On some exact solutions of the three-dimensional non-linear Schrödinger equation

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Some exact solutions of the three-dimensional non-linear Schrödinger equation are found. The formulae for generating solutions of the Schrödinger-invariant equations are adduced.

The linear heat equation and its complex generalisation, i.e. the Schrödinger equation

$$(P_0 - P_a^2/2m)u = 0, \qquad P_0 = i\partial/\partial x_0, \quad P_a = -i\partial/\partial x_a, \quad a = \overline{1,3}, \tag{1}$$

where

$$u = u(x_0, \boldsymbol{x}), \qquad x_0 \equiv t, \qquad \boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$$

and m is the particle mass, is invariant under the generalised Galilei group $G_2(1,3)$. The basis elements of the Lie algebra $AG_2(1,3)$ have the following form:

$$P_0 = i\partial/\partial x_0, \qquad P_a = -i\partial/\partial x_a, \qquad J_{ab} = x_a P_b - x_b P_a, \tag{2}$$

$$G_a = x_0 P_a + m x_a, \qquad I = u\partial/\partial u, \qquad a, b = \overline{1,3}, \tag{3}$$

$$D = 2x_0 P_0 - \boldsymbol{x}\boldsymbol{P} + \frac{3}{2}i, \tag{4}$$

$$A = x_0\left(x_0 P_0 - \boldsymbol{x}\boldsymbol{P} + \frac{3}{2}i\right) + \frac{1}{2}m\boldsymbol{x}^2. \tag{5}$$

The same algebra for the one-dimensional equation had been found over a hundred years ago by S. Lie [8]. For the three-dimensional equation (1) this algebra had been found by Hagen [7] and Niederer [9] (see also Fushchych and Nikitin [4, 5]). The elements D and A generate the scale and projective transformations respectively. We denote the group generated by operators (2)–(4) and its Lie algebra by symbols $G_1(1,3)$ and $AG_1(1,3)$. The group and the algebra generated by (2)–(5) are denoted as $G_2(1,3)$ and $AG_2(1,3)$.

We now consider the following non-linear generalisation of (1):

$$(P_0 - P_a^2/2m)u + F(x, u, u^*) = 0, \tag{6}$$

where F is an arbitrary differentiable function. To construct the families of exact solutions of (6) we have to know the symmetry of (6) which obviously depends on the structure of the non-linearity.

J. Phys. A: Math. Gen., 1987, **20**, L929–L933.

Theorem. *Equation (6) is invariant under the following algebras:*

$$AG(1,3) \qquad iff \quad F = \phi(|u|)u, \tag{7}$$

where ϕ is an arbitrary smooth function, and

$$AG_1(1,3) \qquad iff \quad F = \lambda|u|^k u, \tag{8}$$

where λ, k are arbitrary parameters, the operator of scale transformations D having the form $D = x_0P_0 - \boldsymbol{xP} + 2i/k$, $k \neq 0$, and

$$AG_2(1,3) \qquad iff \quad F = \lambda|u|^{4/n} u, \tag{9}$$

where $n = 3$ is the number of spatial variables in the Schrödinger equation [2, 3].

To give the proof of the theorem, which we omit because of its clumsiness, it is necessary to apply the Lie method to (6). The detailed account of this method is given by Ovsyannikov [10] and Bluman and Cole [1]. We can make sure that (6) with non-linearities (7)–(9) admits the groups G, G_1 and G_2 by direct verification.

Later on we shall construct the exact solutions of the Schrödinger equation with non-linearity (9), i.e.

$$(P_0 - P_a^2/2m)u + \lambda|u|^{4/3}u = 0. \tag{10}$$

It follows from the theorem that only the equation with fractional non-linearity is invariant under the group $G_2(1,3)$.

Following Fushchych [2] we seek solutions of (10) with the help of the ansatz

$$u(x) = f(x)\varphi(\omega_1, \omega_2, \omega_3), \tag{11}$$

where φ is the function to calculate. This function depends only on three invariant variables ω_1, ω_2 and ω_3 being the first integrals of the Euler–Lagrange system of equations:

$$\frac{dx_0}{\xi^0(x,u)} = \frac{dx_1}{\xi^1(x,u)} = \frac{dx_2}{\xi^2(x,u)} = \frac{dx_3}{\xi^3(x,u)} = \frac{du}{\eta(x,u)}, \tag{12}$$

where ξ^0, ξ^1, ξ^2, ξ^3 and η are coordinates of the infinitesimal operator of the group $G_2(1,3)$, i.e. the following functions:

$$\xi^0 = ax_0^2 + 2bx_0 + d_0, \qquad \boldsymbol{\xi} = (ax_0 + b)\boldsymbol{x} + \boldsymbol{g}x_0 + \boldsymbol{\alpha} \times \boldsymbol{x} + \boldsymbol{d},$$
$$\eta = -\left[\mathrm{Im}\left(\frac{1}{2}a\boldsymbol{x}^2 + \boldsymbol{gx}\right) + \frac{3}{2}(ax_0 + b)\right]u,$$

where a, b, $\boldsymbol{g}$, $\boldsymbol{\alpha}$, d_0 and $\boldsymbol{d}$ are parameters of the group $G_2(1,3)$.

Functions $f(x)$ also are found from the system (12). The method of seeking $f(x)$ and variables ω is given in more detail in [2, 6].

Ansatz (11) reduces (10) to the equations for function φ which depends only on three variables ω_1, ω_2 and ω_3. Thus to construct solutions of (10) using ansatz (11) it is necessary to have the explicit form of the function $f(x)$ and the new invariant variables ω_1, ω_2 and ω_3. Not going into details we write them. Depending on relations between parameters of the group $G_2(1,3)$ there are nine sets $f(x)$ and $\boldsymbol{\omega}(x)$:

1) $f(x) = \left(1-x_0^2\right)^{-3/4}\exp\left[\frac{1}{2}imx_0\boldsymbol{x}^2/\left(1-x_0^2\right)\right]$, $\quad \omega_1 = (\boldsymbol{\alpha x})\left(1-x_0^2\right)^{-1/2}$,
$\omega_2 = \boldsymbol{x}^2\left(1-x_0^2\right)^{-1}$, $\quad \omega_3 = \tanh^{-1}x_0 + \tan^{-1}(\boldsymbol{\beta x}/\boldsymbol{\gamma x})$;

2) $f(x) = x_0^{-3/2}\exp\left(-\frac{1}{2}i\boldsymbol{x}^2x_0^{-1}\right)$, $\quad \omega_1 = (\boldsymbol{\alpha x})x_0^{-1}$,
$\omega_2 = \boldsymbol{x}^2x_0^{-2}$, $\quad \omega_3 = x_0^{-1} + \tan^{-1}(\boldsymbol{\beta x}/\boldsymbol{\gamma x})$;

3) $f(x) = \left(1+x_0^2\right)^{-3/4}\exp\left(-\frac{1}{2}imx_0\boldsymbol{x}^2\left(1+x_0^2\right)^{-1}\right)$,
$\omega_1 = (\boldsymbol{\alpha x})\left(1+x_0^2\right)^{-1/2}$, $\quad \omega_2 = \boldsymbol{x}^2\left(1+x_0^2\right)^{-1}$,
$\omega_3 = -\tan^{-1}x_0 + \tan^{-1}(\boldsymbol{\beta x}/\boldsymbol{\alpha x})$;

4) $f(x) = x_0^{-3/4}$, $\quad \omega_1 = (\boldsymbol{\alpha x})x_0^{-1/2}$, $\quad \omega_2 = \boldsymbol{x}^2x_0^{-1}$,
$\omega_3 = -\ln x_0 + \tan^{-1}(\boldsymbol{\beta x}/\boldsymbol{\gamma x})$;

5) $f(x) = x_0^{-3/4}$, $\quad \omega_1 = (\boldsymbol{\alpha x})x_0^{-1/2}$, $\quad \omega_2 = (\boldsymbol{\beta x})x_0^{-1/2}$, $\quad \omega_3 = (\boldsymbol{\gamma x})x_0^{-1/2}$;

6) $f(x) = 1$, $\quad \omega_1 = \boldsymbol{\alpha x}$, $\quad \omega_2 = \boldsymbol{x}^2$, $\quad \omega_3 = -x_0 + \tan^{-1}(\boldsymbol{\beta x}/\boldsymbol{\gamma x})$;

7) $f(x) = 1$, $\quad \omega_1 = \boldsymbol{\alpha x}$, $\quad \omega_2 = \boldsymbol{x}^2$, $\quad \omega_3 = x_0$;

8) $f(x) = \exp\left(-\frac{1}{2}im\boldsymbol{\alpha x}/x_0\right)$, $\quad \omega_1 = \boldsymbol{\alpha x} + x_0\boldsymbol{\beta x}$,
$\omega_2 = \boldsymbol{\alpha x} + x_0\boldsymbol{\gamma x}$, $\quad \omega_3 = x_0$;

9) $f(x) = 1$, $\quad \omega_1 = \boldsymbol{\alpha x}$, $\quad \omega_2 = \boldsymbol{\beta x}$, $\quad \omega_3 = x_0$,

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ are constant vectors satisfying the conditions

$$\boldsymbol{\alpha}^2 = \boldsymbol{\beta}^2 = \boldsymbol{\gamma}^2 = 1, \qquad \boldsymbol{\alpha\beta} = \boldsymbol{\beta\gamma} = \boldsymbol{\gamma\alpha} = 0.$$

We adduce the explicit form of the reduced equations for the function φ, obtained from ansatz (11) in all nine cases:

1) $L\varphi + 6\varphi_2 - 2im\varphi_3 + m^2\omega_2\varphi - 2\lambda m|\varphi|^{4/3}\varphi = 0$,
$L\varphi \equiv \varphi_{11} + 4\omega_2\varphi_{22} + (\omega_2-\omega_1^2)^{-1}\varphi_{33} + 4\omega_1\varphi_{12}$,

2) $L\varphi + 6\varphi_2 + 2im\varphi_3 - 2\lambda m|\varphi|^{4/3}\varphi = 0$,

3) $L\varphi + 6\varphi_2 + 2im\varphi_3 - m^2\omega_2\varphi - 2\lambda m|\varphi|^{4/3}\varphi = 0$,

4) $L\varphi + im\omega_1\varphi_1 + 2(im\omega_2+3)\varphi_2 + 2im\varphi_3 + \frac{3}{2}im\varphi - 2\lambda m|\varphi|^{4/3}\varphi = 0$,

5) $\varphi_{11} + \varphi_{22} + \varphi_{33} + im(\omega_1\varphi_1 + \omega_2\varphi_2 + \omega_3\varphi_3) + \frac{3}{2}im\varphi - 2\lambda m|\varphi|^{4/3}\varphi = 0$,

6) $L\varphi + 6\varphi_2 + 2im\varphi_3 - 2\lambda m|\varphi|^{4/3}\varphi = 0$,

7) $\varphi_{11} + 4\omega_2\varphi_{22} + 4\omega_1\varphi_{12} + 6\varphi_2 - 2im\varphi_3 - 2\lambda m|\varphi|^{4/3}\varphi = 0$,

8) $\omega_3\left[(1+\omega_3^2)(\varphi_{11}+\varphi_{22}) + \varphi_{12}\right] - 2im\left(\omega_1\varphi_1 + \omega_2\varphi_2 + \omega_3\varphi_3 + \frac{1}{2}\varphi\right) -$
$-2\lambda m\omega_3|\varphi|^{4/3}\varphi = 0$,

9) $2im\varphi_3 - \varphi_{11} - \varphi_{22} - 2\lambda m|\varphi|^{4/3}\varphi = 0$,

where

$$\varphi_a = \partial\varphi/\partial\omega_a, \qquad \varphi_{ab} = \partial^2\varphi/\partial\omega_a\partial\omega_b, \qquad a,b=\overline{1,3}.$$

We did not succeed in finding the exact solutions of all of the reduced equations. However, some of them had been solved. Let us write the final form of several exact solutions of (10).

$$u(x) = \left(1-x_0^2\right)^{-3/4}\exp\left[\frac{1}{2}im\boldsymbol{x}^2\left(1-x_0\right)^{-1}\right], \qquad \lambda=\frac{3}{2}i. \tag{13}$$

$$u(x) = (c_0x_0-\boldsymbol{c}\boldsymbol{x})^{-3/2}\exp\left\{-\frac{1}{2}im\boldsymbol{x}^2x_0^{-1}\right\}, \tag{14}$$

where c_0, $\boldsymbol{c}=(c_1,c_2,c_3)$ are arbitrary constants, satisfying the condition $\boldsymbol{c}^2=\frac{8}{15}\lambda m$.

$$u(x) = x_0^{-3/2}\exp\left[-\frac{1}{2}im\left(\boldsymbol{x}^2-\boldsymbol{r}\boldsymbol{x}\right)x_0^{-1}\right], \qquad \boldsymbol{r}^2=-8\lambda/m. \tag{15}$$

$$u(x) = \left(\frac{8}{3}\lambda\boldsymbol{x}^2\right)^{-3/4}\exp\left(-\frac{1}{2}im\boldsymbol{x}^2x_0^{-1}\right). \tag{16}$$

$$u(x) = x_0^{-3/2}\varphi(\omega_1)\exp\left(-\frac{1}{2}im\boldsymbol{x}^2x_0^{-1}\right), \qquad \omega_1=\boldsymbol{\alpha}\boldsymbol{x}/x_0, \tag{17}$$

where function $\varphi(\omega_1)$ is defined by the elliptic integral

$$\int_0^\varphi d\tau\left(k_1+\tau^{10/3}\right)^{-1/2} = \left(\frac{6}{5}\lambda m\right)^{1/2}(\omega_1+k_2), \tag{18}$$

where k_1, k_2 are arbitrary constants.

$$u(x) = x_0^{-3/2}\exp\left(-\frac{1}{2}im\boldsymbol{x}^2x_0^{-1}\right)\varphi(\omega_2), \qquad \omega_2=\boldsymbol{x}^2/x_0, \tag{19}$$

where function $\varphi(\omega_2)$ is the solution of the Emden–Fauler equation

$$2\omega_2\varphi_{22}+3\varphi_2-\lambda m\varphi^{7/3}=0. \tag{20}$$

$$u(x) = x_0^{-3/4}\varphi(\omega_1), \qquad \omega_1=(\boldsymbol{\alpha}\boldsymbol{x})x_0^{-1/2}, \tag{21}$$

where function $\varphi(\omega_1)$ is defined by elliptic integral (18).

$$u(x) = \varphi(\omega_2), \qquad \omega_2=\boldsymbol{x}^2, \tag{22}$$

where $\varphi(\omega_2)$ is the solution of (20).

$$u(x) = (c_0/3\lambda)^{3/4}x_0^{-1/2}\exp\left(ic_0x_0^{-1/3}-\frac{1}{2}im\boldsymbol{c}\boldsymbol{x}/x_0\right), \tag{23}$$

where $\boldsymbol{c}^2=1$ and $c_0=\text{const}$.

$$u(x) = (c_0/\lambda)^{3/4}\exp(ic_0x_0), \tag{24}$$

$$u(x) = (\lambda_2 x_0)^{-3/4} \exp\left(-i\lambda_1\lambda_2(\lambda_2 x_0)^{-3/4}\right), \tag{25}$$

where $\lambda = \frac{3}{4}(\lambda_1 + i\lambda_2)$ and λ_1, λ_2 are arbitrary real constants.

$$u(x) = (\boldsymbol{c}\boldsymbol{x})^{-3/2}, \qquad \boldsymbol{c}^2 = \frac{8}{15}\lambda m. \tag{26}$$

Formulae (13)–(26) give multiparameter families of exact solutions of the nonlinear Schrödinger equation (10). Some of them are of non-perturbative type due to a singularity with respect to the coupling constant λ. Obtained solutions may be used in quantum field theory, and in many non-linear problems of solid state and plasma physics.

In conclusion we adduce the formulae of extension of solutions of (10). If $u = u_1(x)$ is a given solution of (10) then the new solutions u_2, u_3 may be found by formulae

$$u_2 = u_1(x_0, \boldsymbol{x} + \boldsymbol{v}x_0) \exp\left[im\left(\frac{1}{2}\boldsymbol{v}^2 x_0 + \boldsymbol{v}\boldsymbol{x}\right)\right],$$

$$u_3 = u_1\left(\frac{x_0}{1-ax_0}, \frac{\boldsymbol{x}}{1-ax_0}\right)(1-ax_0)^{-3/2} \exp\left(\frac{1}{2}im\frac{a\boldsymbol{x}^2}{1-ax_0}\right),$$

where a, $\boldsymbol{v}$ are arbitrary constants. These formulae follow from the fact that (10) admits both groups $G(1,3)$ and $G_2(1,3)$.

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