Non-local ansätze for the Dirac equation

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Using non-local (non-Lie) symmetry of the linear Dirac equation we have constructed a number of new ansätze reducing it to systems of ordinary differential equations.

It is well known (see e.g. [1]) that the Poincaré group $P(1,3)$ is a maximal local (in Lie's sense) invariance group of the linear Dirac equation

$$(i\gamma_\mu\partial_\mu + m)\psi(x) = 0, \quad m = \text{const}, \tag{1}$$

where $\psi = \psi(x_0, \boldsymbol{x})$ is a four-component spinor, $\partial_\mu \equiv \partial/\partial x_\mu$, $\mu = \overline{0,3}$ and γ_μ are imaginary 4×4 matrices satisfying the Clifford algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}I \equiv 2I \begin{cases} 1, & \mu = \nu = 0, \\ -1, & \mu = \nu = \overline{1,3}, \\ 0, & \mu \neq \nu. \end{cases}$$

In [2, 3] ansätze reducing the Dirac equation to systems of ordinary differential equations (ODE) were constructed, the subgroup structure of the group $P(1,3)$ investigated in detail by Patera et al [4, 5] being used.

As shown in [1, 6, 7] equation (1) possesses non-local (non-Lie) symmetry. So far this additional non-local symmetry has not been used to construct ansatze reducing the Dirac equation to systems of ODE. In the present paper we construct a number of such ansätze following an approach suggested in [3, 8].

If one puts

$$\Gamma_\mu = \operatorname{diag}(-i\gamma_\mu, -i\gamma_\mu), \quad \Psi^T = (\operatorname{Re}\psi, \operatorname{Im}\psi)^T$$

then equation (1) becomes

$$(\Gamma_\mu\partial_\mu - m)\Psi(x) = 0. \tag{2}$$

It is common knowledge that the complete set of first-order symmetry operators of the Dirac equation (2) is not a Lie algebra. We have succeeded in picking out the subset which forms the Lie algebra of the Poincaré group:

$$P_\mu = [1 + \varepsilon(\Gamma_4 + \Gamma_5)]\partial^\mu + \varepsilon m(\Gamma_4 + \Gamma_5)\Gamma_\mu, \tag{3}$$

$$J_{\mu\nu} = -x_\mu\partial^\nu + x_\nu\partial^\mu - \frac{1}{4}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu), \tag{4}$$

where $\varepsilon = \text{const}$, $\partial^\mu = g^{\mu\nu}\partial_\nu$ for $\mu,\nu = \overline{0,3}$ and

$$\Gamma_4 + \Gamma_5 = 2\begin{pmatrix} 0 & 0 \\ \gamma_0\gamma_1\gamma_2\gamma_3 & 0 \end{pmatrix}.$$

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It is important to note that operators (3) generate a non-local group ot transformations

$$\Psi' = [1 - \varepsilon m(\Gamma_4 + \Gamma_5)\theta^\mu\Gamma_\mu]\Psi + \varepsilon(\Gamma_4 + \Gamma_5)\theta_\mu\Psi_{x_\mu}, \quad x'_\mu = x_\mu + \theta_\mu, \tag{5}$$

where θ_μ are group parameters.

According to [4, 8] there exists a correspondence between three-dimensional subalgebras of the algebra (3) and (4) and ansätze reducing the Dirac equation (2) to ODE. Omitting very cumbersome intermediate calculations we write the final result for the non-local ansätze for the spinor field.

1. $\langle P_0 + P_3, P_1, P_2\rangle$

$$\Psi(x) = \exp\left[\varepsilon m\Gamma_{45}\left(\Gamma_1 x_1 + \Gamma_2 x_2 + \frac{1}{2}\eta\Gamma_{03}\right)\right]\varphi(\xi).$$

2. $\langle P_1, P_2, P_3\rangle$

$$\Psi(x) = \exp\left[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2 + \Gamma_3 x_3)\right]\varphi(x_0).$$

3. $\langle P_0, P_1, P_2\rangle$

$$\Psi(x) = \exp\left[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2 - \Gamma_0 x_0)\right]\varphi(x_3).$$

4. $\langle J_{03}, P_1, P_2\rangle$

$$\Psi(x) = \exp\left[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2)\right]\exp\left(-\frac{1}{2}\Gamma_0\Gamma_3\ln\xi\right)\varphi\left(x_0^2 - x_3^2\right).$$

5. $\langle J_{03}, P_0 + P_3, P_1\rangle$

$$\Psi(x) = \exp\left[\varepsilon m\Gamma_{45}\left(\Gamma_1 x_1 + \frac{1}{2}\eta\Gamma_{03}\right)\right]\exp\left(-\frac{1}{2}\Gamma_0\Gamma_3\ln\xi\right)\varphi(x_2).$$

6. $\langle J_{03} + \alpha P_2, P_0, P_3\rangle$

$$\begin{aligned}\Psi(x) = {} & \exp\left[\varepsilon m\Gamma_{45}(\Gamma_3 x_3 - \Gamma_0 x_0)\right]\times\\ & \times\exp\left\{\left[\varepsilon\Gamma_{45}\Gamma_2 + \tfrac{1}{2\alpha}\Gamma_0\Gamma_3(\varepsilon\Gamma_{45} - 1)\right]x_2\right\}\varphi(x_1).\end{aligned}$$

7. $\langle J_{03} + \alpha P_2, P_0 + P_3, P_1\rangle$

$$\begin{aligned}\Psi(x) = {} & \exp\left[\varepsilon m\Gamma_{45}\left(\Gamma_1 x_1 + \frac{1}{2}\eta\Gamma_{03}\right)\right]\exp\left\{x_2\left[m\xi^2\Gamma_2(1 + 2\varepsilon\Gamma_{45}) - \right.\right.\\ & \left.\left. - \frac{1}{2}\xi\Gamma_2\Gamma_{03} - 3\alpha\varepsilon m\xi\Gamma_{45} - \varepsilon m\xi^2\Gamma_2\Gamma_{45}\Gamma_0\Gamma_3)\right]\right\}\varphi(\xi).\end{aligned}$$

8. $\langle J_{12}, P_0, P_3\rangle$

$$\Psi(x) = \exp\left[\varepsilon m\Gamma_{45}(\Gamma_3 x_3 - \Gamma_0 x_0)\right]\exp\left[\frac{1}{2}\Gamma_1\Gamma_2\tan^{-1}\frac{x_1}{x_2}\right]\varphi\left(x_1^2 + x_2^2\right).$$

9. $\langle J_{12} + \alpha P_0, P_1, P_2\rangle$

$$\begin{aligned}\Psi(x) = {} & \exp\left[\varepsilon m\Gamma_{45}(\Gamma_1 x_1 + \Gamma_2 x_2)\right]\times\\ & \times\exp\left\{\left[\frac{1}{2\alpha}\Gamma_1\Gamma_2(1 - \varepsilon\Gamma_{45}) - \varepsilon m\Gamma_{45}\Gamma_0\right]x_0\right\}\varphi(x_3).\end{aligned}$$

10. $\langle J_{12}+\alpha P_3, P_1, P_2\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}(\Gamma_1x_1+\Gamma_2x_2)\right]\times$$
$$\times\exp\left\{\left[\varepsilon m\Gamma_{45}\Gamma_3+\frac{1}{2\alpha}(1-\varepsilon\Gamma_{45})\Gamma_1\Gamma_2\right]x_2\right\}\varphi(x_0).$$

11. $\langle J_{12}+P_0+P_3, P_1, P_2\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}(\Gamma_1x_1+\Gamma_2x_2)\right]\times$$
$$\times\exp\left\{\frac{1}{2}\eta\left[\varepsilon m\Gamma_{45}\Gamma_{03}+\frac{1}{2}(1-\varepsilon\Gamma_{45})\Gamma_1\Gamma_2\right]\right\}\varphi(\xi).$$

12. $\langle G_1, P_0+P_3, P_2\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}\left(\Gamma_2x_2+\frac{1}{2}\eta\Gamma_{03}\right)\right]\exp\left[-\frac{x_2}{2\xi}\Gamma_{03}(\Gamma_1+\varepsilon mx_2\Gamma_{45})\right]\varphi(\xi).$$

13. $\langle G_1, P_0+P_3, P_1+\alpha P_2\rangle$

$$\Psi(x)=\exp\left\{\varepsilon m\Gamma_{45}\left[x_1(\Gamma_1+\alpha\Gamma_2)+\frac{1}{2}\eta\Gamma_{03}\right]\right\}\times$$
$$\times\exp\left\{\frac{\alpha x_1-x_2}{\alpha\xi}\left[\frac{1}{2}\Gamma_1\Gamma_{03}-\varepsilon m\xi\Gamma_{45}(\Gamma_1+\alpha\Gamma_2)\right]\right\}\varphi(\xi).$$

14. $\langle G_1+P_2, P_0+P_3, P_1\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}\left(x_1\Gamma_1+\frac{1}{2}\eta\Gamma_{03}\right)\right]\times$$
$$\times\exp\left\{x_2\left[\varepsilon m\Gamma_{45}(\Gamma_2-\xi\Gamma_1)+\frac{1}{2}(\varepsilon\Gamma_{45}-1)\Gamma_{03}\Gamma_1\right]\right\}\varphi(\xi).$$

15. $\langle G_1+P_0, P_0+P_3, P_2\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}\left(x_2\Gamma_2+\frac{1}{2}\eta\Gamma_{03}\right)\right]\times$$
$$\times\exp\left[mx_1(\Gamma_1+\xi\Gamma_{03}+3\varepsilon\Gamma_1\Gamma_{45}-4\varepsilon\xi\Gamma_{45}\Gamma_{03})\right]\varphi(\xi).$$

16. $\langle G_1+P_0, P_0+P_3, P_1\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}\left(x_1\Gamma_1+\frac{1}{2}\eta\Gamma_{03}\right)\right]\times$$
$$\times\exp\left[m\Gamma_2x_2(3\varepsilon\Gamma_{45}+\varepsilon\xi\Gamma_{45}\Gamma_{03}\Gamma_1-1)\right]\varphi(\xi).$$

17. $\langle G_1+P_0, P_1+\alpha P_2, P_0+P_3\rangle$

$$\Psi(x)=\exp\left\{\varepsilon m\Gamma_{45}\left[\frac{1}{2}\eta\Gamma_{03}+\frac{x_2}{\alpha}(\Gamma_1+\alpha\Gamma_2)\right]\right\}\times$$
$$\times\exp\left\{\frac{m}{\alpha^1+1}(1-2\varepsilon\Gamma_{45})[(\Gamma_2-\alpha\Gamma_1)-\alpha\xi\Gamma_{03}+\varepsilon(\alpha\Gamma_1-\Gamma_2)\Gamma_{45}]\times\right.$$
$$\left.\times\left\{\varepsilon\Gamma_{45}\left[\Gamma_0\Gamma_3+\frac{1}{\alpha}\Gamma_1\Gamma_2+\Gamma_{03}\Gamma_1-1\right]-1\right\}(\alpha x_1-x_2)\right\}\varphi(\xi).$$

18. $\langle J_{03}+\alpha J_{12}, P_0, P_3\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}(\Gamma_3x_3-\Gamma_0x_0)\right]\times$$
$$\times\exp\left[\frac{1}{2\alpha}(\Gamma_0\Gamma_3+\alpha\Gamma_1\Gamma_2)\tan^{-1}\frac{x_1}{x_2}\right]\varphi\left(x_1^2+x_2^2\right).$$

19. $\langle J_{03}+\alpha J_{12}, P_1, P_2\rangle$

$$\Psi(x)=\exp\left[\varepsilon m\Gamma_{45}(\Gamma_1x_1+\Gamma_2x_2)\right]\exp\left[-\frac{1}{2}(\Gamma_0\Gamma_3+\alpha\Gamma_1\Gamma_2)\ln\xi\right]\varphi\left(x_0^2-x_3^2\right).$$

20. $\langle G_1, G_2, P_0+P_3\rangle$

$$\Psi(x)=\exp\left(\frac{1}{2}\varepsilon m\eta\Gamma_{45}\Gamma_{03}\right)\times$$
$$\times\exp\left\{-\frac{1}{2}\frac{m}{\xi}\Gamma_{03}\left[\varepsilon\left(x_1^2+x_2^2\right)\Gamma_{45}+\Gamma_1x_1+\Gamma_2x_2\right]\right\}\varphi(\xi).$$

21. $\langle G_1+P_2, G_2+\alpha P_1+\beta P_2, P_0+P_3\rangle$

$$\Psi(x)=\exp\left(\frac{1}{2}\varepsilon m\eta\Gamma_{45}\Gamma_{03}\right)[f_1+\varepsilon\Gamma_{45}(g_1\Gamma_1+g_2\Gamma_2+g_3\Gamma_{03})+$$
$$+\Gamma_{03}(h_1\Gamma_1+h_2\Gamma_2)+\varepsilon u\Gamma_{45}\Gamma_{03}\Gamma_1\Gamma_2]\varphi(\xi),$$

where

$$f_1=1,\quad g_1=\frac{\alpha m}{\tau}(\xi x_2-x_1),\quad g_2=\frac{m}{\tau}[\xi x_1+(\beta\xi-\alpha)x_2],$$
$$h_1=\frac{1}{2\tau}[\alpha x_2-(\xi+\beta)x_1],\quad h_2=\frac{1}{2\tau}(x_1-\xi x_2),\quad g_3=-\frac{m}{2\tau}(\alpha+1)x_1x_2,$$
$$u=-\frac{m}{2\tau}(-x_1^2+\alpha x_2^2+\beta x_1x_2),\quad \tau=\xi(\xi+\beta)-\alpha.$$

22. $\langle G_1, G_2+P_1+\beta P_2, P_0+P_3\rangle$

$$\Psi(x)=\exp\left(\frac{1}{2}\varepsilon m\eta\Gamma_{45}\Gamma_{03}\right)\exp\left[-\frac{x_1}{2\xi}\Gamma_{03}(\varepsilon mx_1\Gamma_{45}+\Gamma_1)\right]\times$$
$$\times\exp\left\{\frac{\varepsilon mx_2^2}{2(\xi+\beta)}\Gamma_{45}\Gamma_{03}\right\}\exp\left\{\frac{x_2}{(\xi+\beta)^2}\left[\varepsilon\Gamma_{45}\left(\frac{1}{2\xi}\Gamma_{03}\Gamma_1+\right.\right.\right.$$
$$\left.\left.\left.+m(\Gamma_1+\beta\Gamma_2)\right)(\xi+\beta)+\frac{1-\xi}{2\xi}(\xi+\beta-\varepsilon\beta\Gamma_{45})\Gamma_{03}\Gamma_1\right]\right\}\varphi(\xi).$$

23. $\langle G_1, G_2+P_2, P_0+P_3\rangle$

$$\Psi(x)=\exp\left(\frac{1}{2}\varepsilon m\Gamma_{45}\Gamma_{03}\eta\right)\exp\left[-\frac{x_1}{2\xi}\Gamma_{03}(\varepsilon mx_1\Gamma_{45}+\Gamma_1)\right]\times$$
$$\times\exp\left\{\frac{\varepsilon mx_2^2}{2(\xi+1)}\Gamma_{45}\Gamma_{03}\right\}\times$$
$$\times\exp\left\{\frac{x_2}{(\xi+1)^2}\left[\varepsilon m(\xi+1)\Gamma_{45}\Gamma_2+\frac{1}{2}(\varepsilon\Gamma_{45}-\xi-1)\Gamma_{03}\Gamma_2\right]\right\}\varphi(\xi).$$

24. $\langle J_{03}, G_1, P_2\rangle$

$$\Psi(x) = \exp(\varepsilon m\Gamma_{45}\Gamma_2 x_2)\exp\left[-\frac{x_1}{2\xi}\Gamma_{03}\Gamma_1\right]\exp\left(-\frac{1}{2}\Gamma_0\Gamma_3\ln\xi\right)\varphi\left(x_0^2 - x_1^2 - x_3^2\right).$$

25. $\langle J_{03} + \alpha P_1 + \beta P_2, G_1, P_0 + P_3\rangle$

$$\Psi(x) = \exp\left(\frac{1}{2}\varepsilon m\Gamma_{45}\Gamma_{03}\eta\right)\exp\left[-\frac{x_1}{2\xi}\Gamma_{03}(\varepsilon m x_1\Gamma_{45} + \Gamma_1)\right]\times$$
$$\times\exp\left\{\frac{mx_2}{\xi^2}[-\xi\Gamma_2 + \beta(1+\Gamma_{45})\Gamma_{03}][\xi\Gamma_{45}(\Gamma_0\Gamma_3 - 1) - \xi + \right.$$
$$\left. + \Gamma_{03}\Gamma_{45}(\alpha\Gamma_1 + \beta\Gamma_2) + \Gamma_{03}]\right\}\varphi(\xi).$$

26. $\langle J_{12} + P_0 + P_3, G_1, G_2\rangle$

$$\Psi(x) = \exp\left\{-\frac{1}{2\xi}\Gamma_{03}(\Gamma_1 x_1 + \Gamma_2 x_2)\right\}\times$$
$$\times\exp\left\{\frac{x_0^2 - \boldsymbol{x}^2}{2\xi}\left[\varepsilon m\Gamma_{45}\Gamma_{03} + \frac{1}{2}\Gamma_1\Gamma_2(\varepsilon\Gamma_{45} - 1)\right]\right\}\varphi(\xi).$$

27. $\langle J_{03} + \alpha J_{12}, G_1, G_2\rangle$

$$\Psi(x) = \exp\left[\frac{1}{2\xi}(\Gamma_1 x_1 + \Gamma_2 x_2)\Gamma_{03}\right]\times$$
$$\times\exp\left[-\frac{1}{2}(\Gamma_0\Gamma_3 + \alpha\Gamma_1\Gamma_2)\ln\xi\right]\varphi\left(x_0^2 - \boldsymbol{x}^2\right).$$

The following notations were used in the above ansätze:

$$G_k = J_{0k} + J_{k3},\ k = 1,2, \quad \xi = x_3 + x_0, \quad \eta = x_3 - x_0,$$
$$\Gamma_{03} = \Gamma_0 + \Gamma_3, \quad \Gamma_{45} = \Gamma_4 + \Gamma_5,$$

α and β are constants, $\varphi(z)$ is a new unknown spinor and $\langle Q_1, Q_2, Q_3\rangle$ is a subalgebra of the algebra (3) and (4) having basis elements Q_1, Q_2, Q_3.

Let us adduce an example of reduced ODE. If one substitutes ansatz 8 into (2) then the equation for $\varphi(z)$ becomes

$$\left[2z^{1/2}\Gamma_2\frac{d}{dz} + \frac{1}{2}z^{-1/2}\Gamma_2 - m + 2\varepsilon m(\Gamma_4 + \Gamma_5)\right]\varphi(z) = 0.$$

Note 1. If one puts $\varepsilon = 0$ in (3) then (3) and (4) generate the local Lie group $P(1,3)$. That is why, on putting $\varepsilon = 0$ into the ansätze above, one obtains Poincaré-invariant ansätze for the spinor field constructed in [3].

Note 2. The above non-local ansätze can be applied to the construction of exact solutions of non-linear Lorentz-invariant spinor equations admitting the group (5). One example of such equations is

$$\{\partial_\mu\partial^\mu + \lambda[\bar{\Psi}(\Gamma_4 + \Gamma_5)\Gamma_\mu\partial_\mu\Psi](\Gamma_4 + \Gamma_5)\}\Psi = 0,$$

where λ is constant and $\bar{\Psi} = \Psi^T\Gamma_0\Gamma_4$. This problem will be considered in a future publication.

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