

Non-local ansätze for the Dirac equation

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Using non-local (non-Lie) symmetry of the linear Dirac equation we have constructed a number of new ansätze reducing it to systems of ordinary differential equations.

It is well known (see e.g. [1]) that the Poincaré group $P(1, 3)$ is a maximal local (in Lie's sense) invariance group of the linear Dirac equation

$$(i\gamma_\mu \partial_\mu + m)\psi(x) = 0, \quad m = \text{const}, \quad (1)$$

where $\psi = \psi(x_0, \mathbf{x})$ is a four-component spinor, $\partial_\mu \equiv \partial/\partial x_\mu$, $\mu = \overline{0, 3}$ and γ_μ are imaginary 4×4 matrices satisfying the Clifford algebra

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu} I \equiv 2I \begin{cases} 1, & \mu = \nu = 0, \\ -1, & \mu = \nu = \overline{1, 3}, \\ 0, & \mu \neq \nu. \end{cases}$$

In [2, 3] ansätze reducing the Dirac equation to systems of ordinary differential equations (ODE) were constructed, the subgroup structure of the group $P(1, 3)$ investigated in detail by Patera et al [4, 5] being used.

As shown in [1, 6, 7] equation (1) possesses non-local (non-Lie) symmetry. So far this additional non-local symmetry has not been used to construct ansätze reducing the Dirac equation to systems of ODE. In the present paper we construct a number of such ansätze following an approach suggested in [3, 8].

If one puts

$$\Gamma_\mu = \text{diag}(-i\gamma_\mu, -i\gamma_\mu), \quad \Psi^T = (\text{Re } \psi, \text{Im } \psi)^T$$

then equation (1) becomes

$$(\Gamma_\mu \partial_\mu - m)\Psi(x) = 0. \quad (2)$$

It is common knowledge that the complete set of first-order symmetry operators of the Dirac equation (2) is not a Lie algebra. We have succeeded in picking out the subset which forms the Lie algebra of the Poincaré group:

$$P_\mu = [1 + \varepsilon(\Gamma_4 + \Gamma_5)]\partial^\mu + \varepsilon m(\Gamma_4 + \Gamma_5)\Gamma_\mu, \quad (3)$$

$$J_{\mu\nu} = -x_\mu \partial^\nu + x_\nu \partial^\mu - \frac{1}{4}(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu), \quad (4)$$

where $\varepsilon = \text{const}$, $\partial^\mu = g^{\mu\nu} \partial_\nu$ for $\mu, \nu = \overline{0, 3}$ and

$$\Gamma_4 + \Gamma_5 = 2 \begin{pmatrix} 0 & 0 \\ \gamma_0 \gamma_1 \gamma_2 \gamma_3 & 0 \end{pmatrix}.$$

It is important to note that operators (3) generate a non-local group of transformations

$$\Psi' = [1 - \varepsilon m(\Gamma_4 + \Gamma_5)\theta^\mu \Gamma_\mu] \Psi + \varepsilon(\Gamma_4 + \Gamma_5)\theta_\mu \Psi_{x_\mu}, \quad x'_\mu = x_\mu + \theta_\mu, \quad (5)$$

where θ_μ are group parameters.

According to [4, 8] there exists a correspondence between three-dimensional subalgebras of the algebra (3) and (4) and ansätze reducing the Dirac equation (2) to ODE. Omitting very cumbersome intermediate calculations we write the final result for the non-local ansätze for the spinor field.

1. $\langle P_0 + P_3, P_1, P_2 \rangle$

$$\Psi(x) = \exp \left[\varepsilon m \Gamma_{45} \left(\Gamma_1 x_1 + \Gamma_2 x_2 + \frac{1}{2} \eta \Gamma_{03} \right) \right] \varphi(\xi).$$

2. $\langle P_1, P_2, P_3 \rangle$

$$\Psi(x) = \exp [\varepsilon m \Gamma_{45} (\Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3)] \varphi(x_0).$$

3. $\langle P_0, P_1, P_2 \rangle$

$$\Psi(x) = \exp [\varepsilon m \Gamma_{45} (\Gamma_1 x_1 + \Gamma_2 x_2 - \Gamma_0 x_0)] \varphi(x_3).$$

4. $\langle J_{03}, P_1, P_2 \rangle$

$$\Psi(x) = \exp [\varepsilon m \Gamma_{45} (\Gamma_1 x_1 + \Gamma_2 x_2)] \exp \left(-\frac{1}{2} \Gamma_0 \Gamma_3 \ln \xi \right) \varphi(x_0^2 - x_3^2).$$

5. $\langle J_{03}, P_0 + P_3, P_1 \rangle$

$$\Psi(x) = \exp \left[\varepsilon m \Gamma_{45} \left(\Gamma_1 x_1 + \frac{1}{2} \eta \Gamma_{03} \right) \right] \exp \left(-\frac{1}{2} \Gamma_0 \Gamma_3 \ln \xi \right) \varphi(x_2).$$

6. $\langle J_{03} + \alpha P_2, P_0, P_3 \rangle$

$$\Psi(x) = \exp [\varepsilon m \Gamma_{45} (\Gamma_3 x_3 - \Gamma_0 x_0)] \times \\ \times \exp \left\{ \left[\varepsilon \Gamma_{45} \Gamma_2 + \frac{1}{2\alpha} \Gamma_0 \Gamma_3 (\varepsilon \Gamma_{45} - 1) \right] x_2 \right\} \varphi(x_1).$$

7. $\langle J_{03} + \alpha P_2, P_0 + P_3, P_1 \rangle$

$$\Psi(x) = \exp \left[\varepsilon m \Gamma_{45} \left(\Gamma_1 x_1 + \frac{1}{2} \eta \Gamma_{03} \right) \right] \exp \left\{ x_2 \left[m \xi^2 \Gamma_2 (1 + 2\varepsilon \Gamma_{45}) - \right. \right. \\ \left. \left. - \frac{1}{2} \xi \Gamma_2 \Gamma_{03} - 3\alpha \varepsilon m \xi \Gamma_{45} - \varepsilon m \xi^2 \Gamma_2 \Gamma_{45} \Gamma_0 \Gamma_3 \right] \right\} \varphi(\xi).$$

8. $\langle J_{12}, P_0, P_3 \rangle$

$$\Psi(x) = \exp [\varepsilon m \Gamma_{45} (\Gamma_3 x_3 - \Gamma_0 x_0)] \exp \left[\frac{1}{2} \Gamma_1 \Gamma_2 \tan^{-1} \frac{x_1}{x_2} \right] \varphi(x_1^2 + x_2^2).$$

9. $\langle J_{12} + \alpha P_0, P_1, P_2 \rangle$

$$\Psi(x) = \exp [\varepsilon m \Gamma_{45} (\Gamma_1 x_1 + \Gamma_2 x_2)] \times \\ \times \exp \left\{ \left[\frac{1}{2\alpha} \Gamma_1 \Gamma_2 (1 - \varepsilon \Gamma_{45}) - \varepsilon m \Gamma_{45} \Gamma_0 \right] x_0 \right\} \varphi(x_3).$$

$$10. \langle J_{12} + \alpha P_3, P_1, P_2 \rangle$$

$$\begin{aligned} \Psi(x) &= \exp [\varepsilon m \Gamma_{45} (\Gamma_1 x_1 + \Gamma_2 x_2)] \times \\ &\times \exp \left\{ \left[\varepsilon m \Gamma_{45} \Gamma_3 + \frac{1}{2\alpha} (1 - \varepsilon \Gamma_{45}) \Gamma_1 \Gamma_2 \right] x_2 \right\} \varphi(x_0). \end{aligned}$$

$$11. \langle J_{12} + P_0 + P_3, P_1, P_2 \rangle$$

$$\begin{aligned} \Psi(x) &= \exp [\varepsilon m \Gamma_{45} (\Gamma_1 x_1 + \Gamma_2 x_2)] \times \\ &\times \exp \left\{ \frac{1}{2} \eta \left[\varepsilon m \Gamma_{45} \Gamma_{03} + \frac{1}{2} (1 - \varepsilon \Gamma_{45}) \Gamma_1 \Gamma_2 \right] \right\} \varphi(\xi). \end{aligned}$$

$$12. \langle G_1, P_0 + P_3, P_2 \rangle$$

$$\Psi(x) = \exp \left[\varepsilon m \Gamma_{45} \left(\Gamma_2 x_2 + \frac{1}{2} \eta \Gamma_{03} \right) \right] \exp \left[-\frac{x_2}{2\xi} \Gamma_{03} (\Gamma_1 + \varepsilon m x_2 \Gamma_{45}) \right] \varphi(\xi).$$

$$13. \langle G_1, P_0 + P_3, P_1 + \alpha P_2 \rangle$$

$$\begin{aligned} \Psi(x) &= \exp \left\{ \varepsilon m \Gamma_{45} \left[x_1 (\Gamma_1 + \alpha \Gamma_2) + \frac{1}{2} \eta \Gamma_{03} \right] \right\} \times \\ &\times \exp \left\{ \frac{\alpha x_1 - x_2}{\alpha \xi} \left[\frac{1}{2} \Gamma_1 \Gamma_{03} - \varepsilon m \xi \Gamma_{45} (\Gamma_1 + \alpha \Gamma_2) \right] \right\} \varphi(\xi). \end{aligned}$$

$$14. \langle G_1 + P_2, P_0 + P_3, P_1 \rangle$$

$$\begin{aligned} \Psi(x) &= \exp \left[\varepsilon m \Gamma_{45} \left(x_1 \Gamma_1 + \frac{1}{2} \eta \Gamma_{03} \right) \right] \times \\ &\times \exp \left\{ x_2 \left[\varepsilon m \Gamma_{45} (\Gamma_2 - \xi \Gamma_1) + \frac{1}{2} (\varepsilon \Gamma_{45} - 1) \Gamma_{03} \Gamma_1 \right] \right\} \varphi(\xi). \end{aligned}$$

$$15. \langle G_1 + P_0, P_0 + P_3, P_2 \rangle$$

$$\begin{aligned} \Psi(x) &= \exp \left[\varepsilon m \Gamma_{45} \left(x_2 \Gamma_2 + \frac{1}{2} \eta \Gamma_{03} \right) \right] \times \\ &\times \exp [m x_1 (\Gamma_1 + \xi \Gamma_{03} + 3\varepsilon \Gamma_1 \Gamma_{45} - 4\varepsilon \xi \Gamma_{45} \Gamma_{03})] \varphi(\xi). \end{aligned}$$

$$16. \langle G_1 + P_0, P_0 + P_3, P_1 \rangle$$

$$\begin{aligned} \Psi(x) &= \exp \left[\varepsilon m \Gamma_{45} \left(x_1 \Gamma_1 + \frac{1}{2} \eta \Gamma_{03} \right) \right] \times \\ &\times \exp [m \Gamma_2 x_2 (3\varepsilon \Gamma_{45} + \varepsilon \xi \Gamma_{45} \Gamma_{03} \Gamma_1 - 1)] \varphi(\xi). \end{aligned}$$

$$17. \langle G_1 + P_0, P_1 + \alpha P_2, P_0 + P_3 \rangle$$

$$\begin{aligned} \Psi(x) &= \exp \left\{ \varepsilon m \Gamma_{45} \left[\frac{1}{2} \eta \Gamma_{03} + \frac{x_2}{\alpha} (\Gamma_1 + \alpha \Gamma_2) \right] \right\} \times \\ &\times \exp \left\{ \frac{m}{\alpha^4 + 1} (1 - 2\varepsilon \Gamma_{45}) [(\Gamma_2 - \alpha \Gamma_1) - \alpha \xi \Gamma_{03} + \varepsilon (\alpha \Gamma_1 - \Gamma_2) \Gamma_{45}] \times \right. \\ &\times \left. \left\{ \varepsilon \Gamma_{45} \left[\Gamma_0 \Gamma_3 + \frac{1}{\alpha} \Gamma_1 \Gamma_2 + \Gamma_{03} \Gamma_1 - 1 \right] - 1 \right\} (\alpha x_1 - x_2) \right\} \varphi(\xi). \end{aligned}$$

18. $\langle J_{03} + \alpha J_{12}, P_0, P_3 \rangle$

$$\Psi(x) = \exp[\varepsilon m \Gamma_{45}(\Gamma_3 x_3 - \Gamma_0 x_0)] \times \\ \times \exp\left[\frac{1}{2\alpha}(\Gamma_0 \Gamma_3 + \alpha \Gamma_1 \Gamma_2) \tan^{-1} \frac{x_1}{x_2}\right] \varphi(x_1^2 + x_2^2).$$

19. $\langle J_{03} + \alpha J_{12}, P_1, P_2 \rangle$

$$\Psi(x) = \exp[\varepsilon m \Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2)] \exp\left[-\frac{1}{2}(\Gamma_0 \Gamma_3 + \alpha \Gamma_1 \Gamma_2) \ln \xi\right] \varphi(x_0^2 - x_3^2).$$

20. $\langle G_1, G_2, P_0 + P_3 \rangle$

$$\Psi(x) = \exp\left(\frac{1}{2} \varepsilon m \eta \Gamma_{45} \Gamma_{03}\right) \times \\ \times \exp\left\{-\frac{1}{2} \frac{m}{\xi} \Gamma_{03} [\varepsilon(x_1^2 + x_2^2) \Gamma_{45} + \Gamma_1 x_1 + \Gamma_2 x_2]\right\} \varphi(\xi).$$

21. $\langle G_1 + P_2, G_2 + \alpha P_1 + \beta P_2, P_0 + P_3 \rangle$

$$\Psi(x) = \exp\left(\frac{1}{2} \varepsilon m \eta \Gamma_{45} \Gamma_{03}\right) [f_1 + \varepsilon \Gamma_{45}(g_1 \Gamma_1 + g_2 \Gamma_2 + g_3 \Gamma_{03}) + \\ + \Gamma_{03}(h_1 \Gamma_1 + h_2 \Gamma_2) + \varepsilon u \Gamma_{45} \Gamma_{03} \Gamma_1 \Gamma_2] \varphi(\xi),$$

where

$$f_1 = 1, \quad g_1 = \frac{\alpha m}{\tau}(\xi x_2 - x_1), \quad g_2 = \frac{m}{\tau}[\xi x_1 + (\beta \xi - \alpha)x_2], \\ h_1 = \frac{1}{2\tau}[\alpha x_2 - (\xi + \beta)x_1], \quad h_2 = \frac{1}{2\tau}(x_1 - \xi x_2), \quad g_3 = -\frac{m}{2\tau}(\alpha + 1)x_1 x_2, \\ u = -\frac{m}{2\tau}(-x_1^2 + \alpha x_2^2 + \beta x_1 x_2), \quad \tau = \xi(\xi + \beta) - \alpha.$$

22. $\langle G_1, G_2 + P_1 + \beta P_2, P_0 + P_3 \rangle$

$$\Psi(x) = \exp\left(\frac{1}{2} \varepsilon m \eta \Gamma_{45} \Gamma_{03}\right) \exp\left[-\frac{x_1}{2\xi} \Gamma_{03}(\varepsilon m x_1 \Gamma_{45} + \Gamma_1)\right] \times \\ \times \exp\left\{\frac{\varepsilon m x_2^2}{2(\xi + \beta)} \Gamma_{45} \Gamma_{03}\right\} \exp\left\{\frac{x_2}{(\xi + \beta)^2} \left[\varepsilon \Gamma_{45} \left(\frac{1}{2\xi} \Gamma_{03} \Gamma_1 + \right. \right. \right. \\ \left. \left. \left. + m(\Gamma_1 + \beta \Gamma_2)\right)(\xi + \beta) + \frac{1 - \xi}{2\xi}(\xi + \beta - \varepsilon \beta \Gamma_{45}) \Gamma_{03} \Gamma_1\right]\right\} \varphi(\xi).$$

23. $\langle G_1, G_2 + P_2, P_0 + P_3 \rangle$

$$\Psi(x) = \exp\left(\frac{1}{2} \varepsilon m \Gamma_{45} \Gamma_{03} \eta\right) \exp\left[-\frac{x_1}{2\xi} \Gamma_{03}(\varepsilon m x_1 \Gamma_{45} + \Gamma_1)\right] \times \\ \times \exp\left\{\frac{\varepsilon m x_2^2}{2(\xi + 1)} \Gamma_{45} \Gamma_{03}\right\} \times \\ \times \exp\left\{\frac{x_2}{(\xi + 1)^2} \left[\varepsilon m(\xi + 1) \Gamma_{45} \Gamma_2 + \frac{1}{2}(\varepsilon \Gamma_{45} - \xi - 1) \Gamma_{03} \Gamma_2\right]\right\} \varphi(\xi).$$

24. $\langle J_{03}, G_1, P_2 \rangle$

$$\Psi(x) = \exp(\varepsilon m \Gamma_{45} \Gamma_2 x_2) \exp\left[-\frac{x_1}{2\xi} \Gamma_{03} \Gamma_1\right] \exp\left(-\frac{1}{2} \Gamma_0 \Gamma_3 \ln \xi\right) \varphi(x_0^2 - x_1^2 - x_3^2).$$

25. $\langle J_{03} + \alpha P_1 + \beta P_2, G_1, P_0 + P_3 \rangle$

$$\begin{aligned} \Psi(x) = & \exp\left(\frac{1}{2} \varepsilon m \Gamma_{45} \Gamma_{03} \eta\right) \exp\left[-\frac{x_1}{2\xi} \Gamma_{03} (\varepsilon m x_1 \Gamma_{45} + \Gamma_1)\right] \times \\ & \times \exp\left\{\frac{m x_2}{\xi^2} [-\xi \Gamma_2 + \beta(1 + \Gamma_{45}) \Gamma_{03}] [\xi \Gamma_{45} (\Gamma_0 \Gamma_3 - 1) - \xi + \right. \\ & \left. + \Gamma_{03} \Gamma_{45} (\alpha \Gamma_1 + \beta \Gamma_2) + \Gamma_{03}]\right\} \varphi(\xi). \end{aligned}$$

26. $\langle J_{12} + P_0 + P_3, G_1, G_2 \rangle$

$$\begin{aligned} \Psi(x) = & \exp\left\{-\frac{1}{2\xi} \Gamma_{03} (\Gamma_1 x_1 + \Gamma_2 x_2)\right\} \times \\ & \times \exp\left\{\frac{x_0^2 - \mathbf{x}^2}{2\xi} \left[\varepsilon m \Gamma_{45} \Gamma_{03} + \frac{1}{2} \Gamma_1 \Gamma_2 (\varepsilon \Gamma_{45} - 1)\right]\right\} \varphi(\xi). \end{aligned}$$

27. $\langle J_{03} + \alpha J_{12}, G_1, G_2 \rangle$

$$\begin{aligned} \Psi(x) = & \exp\left[\frac{1}{2\xi} (\Gamma_1 x_1 + \Gamma_2 x_2) \Gamma_{03}\right] \times \\ & \times \exp\left[-\frac{1}{2} (\Gamma_0 \Gamma_3 + \alpha \Gamma_1 \Gamma_2) \ln \xi\right] \varphi(x_0^2 - \mathbf{x}^2). \end{aligned}$$

The following notations were used in the above ansätze:

$$\begin{aligned} G_k &= J_{0k} + J_{k3}, \quad k = 1, 2, \quad \xi = x_3 + x_0, \quad \eta = x_3 - x_0, \\ \Gamma_{03} &= \Gamma_0 + \Gamma_3, \quad \Gamma_{45} = \Gamma_4 + \Gamma_5, \end{aligned}$$

α and β are constants, $\varphi(z)$ is a new unknown spinor and $\langle Q_1, Q_2, Q_3 \rangle$ is a subalgebra of the algebra (3) and (4) having basis elements Q_1, Q_2, Q_3 .

Let us adduce an example of reduced ODE. If one substitutes ansatz 8 into (2) then the equation for $\varphi(z)$ becomes

$$\left[2z^{1/2} \Gamma_2 \frac{d}{dz} + \frac{1}{2} z^{-1/2} \Gamma_2 - m + 2\varepsilon m (\Gamma_4 + \Gamma_5)\right] \varphi(z) = 0.$$

Note 1. If one puts $\varepsilon = 0$ in (3) then (3) and (4) generate the local Lie group $P(1, 3)$. That is why, on putting $\varepsilon = 0$ into the ansätze above, one obtains Poincaré-invariant ansätze for the spinor field constructed in [3].

Note 2. The above non-local ansätze can be applied to the construction of exact solutions of non-linear Lorentz-invariant spinor equations admitting the group (5). One example of such equations is

$$\{\partial_\mu \partial^\mu + \lambda [\bar{\Psi} (\Gamma_4 + \Gamma_5) \Gamma_\mu \partial_\mu \Psi] (\Gamma_4 + \Gamma_5)\} \Psi = 0,$$

where λ is constant and $\bar{\Psi} = \Psi^T \Gamma_0 \Gamma_4$. This problem will be considered in a future publication.

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