On some exact solutions of a system of non-linear differential equations for spinor and vector fields

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The problem of finding ansätze for a non-linear Dirac equation which is invariant under the extended Poincaré group is solved. With the help of these ansätze some multi-parameter families of exact solutions of non-linear Dirac and Dirac-Maxwell equations are constructed.

1. Introduction

In the present work using ideas and methods of S. Lie (see [12, 2]) we have constructed large classes of exact solutions of the non-linear Dirac equation

$$\left(\gamma_\mu p^\mu + \lambda(\bar\psi\psi)^{1/2k}\right)\psi(x) = 0, \quad k \neq 0, \tag{1.1}$$

where γ_μ are 4×4 Dirac matrices, $p_\mu = ig_{\mu\nu}\partial/\partial x_\nu$, $\bar\psi = \psi^\dagger\gamma_0$, $x = (x_0, x_1, x_2, x_3)$, ψ is a four-component spinor and k, λ are parameters, and of the system of eight non-linear equations,

$$\begin{aligned} &(\gamma_\mu p^\mu + \lambda_1\gamma_\mu\mathcal{A}^\mu + m_1)\psi(x) = 0, \\ &p_\nu p^\nu\mathcal{A}_\mu - p_\mu p^\nu\mathcal{A}_\nu = \exp(\bar\psi\gamma_\mu\psi) + \mathcal{A}_\mu(m_2 + \lambda_2\mathcal{A}^\nu\mathcal{A}_\nu), \end{aligned} \tag{1.2}$$

where $\mathcal{A}_\mu(x)$ is the vector potential of the electromagnetic field and e, λ_1, λ_2, m_1, m_2 are constants. If we choose $m_2 = \lambda_2 = 0$, then system (1.2) coincides with equations of the classical electrodynamics describing interaction of electromagnetic and spinor fields.

To construct multiparameter families of exact solutions of (1.1) and (1.2) we essentially use their symmetry properties and the ansatz

$$\psi(x) = A(x)\varphi(\boldsymbol\omega) + B(x) \tag{1.3}$$

suggested by Fushchych [3, 4] and effectively realised by Fushchych and Shtelen [6, 7] and Fushchych and Serov [5] for a number of non-linear wave equations. $A(x)$ is a 4×4 matrix and $B(x)$ is a four-component spinor, algorithms for their construction being cited below, and $\varphi(\boldsymbol\omega)$ is the column vector, components of which depend in general on three invariant variables $\boldsymbol\omega = \{\omega_1, \omega_2, \omega_3\}$ (for more details see Fushchych [3, 4]). Later we shall consider the case when $B(x) = 0$.

On using finite transformations it is established that equation (1.1) is invariant under the extended Poincaré group $\tilde{\mathcal{P}}(1,3)$, i.e. under the Poincaré group $\mathcal{P}(1,3)$ supplemented by a group of scale transformations.

Basis elements of the Lie algebra $A\tilde{\mathcal{P}}(1,3)$ have the form

$$\begin{aligned} &P_\mu = p_\mu, \quad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \\ &D = x_\mu p^\mu - ik, \quad S_{\mu\nu} = (i/4)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu), \quad \mu,\nu = \overline{0,3}. \end{aligned} \tag{1.4}$$

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A general scheme for constructing solutions of the system (1.1) (solutions of the system (1.2) are obtained in an analogous way) is as follows. We look for solutions of equation (1.1) which are invariant under the subgroup of the group $\tilde{\mathcal{P}}(1,3)$ generated by linear combination of all basis elements of $A\tilde{\mathcal{P}}(1,3)$

$$Q = C^{\mu\nu}J_{\mu\nu} + C^{00}D + C^{\mu}P_{\mu}, \tag{1.5}$$

where $C^{\mu\nu}$, C^{00}, C^{μ} are constants and $\mu,\nu=\overline{0,3}$.

The matrix $A(x)$ is a solution of the following system of partial differential equations (PDE):

$$QA(x) = 0. \tag{1.6}$$

Invariant variables are the first integrals of the Euler–Lagrange system of ordinary differential equations (ODE)

$$\frac{dx_0}{\xi^0(x)} = \frac{dx_a}{\xi^a(x)}, \quad a=\overline{1,3}, \tag{1.7}$$

where $\xi^{\mu} = C^{\mu\nu}x_{\nu} + C^{00}x^{\mu} + C^{\mu}$.

If one knows an explicit form of the matrix $A(x)$ then after substituting (1.3) into the corresponding equation we shall obtain an equation for a spinor $\varphi(\boldsymbol{\omega})$ depending on three invariant variables $\{\omega_1,\omega_2,\omega_3\}$ only. This means that ansatz (1.3) with the chosen matrix $A(x)$ provides separation of variables in equation (1.1). Solutions of the corresponding equation for $\varphi(\boldsymbol{\omega})$ being substituted in (1.3) yield the solutions of the initial equation.

To realise this scheme it is necessary first of all to construct in an explicit form matrices $A(x)$ satisfying (1.6). So one has to solve the first-order linear system of 16 PDE with variable coefficients. It is rather difficult to solve such a system by standard methods, which is why we use the following trick. The operator Q is transformed into another operator

$$Q' = WQW^{-1} \tag{1.8}$$

with the help of the invertible operator

$$W(x,p) = \exp(\theta\Sigma), \quad W^{-1}(x,p) = \exp(-\theta\Sigma), \tag{1.9}$$

where

$$\Sigma = \theta^{\mu\nu}J_{\mu\nu} + \theta^{00}D + \theta^{\mu}P_{\mu}. \tag{1.10}$$

Transformation W is so chosen that operator Q' is as simple as possible. This purpose can always be achieved because of the Poincaré invariance of system (1.1). From the physical point of view this means that the non-linear Dirac equation is solved in the fixed reference system. The construction of the solutions which do not depend on the reference system (ungenerable solutions) is the next step.

2. Construction of the matrix $A(x)$

Before proceeding with a direct solution of the system (1.6) let us simplify it using the method described in the introduction. To do this we need the Campbell–Hausdorff formula

$$\exp(\theta Q_1)Q_2\exp(-\theta Q_1)=\sum_{k=0}^{\infty}\frac{\theta^k}{k!}\{Q_1,Q_2\}^k,$$
$$\{Q_1,Q_2\}^0=Q_2,\quad \{Q_1,Q_2\}^n=[Q_1,\{Q_1,Q_2\}^{n-1}], \tag{2.1}$$

where Q_1, Q_2 are operators and $[A,B]=AB-BA$.

A fundamental role is played by the following lemma.

Lemma. *The operator $Q=C^{\mu\nu}J_{\mu\nu}=A_kM_k+B_lN_l$, where $M_k=-\frac{1}{2}\varepsilon_{klm}J_{lm}$, $N_k=J_{0k}$, by a transformation $Q\to Q'=VQV^{-1}$, where $V=\exp(\theta^{\mu\nu}J_{\mu\nu}$, can be reduced to one of the following forms:*

$$(i)\quad Q'=\alpha J_{01}+\beta J_{23},\quad (\boldsymbol{A}\cdot\boldsymbol{B})^2+\left(\boldsymbol{A}^2-\boldsymbol{B}^2\right)^2\neq 0,$$
$$(ii)\quad Q'=\alpha(J_{01}+J_{12}),\quad \boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{A}^2-\boldsymbol{B}^2=0.$$

Proof. Let us introduce new operators

$$J_a=(i/2)(M_a+iN_a),\quad K_a=(i/2)(M_a-iN_a),\quad a=\overline{1,3}.$$

One can easily check that the following commutational relations hold:

$$[J_a,J_b]=i\varepsilon_{abc}J_c,\quad [K_a,K_b]=i\varepsilon_{abc}K_c,\quad [J_a,K_b]=0 \tag{2.2}$$

so $Q=a_kJ_k+b_lK_l$, where $a_k=-B_k-iA_k$ and $b_l=B_l-iA_l$.

Using (2.1) and (2.2) one obtains

$$Q'=V_1QV_1^{-1}=\left(a_1^2+a_2^2+a_3^2\right)^{1/2}J_1+\left[\left(a_1^2+a_2^2+a_3^2\right)^{1/2}\right]^*K_1=\alpha J_{01}+\beta J_{23},$$

where

$$V_1=\exp\left[-i\tan^{-1}(a_2/a_3)J_1\right]\exp\left\{i\tan^{-1}\left[a_1\left(a_2^2+a_3^2\right)^{-1/2}+\pi/2\right]J_2\right\}\times$$
$$\times\exp\left[-i\tan^{-1}(b_2/b_3)K_1\right]\exp\left\{i\tan^{-1}\left[b_1\left(b_2^2+b_3^2\right)^{-1/2}+\pi/2\right]K_2\right\}. \tag{2.3}$$

It is evident that these formulae lose their validity in the case

$$a_1^2+a_2^2+a_3^2=0\ \Leftrightarrow\ \boldsymbol{A}^2=\boldsymbol{B}^2,\quad \boldsymbol{A}\cdot\boldsymbol{B}=0.$$

Therefore one can use this approach only in case (i). Let us now consider case (ii). It follows from (2.1) that

$$\exp(\theta M_a)A_kM_k\exp(-\theta M_a)=$$
$$=A_kM_k\cos\theta+A_aM_a(1-\cos\theta)+\varepsilon_{akl}A_kM_l\sin\theta \tag{2.4}$$

(no summation is performed over a),

$$\exp(\theta M_a)B_lN_l\exp(-\theta M_a)=$$
$$=B_lN_l\cos\theta+B_aN_a(1-\cos\theta)+\varepsilon_{akl}B_kN_l\sin\theta \tag{2.5}$$

(no summation is performed over a).

Using identities (2.4) and (2.5), one can be convinced that the following equality holds:

$$Q' = V_2 Q V_2^{-1} = V_2(A_k M_k + B_l N_l)V_2^{-1} = -|\boldsymbol{A}|\,\mathrm{sgn} A_3(J_{01} + J_{12}),$$

where

$$V_2 = \exp\left[\tan^{-1}(A_1/A_2)M_3\right] \exp\left\{\tan^{-1}\left[\left(A_1^2 + A_2^2\right)^{1/2}/A_3\right] M_1\right\} \times$$
$$\times \exp\left[\left[\left\{\tan^{-1}\left[B_3|\boldsymbol{A}|/(B_2A_1 - B_1A_2)\right] + \pi\theta(B_1A_2 - B_2A_1)\right\} M_3\right]\right],$$
$$\mathrm{sgn}\, x = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0, \end{cases} \qquad \theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

This completes the proof. Let us prove the main statement.

Theorem. *The operator $Q = A_k M_k + B_l N_l + C^{00} D + C^\mu P_\mu$ with the help of transformation (1.8) can be reduced to one of the following forms:*

(A) $\qquad \boldsymbol{A}\cdot\boldsymbol{B} = 0, \quad \boldsymbol{A}^2 = \boldsymbol{B}^2,$

(i) $Q' = J_{01} + J_{12} + aD,$ (2.6)

(ii) $Q' = J_{01} + J_{12} + \beta P_3 - P_0,$ (2.7)

(iii) $Q' = J_{01} + J_{12} + \beta P_3,$ (2.8)

(B) $\qquad (\boldsymbol{A}\cdot\boldsymbol{B})^2 + \left(\boldsymbol{A}^2 - \boldsymbol{B}^2\right)^2 \neq 0,$

(iv) $Q' = J_{23} + aD,$ (2.9)

(v) $Q' = J_{01} + bJ_{23} + aD,$ (2.10)

(vi) $Q' = J_{01} + bJ_{23} + D + \beta P_0,$ (2.11)

(vii) $Q' = J_{01} + P_2,$ (2.12)

(viii) $Q' = J_{23} + \alpha_1 P_0 + \alpha_2 P_1,$ (2.13)

(C) $\qquad \boldsymbol{A} = \boldsymbol{B} = 0,$

(ix) $Q' = D,$ (2.14)

(x) $Q' = P_0 + P_1,$ (2.15)

(xi) $Q' = P_0,$ (2.16)

(xii) $Q' = P_1.$ (2.17)

Proof. If $\boldsymbol{A} \neq 0$, $\boldsymbol{B} \neq 0$ then it follows from the lemma that there exists an operator V_1 (V_2) of the form (1.9) such that

(a) under $\boldsymbol{A}\cdot\boldsymbol{B} = \boldsymbol{A}^2 - \boldsymbol{B}^2 = 0,$
$V_1 Q V_1^{-1} = \alpha(J_{01} + J_{12}) + \theta D + \theta^\mu P_\mu,$

(b) under $(\boldsymbol{A}\cdot\boldsymbol{B})^2 + \left(\boldsymbol{A}^2 - \boldsymbol{B}^2\right)^2 \neq 0,$
$V_2 Q V_2^{-1} = \alpha J_{01} + \beta J_{23} + \theta D + \theta^\mu P_\mu.$

It is clear from (1.6) and (1.7) that operators Q and αQ, $\alpha \neq 0$, generate the same invariant solutions. One may suppose that $\alpha = 1$.

We need the following formulae which are consequences of the Campbell–Hausdorff formula:

$$\exp(i\lambda^\mu P_\mu) J_{\alpha\beta} \exp(-i\lambda^\mu P_\mu) = J_{\alpha\beta} + (\lambda_\beta P_\alpha - \lambda_\alpha P_\beta), \tag{2.18}$$

$$\exp(i\lambda^\mu P_\mu) D \exp(-i\lambda^\mu P_\mu) = D - \lambda^\mu P_\mu, \tag{2.19}$$

$$\exp(i\lambda^\mu P_\mu) P_\alpha \exp(-i\lambda^\mu P_\mu) = P_\alpha. \tag{2.20}$$

Let us consider the case (a):

$$Q' \to Q'' = \exp(i\lambda^\mu P_\mu)(J_{01} + J_{12} + \theta D + \theta^\alpha P_\alpha)(\exp(-i\lambda^\mu P_\mu) =$$
$$= J_{01} + J_{12} + \theta D + \theta^\mu P_\mu + \lambda_1 P_0 - \lambda_2 P_1 - \lambda_1 P_2 - \theta\lambda^\alpha P_\alpha.$$

Under $\theta \neq 0$ one can always choose λ_α that

$$Q'' = J_{01} + J_{12} + \theta D$$

and under $\theta = 0$ so that

$$Q'' = J_{01} + J_{12} + \alpha P_0 + \beta P_3, \quad \alpha \leq 0.$$

If in the last operator $\alpha \neq 0$, then

$$Q''' = \exp(-i\ln|\alpha| D)(J_{01} + J_{12} + \alpha P_0 + \beta P_3)\exp(i\ln|\alpha| D) =$$
$$= J_{01} + J_{12} - P_0 + \beta P_3.$$

If $\alpha = 0$ then

$$Q'' = J_{01} + J_{12} + \beta P_3.$$

Let us now consider case (b). If $\alpha \neq 0$ then on dividing into α and on transforming the operator Q according to (2.18)–(2.20) we obtain

$$Q' = \exp(i\lambda^\mu P_\mu)(J_{01} + bJ_{23} + \theta D + \theta^\mu P_\mu)\exp(-i\lambda^\mu P_\mu) =$$
$$= J_{01} + (\lambda_1 P_0 - \lambda_0 P_1) + bJ_{23} + b(\lambda_3 P_2 - \lambda_2 P_3) + \theta D - \theta\lambda^\mu P_\mu + \theta^\mu P_\mu.$$

Under $\theta \neq \pm 1$, $\theta^2 + b^2 \neq 0$ it is always possible to choose λ_μ so that

$$Q' = J_{01} + bJ_{23} + \theta D.$$

Under $\theta = \pm 1$ it is possible to choose λ_μ so that

$$Q' = J_{01} + bJ_{23} + \delta D + \beta P_0.$$

Under $\theta = b = 0$ there exist such λ_μ that

$$Q' = J_{01} + P_2.$$

Under $\alpha = 0$ using formulae (2.18)–(2.20) one can check that the operator Q can be reduced to one of the following forms:

$$Q' = J_{23} + aD, \quad \theta \neq 0,$$
$$Q' = J_{23} + \alpha_1 P_0 + \alpha_2 P_1, \quad \theta = 0.$$

The only thing left is to consider the case $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{0}$, i.e. $Q = \theta D + \theta^\mu P_\mu$. Using formulae (2.18)–(2.20) it is easy to be convinced that under $\theta = 0$

$$\exp[(i/\theta)\theta^\mu P_\mu](\theta D + \theta^\mu P_\mu)\exp[-(i/\theta)\theta^\mu P_\mu] = \theta D.$$

If $\theta = 0$ then analysing three possibilities $\theta_\mu\theta^\mu = 0$, $\theta_\mu\theta^\mu > 0$, $\theta_\mu\theta^\mu < 0$ we obtain operators (2.15)–(2.17). The theorem is proved.

Note 1. When proving the theorem we used only commutational relations of an algebra $A\tilde{\mathcal{P}}(1,3)$ and we did not use its concrete representation.

Note 2. It is seen from the proof that $\tilde{\mathcal{P}}(1,3)$-invariant solutions are exhausted by solutions generated from ones invariant under operators (2.6)–(2.17) with the help of transformations from $\tilde{\mathcal{P}}(1,3)$.

This theorem essentially simplifies the problem of finding ansätze because instead of integrating the system (1.6) where Q is an operator of the general form (1.5), it is enough to find a partial solution of this system with Q having the form (2.6)–(2.17).

For example, let us consider case (2.9). The matrix $A(x)$ is a solution of the following matrix system of PDE

$$x_2 A_{x_3} - x_3 A_{x_2} + \frac{1}{2}\gamma_2\gamma_3 A + a x_\mu A_{x_\mu} - akA = 0, \tag{2.21}$$

where $A_{x_a} = \partial A/\partial x_a$, $a = \overline{0,3}$.

We look for a partial solution of (2.21) of the form

$$A(x) = f(x)\exp(g(x)\gamma_2\gamma_3). \tag{2.22}$$

Substituting (2.22) into (2.21) we obtain

$$\left[x_2 f_{x_3} - x_3 f_{x_2} + a x_\mu f_{x_\mu} - akf + f\left(x_2 g_{x_3} - x_3 g_{x_2} + a x_\mu g_{x_\mu} + \frac{1}{2}\right)\gamma_2\gamma_3\right]\times$$
$$\times\exp(g(x)\gamma_2\gamma_3) = 0.$$

A partial solution of the last system is given by formulae

$$f(x) = \left(x_2^2 + x_3^2\right)^{-k/2}, \quad g(x) = -\frac{1}{2}\tan^{-1}(x_2/x_3).$$

Finally

$$A(x) = \exp\left[-\frac{1}{2}\gamma_2\gamma_3\tan^{-1}(x_2/x_3)\right]\left(x_2^2 + x_3^2\right)^{-k/2}.$$

In the same way we have obtained matrices $A(x)$ which correspond to operators (2.6)–(2.17)

$$\text{(i)}\ a \neq 0,\ A(x) = (x_0 - x_2)^{-k}\exp\left[\frac{1}{2}a^{-1}\gamma_1(\gamma_0 - \gamma_2)\ln(x_0 - x_2)\right], \tag{2.23}$$

$$a = 0,\ A(x) = \exp\left[\frac{1}{2}x_1(x_0 - x_2)^{-1}\gamma_1(\gamma_0 - \gamma_2)\right], \tag{2.24}$$

$$\text{(ii)}\ A(x) = \exp\left[\frac{1}{2}\gamma_1(\gamma_2 - \gamma_0)(x_2 - x_0)\right], \tag{2.25}$$

$$\text{(iii)}\ A(x) = \exp\left[\frac{1}{2}\beta^{-1}\gamma_1(\gamma_2 - \gamma_0)x_3\right], \tag{2.26}$$

$$\text{(iv)}\ A(x) = \left(x_2^2 + x_3^2\right)^{-k/2} \exp\left[-\frac{1}{2}\gamma_2\gamma_3 \tan^{-1}(x_2/x_3)\right], \tag{2.27}$$

$$\begin{aligned}\text{(v)}\ a \neq -1,\ A(x) = \left(x_0^2 - x_1^2\right)^{-k/2} \times \\ \times \exp\left[\frac{1}{2}(a+1)^{-1}\gamma_0\gamma_1 \ln(x_0 + x_1) - \frac{1}{2}\gamma_2\gamma_3 \tan^{-1}(x_2/x_3)\right],\end{aligned} \tag{2.28}$$

$$\begin{aligned}a = -1,\ A(x) = \left(x_0^2 - x_1^2\right)^{-k/2} \times \\ \times \exp\left[-\frac{1}{4}\gamma_0\gamma_1 \ln(x_0 - x_1) - \frac{1}{2}\gamma_2\gamma_3 \tan^{-1}(x_2/x_3)\right],\end{aligned} \tag{2.29}$$

$$\begin{aligned}\text{(vi)}\ A(x) = (2x_0 + 2x_1 + \beta)^{-k/2} \times \\ \times \exp\left[\frac{1}{4}\gamma_0\gamma_1 \ln(2x_0 + 2x_1 + \beta) - \frac{1}{2}\tan^{-1}(x_2/x_3)\gamma_2\gamma_3\right],\end{aligned} \tag{2.30}$$

$$\text{(vii)}\ A(x) = \exp\left[\frac{1}{2}\gamma_0\gamma_1 \ln(x_0 + x_1)\right], \tag{2.31}$$

$$\text{(viii)}\ A(x) = \exp\left[-\frac{1}{2}\gamma_2\gamma_3 \tan^{-1}(x_2/x_3)\right], \tag{2.32}$$

$$\text{(ix)}\ A(x) = x_0^{-k} I, \tag{2.33}$$

$$\text{(x)}\ A(x) = I, \tag{2.34}$$

$$\text{(xi)}\ A(x) = I, \tag{2.35}$$

$$\text{(xii)}\ A(x) = I, \tag{2.36}$$

where I is a unit 4×4 matrix.

3. Ansätze for the non-linear Dirac equation (1.1)

As pointed out in the introduction, to find invariant variables $\omega_1(x)$, $\omega_2(x)$, $\omega_3(x)$ it is necessary to find all the first integrals of the Euler–Lagrange system of ODE

$$\frac{dx_\mu}{d\tau} = C_{\mu\nu}x^\nu + C_{00}x_\mu + C_\mu. \tag{3.1}$$

Because of the lemma proved above, one can restrict oneself to the following cases of the system (3.1):

(i) $C_{01} = -C_{12} = 1$, $C_{00} = a$, rest coefficients are equal to 0,

(ii) $C_{01} = -C_{12} = 1$, $C_0 = -1$,
$C_3 = -\beta$, rest coefficients are equal to 0,

(iii) $C_{01} = -C_{12} = 1$, $C_3 = -\beta$, rest coefficients are equal to 0,

(iv) $C_{23} = -1$, $C_{00} = a$, rest coefficients are equal to 0,

(v) $C_{01} = 1$, $C_{23} = -b$, $C_{00} = a$, rest coefficients are equal to 0,

(vi) $C_{01} = 1$, $C_{23} = -b$, $C_{00} = 1$,
$C_0 = \beta$, rest coefficients are equal to 0,

(vii) $C_{01} = 1$, $C_2 = -1$, rest coefficients are equal to 0,

(viii) $C_{23}=-1, \quad C_0=\alpha_1, \quad C_1=-\alpha_2,$ rest coefficients are equal to 0,
(ix) $C_{\mu\nu}=0, \quad C_{00}=1, \quad C_\mu=0,$
(x) $C_{\mu\nu}=C_{00}=0, \quad C_0=-C_1=1, \quad C_2=C_3=0,$
(xi) $C_{\mu\nu}=C_{00}=0, \quad C_1=C_2=C_3=0, \quad C_0=1,$
(xii) $C_{\mu\nu}=C_{00}=0, \quad C_0=C_2=C_3=1, \quad C_1=-1.$

Solution of the system (3.1) in cases (i)–(xii) above is carried out in the usual way, so we write down its first integrals omitting intermediate calculations.

$$\text{(i) } a\neq 0,\ \omega_1=\left(x_0^2-x_1^2-x_2^2\right)x_3^{-2},\ \omega_2=(x_0-x_2)x_3^{-1}, \quad \omega_3=ax_1(x_0-x_2)^{-1}-\ln(x_0-x_2), \tag{3.2}$$

$$a=0,\ \omega_1=x_0-x_2,\ \omega_2=x_3, \quad \omega_3=x_0^2-x_1^2-x_2^2, \tag{3.3}$$

$$\text{(ii) } \omega_1=x_3+\beta(x_0-x_2),\ \omega_2=2x_1+(x_0-x_2)^2, \quad \omega_3=3x_3+3x_1(x_0-x_2)+(x_0-x_2)^3, \tag{3.4}$$

$$\text{(iii) } \omega_1=x_0-x_2,\ \omega_2=x_0^2-x_1^2-x_2^2,\ \omega_3=\beta x_1-(x_0-x_2)x_3, \tag{3.5}$$

$$\text{(iv) } \omega_1=x_0x_1^{-1},\ \omega_2=\ln\left(x_2^2+x_3^2\right)+2a\tan^{-1}(x_2/x_3), \quad \omega_3=\left(x_2^2+x_3^2\right)(x_0x_1)^{-1}, \tag{3.6}$$

$$\text{(v) } a\neq -1,\ \omega_3=b\ln\left(x_2^2+x_3^2\right)+2a\tan^{-1}(x_2/x_3), \quad \omega_1=(x_0+x_1)^{2a}\left(x_0^2-x_1^2\right)^{-(a+1)},\ \omega_2=\left(x_0^2-x_1^2\right)\left(x_2^2+x_3^2\right)^{-1}, \tag{3.7}$$

$$a=-1,\ \omega_1=x_0+x_1,\ \omega_2=\left(x_0^2-x_1^2\right)\left(x_2^2+x_3^2\right)^{-1}, \quad \omega_3=b\ln\left(x_2^2+x_3^2\right)-2\tan^{-1}(x_2/x_3), \tag{3.8}$$

$$\text{(vi) } \omega_1=(2x_0+2x_1+\beta)\exp[2\beta^{-1}(x_1-x_0)], \quad \omega_2=(2x_0+2x_1+\beta)\left(x_2^2+x_3^2\right)^{-1}, \quad \omega_3=b\ln\left(x_2^2+x_3^2\right)+2\tan^{-1}(x_2/x_3), \tag{3.9}$$

$$\text{(vii) } \omega_1=x_0^2-x_1^2,\ \omega_2=\ln(x_0+x_1)-x_2,\ \omega_3=x_3, \tag{3.10}$$

$$\text{(viii) } \omega_1=x_2^2+x_3^2,\ \omega_2=\tan^{-1}(x_2/x_3)+\beta_1x_0+\beta_2x_1, \quad \omega_3=\alpha_2x_0+\alpha_1x_1,\ -\alpha_1\beta_1+\alpha_2\beta_2=1, \tag{3.11}$$

$$\text{(ix) } \omega_a=x_ax_0^{-1},\ a=\overline{1,3}, \tag{3.12}$$

$$\text{(x) } \omega_1=x_0+x_1,\ \omega_2=x_2,\ \omega_3=x_3, \tag{3.13}$$

$$\text{(xi) } \omega_a=x_a,\ a=\overline{1,3}, \tag{3.14}$$

$$\text{(xii) } \omega_1=x_0,\ \omega_2=x_2,\ \omega_3=x_3. \tag{3.15}$$

Now substituting (2.23)–(2.36) and (3.2)–(3.15) into (1.3) under $B(x) = 0$ we obtain the following set of ansätze for the non-linear Dirac equation (1.1):

$$\text{(i) } \psi(x) = (x_0 - x_2)^{-k} \exp\left[\frac{1}{2} a^{-1} \gamma_1 (\gamma_2 - \gamma_0) \ln(x_0 - x_2)\right] \varphi(\boldsymbol{\omega}), \tag{3.16}$$

$$\psi(x) = \exp\left[\frac{1}{2} x_1 (x_0 - x_2)^{-1} \gamma_1 (\gamma_2 - \gamma_0)\right] \varphi(\boldsymbol{\omega}), \tag{3.17}$$

$$\text{(ii) } \psi(x) = \exp\left[\frac{1}{2} \gamma_1 (\gamma_2 - \gamma_0)(x_0 - x_2)\right] \varphi(\boldsymbol{\omega}), \tag{3.18}$$

$$\text{(iii) } \psi(x) = \exp\left[\frac{1}{2} \beta^{-1} \gamma_1 (\gamma_2 - \gamma_0) x_3\right] \varphi(\boldsymbol{\omega}), \tag{3.19}$$

$$\text{(iv) } \psi(x) = \left(x_2^2 + x_3^2\right)^{-k/2} \exp\left[-\frac{1}{2} \gamma_2 \gamma_3 \tan^{-1}(x_2/x_3)\right] \varphi(\boldsymbol{\omega}), \tag{3.20}$$

$$\begin{aligned} \text{(v) } \psi(x) &= \left(x_0^2 - x_1^2\right)^{-k/2} \times \\ &\times \exp\left[\frac{1}{2}(a+1)^{-1} \gamma_0 \gamma_1 \ln(x_0 + x_1) - \frac{1}{2} \gamma_2 \gamma_3 \tan^{-1}(x_2/x_3)\right] \varphi(\boldsymbol{\omega}), \end{aligned} \tag{3.21}$$

$$\begin{aligned} \psi(x) &= \left(x_0^2 - x_1^2\right)^{-k/2} \times \\ &\times \exp\left[-\frac{1}{4} \gamma_0 \gamma_1 \ln(x_0 - x_1) - \frac{1}{2} \gamma_2 \gamma_3 \tan^{-1}(x_2/x_3)\right] \varphi(\boldsymbol{\omega}), \end{aligned} \tag{3.22}$$

$$\begin{aligned} \text{(vi) } \psi(x) &= (2x_0 + 2x_1 + \beta)^{-k/2} \times \\ &\times \exp\left[\frac{1}{4} \gamma_0 \gamma_1 \ln(2x_0 + 2x_1 + \beta) - \frac{1}{2} \gamma_2 \gamma_3 \tan^{-1}(x_2/x_3)\right] \varphi(\boldsymbol{\omega}), \end{aligned} \tag{3.23}$$

$$\text{(vii) } \psi(x) = \exp\left[\frac{1}{2} \gamma_0 \gamma_1 \ln(x_0 + x_1)\right] \varphi(\boldsymbol{\omega}), \tag{3.24}$$

$$\text{(viii) } \psi(x) = \exp\left[-\frac{1}{2} \gamma_2 \gamma_3 \tan^{-1}(x_2/x_3)\right] \varphi(\boldsymbol{\omega}), \tag{3.25}$$

$$\text{(ix) } \psi(x) = x_0^{-k} \varphi(\boldsymbol{\omega}), \tag{3.26}$$

$$\text{(x) } \psi(x) = \varphi(\boldsymbol{\omega}), \tag{3.27}$$

$$\text{(xi) } \psi(x) = \varphi(\boldsymbol{\omega}), \tag{3.28}$$

$$\text{(xii) } \psi(x) = \varphi(\boldsymbol{\omega}). \tag{3.29}$$

The problem of finding all the ansätze for $\tilde{\mathcal{P}}(1,3)$-invariant solutions is therefore completely solved. The second step of the algorithm — the reduction of the Dirac equation — will be realised in the next section.

4. Reduction of the non-linear Dirac equation (1.1)

It was pointed out above that substitution of ansatz (1.3) into (1.1) results in a reduction by one of a number of independent variables. This means that the equation obtained will depend on the three independent variables ω_1, ω_2, ω_3. Omitting cumbersome calculations we write down resulting systems of PDE:

$$\begin{aligned}\text{(i) } & k(\gamma_2-\gamma_0)\varphi+\left[(\gamma_0-\gamma_2)\left(\omega_1+a^{-2}\omega_2^2\omega_3^2\right)+\right.\\ & \left.+(\gamma_0+\gamma_2)\omega_2^2-2a^{-1}\gamma_1\omega_3\omega_2^2-2\gamma_3\omega_1\omega_2\right]\varphi_{\omega_1}+\\ & +\left[(\gamma_0-\gamma_2)\omega_2-\gamma_3\omega_2^2\right]\varphi_{\omega_2}+\\ & +\left[a\gamma_1+(\gamma_2-\gamma_0)(\omega_3+1)\right]\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.1}$$

$$\begin{aligned}& \frac12(\gamma_0-\gamma_2)\omega_1^{-1}\varphi+(\gamma_0-\gamma_2)\varphi_{\omega_1}+\gamma_3\varphi_{\omega_2}+\\ & +\left[(\gamma_0+\gamma_2)\omega_1+(\gamma_0-\gamma_2)\omega_3\omega_1^{-1}\right]\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.2}$$

$$\begin{aligned}\text{(ii) } & [\gamma_3+\beta(\gamma_0-\gamma_2)]\varphi_{\omega_1}+2\gamma_1\varphi_{\omega_2}+\\ & +\frac32(2\gamma_2)+(\gamma_0-\gamma_2)\omega_2)\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.3}$$

$$\begin{aligned}\text{(iii) } & \frac12\beta^{-1}\gamma_4(\gamma_0-\gamma_2)\varphi+(\gamma_0-\gamma_2)\varphi_{\omega_1}+\left[(\gamma_0+\gamma_2)\omega_1-2\beta^{-1}\gamma_1\omega_3+\right.\\ & \left.+(\gamma_0-\gamma_2)\left(\beta^{-2}\omega_3^2+\omega_2\right)\omega_1^{-1}\right]\varphi_{\omega_2}+(\beta\gamma_1-\gamma_3\omega_1)\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.4}$$

$$\begin{aligned}\text{(iv) } & \frac12(1-2k)\gamma_3\varphi+(\omega_1\omega_3)^{1/2}(\gamma_0-\gamma_1\omega_1)\varphi_{\omega_1}+2(\gamma_3+a\gamma_2)\varphi_{\omega_2}+\\ & +\left[2\gamma_3-(\gamma_0+\gamma_1\omega_1)\omega_3^{1/2}\omega_1^{-1/2}\right]\omega_3\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.5}$$

$$\begin{aligned}\text{(v) } & \left[-k\left(\gamma_0\cosh\ln\omega_1^{-1/2(a+1)}-\gamma_1\sinh\ln\omega_1^{1/2(a+1)}\right)+\right.\\ & \left.+\frac12(a+1)^{-1}(\gamma_0+\gamma_1)\omega_1^{-1/2(a+1)}+\frac12\gamma_3\omega_2^{1/2}\right]\varphi-\\ & -2(a+1)\omega_1\left(\gamma_0\cosh\ln\omega_1^{1/2(a+1)}-\gamma_1\sinh\ln\omega_1^{1/2(a+1)}\right)\varphi_{\omega_1}+\\ & +2\left[\gamma_0\cosh\ln\omega_1^{1/2(a+1)}-\gamma_1\sinh\ln\omega_1^{1/2(a+1)}-\gamma_3\omega_2^{1/2}\right]\omega_2\varphi_{\omega_2}+\\ & +2(a\gamma_2+b\gamma_3)\omega_2^{1/2}\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.6}$$

$$\begin{aligned}& \left[-k\left(\gamma_0\cosh\ln\omega_1^{1/2}-\gamma_1\sinh\ln\omega_1^{1/2}\right)+\right.\\ & \left.+\frac14(\gamma_0-\gamma_1)\omega_1^{1/2}+\frac12\gamma_3\omega_2^{1/2}\right]\varphi+(\gamma_0+\gamma_1)\omega_1^{1/2}\varphi_{\omega_1}+\\ & +2\omega_2\left(\gamma_0\cosh\ln\omega_1^{1/2}-\gamma_1\sinh\ln\omega_1^{1/2}-2\gamma_3\omega_2^{1/2}\right)\varphi_{\omega_2}+\\ & +2(b\gamma_3-\gamma_2)\omega_2^{1/2}\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.7}$$

$$\begin{aligned}\text{(vi) } & \frac12[(1-2k)(\gamma_0+\gamma_1)+\gamma_3\omega_2]\varphi+2[(\beta-1)\gamma_0+(\beta+1)\gamma_1]\omega_1\varphi_{\omega_1}+\\ & +2\omega_2\left(\gamma_0+\gamma_1-\omega_2^{1/2}\gamma_3\right)\varphi_{\omega_2}+2(\gamma_2+b\gamma_3)\omega_2^{1/2}\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi,\end{aligned}\tag{4.8}$$

$$\text{(vii)}\ \frac{1}{2}(\gamma_0+\gamma_1)\varphi+[\gamma_0(\omega_1+1)+\gamma_1(\omega_1-1)]\varphi_{\omega_1}+ \\ +(\gamma_0+\gamma_1-\gamma_2)\varphi_{\omega_2}+\gamma_3\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.9}$$

$$\text{(viii)}\ \frac{1}{2}\omega_1^{-1/2}\varphi+2\omega_1^{1/2}\gamma_3\varphi_{\omega_1}+\left(\omega_1^{-1/2}\gamma_2+\beta_1\gamma_0+\beta_2\gamma_1\right)\varphi_{\omega_2}+ \\ +(\alpha_2\gamma_0+\alpha_1\gamma_1)\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.10}$$

$$\text{(ix)}\ -k\gamma_0\varphi+(\gamma_a-\omega_a\gamma_0)\varphi_{\omega_a}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.11}$$

$$\text{(x)}\ (\gamma_0+\gamma_1)\varphi_{\omega_1}+\gamma_2\varphi_{\omega_2}+\gamma_3\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.12}$$

$$\text{(xi)}\ \gamma_a\varphi_{\omega_a}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.13}$$

$$\text{(xii)}\ \gamma_0\varphi_{\omega_1}+\gamma_2\varphi_{\omega_2}+\gamma_3\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.14}$$

wher $\varphi_{\omega_a}=\partial\varphi/\partial\omega_a$ and $a=\overline{1,3}$.

A partial solution of one of the equations (4.1)–(4.14) through formulae (3.16)–(3.29) gives a partial solution of the non-linear Dirac equation. To obtain a partial solution of the reduced equation one can again apply the reduction procedure. But it demands a knowledge of the symmetry of equations (4.1)–(4.14). Investigation of symmetrical properties of equations in question is a very interesting problem (for example, equation (4.12) possesses an infinite-parameter symmetry group) and it will be considered in a future paper. We shall perform the direct reduction (if it is possible) of systems (4.1)–(4.14) to systems of ODE.

Let us suppose that in (4.1) $\varphi=\varphi(\omega_2)$. It follows that

$$k(\gamma_2-\gamma_0)\varphi+\omega_2(\gamma_0-\gamma_2-\omega_2\gamma_3)\varphi_{\omega_2}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi. \tag{4.15}$$

Similarly, if one chooses $\varphi=\varphi(\omega_3)$ then

$$k(\gamma_2-\gamma_0)\varphi+[(\gamma_2-\gamma_0)(1+\omega_3)+a\gamma_1)\varphi_{\omega_3}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi. \tag{4.16}$$

(4.15) and (4.16) are non-linear systems of ODE.

Equation (4.2) gives the following system of ODE:

$$(\gamma_0-\gamma_2)\varphi_{\omega_1}+\frac{1}{2}\omega_1^{-1}(\gamma_0-\gamma_2)\varphi=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi. \tag{4.17}$$

From (4.3) it follows that

$$[\gamma_3+\beta(\gamma_0-\gamma_2)]\varphi_{\omega_1}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.18}$$

$$2\gamma_1\varphi_{\omega_2}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi. \tag{4.19}$$

Systems (4.4) and (4.5) can be reduced to the systems of ODE of the form

$$2\beta(\gamma_0-\gamma_2)\varphi_{\omega_1}+(\gamma_0-\gamma_2)\gamma_4\varphi=2i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.20}$$

$$\frac{1}{2}(1-2k)\gamma_3\varphi+2(\gamma_3+a\gamma_2)\varphi_{\omega_2}=i\lambda(\bar\varphi\varphi)^{1/2k}\varphi. \tag{4.21}$$

We did not succeed in reducing systems (4.6)–(4.8) to ODE. From (4.9) one can obtain three systems of ODE:

$$\frac{1}{2}(\gamma_0+\gamma_1)\varphi + [(\gamma_0+\gamma_1)\omega_1+\gamma_0-\gamma_1]\varphi_{\omega_1} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.22}$$

$$\frac{1}{2}(\gamma_0+\gamma_1)\varphi + (\gamma_0+\gamma_1-\gamma_2)]\varphi_{\omega_2} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.23}$$

$$\frac{1}{2}(\gamma_0+\gamma_1)\varphi + \gamma_3\varphi_{\omega_3} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi. \tag{4.24}$$

Equation (4.10) gives the system

$$\frac{1}{2}\gamma_3\omega_1^{-1/2} + 2\gamma_3\omega_1^{1/2}\varphi_{\omega_1} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi. \tag{4.25}$$

Equations (4.11)–(4.14) are reduced to the following systems of ODE:

$$-k\gamma_0\varphi + (\gamma_a-\omega_a\gamma_0)\varphi_{\omega_a} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.26}$$

$$(\gamma_0+\gamma_1)\varphi_{\omega_1} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.27}$$

$$\gamma_a\varphi_{\omega_a} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi, \tag{4.28}$$

$$\gamma_0\varphi_{\omega_1} = i\lambda(\bar\varphi\varphi)^{1/2k}\varphi \tag{4.29}$$

(no summation is carried over a).

Symmetry properties of the non-linear Dirac equation therefore enable us to reduce the problem of finding its partial solution to an essentially simpler one of integration of systems of ode (4.15)–(4.29). To solve these systems one can apply various methods including numerical ones.

5. Construction of exact solutions of the non-linear Dirac equation (1.1)

We shall consider only systems of ODE solvable in quadratures, but we shall not consider cases which give already known solutions. The general solution of (4.19) has the form

$$\varphi(\omega_2) = \exp[-(i\lambda/2)(\bar\chi\chi)^{1/2k}\gamma_1\omega_2]\chi,$$

where χ is an arbitrary constant spinor.

Substituting the above result into (3.18), we obtain a solution of the initial equation (1.1):

$$\begin{aligned}\psi(x) = \exp\left[\frac{1}{2}(\gamma_0-\gamma_2)(x_0-x_2)\right]\times\\ \times\exp\left\{-(i\lambda/2)(\bar\chi\chi)^{1/2k}\gamma_1\left[2x_1+(x_0-x_2)^2\right]\right\}\chi.\end{aligned} \tag{5.1}$$

Let us next consider equation (4.21). Under $k=\frac{1}{2}$ its general solution has the form

$$\varphi(\omega_2) = \exp\left[-\frac{1}{2}i\lambda\bar\chi\chi\left(1+a^2\right)^{-1}(\gamma_3+a\gamma_2)\omega_2\right]\chi, \tag{5.2}$$

where χ is a constant spinor.

Under $k \neq \frac{1}{2}$, $a \neq 0$, we did not succeed in integrating the corresponding equation. If $a = 0$, then making a change of variables we obtain

$$\varphi(\omega_2) = \exp\left[\frac{1}{4}(2k-1)\omega_2\right]\phi(\omega_2),$$

$$2\exp\left[\frac{1}{4}(1-2k)k^{-1}\omega_2\right]\gamma_3\phi_{\omega_2} = i\lambda(\bar{\phi}\phi)^{1/2k}\phi.$$

The general solution of the last equation is given by the formula

$$\phi = \exp\left\{(2i\lambda k)(1-2k)^{-1}(\bar{\chi}\chi)^{1/2k}\exp\left[\frac{1}{4}(2k-1)k^{-1}\omega_2\right]\gamma_3\right\}\chi,$$

where χ is the arbitrary constant spinor.

Substituting the above results into (3.20) we obtain the following solutions of the non-linear Dirac equation.

If $k = \frac{1}{2}$

$$\psi(x) = \left(x_2^2 + x_3^2\right)^{-1/4}\exp\left\{-\frac{1}{2}\gamma_2\gamma_3\tan^{-1}(x_2/x_3)\right\}\times$$
$$\times\exp\left\{-\frac{1}{2}i\lambda\bar{\chi}\chi\left(1+a^2\right)^{-1}(\gamma_3 + a\gamma_2)\left[\ln\left(x_2^2+x_3^2\right) + 2a\tan^{-1}(x_2/x_3)\right]\right\}\chi. \tag{5.3}$$

If $k \neq \frac{1}{2}$

$$\psi(x) = \left(x_2^2 + x_3^2\right)^{-1/4}\exp\left[-\frac{1}{2}\gamma_2\gamma_3\tan^{-1}(x_2/x_3)\right]\times$$
$$\times\exp\left[2i\lambda k(1-2k)^{-1}(\bar{\chi}\chi)^{1/2k}\left(x_2^2+x_3^2\right)^{(2k-1)/4}\gamma_3\right]\chi. \tag{5.4}$$

It is important to note that equation (4.3) can be reduced to the two-dimensional Dirac equation. This fact can be used for obtaining new non-trivial classes of solutions of (1.1). If we choose in (4.3), $\varphi = \varphi(\omega_1, \omega_2)$ then

$$[\gamma_3 + \beta(\gamma_0 - \gamma_2)]\varphi_{\omega_1} + 2\gamma_1\varphi_{\omega_2} = i\lambda(\bar{\varphi}\varphi)^{1/2k}\varphi. \tag{5.5}$$

Having made a change of variables

$$z_1 = \omega_1, \quad z_2 = \frac{1}{2}\omega_2$$

and denoting

$$\Gamma_1 = \gamma_3 + \beta(\gamma_0 - \gamma_2), \quad \Gamma_2 = \gamma_1$$

we obtain

$$\Gamma_1\varphi_{z_1} + \Gamma_2\varphi_{z_2} = i\lambda(\bar{\varphi}\varphi)^{1/2k}\varphi, \tag{5.6}$$

where $\Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2g_{ab}$ and $a, b = \overline{1,2}$.

(i) We look for a solution of (5.6) in the form

$$\varphi(z) = (\Gamma_a z_a f(z_b z_b) + ig(z_b z_b))\chi, \tag{5.7}$$

where χ is a constant spinor and f, g are unknown scalar functions. Substitution of (5.7) into (5.6) gives the system of ODE

$$f+\omega\frac{df}{d\omega}=\frac{1}{2}\lambda(\bar{\chi}\chi)^{1/2k}\left(g^2-\omega f^2\right)^{1/2k}g,$$
$$\frac{dg}{d\omega}=\frac{1}{2}\lambda(\bar{\chi}\chi)^{1/2k}\left(g^2-\omega f^2\right)^{1/2k}f.$$

The partial solution of this system is given by the formulae ($k<0$)

$$\begin{aligned} f&=|k|^{1/2}\left(\mp\frac{\left(k^2+|k|\right)^{1/2}}{\lambda(\bar{\chi}\chi)^{1/2k}}\right)^k\omega^{-(k+1)/2},\\ g&=\mp\left(1+|k|^{-1}\right)^{-1/2}\left(\mp\frac{\left(k^2+|k|\right)^{1/2}}{\lambda(\bar{\chi}\chi)^{1/2k}}\right)^k\omega^{-k/2}. \end{aligned}\tag{5.8}$$

(ii) We shall look for a solution of (5.6) in the form

$$\varphi(z)=\Gamma_a z_a(z_b z_b)^{-1}\phi(\beta_a z_a/z_b z_b),\quad a,b=\overline{1,2},\tag{5.9}$$

where $\phi=\phi(\omega)$ is a four-component spinor, $\omega=(\beta_a z_a)/(z_b z_b)$ and $k=\frac{1}{2}$. It follows from (5.6) that $\phi(\omega)$ satisfies the system of ODE of the form

$$(\Gamma_a\beta_a)\frac{d\phi}{d\omega}=i\lambda(\bar{\phi}\phi)\phi,$$

whose general solution has the form

$$\phi(\omega)=\exp\left[-i\lambda(\bar{\chi}\chi)\left(\beta_1^2+\beta_2^2\right)^{-1}(\Gamma_a\beta_a)\omega\right]\chi.\tag{5.10}$$

Using formulae (3.18), (5.7)–(5.10) we obtain the following solutions of the nonlinear Dirac equation (1.1).

If $k<0$

$$\begin{aligned} \psi(x)=\exp\left[\frac{1}{2}\gamma_1(\gamma_0-\gamma_2)(x_0-x_2)\right]\Bigg[\Bigg\{&[\gamma_3+\beta(\gamma_0-\gamma_2)]\times\\ \times[x_3+\beta(x_0-x_2)]+\frac{1}{2}\gamma_1&\left[2x_1+(x_0-x_2)^2\right]\Bigg\}f(\omega)+ig(\omega)\Bigg]\chi, \end{aligned}\tag{5.11}$$

where

$$\omega=[x_3+\beta(x_0-x_2)]^2+\frac{1}{4}\left[2x_1+(x_0-x_2)^2\right]^2$$

and $f(\omega)$, $g(\omega)$ are defined by (5.8).

If $k = \frac{1}{2}$

$$
\begin{aligned}
\psi(x) = \exp\left[\frac{1}{2}\gamma_1(\gamma_0-\gamma_2)(x_0-x_2)\right]\Big\{&[\gamma_3+\beta(\gamma_0-\gamma_2)]\times\\
&\times[x_3+\beta(x_0-x_2)]+\frac{1}{2}\gamma_1\left[2x_1+(x_0-x_2)^2\right]\Big\}\omega^{-1}\times\\
&\times\exp\Big[-i\lambda(\bar\chi\chi)\left(\beta_1^2+\beta_2^2\right)^{-1}\left\{\beta_1[\gamma_3+\beta(\gamma_0-\gamma_2)]+\frac{1}{2}\beta_2\gamma_1\right\}\times\\
&\times\left\{\beta_1[x_3+\beta(x_0-x_2)]+\frac{1}{2}\beta_2\left[2x_1+(x_0-x_2)^2\right]\right\}\omega^{-1}\Big]\chi,
\end{aligned}
\tag{5.12}
$$

where

$$
\omega = [x_3+\beta(x_0-x_2)]^2+\frac{1}{4}\left[2x_1+(x_0-x_2)^2\right]^2.
$$

Let us point out one of the possible ways of obtaining ungenerable families of solutions. On applying the procedure of generation of solutions by Lorentz rotations in the plane (x_0, x_1) to the solution (5.1) one obtains

$$
\begin{aligned}
\psi_2(x) = \exp&\left(-\frac{1}{2}\theta\gamma_0\gamma_1\right)\exp\left[\frac{1}{2}\gamma_1(\gamma_0-\gamma_2)(x_0'-x_2')\right]\times\\
&\times\exp\left\{-\frac{1}{2}i\lambda(\bar\chi\chi)^{1/2k}\gamma_1\left[2x_1'+(x_0'-x_2')^2\right]\right\}\chi,\\
x_0' = x_0&\cosh\theta+x_1\sinh\theta,\quad x_1'=x_1\cosh\theta+x_0\sinh\theta,\quad x_2'=x_2,\quad x_3'=x_3.
\end{aligned}
$$

Let us rewrite this expression in the equivalent form

$$
\begin{aligned}
\psi_2(x) = \exp\left(-\frac{1}{2}\theta\gamma_0\gamma_1\right)&\exp\left[\frac{1}{2}\gamma_1(\gamma_0-\gamma_2)(x_0\cosh\theta+x_1\sinh\theta-x_2)\right]\times\\
&\times\exp\left(\frac{1}{2}\theta\gamma_0\gamma_1\right)\exp\left(-\frac{1}{2}\theta\gamma_0\gamma_1\right)\Big\{-\frac{1}{2}i\lambda(\bar\chi\chi)^{1/2k}\gamma_1\times\\
&\times\left[2x_1\cosh\theta+2x_0\sinh\theta+(x_0\cosh\theta+x_1\sinh\theta-x_2)^2\right]\Big\}\times\\
&\times\exp\left(\frac{1}{2}\theta\gamma_0\gamma_1\right)\exp\left(-\frac{1}{2}\theta\gamma_0\gamma_1\right)\chi.
\end{aligned}
$$

On taking into consideration the identities

$$
\exp\left(-\frac{1}{2}\theta\gamma_0\gamma_1\right)\gamma_\alpha\exp\left(\frac{1}{2}\theta\gamma_0\gamma_1\right)=\begin{cases}\gamma_0\cosh\theta+\gamma_1\sinh\theta, & \alpha=0,\\ \gamma_1\cosh\theta+\gamma_0\sinh\theta, & \alpha=1,\\ \gamma_\alpha, & \alpha=\overline{2,3},\end{cases}
$$

we obtain the following expression:

$$\psi_2(x) = \exp\left[\frac{1}{2}(\gamma_1\cosh\theta + \gamma_0\sinh\theta)(\gamma_0\sinh\theta + \gamma_1\cosh\theta - \gamma_2)\times\right.$$
$$\left.\times\,(x_0\cosh\theta + x_1\sinh\theta - x_2)\right]\times$$
$$\times\exp\left\{-\frac{1}{2}i\lambda(\bar\chi'\chi')^{1/2k}(\gamma_1\cosh\theta + \gamma_0\sinh\theta)\times\right.$$
$$\left.\times\left[2x_1\cosh\theta + 2x_0\sinh\theta + (x_0\cosh\theta + x_1\sinh\theta - x_2)^2\right]\right\}\chi',$$

where $\chi' = \exp\left(-\frac{1}{2}\theta\gamma_0\gamma_1\right)\chi$.

Using rest transformations from $O(1,3)\subset\tilde{\mathcal{P}}(1,3)$ in the same way one can find a family of solutions of equation (1.1) of the form

$$\psi(x) = \exp\left[\frac{1}{2}(\gamma a)(\gamma b)bx\right]\exp\left\{-\frac{1}{2}i\lambda(\bar\chi\chi)^{1/2k}(\gamma a)\left[2ax + (bx)^2\right]\right\}\chi, \tag{5.13}$$

where parameters a_μ, b_μ satisfy the conditions

$$aa = -1, \quad bb = ab = 0, \quad \gamma a = \gamma_\mu a^\mu, \quad bx = b_\mu x^\mu, \quad ab = a_\mu b^\mu.$$

Applying the formula for generating solutions by scale transformations

$$\psi_2(x) = e^{-k\alpha}\psi_1(x'), \quad x'_\mu = e^\alpha x_\mu, \quad \alpha = \mathrm{const}$$

one can obtain

$$\psi(x) = \exp\left[\frac{1}{2}\theta(\gamma a)(\gamma b)bx\right]\exp\left\{-\frac{1}{2}i\lambda(\bar\chi\chi)^{1/2k}(\gamma a)\left[2ax + \theta(bx)^2\right]\right\}\chi. \tag{5.14}$$

At last, generating from (5.14) new solutions by the group of translations, we obtain an ungenerable family of solutions of the non-linear Dirac equation (1.1).

(i) $k\in\mathbb{R}^1$, $k\neq 0$,

$$\psi(x) = \exp\left[\frac{1}{2}\theta(\gamma a)(\gamma b)bz\right]\exp\left\{-\frac{1}{2}i\lambda(\bar\chi\chi)^{1/2k}(\gamma a)\left[2az + \theta(bz)^2\right]\right\}\chi,$$
$$z_\mu = x_\mu + \theta_\mu, \quad \gamma a = \gamma_\mu a^\mu, \quad bz = b_\mu z^\mu, \quad az = a_\mu z^\mu,$$

where χ is an arbitrary constant spinor and θ, θ_μ, a_μ, b_μ are constants satisfying the following constraints:

$$aa = -1, \quad bb = 0, \quad ab = 0. \tag{5.15}$$

The same procedure when applied to (5.3), (5.4), (5.11) and (5.12) gives ungenerable families of the form

(ii) $k\in\mathbb{R}^1$, $k\neq 0, \frac{1}{2}$,

$$\begin{aligned}\psi(x) = {} & \left[(az)^2 + (bz)^2\right]^{-1/4}\exp\left[-\frac{1}{2}(\gamma a)(\gamma b)\tan^{-1}(az/bz)\right]\times\\ & \times\exp\left\{i2\lambda k(2k-1)^{-1}(\bar\chi\chi)^{1/2k}(\gamma b)\left[(az)^2 + (bz)^2\right]^{(2k-1)/4k}\right\}\chi,\end{aligned} \tag{5.16}$$

where $aa = -1$, $bb = -1$, $ab = 0$, $z_\mu = x_\mu + \theta_\mu$, θ_μ being arbitrary constants, and χ is an arbitrary constant spinor.

(iii) $k = \frac{1}{2}$,

$$\psi(x) = \left[(az)^2 + (bz)^2\right]^{-1/4} \exp\left[-\frac{1}{2}(\gamma a)(\gamma b)\tan^{-1}(az/bz)\right] \times$$
$$\times \exp\left[-\frac{1}{2}i\lambda\bar{\chi}\chi\left(1+\theta^2\right)^{-1}(\gamma b + \theta\gamma a) \times \right.$$
$$\left. \times \left\{\ln\left[(az)^2 + (bz)^2\right] + 2\theta\tan^{-1}(az/bz)\right\}\right]\chi,$$

where $z_\mu = x_\mu + \theta_\mu$ and a_μ, b_μ, θ_μ, θ are arbitrary constants satisfying conditions (5.16).

(iv) $k = \frac{1}{2}$,

$$\psi(x) = \exp\left[\frac{1}{4}(\gamma c)(\gamma b)bz\right]\left\{(\gamma a + \beta\gamma b)(az + \beta bz) + \frac{1}{4}\gamma c\left[cz + (bz)^2\right]\right\}\omega^{-1} \times$$
$$\times \exp\left\{-i\lambda\bar{\chi}\chi\left(\beta_1^2 + \beta_2^2\right)^{-1}\left[\beta_1(\gamma a + \beta\gamma b) + \frac{1}{2}\beta_2\gamma c\right] \times \right.$$
$$\left. \times \left[\beta_1(az + \beta bz) + \frac{1}{2}\beta_2\left(cz + (bz)^2\right)\right]\omega^{-1}\right\}\chi,$$
$$\omega = (az + \beta bz)^2 + \frac{1}{4}\left[cz + (bz)^2\right]^2, \quad z_\mu = x_\mu + \theta_\mu,$$

and θ_μ, a_μ, b_μ, c_μ, β, β_i are arbitrary constants satisfying the conditions

$$ab = bc = ca = bb = 0, \quad aa = -1, \quad cc = -4. \tag{5.17}$$

(v) $k < 0$,

$$\psi(x) = \exp\left[\frac{1}{4}(\gamma c)(\gamma b)bz\right]\left[\left\{(\gamma a + \beta\gamma b)(az + \beta bz) + \right.\right.$$
$$\left.\left. + \frac{1}{4}(\gamma c)\left[cz + (bz)^2\right]\right\}f(\omega) + ig(\omega)\right]\chi,$$
$$z_\mu = x_\mu + \theta_\mu, \quad \omega = (az + \beta bz)^2 + \frac{1}{4}\left[cz + (bz)^2\right]^2$$

with $f(\omega)$, $g(\omega)$ from (5.8). Parameters a_μ, b_μ, c_μ, θ_μ satisfy conditions (5.17) and χ is an arbitrary constant spinor.

(vi) $k \in \mathbb{R}^1$, $k \neq 0$,

$$\psi(x) = \exp\left[\frac{1}{2}(\gamma a)(\gamma b)\ln(az + bz)\right]\exp\left\{\left[\frac{1}{2}(\gamma c)(\gamma a + \gamma b) + \right.\right.$$
$$\left.\left. + i\lambda(\bar{\chi}\chi)^{1/2k}(\gamma c - \gamma a - \gamma b)\right]\left[\ln(az + bz) - cz\right]\right\}\chi, \tag{5.18}$$

$$\psi(x) = \exp\left[\frac{1}{2}(\gamma a)(\gamma b)\ln(az+bz)\right]\times$$
$$\times\exp\left\{\left[\frac{1}{2}(\gamma c)(\gamma a+\gamma b) - i\lambda(\bar\chi\chi)^{1/2k}\gamma c\right](cz)\right\}\chi, \tag{5.19}$$

where $z_\mu = x_\mu + \theta_\mu$, χ is an arbitrary constant spinor and a_μ, b_μ, c_μ are arbitrary constants satisfying conditions

$$-aa = bb = -1, \quad cc = -1, \quad ab = bc = ca = 0.$$

(vii) $k \in \mathbb{R}^1$, $k \neq 0$,

$$\psi(x) = \exp\left[\frac{1}{2}(\gamma a)(\gamma b)bz\right]\exp\left[-i\lambda(\gamma c+\beta\gamma b)(\bar\chi\chi)^{1/2k}(cz+\beta bz)\right]\chi, \tag{5.20}$$

where $z_\mu = x_\mu + \theta_\mu$, χ is an arbitrary constant spinor and a_μ, b_μ, c_μ, θ_μ are arbitrary constants satisfying the conditions

$$aa = cc = -1, \quad ab = bc = ca = bb = 0. \tag{5.21}$$

In conclusion of this section, let us consider the special case of equation (1.1) when $k = \frac{3}{2}$. It is common knowledge that the corresponding non-linear Dirac equation is conformally invariant [10, 13]. This enables us to obtain a larger family of solutions with the help of a procedure of generating solutions by special conformal transformations, corresponding formulae having the form [7]

$$\psi_2(x) = \sigma^{-2}(x)(1-(\gamma x)(\gamma\theta))\psi_1(x'),$$
$$x'_\mu = (x_\mu - \theta_\mu(xx))\sigma^{-1}(x), \quad \sigma(x) = 1 - 2\theta x + (\theta\theta)(xx). \tag{5.22}$$

Using solutions (5.14) under $k = \frac{3}{2}$ as $\psi_1(x)$ we obtain a new solution of the conformally invariant equation (1.1)

$$\psi(x) = [1-(\gamma x)(\gamma\theta)]\sigma^{-2}(x)\exp\left\{\frac{1}{2}\tilde\theta(\gamma a)(\gamma b)(bx-(b\theta)(xx))\sigma^{-1}(x)\right\}\times$$
$$\times\exp\left\{-\frac{1}{2}i\lambda(\bar\chi\chi)^{1/3}\gamma a[2(ax-(a\theta)(xx))\sigma(x) + \right. \tag{5.23}$$
$$\left. + \tilde\theta(bx-(b\theta)(xx))^2]\sigma^{-2}(x)\right\}\chi,$$

where $aa = -1$, $bb = 0$, $ab = 0$ and θ_μ, $\tilde\theta$ are arbitrary constants.

The same procedure when applied to solutions (5.18)–(5.20) under $k = \frac{3}{2}$ give some new solutions of the non-linear Dirac equation.

6. Exact solutions of the system (1.2)

We shall seek solutions of (1.2) when $m_1 = 0$, $m_2 = 0$, the following ansatz being used:

$$\psi(x) = \gamma b\exp(if(ax))\chi,$$
$$\mathcal{A}_\mu(x) = b_\mu g_1(ax) + a_\mu g_2(ax), \tag{6.1}$$

where $bb = 0$, $ax = a_\mu x^\mu$ and f, g_1, g_2 are arbitrary differentiable functions.

Substitution of (6.1) into (1.2) gives the system of ODE

$$\begin{aligned} &\lambda_1 g_2 = \dot{f}, \\ &(aa)\ddot{g}_1 = -2eb\theta - \lambda_2 g_1 \left(2ab g_1 g_2 + (aa) g_2^2\right), \\ &-(ab)\ddot{g}_1 = -\lambda_2 g_2 \left(2ab g_1 g_2 + (aa) g_2^2\right), \end{aligned} \tag{6.2}$$

where a dot means differentiation with respect to $\omega = ax$, $b\theta = b_\mu \theta^\mu$, $aa = a_\mu a^\mu$, $ab \neq 0$, $\theta_\mu = \bar{\chi}\gamma_\mu \chi$, $\mu = \overline{0,3}$.

We have succeeded in integrating the system (6.2) in the case $aa = 0$, $ab \neq 0$, i.e.

$$\begin{aligned} &\lambda_1 g_2 = \dot{f}, \\ &g_2 g_1^2 = -(eb\theta)/(\lambda_2 ab), \\ &\ddot{g}_1 = 2\lambda_2 g_1 g_2^2. \end{aligned} \tag{6.3}$$

From the second equation it follows that

$$g_2 = -(eb\theta)/(\lambda_2 ab) g_1^{-2}. \tag{6.4}$$

Substituting (6.4) into (6.3) we obtain ODE for determination of $g_1(\omega)$

$$\ddot{g}_1 = \left(k^2/\lambda_2\right) g_1^{-3}, \quad k = \sqrt{2}(eb\theta)/(ab). \tag{6.5}$$

Integration of the last ODE yields

$$\omega + C_2 = \begin{cases} 2|\lambda_2|^{1/2}|k|^{-1} g_1^2, & \lambda_2 < 0, \\ C_1^{-1}\left(C_1 g_1^2 - k^2/\lambda_2\right)^{1/2}, & C_1 \neq 0. \end{cases} \tag{6.6}$$

Finally

$$C_1 \neq 0, \quad g_1 = \pm C_1^{-1/2}\left[(C_1\omega + C_2)^2 + k^2/\lambda_2\right]^{-1}, \tag{6.7}$$

$$\lambda_2 < 0, \quad g_1 = \pm(k/|\lambda_2|)\left(2|k||\lambda_2|^{-1/2}\omega + C_2\right)^{-1}. \tag{6.8}$$

Substituting the above results into (6.4) we find expressions for $g_2(\omega)$

$$C_1 \neq 0, \quad g_2 = -(kC_1/\lambda_2)\left[(C_1\omega + C_2)^2 + k^2/\lambda_2\right]^{-1}, \tag{6.9}$$

$$\lambda_2 < 0, \quad g_2 = -(k/|\lambda_2|)\left(2|k||\lambda_2|^{-1/2}\omega + C_2\right)^{-1}. \tag{6.10}$$

Substituting these expressions into the first equation from (6.3) we obtain $f(\omega)$

$$C_1 \neq 0, \quad f(\omega) = -\lambda_1 \lambda_2^{-1/2} \tan^{-1}\left[k^{-1}\lambda_2^{1/2}(C_1\omega + C_2)\right], \tag{6.11}$$

$$\lambda_2 < 0, \quad f(\omega) = \lambda_1 |\lambda_2|^{-1/2} \ln\left(2k|\lambda_2|^{-1/2}\omega + C_2\right), \tag{6.12}$$

where C_1, C_2 are arbitrary constants.

Substitution of (6.7)–(6.12) into (6.1) gives two families of solutions of the initial equation (1.2)

(i) $\lambda_2 \neq 0$, $C_1 \neq 0$,

$$\begin{aligned}
&\psi(x) = \gamma b \exp\left\{-i\lambda_1\lambda_2^{-1/2}\tan^{-1}\left[\lambda_2^{1/2}k^{-1}(C_1ax+C_2)\right]\right\}\chi,\\
&\mathcal{A}_\mu(x) = \pm b_\mu C_1^{-1/2}\left[(C_1ax+C_2)^2-k^2\lambda_2^{-1}\right]^{1/2} -\\
&\qquad\qquad - a_\mu(kC_1/\lambda_2)\left[(C_1ax+C_2)^2-k^2/\lambda_2\right]^{-1},
\end{aligned} \tag{6.13}$$

(ii) $\lambda_2 < 0$,

$$\begin{aligned}
&\psi(x) = \gamma b \exp\left[-i\lambda_1|\lambda_2|^{-1/2}\ln\left(2k|\lambda_2|^{-1/2}ax+C_3\right)\right]\chi,\\
&\mathcal{A}_\mu(x) = \pm b_\mu\left(2k|\lambda_2|^{-1/2}ax+C_3\right)^{1/2} - a_\mu(k/|\lambda_2|)\left(2k|\lambda_2|^{-1/2}ax+C_3\right)^{-1},
\end{aligned} \tag{6.14}$$

where $k = \sqrt{2}eb^\mu(\bar{\chi}\gamma_\mu\chi)/(ab)$, C_1, C_2, C_3 are arbitrary constants and χ is an arbitrary constant spinor.

Let us note that the solutions obtained depend analytically on parameters λ_1, e while parameter λ_2 is included in a singular way. It means that solutions (6.13) and (6.14) cannot be obtained in the framework of perturbation theory by expanding in a series with respect to a small parameter λ_2.

On introducing as usual the tensor of the electromagnetic field $F_{\mu\nu} = \partial\mathcal{A}_\nu/\partial x_\mu - \partial\mathcal{A}_\mu/\partial x_\nu$ we obtain

$$\begin{aligned}
&F_{\mu\nu} = \pm(a_\mu b_\nu - a_\nu b_\mu)C_1^{1/2}\left[(C_1ax+C_2)^2-k^2/\lambda_2\right]^{-1/2},\\
&F_{\mu\nu} = \pm(a_\mu b_\nu - a_\nu b_\mu)k|\lambda_2|^{-1/2}\left(2k|\lambda_2|^{-1/2}ax+C_3\right)^{-1/2}
\end{aligned}$$

for solutions (6.13) and (6.14) respectively.

To obtain new families of solutions of the system (1.2) one can use its symmetry under conformal group $C(1,3)$ [9]. The formula for generating solutions by special conformal transformations has the form [8]

$$\begin{aligned}
&\psi_2(x) = \sigma^{-2}[1-(\gamma x)(\gamma\theta)]\psi_1(x'),\\
&\mathcal{A}_\mu^{(2)}(x) = \sigma^{-2}(x)[g_{\mu\nu}\sigma(x) + 2(\theta_\mu x_\nu - \theta_\nu x_\mu + 2\theta x x_\mu\theta_\nu -\\
&\qquad\qquad - xx\theta_\mu\theta_\nu - \theta\theta x_\mu x_\nu)]\mathcal{A}^\nu_{(1)}(x'),\\
&x'_\mu = (x_\mu - \theta_\mu xx)\sigma^{-1}(x), \quad \sigma(x) = 1 - 2\theta x + (\theta\theta)(xx).
\end{aligned}$$

Using (6.13) and (6.14) as $\psi_1(x)$ and $\mathcal{A}_\mu^{(1)}(x)$ one can construct new multiparameter families of exact solutions of (1.2) but we omit corresponding formulae because of their cumbersome character.

7. Conclusion

In the present work, large classes of exact solutions of the non-linear Dirac equation and of the system of non-linear equations of quantum electrodynamics were constructed. Solutions obtained by Akdeniz [1], Fushchych and Shtelen [6, 7], Kortel [11], Merwe [14] and Takahashi [15] can be obtained with the help of ansätze (3.16)–(3.29).

Most of the solutions depend analytically on constants λ, λ_i, e. However solutions (6.13) and (6.14) have a non-perturbative character because of their singular dependence on the parameter λ_2.

We have constructed ansätze which reduced the four-dimensional systems (1.1) and (1.2) to three-, two- and one-dimensional systems of PDE. It is important to note that these ansaẗze can be applied to any spinor equations which are invariant under the extended Poincaré group $\mathcal{P}(1,3)$.

Appendix

It is important to note that the ansätze (3.16)–(3.29) do not exhaust all possible ansätze for the Dirac equation (1.1). To reduce (1.1) to ODE one can use the following ansatz:

$$\psi(x) = [ig(\omega) + f(\omega)\gamma_\mu \partial\omega/\partial x_\mu]\chi, \tag{A1}$$

where g, f are unknown real-valued functions, χ is an arbitrary constant spinor and $\omega = \omega(x)$ is a real-valued function satisfying conditions of the form

$$\begin{aligned} &p_\mu p^\mu \omega + A(\omega) = 0, \\ &(p_\mu\omega)(p^\mu\omega) + B(\omega) = 0, \quad A, B: \ \mathbb{R}^1 \to \mathbb{R}^1. \end{aligned} \tag{A2}$$

Substitution of (A1) into (1.1) gives a system of ode for determination of f and g. We now list some multiparameter families of exact solutions of the non-linear Dirac equation (1.1) obtained in this way.

(i) $k \in \mathbb{R}$, $k \neq 0$

$$\psi(x) = \left[-i\sinh\left(\lambda(\bar\chi\chi)^{1/2k}\omega\right) + \gamma_\mu(\partial\omega/\partial x_\mu)\cosh\left(\lambda(\bar\chi\chi)^{1/2k}\omega\right)\right]\chi,$$

where $\omega(x)$ is determined by the following equalities:

$$(a)\ \omega = bx\cos\varphi_1 + cx\sin\varphi_1 + \varphi_2, \tag{A3}$$

$$(b)\ ax + bx\cos\phi_1 + cx\sin\phi_1 + \phi_2 = 0, \tag{A4}$$

and $\varphi_i = \varphi_i(ax+dx)$, $\phi_i = \phi_i(\omega + dx)$ are arbitrary differentiable functions.

(ii) $k > 1$,

$$\begin{aligned} \psi(x) = \omega^{-k}\Big\{&\pm\left(1-k^{-1}\right)^{1/2} + \omega^{-1}[(bx+\varphi_1)(\gamma b + (\gamma a + \gamma d)\dot\varphi_1) + \\ &+ (cx+\varphi_2)(\gamma c + (\gamma a + \gamma d)\dot\varphi_2)]\Big\}\chi, \\ \omega = \big[(bx&+\varphi_1)^2 + (cx+\varphi_2)^2\big]^{1/2}, \end{aligned} \tag{A5}$$

where $\varphi_i = \varphi_i(ax+dx)$ are arbitrary differentiable functions and the dot means differentiation with respect to $ax+dx$.

(iii) $k = 1$,

$$\begin{aligned} &\psi(x) = \left(1+\theta^2\omega^2\right)^{-3/2}[i - \theta((\gamma a)(ax) - (\gamma b)(bx) - (\gamma c)(cx))]\chi, \\ &\omega = \left[(ax)^2 - (bx)^2 - (cx)^2\right]^{1/2} \end{aligned} \tag{A6}$$

and the following condition holds:

$$3\theta - \lambda(\bar\chi\chi)^{1/2k} = 0.$$

In (A3)–(A6) a_μ, b_μ, c_μ, d_μ are arbitrary parametrs satisfying the following conditions:

$$-aa = bb = cc = dd = -1, \qquad ab = ac = ad = bc = bd = cd = 0.$$

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