x)\phi(\omega), \quad (2.2)$$

where $A(x)$ is 4×4 matrix, $\phi(\omega)$ is 4-component column-function depending on three new variables $\omega = \{\omega_1, \omega_2, \omega_3\}$.

For the Ansatz (2.2) to work effectively it is necessary to find $A(x)$, ω in a form which after a substitution of (2.2) into (1) would yield an equation for $\phi(\omega)$ depending only on new variables ω .

This requirement is met if the following equalities are satisfied:

$$QA(x) \equiv \left(\zeta^\mu \frac{\partial}{\partial x_\mu} + \eta \right) A(x) = 0, \quad (2.3)$$

$$\zeta^\mu \frac{\partial \omega^l}{\partial x_\mu} = 0, \quad l = 1, 2, 3, \quad (2.4)$$

where $\zeta^\mu(x)$, $\eta(x)$ are the coefficients of the infinitesimal operators $Q = \{Q_1, Q_2, \dots\}$ of the group $\tilde{P}(1, 3)$. In our case the generators of the $P(1, 3)$ have the form

$$P_\mu = p_\mu, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}. \quad (2.5)$$

Thus the problem of describing Ansatzes of the form (2.2) reduces to the construction of the general solution to the system of equations (2.3), (2.4) with the given ζ^μ , η .

As an example let us consider the case where in (2.3), (2.4) the operators Q have the simple form

$$Q = \{Q_1 = J_{03}, Q_2 = P_1, Q_3 = P_2\}.$$

Then the system (2.3), (2.4) has the form

$$\left(x_0 p_3 - x_3 p_0 + \frac{i}{2} \gamma_0 \gamma_3 \right) A(x) = 0, \quad p_1 A(x) = 0, \quad p_2 A(x) = 0, \quad (2.6)$$

$$(x_0 p_3 - x_3 p_0)\omega = 0, \quad p_1 \omega = 0, \quad p_2 \omega = 0. \quad (2.7)$$

It follows from (2.7) that $\omega = \omega(x_0, x_3) = x_0^2 - x_3^2$. We look for the solutions to (2.6) in the form

$$A(x) = \exp\{\gamma_0\gamma_3g(x)\}. \quad (2.8)$$

After a substitution of (2.8) into (2.6), we obtain

$$x_0 \frac{\partial g}{\partial x_3} + x_3 \frac{\partial g}{\partial x_0} - \frac{1}{2} = 0. \quad (2.9)$$

The particular solution of eq. (2.9) is given by the expression

$$g(x) = \frac{1}{2} \ln(x_0 + x_3).$$

Thus we have

$$A(x) = \exp\left\{\frac{1}{2}\gamma_0\gamma_3 \ln(x_0 + x_3)\right\}. \quad (2.10)$$

Without going into the technical details on solving the system (2.3), (2.4) we give some expressions for the matrix $A(x)$ and ω .

Example 2.1.

$$A(x) = (x_0 - x_2)^{-k} \exp\left\{\frac{1}{2a}\gamma_1(\gamma_2 - \gamma_0) \ln(x_0 - x_2)\right\}, \quad (2.11)$$

$$\begin{aligned} \omega_1 &= (x_0^2 - x_1^2 - x_2^2)x_3^{-2}, & \omega_2 &= (x_0 - x_2)x_3^{-2}, \\ \omega_3 &= ax_1(x_0 - x_2)^{-1} - \ln(x_0 - x_2), & a &\neq 0. \end{aligned} \quad (2.12)$$

If the parameter $a = 0$, then

$$A(x) = \exp\left\{\frac{x_1}{2(x_0 - x_3)}\gamma_1(\gamma_2 - \gamma_0)\right\}. \quad (2.13)$$

Example 2.2.

$$\begin{aligned} A(x) &= 2(x_0 + 2x_1 + \beta)^{-k/2} \times \\ &\times \exp\left\{\frac{1}{4}\gamma_0\gamma_2 \ln(2x_0 + 2x_1 + \beta) - \frac{1}{2}\gamma_2\gamma_3 \operatorname{tg}^{-1}\frac{x_2}{x_3}\right\}, & \beta &\neq 0, \end{aligned} \quad (2.14)$$

$$\begin{aligned} \omega_1 &= (2x_0 + 2x_1 + \beta) \exp\{2(x_1 - x_0)\beta^{-1}\}, \\ \omega_2 &= (2x_0 + 2x_1 + \beta) (x_2^2 + x_3^2)^{-1}, & \omega_3 &= b \ln(x_2^2 + x_3^2) + 2 \operatorname{tg}^{-1}\frac{x_2}{x_3}. \end{aligned} \quad (2.15)$$

3. 3.1. Reduced equations. The Ansatz (2.2) with the matrices (2.10), (2.11), (2.14) and new variables ω gives the following reduced equations

$$\begin{aligned} &k(\gamma_2 - \gamma_0)\phi + [(\gamma_0 - \gamma_2)(\omega_1 + a^{-2}\omega_2^2\omega_3^2) + (\gamma_0 + \gamma_2)\omega_2^2 - \\ &- 2a^{-1}\gamma_1\omega_3\omega_2^2 - 2\gamma_3\omega_1\omega_2] \frac{\partial \phi}{\partial \omega_1} + [(\gamma_0 - \gamma_2)\omega_2 - \gamma_3\omega_2^2] \frac{\partial \phi}{\partial \omega_2} + \\ &+ [a\gamma_1 + (\gamma_2 - \gamma_0)(\omega_3 + 1)] \frac{\partial \phi}{\partial \omega_3} = i\lambda(\bar{\phi}\phi)^{1/2k}\phi, \end{aligned} \quad (3.1)$$

$$(\gamma_0 + \gamma_1) \frac{\partial \phi}{\partial \omega_1} + \gamma_2 \frac{\partial \phi}{\partial \omega_2} + \gamma_3 \frac{\partial \phi}{\partial \omega_3} = i\lambda(\bar{\phi}\phi)^{1/2k} \phi, \quad (3.2)$$

$$\frac{1}{2}(1 - 2k)\gamma_3\phi + 2(\gamma_3 + a\gamma_2) \frac{\partial \phi}{\partial \omega_2} = i\lambda(\bar{\phi}\phi)^{1/2k} \phi. \quad (3.3)$$

The Ansatz (2.2) with the matrix $A(x) = 1$, $\omega_1 = x_0 + x_3$, $\omega_2 = x_1$, $\omega_3 = x_2$ leads to the equation

$$(\gamma_0 + \gamma_3) \frac{\partial \phi}{\partial \omega_1} + \gamma_1 \frac{\partial \phi}{\partial \omega_2} + \gamma_2 \frac{\partial \phi}{\partial \omega_3} = i\lambda(\bar{\phi}\phi)^{1/2k} \phi. \quad (3.4)$$

It turns out that some of the reduced equations possess substantially more symmetries than the initial equation (2.1). For example the equation (3.4) is invariant under infinite-dimensional Lie algebras. More exactly, the following statement is true:

Theorem 4. *The system (3.4) is invariant under the infinitely dimensional algebra whose basis elements have the form*

For the case $k = 1$

$$\begin{aligned} Q_1 &= \Phi_1(\omega_1) \frac{\partial}{\partial \omega_2} + \Phi_2(\omega_2) \frac{\partial}{\partial \omega_3} + \frac{1}{2}[\Phi_1\gamma_1 + \Phi_2\gamma_2](\gamma_0 + \gamma_3), \\ Q_2 &= -\omega_2 \frac{\partial}{\partial \omega_3} + \omega_3 \frac{\partial}{\partial \omega_2}, \\ Q_3 &= \Phi_0(\omega_1) \frac{\partial}{\partial \omega_1} + \dot{\Phi}_0(\omega_1) \left(\omega_2 \frac{\partial}{\partial \omega_3} + \omega_3 \frac{\partial}{\partial \omega_2} \right) + \\ &\quad + \dot{\Phi}_0 + \frac{1}{2}\ddot{\Phi}_0(\omega_1)(\gamma_1\omega_2 + \gamma_2\omega_1)(\gamma_0 + \gamma_3), \\ Q_4 &= \Phi_3(\omega_1)\gamma_4(\gamma_0 + \gamma_3). \end{aligned}$$

For the case $k \neq 1$

$$\begin{aligned} Q_1 &= \frac{\partial}{\partial \omega_1}, \quad Q_2 = -\omega_2 \frac{\partial}{\partial \omega_3} + \omega_3 \frac{\partial}{\partial \omega_2} + \frac{1}{2}\gamma_2\gamma_3, \\ Q_3 &= \Phi_1(\omega_1) \frac{\partial}{\partial \omega_2} + \Phi_2(\omega_1) \frac{\partial}{\partial \omega_3} + \frac{1}{2}[\dot{\Phi}_1(\omega_2)\gamma_1 + \dot{\Phi}_2(\omega_1)\gamma_2](\gamma_2 + \gamma_3), \\ Q_4 &= \omega_1 \frac{\partial}{\partial \omega_1} + \omega_2 \frac{\partial}{\partial \omega_2} + \omega_3 \frac{\partial}{\partial \omega_3} + k, \\ Q_5 &= \Phi_3(\omega_1)\gamma_4(\gamma_0 + \gamma_3), \end{aligned}$$

where $\Phi_0(\omega_1)$, $\Phi_1(\omega_1)$, $\Phi_2(\omega_1)$, $\Phi_3(\omega_1)$ are arbitrary smooth functions, a dot designates differentiation with respect to ω_1 .

3.2. The reduction of the nonlinear spinor equation to a system of ODE. Here we give the explicit form of some ODE's to which the Dirac equation is reduced:

$$i\gamma_2 \dot{\phi}(\omega) = \lambda(\bar{\phi}\phi)^{1/2k} \phi, \quad (3.5)$$

$$\frac{i}{2}(\gamma_0 + \gamma_3)\phi + i[\omega(\gamma_0 - \gamma_3)]\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k} \phi, \quad (3.6)$$

$$\frac{i}{2}\omega^{-1/2}\gamma_2\phi + 2i\omega^{1/2}\gamma_2\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k}\phi, \quad (3.7)$$

$$\frac{i}{2}[\omega(\omega + 1)]^{-1}(2\omega + 1)[\gamma_0 + \gamma_3]\phi + i(\gamma_0 + \gamma_3)\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k}\phi, \quad (3.8)$$

$$i(\gamma_0 + \gamma_3)\dot{\phi} + i\left[(\gamma_0 + \gamma_3)\omega^{-1} + \frac{1}{4}(\gamma_0 - \gamma_3)\gamma_4\right]\phi = \lambda(\bar{\phi}\phi)^{1/2k}\phi, \quad (3.9)$$

$$i\omega^{-1}(\gamma_0 + \gamma_3)\phi + i(\gamma_0 + \gamma_3)\dot{\phi} = \lambda(\bar{\phi}\phi)^{1/2k}\phi, \quad \dot{\phi} \equiv \frac{d\phi}{d\omega}. \quad (3.10)$$

The full list of the systems of ordinary differential equations is given in Fushchych and Zhdanov (1988).

4. The explicit solutions of the Dirac equation. Some of the ODE's can be integrated in quadratures.

I. Case $k = 1/2$, with the nonlinearity $\lambda(\bar{\Psi}\Psi)$, $\omega = x_2^2 + x_3^2$,

$$\begin{aligned} \Psi(x) = & (x_2^2 + x_3^2)^{-1/4} \exp\left\{-\frac{1}{2}\gamma_2\gamma_3 \operatorname{tg}^{-1}\frac{x_2}{x_3}\right\} \times \\ & \times \exp\left\{-i\lambda\frac{\bar{\chi}\chi}{2(1+a^2)}(\gamma_3 + a\gamma_2) \ln(x_2^2 + x_3^2) + 2\operatorname{tg}^{-1}\frac{x_2}{x_3}\right\} \chi, \end{aligned} \quad (4.1)$$

χ is constant spinor, a is a real parameter.

II. Case $k \neq 1/2$, nonlinearity $\lambda(\bar{\Psi}\Psi)^{1/2k}$, $\omega = x_2^2 + x_3^2$,

$$\Psi(x) = (x_2^2 + x_3^2) \exp\left\{-\frac{1}{2}\gamma_2\gamma_3 \operatorname{tg}^{-1}\frac{x_2}{x_3}\right\} \exp\left\{\frac{2i\lambda k}{1-2k}(x_2^2 + x_3^2)^{\frac{2k-1}{4k}}\gamma_3\right\} \chi. \quad (4.2)$$

III. Case $k \neq 1/2$, nonlinearity $\lambda(\bar{\Psi}\Psi)^{1/2k}$, $\omega = x_0 + x_3$,

$$\begin{aligned} \Psi(x) = & \exp\left\{\left[-\frac{1}{2}(\dot{\Phi}_1\gamma_1 + \dot{\Phi}_2\gamma_2) + \Phi_3\gamma_4\right](\gamma_0 + \gamma_3)\right\} \times \\ & \times \exp\left\{i\lambda(\bar{\chi}\chi)^{1/2k}\gamma_1(x_1 + \Phi_1)\right\} \chi, \end{aligned} \quad (4.3)$$

where Φ_1, Φ_2, Φ_3 arbitrary smooth functions of ω .

IV. Case $k = 1$, nonlinearity $\lambda(\bar{\Psi}\Psi)^{1/2}$,

$$\Psi(x) = \frac{\gamma_0 x_0 - \gamma_1 x_1 - \gamma_2 x_2}{(x_0^2 - x_1^2 - x_2^2)^{3/2}} \exp\left\{i\lambda(\bar{\chi}\chi)^{1/2}\frac{\gamma_0 x_0}{x_0^2 - x_1^2 - x_2^2}\right\} \chi. \quad (4.4)$$

Now, making use of the Poincaré invariance of the Dirac equation, is not difficult to construct new multiparameter families of exact solutions of the equation starting from those obtained above. The two following examples illustrate the procedure of generating solutions.

V. Case $k = 1/2$, nonlinearity $\lambda(\bar{\Psi}\Psi)$,

$$\begin{aligned} \Psi = & [(a \cdot z)^2 + (b \cdot z)^2]^{-1/4} \exp\left\{-\frac{1}{2}(\gamma \cdot a)(\gamma \cdot b) \operatorname{tg}^{-1}\frac{a \cdot z}{b \cdot z}\right\} \times \\ & \times \exp\left\{-i\lambda\frac{\bar{\chi}\chi}{2(1+\theta^2)}(\gamma \cdot b + \theta\gamma \cdot a) \times \right. \\ & \left. \times \left[\ln((a \cdot z)^2 + (b \cdot z)^2) + 2\theta \operatorname{tg}^{-1}\frac{a \cdot z}{b \cdot z}\right]\right\} \chi, \end{aligned} \quad (4.5)$$

$z_\mu = x_\mu + \theta_\mu$, a_μ , b_μ , θ , θ_μ are arbitrary parameters satisfying conditions

$$\begin{aligned} a \cdot a &= a_\mu a^\mu = -1, & b \cdot b &= b_\mu b^\mu = -1, & a \cdot b &= 0, \\ \gamma \cdot a &\equiv \gamma_0 a_0 - \gamma_1 a_1 - \gamma_2 a_2 - \gamma_3 a_3, & \theta &= (\theta_0^2 - \theta_1^2 - \theta_2^2 - \theta_3^2)^{1/2}. \end{aligned}$$

VI. Case $k \neq 1/2$, nonlinearity $\lambda(\bar{\Psi}\Psi)^{1/2k}$,

$$\begin{aligned} \Psi(x) &= [(a \cdot z)^2 + (b \cdot z)^2]^{-1/4} \exp \left\{ -\frac{1}{2}(\gamma \cdot a)(\gamma \cdot b) \operatorname{tg}^{-1} \frac{a \cdot z}{b \cdot z} \right\} \times \\ &\times \exp \left\{ -i \frac{2\lambda k}{1-2k} (\gamma \cdot b)(\bar{\chi}\chi)^{1/2} [(a \cdot z)^2 + (b \cdot z)^2]^{\frac{1-2k}{4k}} \right\} \chi. \end{aligned} \quad (4.6)$$

Formulas (4.5), (4.6) give multiparameter families of exact solutions of the Dirac equations. These families are nongenerating with respect to the group $\tilde{P}(1,3)$ in the sense that solutions (4.5), (4.6) have the same symmetries as the equation (2.1).

5. Conformally invariant solutions. Conformally invariant Dirac–Gürsey equation has the form

$$\left\{ \gamma_\mu p^\mu + (\bar{\Psi}\Psi)^{1/3} \right\} \Psi = 0. \quad (5.1)$$

With the help of a conformally invariant ansatz we can construct the following solutions

$$\Psi(x) = \frac{\gamma \cdot x}{(x \cdot x)^2} \exp\{i\lambda k(\gamma \cdot \beta)\omega\}\chi, \quad k = 1/3, \quad (5.2)$$

$$\omega = \frac{\beta_\mu x^\mu}{x_\nu x^\nu}, \quad x_\nu x^\nu \neq 0, \quad \beta_\mu \beta^\mu > 0;$$

$$\begin{aligned} \Psi(x) &= \sigma^2(x)[1 - (\gamma \cdot x)(\gamma \cdot \beta)] \times \\ &\times \exp \left\{ \frac{\theta}{2} (\gamma \cdot a)(\gamma \cdot b)[b \cdot x - (b \cdot \theta)(x \cdot x)]\sigma^{-1}(x) \right\} \times \\ &\times \exp \left\{ -\frac{i}{2} \lambda (\bar{\chi}\chi)^{1/3} (\gamma \cdot a)[2(a \cdot x) - (a \cdot \theta)(x \cdot x)]\sigma(x) + \right. \\ &\left. + \theta(b \cdot x - (b \cdot \theta)(x \cdot x))^2 \sigma^{-2}(x) \right\} \chi, \end{aligned} \quad (5.3)$$

$$a \cdot b = b \cdot b = 0, \quad a \cdot a = a_0^2 - a_1^2 - a_2^2 - a_3^2 = -1,$$

$$c^2 = c_\alpha c^\alpha, \quad x^2 = x_\mu x^\mu.$$

Formulae (5.2), (5.3) give multiparameter families of exact solutions of the equation (5.1). The family (5.3) is nongenerating with respect to the group $C(1,3)$.

If $\Psi_1(x)$ is a solution of equation (5.1) then

$$\begin{aligned} \Psi_2(x) &= \sigma^{-2}(x)[1 - (\gamma \cdot x)(\gamma \cdot c)]\Psi_1(x'), \\ x' &= \left\{ x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(x)} \right\} \end{aligned} \quad (5.4)$$

will also be a solution (5.4) is the formula for multiplication of solutions of Dirac equation.

6. How to construct solutions of the nonlinear d'Alambert equation via solutions of the Dirac equation? Complex scalar field can be represented as

$$u(x) = \bar{\Psi}\Psi \exp\{i\theta(x)\}, \quad (6.1)$$

where $\Psi(x)$ is a solution of the Dirac equation, $\theta(x)$ is a phase. In the simplest case, when

$$\bar{\Psi}\Psi = c = \text{const}, \quad \theta(x) = \tau_\mu x^\mu \quad (6.2)$$

formula (6.1) gives a plane-wave solution of the linear d'Alambert equation. In most cases solutions of the nonlinear Dirac equation generate a scalar field (6.1) which satisfies the nonlinear d'Alambert equation

$$p_\mu p^\mu u = k|u|^r u, \quad (6.3)$$

k, r are constants.

Let us exhibit some exact solutions of the equation (6.3) obtained this way

$$u(x) = c (x_1^2 + x_2^2) \exp\{i\phi_0(x_0 + x_3)\}, \quad r = 2, \quad (6.4)$$

$$u(x) = c [(x_1 + \phi_1(x_0 + x_3))^2 + (x_2 + \phi_2(x_0 + x_3))^2]_x^{-1/2} \times \\ \times \exp\{i\phi_0(x_0 + x_3)\}, \quad r = 2, \quad (6.5)$$

$$u(x) = c (x_0^2 - x_1^2 - x_2^2 - x_3^2)^{-2}, \quad r = 1/2, \quad (6.6)$$

$$u(x) = c (x_2^2 + x_3^2)^{-1/2} \exp\{i\phi(x_0 + x_1)\}, \quad r = 2. \quad (6.7)$$

In formulae (6.4)–(6.7) $\phi_0, \phi_1, \phi_2, \phi$ are arbitrary smooth functions.

So, solutions of the nonlinear Dirac spinor equation give a possibility to construct solutions to the nonlinear d'Alambert equation.

All these ideas and results were considered in more detail by Fushchych (1981, 1987), Fushchych and Shtelen (1983, 1987), Fushchych, Shtelen and Zhdanov (1985), Fushchych and Zhdanov (1987, 1988, 1989), Fushchych and Nikitin (1987) (see Appendix).

7. The solutions of the multidimensional Schrödinger equation. Let us consider the following nonlinear equation

$$\left(i \frac{\partial}{\partial x_0} - \frac{1}{2m} \Delta\right) u + F(x, u, u^*) = 0, \quad u \equiv u(x_0 \equiv t, x_1, x_2, x_3). \quad (7.1)$$

It is well known that if $F = 0$, then linear Schrödinger equation (see Sofus Lie (1881), Hagen (1972), Niederere (1972), Kalnins and Miller (1987)) is invariant under the generalised Galilei group, which will be denoted by the symbols $G_2(1, 3)$. The basis elements of the Lie algebra $AG_2(1, 3) = \langle P_0, P_a, J_{ab}, G_a, D, A, I \rangle$ have the following form:

$$P_0 = i \frac{\partial}{\partial x_0}, \quad P_a = -i \frac{\partial}{\partial x_a}, \quad a = 1, 2, 3, \quad J_{ab} = x_a P_b - x_b P_a, \quad I = u \frac{\partial}{\partial u}, \quad (7.2) \\ G_a = x_0 P_a + m x_a, \quad D = 2x_0 P_0 - x_a P_a + \frac{3}{2} i, \quad A = x_0 D + \frac{m}{2} x_a^2.$$

Symbols $AG_1(1, 3)$, $AG(1, 3)$ denote the following Lie algebras

$$\begin{aligned} AG_1(1, 3) &= \langle P_0, P_a, J_{ab}, G_a, D, I \rangle, \\ AG(1, 3) &= \langle P_0, P_a, J_{ab}, G_a, I \rangle. \end{aligned}$$

To construct families of exact solutions of (7.1) in explicit form we have to know the symmetries of (7.1) which obviously depends on the structure of the function F .

By Lie's algorithm (as given by Ovsyannikov (1978), Olver (1986)), the following statement can be proved.

Theorem 5. Equation (7.1) is invariant under the following algebras:

$$AG(1, 3) \quad \text{iff} \quad F = \Phi(|u|)u, \quad (7.3)$$

where Φ is arbitrary smooth function, and

$$AG_1(1, 3) \quad \text{iff} \quad F = \lambda|u|^k u, \quad (7.4)$$

where λ , k are arbitrary parameters, the operator of scale transformations D having the form $D = x_0 P_0 - x_a P_a + 2i/k$, $k \neq 0$, and

$$AG_2(1, 3) \quad \text{iff} \quad F = \lambda|u|^{4/n} u. \quad (7.5)$$

Later on we shall construct the exact solutions of the equation (7.1) with nonlinearity (7.5), i.e.

$$\left(p_0 - \frac{p_a^2}{2m} \right) u + \lambda|u|^{4/3} u = 0. \quad (7.6)$$

Following Fushchych (1981) we seek solutions of (7.6) with the help of the ansatz

$$u = f(x)\phi(\omega_1, \omega_2, \omega_3), \quad (7.7)$$

where ϕ is the function to be calculated. To construct solutions of (7.6) using ansatz (7.7) it is necessary to have the explicit form of the function $f(x)$ and the new invariant variables ω_1 , ω_2 and ω_3 . Next I shall present two Ansätze of the type (7.7).

$$\begin{aligned} 1. \quad f(x) &= (1 - x_0^2)^{-3/4} \exp \left\{ \frac{im}{2} \frac{x_0 \vec{x}^2}{1 - x_0^2} \right\}, \quad \omega_1 = \vec{\alpha} \vec{x} (1 - x_0^2)^{-1/2}, \\ \omega_2 &= \vec{x}^2 (1 - x_0^2)^{-1}, \quad \omega_3 = \arctan x_0 + \arctan \frac{\vec{\beta} \vec{x}}{\vec{\gamma} \vec{x}}, \end{aligned}$$

where $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ are constant vectors satisfying the conditions

$$\vec{\alpha}^2 = \vec{\beta}^2 = \vec{\gamma}^2 = 1, \quad \vec{\alpha} \vec{\beta} = \vec{\beta} \vec{\gamma} = \vec{\gamma} \vec{\alpha} = 0.$$

The Ansätze (7.6), (7.7) give the following reduced equation

$$\begin{aligned} L\phi + 6 \frac{\partial \phi}{\partial \omega_2} - 2im \frac{\partial \phi}{\partial \omega_3} + m^2 \omega_2 \phi - 2\lambda m |\phi|^{4/3} \phi &= 0, \\ L\phi \equiv \frac{\partial^2 \phi}{\partial \omega_1^2} + 4\omega_2 \frac{\partial^2 \phi}{\partial \omega_2^2} + (\omega_2 - \omega_1^2)^{-1} \frac{\partial^2 \phi}{\partial \omega_3^2} + 4\omega_1 \frac{\partial^2 \phi}{\partial \omega_1 \partial \omega_2}. \end{aligned} \quad (7.8)$$

2. The second Ansatz has the form

$$f(x) = x_0^{-3/2} \exp \left\{ -\frac{im}{2} x^2 x_0^{-1} \right\}, \quad (7.9)$$

$$\omega_1 = (\vec{\alpha}\vec{x})x_0^{-1}, \quad \omega_2 = \vec{x}^2 x_0^{-2}, \quad \omega_3 = x_0^{-1} + \arctan \frac{\vec{\beta}\vec{x}}{\vec{\gamma}\vec{x}}. \quad (7.10)$$

The Ansätze (7.9), (7.10) reduce the equation (7.6) to

$$L\phi + 6 \frac{\partial \phi}{\partial \omega_2} + 2im \frac{\partial \phi}{\partial \omega_3} - 2m|\phi|^{4/3}\phi = 0.$$

8. Solutions of the equation (7.6). In this paragraph I present some explicit solutions of the equation (7.6)

$$u = (1 - x_0^2)^{-3/4} \exp \left\{ \frac{im}{2} \vec{x} (1 - x_0)^{-1} \right\}, \quad \lambda = \frac{3}{2}i; \quad (8.1)$$

$$u = (c_0 x_0 - \vec{c}\vec{x})^{-3/2} \exp \left\{ -\frac{im}{2} \vec{x}^2 x_0^{-1} \right\}, \quad (8.2)$$

where c_0, \vec{c} are arbitrary parameters satisfying the following condition $\vec{c}^2 = \frac{8}{15}\lambda m$;

$$u = x_0^{-3/2} \exp \left\{ -\frac{im}{2} (\vec{x}^2 - \vec{r}\vec{x}) x_0^{-1} \right\}, \quad \vec{r}^2 = -\frac{8\lambda}{m}; \quad (8.3)$$

$$u = \left(\frac{8}{3} \lambda m \vec{x}^2 \right)^{-3/4} \exp \left\{ -\frac{im}{2} \vec{x}^2 x_0^{-1} \right\}; \quad (8.4)$$

$$u = x_0^{-3/2} \phi(\omega_1) \exp \left\{ -\frac{im}{2m} \vec{x}^2 x_0^{-1} \right\}, \quad \omega_1 = \frac{\vec{\alpha}\vec{x}}{x_0}, \quad (8.5)$$

function $\phi(\omega_1)$ is defined by the elliptic integral

$$\int_0^\phi \frac{d\tau}{(k_1 + \tau^{10/3})^{1/2}} = \left(\frac{6}{5} \lambda m \right)^{1/2} (\omega_1 + k_2),$$

where k_1, k_2 are arbitrary parameters.

$$u = x_0^{-3/2} \exp \left\{ -\frac{im}{2} \vec{x}^2 x_0^{-1} \right\} \phi(\omega_2), \quad \omega_2 = \vec{x}^2 x_0^{-1}, \quad (8.6)$$

where function $\phi(\omega_2)$ is a solution of Emden–Fauler equation

$$2\omega_2 \frac{d^2 \phi}{d\omega_2^2} + 3 \frac{d\phi}{d\omega_2} - \lambda m \phi^{7/3} = 0.$$

The formulae (8.1)–(8.6) give multiparameter families of exact solutions of the equation (7.6). Some of them are of non-perturbative type due to a singularity with respect to the coupling constant λ .

In conclusion we give formulae for multiplication of solutions. If u_1 is a solution of the equation with the nonlinearity $4/3$ then the functions u_2, u_3 defined by

$$u_2 = u_1(x_0, \vec{x} + \vec{v}x_0) \exp \left\{ im \left(\frac{\vec{v}x_0}{2} + \vec{v}\vec{x} \right) \right\},$$

$$u_3 = u_2 \left(\frac{x_0}{dx_0 - 1}, \frac{\vec{x}}{1 - dx_0} \right) (1 - dx_0)^{-3/2} \exp \left\{ \frac{im}{2} \frac{d\vec{v}^2}{1 - dx_0} \right\},$$

also satisfy equation (7.6). Here d, \vec{v} are arbitrary parameters.

The Ansätze that have been presented here may also be applied to the equation

$$i \frac{\partial u}{\partial t} + \frac{1}{2m} \Delta u + \lambda \left\{ \frac{\partial |u|^2}{\partial x_a} \frac{\partial |u|^{-1}}{\partial x_a} \right\} u = 0,$$

which is also invariant under the group $G_2(1, 3)$.

More full consideration solutions of equation (7.6) was given Fushchych and Serov (1987), Fushchych and Cherniha (1986).

Acknowledgement. Author wishes to acknowledge Peter Olver and Avner Friedman for the invitation and the hospitality of the Institute for Mathematics and its Applications at the University of Minnesota.

1. Akdeniz K.G., Smailagic A., *Lett. Math. Phys.*, 1984, **8**, 175.
2. Barut A.O., Xu B.W., *Physica D*, 1982, **6**, 137.
3. Finkelstein R., Fronsdal, Kaust P., *Phys. Rev.*, 1956, **103**, 1571.
4. Fushchych W., The symmetry of mathematical physics problems, in Algebraic-theoretical studies in mathematical physics, Kiev, Institute of Mathematics, 1981, 6–28.
5. Fushchych W., Serov N., The symmetry and some exact solutions of the nonlinear multidimensional Liouville, d'Alambert and eikonal equations, *J. Phys. A: Math. Gen.*, 1983, **16**, 3645–3656.
6. Fushchych W., Serov N., On some exact solutions of the three-dimensional non-linear Schrödinger equation, *J. Phys. A: Math. Gen.*, 1987, **20**, L929–L933.
7. Fushchych W., On symmetry and exact solutions of some multidimensional equations of mathematical physics, in Algebraic-theoretical methods in mathematical physics problems, Kiev, Institute of Mathematics, 1983, 4–23.
8. Fushchych W., On symmetry and exact solutions of the multidimensional wave equations, *Ukr. Math. Zhurn.*, 1987, **39**, 116–123.
9. Fushchych W., Shtelen W., On some exact solutions of the nonlinear Dirac equation, *J. Phys. A: Math. Gen.*, 1983, **16**, 271–277.
10. Fushchych W., Shtelen W., On reduction and exact solutions of the nonlinear Dirac equation, *Teoret. and Math. Fiziks (USSR)*, 1987, **72**, 142–154.
11. Fushchych W., Shtelen W., Zhdanov R.Z., On the new conformally invariant equations for spinor fields and their exact solutions, *Phys. Lett. B*, 1985, **159**, 189–191.
12. Fushchych W., Zhdanov R.Z., On some exact solutions of the systems of nonlinear differential equations for spinor and vector fields, *J. Phys. A: Math. Gen.*, 1987, **20**, 4173–4190.
13. Fushchych W., Zhdanov R.Z., On the reduction and some new exact solutions of the non-linear Dirac and Dirac–Klein–Gordon equations, *J. Phys. A: Math. Gen.*, 1988, **21**, L5–L9.
14. Fushchych W., Zhdanov R.Z., Symmetry and exact solutions of nonlinear spinor equations, (to appear).
15. Fushchych W., Nikitin A., Symmetries of Maxwell's equations, Dordrecht, D. Reidel Publ. Comp., 1987.

16. Gagnon L., Winternitz P., Lie symmetries of a generalized non-linear Schrödinger equation, *J. Phys. A: Math. Gen.*, 1988, **21**, 1493–1513.
17. Grundland A.M., Harnad J., Winternitz P., Symmetry reduction for nonlinear relativistically invariant equations, *J. Math. Phys.*, 1984, **25**, 491.
18. Hagen C., *Phys. Rev. D*, 1972, **5**, 377.
19. Heisenberg W., *Rev. Mod. Phys.*, 1957, **29**, 269.
20. Ivanenko D., *Sov. Phys.*, 1938, **13**, 141.
21. Kalnins E.G., Miller W., *R*-separation of variables for the time-dependent Hamilton–Jacobi and Schrödinger equations, *J. Math. Phys.*, 1987, **28**, 1005–1015.
22. Kortel F., *Nuovo Cimento*, 1956, **4**, 729.
23. Kurdgelaidze D., *Jurn. Exp. Theor. Fiz. (USSR)*, 1957, **32**, 1156.
24. Lie S., *Arch. Math.*, 1881, **6**, 377.
25. Niederere U., *Helv. Phys. Acta*, 1972, **45**, 808.
26. Olver P., *Application of Lie groups to differential equations*, Springer-Verlag, New York, 1986.
27. Ovsyannikov L., *Group analysis of differential equations*, Academic, New York, 1982.
28. Tajiri M., *J. Phys. Soc. Japan*, 1983, **52**, 1908–1917.
29. Takahashi K., *J. Math. Phys.*, 1979, **20**, 1232.