The symmetry and exact solutions of some multidimensional nonlinear equations of mathematical physics

W.I. FUSHCHYCH

We discuss exact solutions of some nonlinear equations obtained in collaboration with Shtelen W.M., Serov M.I., Zhdanov R.Z. (Institute of Mathematics, Kiev).

We consider the following equations:

— The nonlinear wave equation

$$p_\mu p^\mu u + F_1(u) = 0, \tag{0.1}$$

$u = u(x)$ scalar, $x = (x_0, x_1, \dots, x_n) \in R(1,n)$, $F_1(u)$ twice differentiable, $p_\mu = i\partial/\partial x_\mu$.

— The generalized Monge–Ampere equation

$$\begin{gathered}\det(u_{\mu\nu}) = F_2(x, u, \underset{1}{u}),\\ u_{\mu\nu} = \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \qquad \underset{1}{u} = \left\{ \frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right\},\end{gathered} \tag{0.2}$$

F_2 smooth. With $F_2 = 0$, we get the Monge–Ampere equation used in differential geometry and, especially at present, in quantum field theory.

— The multidimensional hyperbolic analog of the Euler–Lagrange equation for minimal surfaces or the Born–Infeld equation

$$\begin{gathered}\Box u(1 - u_\nu u^\nu) + u_{\mu\nu} u^\mu u^\nu = 0,\\ u_\nu u^\nu = \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \dots - \left(\frac{\partial u}{\partial x_n}\right)^2.\end{gathered} \tag{0.3}$$

— The nonlinear Schrödinger equation

$$\left(p_0 - \frac{p_a^2}{2m}\right) u + F_3(u)u = 0, \tag{0.4}$$

m is a parameter, $u = u(x_0, \dots, x_n)$ is a complex function.

– The nonlinear Dirac equation

$$\left[i\gamma^\mu \partial_\mu + F_4(\bar\Psi\Psi)\right]\Psi = 0, \tag{0.5}$$

γ^μ are 4×4 Dirac matrices, Ψ, $\bar\Psi$ spinors, F_4 smooth and depending on $\Psi\bar\Psi$. The special case

$$\left[i\gamma^\mu \partial_\mu + \lambda \frac{(\bar\Psi\gamma_\mu\Psi)\gamma^\mu}{[(\bar\Psi\gamma_\nu\Psi)\bar\Psi\gamma^\nu\Psi]^{1/3}}\right]\Psi = 0 \tag{0.6}$$

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can be considered as a conformally invariant analog of the Dirac–Heisenberg equation for a spinor field.

If we require these equations to be invariant with respect to a group larger than the Poincaré or the Galilei group, a special form is imposed on F and we can construct a whole family of solutions from known ones from such a symmetry.

We denote by $\tilde{P}(1,n)$ the extended Poincaré group, i.e. the Poincaré group $P(1,n)$ and scale transformations, and by $A\tilde{P}(1,n)$ its Lie algebra.

§ 1. The symmetry

To construct solutions of (0.1)–(0.6) we need to know their symmetry properties.

Theorem 1. *The wave equation (0.1) is invariant under* $\tilde{P}(1,n)$ *iff*

$$F_1(u) = \lambda_1 u^r, \qquad r \neq 1, \tag{1.1}$$

$$F_1(u) = \lambda_2 \exp(u), \tag{1.2}$$

λ_1, λ_2, r *constants.*

The proof is given in [3]. (0.1) is in the case (1.1) and for $r = \frac{n+3}{n-1}$ invariant under the conformal group $C(1,n) \supset P(1,n)$.

Theorem 2. *For* $F_2 = 0$ *equation (0.2) is invariant under* $IGL(1,n+1)$*, the group of linear inhomogeneous transformations of* $R(1,n+1)$*, and* $C(1,n+1)$*, the conformal group of* $R(1,n+1)$*. The basis elements of the corresponding Lie algebra have the form*

$$P_A = ig^{AB}\partial/\partial x_B, \qquad L_{AB} = x_A P_B, \qquad A,B = 0,\dots,n+1, \tag{1.3}$$

$$K_A = x_A D, \qquad D = ix_A P_a, \qquad x_{n+1} \equiv u, \tag{1.4}$$

g^{AB} *is the metric tensor in* $R(1,n+1)$.

The Monge–Ampere equation is invariant, in particular under linear transformations which preserve the quadratic form

$$s^2 = x_0^2 - x_1^2 - \cdots - x_n^2 - u^2$$

containing independing variables x and the depending variable u equally.

Theorem 3. *Equation (0.2) with* $F_2 \neq 0$ *is invariant under* $\tilde{P}(1,n+1)$ *iff*

$$F_2(u) = \lambda(1 - u_\nu u^\nu)^{\frac{n-1}{2}}. \tag{1.5}$$

Theorem 4. *The maximal, in the sense of Lie, invariance group of the equation (0.3) is* $\tilde{P}(1,n+1)$.

Theorem 5. *(0.4) is invariant under the extended Galilei group* $\tilde{G}(1,n)$ *which includes* $G(1,n)$*, scale and projective transformations, iff*

$$F_3 = \lambda |u|^{4/n}. \tag{1.6}$$

where n *is the number of spatial variables.*

Theorem 6. *The nonlinear Dirac equation (0.5) is invariant under the conformal group* $C(1,n)$ *iff*

$$F_4(\bar{\Psi}\Psi) = \lambda(\bar{\Psi}\Psi)^{4/n}. \tag{1.7}$$

All theorems listed above can be proved by Lie's method. The proofs are as a rule cumbersome, so we omit them.

§ 2. Solutions of nonlinear equations

To construct exact solutions of (0.1)–(0.6) we use the symmetry properties of the equations. The solutions in question are multiparametrical and due to their symmetry we use the following ansatz

$$u(x) = f(x)\varphi(\omega) + g(x), \tag{2.1}$$

where $\varphi(\omega)$ an unknown function depending on new variables ($m = n-1$)

$$\omega(x) = \{\omega_1, \omega_2, \ldots, \omega_m\}$$

chosen from the invariants of the symmetry group of the equation. More precisely ω and f, g are determined from the equations

$$\frac{dx_0}{A_0} = \frac{dx_1}{A_1} = \cdots = \frac{dx_n}{A_n} = \frac{du}{B}, \tag{2.2}$$

where A_μ, B are functions defining infinitesimal transformations of the invariance group

$$x'_\mu = x_\mu + \varepsilon A_\mu, \qquad u' = u + \varepsilon B,$$
$$A_\mu = c_{\mu\nu}x^\nu + d_\mu, \qquad B = au + b,$$

where $c_{\mu\nu}$, d_μ, a, b are group parameters. Variables ω are just the first integrals of (2.2).

In the special case that φ depends on one variable, the partial differential equation for u reduces to an ordinary differential equation for φ. Solutions of this ODE give through (2.1) solutions of the original PDE.

Below we list some simple solutions of (0.1)–(0.6).

1. *The nonlinear wave equation*

$$\Box u + \lambda u^r = 0, \qquad r \neq 1, \tag{2.3}$$

$$u(x) = \left\{ -\frac{\lambda}{2}\left(1-k^2\right)\left[(\beta_\nu y^\nu)^2 + y^\nu y_\nu\right] \right\}^{\frac{1}{1-k}}, \qquad \beta^\nu\beta_\nu = -1, \qquad y_\nu = x_\nu + a_\nu; \tag{2.4}$$

$$u(x) = \left\{ \frac{\lambda}{2}\left(1-k^2\right)\alpha_\nu y^\nu \beta^\sigma y_\sigma \right\}^{\frac{1}{1-k}}, \qquad \alpha^\nu\alpha_\nu = \beta^\nu\beta_\nu = 0, \qquad \alpha^\nu\beta_\nu = -1; \tag{2.5}$$

$$u(x) = \left[F(\alpha_\nu x^\nu) + \beta^\nu x_\nu\right]^{\frac{2}{1-k}}, \qquad \beta_\nu\beta^\nu = -\frac{\lambda}{2}\frac{(1-k)^2}{1+k} \neq 0, \tag{2.6}$$

F arbitrary smooth, a_ν, α_ν, β_ν are constants satisfying the above conditions. (2.4)–(2.6) give a family of solutions of equation (2.3). As it is seen from (2.4)–(2.6), for

$k > 1$ the solutions have a singularity at $\lambda = 0$ and cannot be obtained by a standard perturbation method.

2. *Solutions of the Monge–Ampere equation*

For $\det(u_{\mu\nu}) = 0$ arbitrary smooth functions

$$u = \varphi(\omega_1, \omega_2, \ldots, \omega_{n-1}), \qquad \omega_k = \alpha_\nu^k x^\nu, \tag{2.7}$$

$\alpha^k = (\alpha_0^k, \ldots, \alpha_n^k)$ arbitrary constant vectors, are solutions. Additional solutions in explicit and in implicit form are

$$u = (\alpha_\mu x^\mu)^2 - \alpha^2 x^2, \qquad \alpha^2 \equiv \alpha_0^2 - \alpha_1^2 - \cdots - \alpha_n^2; \tag{2.8}$$

$$u = x^2/(\alpha \cdot x), \qquad \alpha \cdot x \equiv \alpha_\nu x^\nu; \tag{2.9}$$

$$\alpha_\nu x^\nu - \alpha_{n+1} u = \varphi_2(\beta_\nu x^\nu - \beta_{n+1} u), \tag{2.10}$$

φ_2 is smooth, β_ν are parameters;

$$u = \sigma^{-1}(x,u)\left[(\alpha \cdot x)^2 - \alpha^2 x^2\right], \qquad \sigma(x,u) = 1 + b_\mu x^\mu - b_{n+1} u. \tag{2.11}$$

3. *Solutions of the generalized Euler–Lagrange equation*

For $\Box u(1 - u_\nu u^\nu) + u_{\mu\nu} u^\mu u^\nu = 0$ the function

$$u = \varphi(\alpha_\nu x^\nu) + \beta_\nu x^\nu \tag{2.12}$$

is a solution, where φ is smooth, and the parameters satisfy the following conditions

$$\alpha_\nu \beta^\nu + \alpha_\nu \alpha^\nu (1 - \beta_\sigma \beta^\sigma) = 0.$$

A solution in implicit form is

$$\begin{gathered} \alpha_\nu x^\nu - \alpha_{n+1} u = \varphi(\beta_\nu x^\nu - \beta_{n+1} u), \\ \left(\alpha_\nu \alpha^\nu - \alpha_{n+1}^2\right)\left(\beta_\sigma \beta^\sigma - \beta_{n+1}^2\right) - (\alpha_\mu \beta^\mu - \alpha_{n+1}\beta_{n+1})^2 = 0. \end{gathered} \tag{2.13}$$

4. *Solutions of the nonlinear Dirac equation*

$$\left[i\gamma^\mu \partial_\mu + \lambda(\bar{\Psi}\Psi)^k\right] = 0. \tag{2.14}$$

Consider the case $k = 1/3$, then eq. (2.14) is invariant under $C(1,3)$. To reduce eq. (2.14) to the system of ODE we use the ansatz

$$\Psi(x) = A(x)\varphi(\omega), \tag{2.15}$$

where $A(x)$ is a 4×4 matrix, $\varphi(\omega)$ is a four-component function, depending on one invariant variable ω. More specifically

$$A(x) = (\gamma^\nu x_\nu)(x_\mu x^\mu)^{-2}, \tag{2.16}$$

$$\omega = \beta^\nu x_\nu (x_\mu x^\mu)^{-1}, \qquad \beta^\nu \beta_\nu > 0. \tag{2.17}$$

(2.15)–(2.17) reduces (2.14) to the following system of ODE

$$\frac{d\varphi}{d\omega} = i\lambda(\beta^\nu \beta_\nu)^{-1}(\bar{\varphi}\varphi)^{1/3}(\gamma \cdot \beta)\varphi. \tag{2.18}$$

Solving eq. (2.18), we get the following solution of (2.14)

$$\Psi(x) = (\gamma\cdot x)(x^\nu x_\nu)^{-2}\exp\{i\lambda\varkappa(\gamma\cdot\beta)\omega)\chi = \\ = (\gamma\cdot x)(x^\nu x_\nu)^{-2}\left[\cos(\lambda\varkappa\beta\omega) + i\frac{\gamma\cdot\beta}{\beta}\sin(\lambda\varkappa\beta\omega)\right]\chi, \tag{2.19}$$

where χ is a constant spinor, $\beta=(\beta^\nu\beta_\nu)^{1/2}$, $\varkappa=(\bar\chi\chi)^{1/3}(\beta^\nu\beta_\nu)^{-1}$. (2.19) is conformally invariant.

In the same way solutions of nonlinear Schrödinger, Navier–Stokes, Liouville equations have been constructed [1, 2, 3, 6]. We can even solve PDE's which are noninvariant with respect to $P(1,3)$, e.g.

$$\Box u = \left(\frac{\lambda_0}{x_0}\right)^2\left(\frac{\partial u}{\partial x_0}\right)^2 + \left(\frac{\lambda_1}{x_1}\right)^2\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\lambda_2}{x_2}\right)^2\left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\lambda_3}{x_3}\right)^2\left(\frac{\partial u}{\partial x_3}\right)^2, \tag{2.20}$$

where λ_0, λ_1, λ_2, λ_3 are parameters, $x_\mu\neq 0$. This reduces with the Lorentz-invariant ansatz $u=\varphi(x^2)$ to an ODE

$$\omega\frac{d^2\varphi}{d\omega^2} + 2\frac{d\varphi}{d\omega} = \lambda^2\left(\frac{d\varphi}{d\omega}\right)^2, \qquad \omega = x^2 \equiv x^\nu x_\nu.$$

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