On subalgebras of the Lie algebra of the extended Poincaré group $\tilde{P}(1,n)$

L.F. BARANNIK, W.I. FUSHCHYCH

Some general results on the subalgebras of the Lie algebra $A\tilde{P}(1,n)$ of the extended Poincaré group $\tilde{P}(1,n)$ $(n\geq 2)$ with respect to $\tilde{P}(1,n)$ conjugation have been obtained. All subalgebras of $A\tilde{P}(1,4)$ that are nonconjugate to the subalgebras of $AP(1,4)$ are classified with respect to $\tilde{P}(1,4)$ conjugation. The list of representatives of each conjugacy class is presented.

1. Introduction

The systematic study of subalgebras of quantum mechanics transformation algebras was begun in the fundamental paper by Patera, Winternitz and Zassenhaus (PWZ) [1] in which the general method for classifying the subalgebras of a finite-dimensional Lie algebra with a nontrivial solvable ideal with respect to some group of automorphisms was suggested. This method is applied to classify all subalgebras of Lie algebras of the following groups: the Poincaré group $P(1,3)$ [1], the extended Poincaré groups $\tilde{P}(1,2)$ [2], $\tilde{P}(1,3)$ [3], the de Sitter groups $O(1,4)$ [4], $O(2,3)$ [5], the optical groups $Opt(1,2)$ [5], $Opt(1,3)$ [6], the Euclidean group $E(3)$ [7], the Schrödinger group $Sch(2)$ [8], and the extended Schrödinger group $\widetilde{Sch}(2)$ [8], the Poincaré group $P(1,4)$ [9–11], the Euclidean group $E(5)$ [12, 13], the Galilei group $G(3)$ [12], and the extended Galilei group $\tilde{G}(3)$ [12]. The application of the general method had allowed us to study the subalgebras structure of the Lie algebra of the generalized Euclidean group $E(n)$ $(n\geq 2)$ [13]. The subalgebras of the algebras $AP(1,3)$, $AG(3)$, and $A\tilde{G}(3)$ were described by another method [14–17].

The PWZ method needs the development for particular classes of algebras of its generality. In the present paper we give the further development of the PWZ method for extended Poincaré algebras $A\tilde{P}(1,n)$ $(n\geq 2)$, denoted also by $ASim(1,n)$. The necessity in the description of subalgebras of $A\tilde{P}(1,n)$ follows from certain problems of theoretical and mathematical physics [1]. In particular, knowledge of the algebra $A\tilde{P}(1,n)$ subalgebras gives us the possibility to study the symmetry reduction for the relativistically invariant scalar differential equation

$$\Phi\left(\Box u,(\nabla u)^2,u\right)=0,$$

where

$$\Box u=u_{x_0x_0}-u_{x_1x_1}-\cdots-u_{x_nx_n},$$
$$(\nabla u)^2=(u_{x_0})^2-(u_{x_1})^2-\cdots-(u_{x_n})^2,$$

and Φ is a sufficiently smooth function [18–20]. The description of the algebra $A\tilde{P}(1,n)$ subalgebras allows us to solve the problem of the reduction of $A\tilde{P}(1,n)$ algebra representations on its subalgebras [21, 22].

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In Sec. 2 we describe the maximal reducible subalgebras of the algebra $A\tilde{O}(1,n)$, and in Sec. 3 we describe the completely reducible subalgebras of the algebra $A\tilde{O}(1,n) = AO(1,n) \oplus \langle \mathbb{D} \rangle$, where $\mathbb{D}$ is the dilatation generator. Section 4 is devoted to study of the subalgebras of the extended Galilei algebra $A\tilde{G}(n-1)$, which is one of the important subalgebras of the $AP(1,n)$ algebra. In Sec. 5 which is the logical sequel to Sec. 4, a number of assertions on subalgebras of the normalizer of isotropic subspace of the Minkowski space $M(1,n)$ in algebra $A\tilde{P}(1,n)$ are conceived. Classification of the $A\tilde{P}(1,n)$ algebra subalgebras with respect to the $\tilde{P}(1,4)$ conjugation is carried out in Sec. 6. The conclusions are summarized in Sec. 7.

2. Maximal reducible subalgebras of the algebra $A\tilde{O}(1,n)$

In this section we describe the maximal reducible subalgebras and the maximal Abelian subalgebras of the algebra $A\tilde{O}(1,n)$.

Let R be the real number field; $\langle Y_1, \ldots, Y_s \rangle$ is a vector space or Lie algebra over R with the generators $Y_1, \ldots, Y_s$; R^m is the m-dimensional arithmetical vector space over R; $U = M(1,n)$ is $(1+n)$-dimensional pseudo-Euclidean space with the scalar product

$$(X, Y) = x_0 y_0 - x_1 y_1 - \cdots - x_n y_n; \tag{2.1}$$

$O(1,n)$ is the group of the linear transformations of $M(1,n)$ which conserve (X,X) for every $X \in M(1,n)$; E_q is the unit matrix of degree q. We suppose that $O(1,n)$ is realized as the group of the real matrices of degree $n+1$.

We call the extended Poincaré group $\tilde{P}(1,n)$ the multiplicative group of the matrices

$$\begin{pmatrix} \lambda\Delta & Y \\ 0 & 1 \end{pmatrix},$$

where $\Delta \in O(1,n)$, $\lambda \in R$, $\lambda > 0$, $Y \in R^{n+1}$.

We denote by AG the Lie algebra of the Lie group G. Using the definition of Lie algebra, we find that $AO(1,n)$ consists of matrices

$$X = \begin{pmatrix} 0 & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0,n-1} & \alpha_{0n} \\ \alpha_{01} & 0 & \alpha_{12} & \cdots & \alpha_{1,n-1} & \alpha_{1n} \\ \alpha_{02} & -\alpha_{12} & 0 & \cdots & \alpha_{2,n-1} & \alpha_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{0,n-1} & -\alpha_{1,n-1} & -\alpha_{2,n-1} & \cdots & 0 & \alpha_{n-1,n} \\ \alpha_{0n} & -\alpha_{1n} & -\alpha_{2n} & \cdots & -\alpha_{n-1,n} & 0 \end{pmatrix}. \tag{2.2}$$

Let E_{ik} be the matrix of degree $n+2$ which has the unity on the cross of ith line and kth column and zeros on the other places ($i, k = 0, 1, \ldots, n+1$). It is easy to get that the basis of the algebra $A\tilde{P}(1,n)$ is formed by the matrices

$$\mathbb{D} = E_{00} + E_{11} + \cdots + E_{nn}, \qquad J_{0a} = -E_{0a} - E_{a0}, \quad J_{ab} = -E_{ab} + E_{ba},$$
$$P_0 = E_{0,n+1}, \quad P_a = E_{a,n+1} \quad (a < b,\ a, b = 1, \ldots, n).$$

The basis elements satisfy the following commutation relations:

$$\begin{aligned} &[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta} J_{\beta\gamma} + g_{\beta\gamma} J_{\alpha\delta} - g_{\alpha\gamma} J_{\beta\delta} - g_{\beta\delta} J_{\alpha\gamma}, \quad J_{\beta\alpha} = -J_{\alpha\beta}, \\ &[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta} P_\gamma - g_{\alpha\gamma} P_\beta, \quad [P_\alpha, P_\beta] = 0, \quad [\mathbb{D}, J_{\alpha\beta}] = 0, \quad [\mathbb{D}, P_\alpha] = P_\alpha, \end{aligned} \tag{2.3}$$

where $g_{00} = -g_{11} = \cdots = -g_{nn} = 1$, $g_{\alpha\beta} = 0$, when $\alpha \neq \beta$ $(\alpha, \beta = 0, 1, \ldots, n)$.

The generators of turning $J_{\alpha\beta}$ generate the algebra $AO(1,n)$ and the translation P_α the commutative ideal N, and moreover $A\tilde{P}(1,n) = N \subset\!\!\!\!+ (AO(1,n) \oplus \langle \mathbb{D} \rangle)$. Let $\tilde{O}(1,n) = \{\lambda E_{n+1} | \lambda \in R, \lambda > 0\} \times O(1,n)$. Evidently, $A\tilde{O}(1,n) = AO(1,n) \oplus \langle \mathbb{D} \rangle$. It is easy to see that $[X,Y] = X \cdot Y$ for all $X \in A\tilde{O}(1,n)$, $Y \in N$. Let us identify N and $M(1,n)$ establishing correspondence between P_i and the $(n+1)$-dimensional column with unity on the ith place and zeros on the others $(i = 0, 1, \ldots, n)$.

Let C be such matrix of degree $n+2$ over R that mapping $\varphi_C : X \to CXC^{-1}$ is an automorphism of the algebra $A\tilde{P}(1,n)$. If $C \in G$, where G is a subgroup of $\tilde{P}(1,n)$, then φ_C is called G automorphism. The subalgebras L and L' of algebra $A\tilde{P}(1,n)$ are called $\tilde{P}(1,n)$ conjugated if $\varphi_C(L) = L'$ for some $\tilde{P}(1,n)$ automorphism φ_C of algebra $A\tilde{P}(1,n)$. Let us identify φ_C and C.

Let W a nondegenerate subspace of the space U. This subspace we also consider to be pseudo-Euclidean relative to scalar product defined in U. Let $O(W)$ be the group of isometries of the space W, $\tilde{O}(W) = O(W) \times \{\lambda E_{n+1} | \lambda \in R, \lambda > 0\}$. A subalgebra $F \subset A\tilde{O}(W)$ is called irreducible if in W there does not exist any F-invariant subspace different from O and W. Otherwise F is called reducible. If for every F-invariant subspace W' in W there exists an F-invariant subspace W'' in W such that $W = W' \oplus W''$ then it is called completely reducible.

Theorem 2.1. *The maximal reducible subalgebras of algebra $A\tilde{O}(1,n)$ are exhausted with respect to $\tilde{O}(1,n)$ conjugation by the following algebras: (1) $AO(1,n-1) \oplus \langle \mathbb{D} \rangle$; (2) $AO(n) \oplus \langle \mathbb{D} \rangle$; (3) $AO(1,k) \oplus AO'(n-k) \oplus \langle \mathbb{D} \rangle$, where $AO'(n-k) = \langle J_{ab} | a,b = k+1, \ldots, n \rangle$ $(k = 2, \ldots, n-2)$; (4) $\langle G_1, \ldots, G_{n-1} \rangle \subset\!\!\!\!+ (AO(n-1) \oplus \langle J_{0n}, \mathbb{D} \rangle)$, where $G_a = J_{0a} - J_{an}$ $(a = 1, \ldots, n-1)$.*

Proof. If L is a maximal subalgebra of the algebra $A\tilde{O}(1,n)$ then $L = AO(1,n)$ or $L = L_1 \oplus \langle \mathbb{D} \rangle$, where L_1 is a maximal subalgebra of the algebra $AO(1,n)$. Let F be a maximal reducible subalgebra of the algebra $AO(1,n)$, U' a subspace of the space U invariant under F. If U' is a degenerate space then it contains one-dimensional F-invariant isotropic space W conjugated under $O(1,n)$ to the space $\langle P_0 + P_n \rangle$. In this case

$$F = \{X \in AO(1,n) | X(P_0 + P_n) \in \langle P_0 + P_n \rangle\}.$$

It is not difficult to show that

$$F = \langle G_1, \ldots, G_{n-1} \rangle \subset\!\!\!\!+ (AO(n-1) \oplus \langle J_{0n} \rangle).$$

If U' is a nondegenerate space of dimension r then it possesses an orthogonal basis consisting of r vectors with nonzero length. Let r_+, r_- be numbers of positive and negative length vectors, in the given basis of the space U', respectively. These numbers are independent of the choice of basis. In accordance with Witt's mapping theorem any two spaces U' and U'_1, for which $r_+ = r_+^1$, $r_- = r_-^1$ are mutually conjugate under the group $O(1,n)$. Obviously, $r_+ \in \{0, 1\}$. Since $U = U' \oplus U'^\perp$ and $U'^\perp$ is invariant under F therefore F is $O(1,n)$ conjugated to one of the algebras,

$$AO(1,n-1), \; AO(n), \; AO(1,k) \oplus AO'(n-k).$$

The theorem is proved.

Let

$$A\tilde{E}(n) = \langle P_1, \ldots P_n \rangle \subset\!\!\!\!+ (AO(n) \oplus \langle \mathbb{D} \rangle),$$
$$AE'(n-k) = \langle P_{k+1}, \ldots, P_n \rangle \subset\!\!\!\!+ AO'(n-k),$$

and $A\tilde{G}(n-1)$ is the extended Galilei algebra with the basis

$$M = P_0 + P_n, P_0, P_1, \ldots, P_{n-1}, G_1, \ldots, G_{n-1}, J_{ab} \quad (a, b = 1, \ldots, n-1).$$

According to Theorem 2.1, the description of subalgebras of the algebra $A\tilde{P}(1,n)$ is reduced to the description with respect to the $\tilde{P}(1,n)$ conjugation of irreducible subalgebras of the algebra $AO(1,n)$ and subalgebras of the following algebras:

$$\langle P_0 \rangle \Subset A\tilde{E}(n), \quad (AP(1,k) \oplus AE'(n-k) \Subset \langle \mathbb{D} \rangle,$$
$$A\tilde{G}(n-1) \Subset \langle J_{0n}, \mathbb{D} \rangle \quad (k = 2, \ldots, n-1).$$

Let π be the projection of the algebra $A\tilde{P}(1,n)$ onto $A\tilde{O}(1,n)$, F a nonzero subalgebra of $A\tilde{O}(1,n)$, and $\hat{F}$ such subalgebra of $A\tilde{P}(1,n)$ that $\pi(\hat{F}) = F$. If the algebra $\hat{F}$ is $\tilde{P}(1,n)$ conjugated to the algebra $W \subset\!\!\!+ F$, where W is an F-invariant subspace of the space U, then we shall assume $\hat{F}$ to be splitting. If every subalgebra $\hat{F} \subset A\tilde{P}(1,n)$ satisfying $\pi(\hat{F}) = F$ is splitting, we shall say that subalgebra F possesses only splitting extensions in the algebra $A\tilde{F}(1,n)$. The splittability of subalgebras for other algebras of inhomogeneous transformations is defined by analogy. If nothing is reserved, then the investigation of subalgebras of given algebra for conjugation is carried out with respect to the group of inner automorphisms.

The affine group $IGL(n,R)$ is defined as a group of matrices

$$\begin{pmatrix} B & Y \\ 0 & 1 \end{pmatrix}, \tag{2.4}$$

where $B \in GL(n,R)$, $Y \in R^n$. The Lie algebra $AIGL(n,R)$ of this group consists of matrices

$$\begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix},$$

where X is a square matrix of degree n over R. Let 0_a be the zero matrix of degree a, $P_a = E_{a,n+1}$. Let us identify X and $\operatorname{diag}[X, 0_1]$, then $AIGL(n,R) = \langle P_1, \ldots, P_n \rangle \Subset AGL(n,R)$. If $m < n$, then we shall assume that $AGL(m,R)$ consists of the matrices $\operatorname{diag}[\bar{X}, 0_{n+1-m}]$, where $\deg \bar{X} = m$.

Lemma 2.1. *Let F be a completely reducible subalgebra of the Lie algebra $AGL(m,R)$ ($m < n$), which is not semisimple. If Z is a nonzero central element of the algebra F and $\hat{F}$ is the Lie algebra, which is obtained from F by replacing Z by $Z + P_{m+1}$, then the algebra $\hat{F}$ is nonsplitting in $AIGL(n,R)$ with respect to $IGL(n,R)$ conjugation.*

Proof. Let X_0 be a square matrix of the degree m, $T = \operatorname{diag}[X_0, 0_{n-m}]$, $Z = \operatorname{diag}[T, 0_1]$,

$$P_{m+1} = \begin{pmatrix} 0_n & Y_{m+1} \\ 0 & 0_1 \end{pmatrix}.$$

If $\hat{F}$ is a splitting algebra, then there exists the matrix C of the form (2.4) such that $C(Z + P_{m+1})C^{-1} = \operatorname{diag}[T', 0_1]$. It follows that $-BTB^{-1}Y + BY_{m+1} = 0$, which implies that $Y_{m+1} = \left(TB^{-1}\right)Y$. However,

$$TB^{-1} = \begin{pmatrix} X_0 & 0 \\ 0 & 0_{n-m} \end{pmatrix} \cdot \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} X_0 B_1 & X_0 B_2 \\ 0 & 0_{n-m} \end{pmatrix},$$

and therefore

$$\left(TB^{-1}\right)\cdot Y=\begin{pmatrix}\alpha_1\\ \vdots\\ \alpha_m\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

This contradiction proves the lemma.

Proposition 2.1. *Let F be a completely reducible Lie algebra of linear transformations of vector space V over the field R, W is an irreducible F submodule of module V. If $FW\neq 0$, then algebra F possesses only splitting extensions in algebra $W \subset\!\!\!\!+ F$.*

Proof. Since F is a completely reducible subalgebra of the algebra $\mathfrak{gl}(V)$, then $F=Q\oplus Z(F)$, where Q is Levy's factor and $Z(F)$ is the center of F [23]. Using Jacobi identity it is not difficult to conceive that $F=F_1\oplus F_2$, where $F_1W=0$ and every direct surnmand of algebra F_2 annuls in W only zero subspace. Further we may restrict ourselves only with the case when $F=F_2$.

Let $Q\neq 0$; $\hat{F}$ such a subalgebra of the algebra $W \subset\!\!\!\!+ F$ that its projection onto F coincides with F. According to Whitehead's theorem [23] $H^1(Q,W)=0$. From this it follows that the algebra $\hat{F}$ contains Q. Let $J\in Z(F)$, $Y\in W$, $Y\neq 0$, and $J+Y\in\hat{F}$. Since $[Q,Y]\neq 0$, then there exists such an element $X\in Q$ that $[X,Y]\neq 0$. Let $Y_1=[X,Y]$, W_1 be the F submodule of module W, generated by Y_1. Because of the fact that $W_1\neq 0$ and W is the irreducible F module we have $W_1=W$. Hence $J\in\hat{F}$. Therefore, if $Q\neq 0$ then $F\subset\hat{F}$, i.e., $\hat{F}$ is a splitting algebra.

Let $Q=0$, $J\in Z(F)$. Since J annuls in W the only zero subspace is then $[J,W]=W$. Whence for every $Y\in W$ there exists such element $Y'\in W$ that $[J,Y']=Y$. Consequently we may suppose that $J\in\hat{F}$. If $\hat{F}$ contains J_1+Y_1, where $Y_1\in W$ and $Y_1\neq 0$, then $[J,Y_1]\in\hat{F}$ and $[J,Y_1]\neq 0$. Arguing as in the case $Q\neq 0$, we get that $J_1\in\hat{F}$, i.e., $\hat{F}$ is a splitting algebra. The proposition is proved.

Proposition 2.2. *Let*

$$A\tilde{E}(n-1)=\langle G_1,\dots,G_{n-1}\rangle \subset\!\!\!\!+ (AO(n-1)\oplus\langle J_{0n}\rangle),$$

where $G_a=J_{0a}-J_{an}$ $(a=1,\dots,n-1)$. *The subalgebra $F\subset AO(n-1)\oplus\langle J_{0n}\rangle$ possesses only splittable extensions in $A\tilde{E}(n-1)$ if and only if F is a semisimple algebra or F not conjugated to a subalgebra of the algebra $AO(n-2)$.*

Proof. Let $W=\langle G_1,\dots,G_{n-1}\rangle$. Since every subalgebra of the algebra $AO(n-1)$ is completely reducible and $[J_{0n},G_a]=-G_a$, then every subalgebra F of the $AO(n-1)\oplus\langle J_{0n}\rangle$ algebra is also a completely reducible algebra of linear transformations of space W.

Let $W=W_1\oplus\cdots\oplus W_s$ be the decomposition of W into the direct sum of irreducible F modules. If projection F onto $\langle J_{0n}\rangle$ is nonzero, then $[F,W_i]=W_i$ for every $i=1,\dots,s$. Whence according to Proposition 2.1 F has only splittable extensions in $A\tilde{E}(n-1)$. Let us assume that projection of F onto $\langle J_{0n}\rangle$ is equal to 0. If F is a semisimple algebra then by Whitehead's theorem every extension of F

in $A\tilde{E}(n-1)$ is splitting. Let F not be a semisimple algebra. When $\dim W_i \geq 2$ for every $i = 1, \ldots, s$ we have $[F, W_i] \neq 0$ and in view of Proposition 2.1 F possesses only splitting extensions in $A\tilde{E}(n-1)$. When $\dim W_i = 1$ $(1 \leq i \leq s)$, the module W_i is annuled by the algebra F and the algebra F is conjugated to a subalgebra of the algebra $AO(n-2)$. If $Z(F)$ is the center of F and X is a nonzero element of $Z(F)$ then for every nonzero $Y \in W_i$ there exists a subalgebra $\hat{F}$ of the algebra $A\tilde{E}(n-1)$, which is obtained from F by replacing X by $X+Y$. By Lemma 2.1 $\hat{F}$ is not splitting. The proposition is proved.

From Theorem 2.1 and properties of solvable subalgebras of algebra $AO(n)$ it follows that if n is odd then $AO(1,n)$ possesses with respect to $O(1,n)$ conjugation only one maximal solvable subalgebra

$$\langle G_1, \ldots, G_{n-1}, J_{12}, J_{34}, \ldots, J_{n-2,n-1}, J_{0n}\rangle.$$

If n is even then $AO(1,n)$ possesses two maximal solvable subalgebras

$$\langle J_{12}, J_{34}, \ldots, J_{n-1,n}\rangle, \quad \langle G_1, \ldots, G_{n-1}, J_{12}, J_{34}, \ldots, J_{n-3,n-2}, J_{0n}\rangle.$$

Since an extension of an Abelian algebra with the help of a solvable algebra is a solvable algebra itself then maximal solvable subalgebras of the algebra $AP(1,n)$ are of the form $U \Subset F$, where F is the maximal solvable subalgebra of the algebra $AO(1,n)$. Maximal solvable subalgebras of the $A\tilde{P}(1,n)$ are exhausted by algebras $U \Subset (F \oplus \langle \mathbb{D}\rangle)$.

Proposition 2.3. *Let $AH(t)$ be the Cartan subalgebra of the algebra $AO(t)$. The maximal Abelian subalgebras of the algebra $A\tilde{O}(1,n)$ are exhausted with respect to $\tilde{O}(1,n)$ conjugation by the following algebras:* $AH(n-1) \oplus \langle J_{0n}, \mathbb{D}\rangle$; $AH(n) \oplus \langle \mathbb{D}\rangle$ $[n \equiv 0 \pmod 2]$; $\langle G_1, \ldots, G_{n-1}, \mathbb{D}\rangle$; $AH(2a) \oplus \langle G_{2a+1}, \ldots, G_{n-1}, \mathbb{D}\rangle$ $(a = 1, \ldots, [n-2/2])$. *The written algebras are pairwise nonconjugated.*

Proof. If F is a maximal Abelian subalgebra of the algebra $A\tilde{O}(1,n)$ then from Proposition 2.2 $F = \Omega \oplus L \oplus \langle \mathbb{D}\rangle$, where L is a subalgebra of the algebra $AO(l) \oplus \langle J_{0n}\rangle$ or the algebra $AO(n)$ and Ω is a subalgebra of the algebra $\langle G_1, \ldots, G_{n-1}\rangle$. If projection L onto $\langle J_{0n}\rangle$ is different from 0 then $\Omega = 0$. Let projection L onto $\langle J_{0n}\rangle$ be equal to 0. If $L = AH(n)$, then $\Omega = 0$. If $L = AH(2a)$, $1 \leq a \leq [n-2/2]$, then $\Omega = \langle G_{2a+1}, \ldots, G_{n-1}\rangle$. The proposition is proved.

3. Completely reducible subalgebras of the algebra $A\tilde{O}(1,n)$

In this section we shall prove a number of general results on completely reducible subalgebras of the algebra $A\tilde{O}(1,n)$ and shall indicate how to search invariant subspaces of space U for these algebras. The main results of this section are Proposition 3.3 and Theorem 3.1.

Proposition 3.1. *If $n \geq 2$ then any irreducible subalgebra of the algebra $AO(1,n)$ is semisimple and noncompact.*

Proof. Let F be an irreducible subalgebra of the algebra $AO(1,n)$, $Z(F)$ the center of F. If $Z(F) \neq 0$ then $Z(F) = \langle J\rangle$, where $J^2 = -E_{n+1}$. Let X be an arbitrary element of the form (2.2) of the algebra $AO(1,n)$. If $X^2 = -E_{n+1}$, then $\alpha_{01}^2 + \alpha_{02}^2 + \cdots + \alpha_{0n}^2 = -1$. This contradiction proves that $Z(F) = 0$.

If F is a compact algebra then there exists such symmetric matrix C that $C^{-1}FC \subset AO(n+1)$ [24]. Since

$$\exp\left(C^{-1}FC\right) = C^{-1} \cdot \exp F \cdot C$$

then in $O(n+1)$ there exists an irreducible subgroup conserving simultaneously

$$x_0^2 + x_1^2 + \cdots + x_n^2 \quad \text{and} \quad \lambda_0^2 x_0^2 - \lambda_1^2 x_1^2 - \cdots - \lambda_n^2 x_n^2$$

($\lambda_0, \lambda_1, \ldots, \lambda_n$ are nonzero real numbers). This contradiction proves the second part of the proposition.

Proposition 3.2. *A reducible subalgebra of the algebra $A\tilde{O}(1,n)$ is completely reducible if and only if it is conjugated to $L_1 \oplus L_2$ or a subalgebra of algebra $L \oplus \langle \mathbb{D} \rangle$, where L_1 is an irreducible subalgebra of the algebra $AO(1,k)$ ($k \geq 2$), L_2 is a subalgebra of $AO'(n-k) \oplus \langle \mathbb{D} \rangle$ and L is one of the algebras, $AO(n)$, $AO(n-1) \oplus \langle J_{0n} \rangle$.*

Proposition 3.2 follows from Theorem 2.1, Propositions 2.2 and 3.1, and the fact that G_a acts noncompletely reducible onto the space $\langle P_0 + P_n, P_a \rangle$.

Let L be a direct sum of the Lie algebras $L_1, \ldots, L_s$, B a Lie subalgebra of L, and π_i the projection L onto L_i. If $\pi_i(B) = L_i$ for $i = 1, \ldots, s$ then B is called a subdirect sum of $L_1, \ldots, L_s$.

Proposition 3.3. *A completely reducible subalgebra $F \subset A\tilde{O}(1,n)$ has only splitting extensions in $A\tilde{P}(1,n)$ if and only if F is semisimple or F is nonconjugate to subalgebra of one of the algebras, $AO(n)$ or $AO(1,n-1)$.*

The proof of Proposition 3.3 is analogous to that of Proposition 2.2.

Let A_i be a Lie algebra over R ($i = 1,2$), $f : A_1 \to A_2$ is an isomorphism, $B = \{(X, f(X) | X \in A_1\}$. Here B is the Lie algebra over R with "componentwise" operational rules,

$$\begin{aligned}
&[(X, f(X)), (X', f(X'))] = ([X, X'], f([X, X'])), \\
&(X, f(X)) + (X', f(X')) = (X + X', f(X + X')), \\
&\lambda(X, F(X)) = (\lambda X, f(\lambda X)),
\end{aligned}$$

where $X, X' \in A_1$, $\lambda \in R$. Let us denote it as (A_1, A_2, φ). Evidently (A_1, A_2, φ) is the subdirect sum of the algebras A_1 and A_2.

Let W_i be a left A_i module ($i = 1,2$). It is easy to see that W_i is the B module if we put

$$(X, f(X)) \cdot Y_1 = X \cdot Y_1, \quad (X, f(X)) \cdot Y_2 = f(X) \cdot Y_2,$$

for every $X \in A_1$, $Y_i \in W_i$ ($i = 1,2$). Let W be a B submodule of the module $W_1 \oplus W_2$. If $W = W_1' \oplus W_2'$, where $W_i' \subset W_i$ ($i = 1,2$) then W is called a splitting B module. Otherwise the module W is called nonsplitting B module.

Lemma 3.1. *Let $B = (A_1, A_2, \varphi)$ and V_i be a left A_i module ($i = 1,2$). In the B module $V_1 \oplus V_2$ exists a nonsplitting B submodule if and only if the B modules V_1 and V_2 have isomorphic composition factors.*

Proof. Let W be a nonsplitting B submodule of the module $V_1 \oplus V_2$. Then W is the subdirect sum of the modules W_1 and W_2, where $W_i \subset V_i$ ($i = 1,2$). Let $S_i = W \cap V_i$ ($i = 1,2$). Evidently, S_i is the B submodule of the module W. The module $W/(S_1 \oplus S_2)$ is nonsplitting B submodule of the module $V_1/S_1 \oplus V_2/S_2$. Whence we shall assume that $W \cap V_i = 0$ ($i = 1,2$).

For every element $Y_1 \in W_1$ there exists only one such element $Y_2 \in W_2$ such that $(Y_1, Y_2) \in W$. We put $\varphi(Y_1) = Y_2$. The mapping φ is the isomorphism of B modules

W_1 and W_2. In this case modules W_1 and W_2 have isomorphic composition factors. The necessity is proved.

Let W_i be a left B submodule of the module V_i $(i = 1, 2)$ and let the composition factor W_1/N_1 of the module W_1 be isomorphic to the composition factor W_2/N_2 of the module W_2. We denote as W the vector space over the field R generated by the pairs $(Z_1, 0)$, $(0, Z_2)$, (Y_1, Y_2), where $Z_i \in N_i$, $Y_i \in W_i$ $(i = 1, 2)$ and $\varphi(Y_1 + N_1) = Y_2 + N_2$ for the isomorphism $\varphi : W_1/N_1 \to W_2/N_2$. It is easy to see that W is a nonsplitting B module. The sufficiency of the lemma is proved.

Let $\Gamma : X \to X$ be the trivial representation of the completely reducible algebra $F \subset A\tilde{O}(1,n)$, the projection of which onto $AO(1,n)$ has not any invariant isotropic subspaces in the space U or annuls the isotropic subspaces. Then Γ is $O(1,n)$ equivalent to $\operatorname{diag}[\Gamma_1, \ldots, \Gamma_m]$, where Γ_i is an irreducible subrepresentation $(i = 1, \ldots, m)$. One may suppose that algebra $F_i = \{\operatorname{diag}[0, \ldots, \Gamma_i(X), \ldots, 0] | X \in F\}$ is an irreducible subalgebra $A\tilde{O}(W_i)$, where

$$W_i = \langle P_{k_{i-1}+1}, P_{k_{i-1}+2}, \ldots, P_{k_i} \rangle \quad (k_0 = -1,\ k_m = n,\ i = 1, \ldots, m).$$

If $F_i \neq 0$ then we shall call algebra F_i an irreducible part of the algebra F. It is well known that if representations Δ and Δ' of the Lie algebra L by skew-symmetric matrices are equivalent over R, then $C \cdot \Delta(X) \cdot C^{-1} = \Delta'(X)$ for some orthogonal matrix C $(X \in L)$. Whence and from Proposition 3.1 we conclude that if Γ_i and Γ_j are equivalent representations then we can assume that for every $X \in F$ the equality $\Gamma_i(X) = \Gamma_j(X)$ takes place. Having united equivalent nonzero irreducible subrepresentations we shall get a nonzero disjunctive primary subrepresentation of the representation Γ. Corresponding to those subalgebras of the algebra $A\tilde{O}(1,n)$, built by the same rule as the irreducible parts of F_i, we shall call them primary parts of the algebra F. If F coincides with its primary part then F is called a primary algebra.

Theorem 3.1. *Let $K_1, K_2, \ldots, K_q$ be primary parts of a subalgebra F of the algebra $A\tilde{O}(1,n)$, and V a subspace of the space U invariant under F. Then $V = V_1 \oplus \cdots \oplus V_q \oplus \tilde{V}$, where $V_i = [K_i, V] = [K_i, V_i]$, $[K_j, V_i] = 0$ when $j \neq i$ $(i, j = 1, \ldots, q)$, $\tilde{V} = \{X \in V | [F, X] = 0\}$. If the primary algebra K is the subdirect sum of the irreducible subalgebras of the algebras $A\tilde{O}(W_1), A\tilde{O}(W_2), \ldots, A\tilde{O}(W_r)$, respectively, then nonzero subspaces W of the space U with the condition $[K, W] = W$ are exhausted with respect to $O(1,n)$ conjugation by the spaces $W_1, W_1 \oplus W_2, \ldots, W_1 \oplus W_2 \oplus \cdots \oplus W_r$.*

Proof. From the complete reducibility of algebra F it follows that $V = V' \oplus V''$, where V'' is the maximal subspace of the space V, annulled by F. Further we shall suppose that $V = V'$. From Proposition 3.1 one can suppose that $F \subset A\tilde{O}(m)$, $m \leq n$. Let K_i be a subdirect sum of irreducible parts $K_{i1}, \ldots, K_{is_i}$, $V_{ij} = [K_{ij}, V]$, π_a be a projection of V onto $\sum\limits_{j=1}^{s_a} \oplus V_{aj}$.

In view of Lemma 3.1 $\pi_a(V) \subset V$ and that is why

$$V = \sum_{a=1}^{q} \oplus \pi_a(V).$$

Since K_a annuls in $\pi_a(V)$ only the zero subspace, then $[K_a, V] = [K_a, \pi_a(V)] = \pi_a(V)$.

Let primary algebra K be a subdirect sum of irreducible subalgebras of algebras $A\tilde{O}(W_1), A\tilde{O}(W_2), \ldots, A\tilde{O}(W_r)$, respectively. If W is a nonzero subspace of the space

$$\Omega = \sum_{j=1}^{r} \oplus W_j$$

and $[K, W] = W$ then in view of Witt's mapping theorem there exists such isometry $B \in O(\Omega)$ that $B(W) = W_1 \oplus \cdots \oplus W_s$ $(1 \leq s \leq r)$ and the space W_i is invariant under BKB^{-1} $(i = 1, \ldots, s)$. Whence BKB^{-1} is a subdirect sum of irreducible subalgebras of algebras $A\tilde{O}(W_1), A\tilde{O}(W_2), \ldots, A\tilde{O}(W_r)$, respectively. Since irreducible parts of the algebra $L \subset A\tilde{O}(n)$ are defined uniquely up to conjugation then one may consider that $BKB^{-1} = K$. The theorem is proved.

On the basis of Theorem 3.1 the description of splitting subalgebras $\hat{F} \subset A\tilde{P}(1, n)$, for which $\pi(\hat{F})$ is a completely reducible algebra and has no isotropic invariant subspaces in the space U, reduces to the description of irreducible subalgebras of the algebras $AO(1, k)$ and $AO(k)$ $(k = 2, 3, \ldots, n)$. The rest of the cases can be reduced to the case of the algebra $A\tilde{G}(n-1) \Subset \langle J_{0n}, \mathbb{D}\rangle$.

4. On the subalgebras of the extended Galilei algebra

The aim of this section is to study subalgebras of the algebra $A\tilde{G}(n-1)$ with respect to $\tilde{P}(1, n)$ conjugation. The main result concerning this problem is contained in Theorem 4.1. Theorem 4.2 gives a description of all Abelian subalgebras of the algebra $A\tilde{G}(n-1)$. As a corollary, we obtain the list of maximal Abelian subalgebras and one-dimensional subalgebras of the algebra $A\tilde{G}(n-1)$.

The basis elements of the extended Galilei algebra $A\tilde{G}(n-1)$ satisfy the following commutation relations:

$$\begin{aligned}
&[J_{ab}, J_{cd}] = g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} - g_{bd}J_{ac}, \quad [P_a, J_{bc}] = g_{ab}P_c - g_{ac}P_b,\\
&[P_a, P_b] = 0, \quad [G_a, J_{bc}] = g_{ab}G_c - g_{ac}G_b, \quad [G_a, G_b] = 0, \quad [P_a, G_b] = \delta_{ab}M,\\
&[P_a, M] = [G_a, M] = [J_{ab}, M] = 0, \quad [P_0, J_{ab}] = [P_0, M] = [P_0, P_a] = 0,\\
&[P_0, G_a] = P_a \quad (a, b, c, d = 1, \ldots, n-1).
\end{aligned}$$

Let $V_1 = \langle G_1, \ldots, G_{n-1}\rangle$ be a Euclidean space with orthonormal basis $G_1, \ldots,$ G_{n-1}, $V_2 = [P_0, V_1]$ $(n \geq 3)$, $\mathfrak{M} = V_1 + V_2 + \langle P_0, M\rangle$. We settle on identifying the group $O(n-1)$ with the isometry group $O(V_1)$, $O(V_2)$. If W is a subspace of V_1 and dim $W = k$ then according to Witt's theorem for every a, $0 \leq a \leq n-k-1$, there exists an isometry $B_a \in O(V_1)$ such that

$$B_a(W) = V_1(a+1, a+k) = \langle G_{a+1}, G_{a+2}, \ldots, G_{a+b}\rangle.$$

Further, in spaces V_1, V_2 we shall consider only subspaces $V_1(a, b)$, $V_2(a, b) = [P_0,$ $V_1(a, b)]$. We call them elementary spaces. The basis $G_a, G_{a+1}, \ldots, G_b$ of the space $V_1(a, b)$ and the basis $P_a, P_{a+1}, \ldots, P_b$ of the space $V_2(a, b)$ we shall call canonical.

Let W_1, W_2 be subspaces of some vector space W over the field R and $W_1 \cap W_2 =$ 0. If $\varphi : W_1 \to W_2$ is an isomorphism then we denote as (W_1, W_2, φ) the space $\{Y + \varphi(Y) | Y \in W_1\}$. As $I(W_1, W_2)$ we denote the isomorphism of elementary spaces W_1 and W_2, by which the canonical basis of W_1 is mapped to the canonical basis of W_2 with numeration of the basis of elements maintained.

Let $AG(n-1) = A\tilde{G}(n-1)/\langle M\rangle$. For the generators of the $AG(n-1)$ we preserve the notation of the generators of the algebra $A\tilde{G}(n-1)$. By τ, τ_0, τ_1, and τ_2 we denote the projection of $A\tilde{G}(n-1)$ and $AG(n-1)$ onto $AO(n-1)\oplus\langle P_0\rangle$, P_0, V_1, and V_2, respectively.

Let F be a subalgebra of the $AO(n-1)\oplus\langle P_0\rangle$, $\hat{F}$ an subalgebra of the $AG(n-1)$ such that $\tau(\hat{F}) = F$. If algebra $\hat{F}$ is conjugated to the algebra $W \subset\!\!\!\!+ F$, where W is the F-invariant subspace of space $V_1 + V_2$, then $\hat{F}$ is called splitting in the algebra $AG(n-1)$. The notion of a splitting subalgebra of the algebra $A\tilde{G}(n-1)$ is defined analogously.

Proposition 4.1. *Let L_1 be a subalgebra of the $AO(n-1)$, L_2 be a subalgebra of the $\langle P_0\rangle$, and F be the subdirect sum of L_1 and L_2. If $P_0 \notin F$ then the algebra F only has splitting extensions in the algebra $AG(n-1)$ if and only if L_1 is a semisimple algebra or L_1 is not conjugated to any subalgebra of the algebra $AO(n-2)$. When $P_0 \in F$, the algebra F only has splitting extensions in the $AG(n-1)$ if and only if L_1 is not conjugated to any subalgebra of the algebra $AO(n-2)$.*

Proof. If L_1 is a semisimple algebra and $L_2 = \langle P_0\rangle$ then by Whitehead's theorem [23] $P_0 \in F$. Let us assume that $L_2 = \langle P_0\rangle$ and $P_0 \notin F$. Let $\hat{F}$ be an subalgebra of the $AG(n-1)$ such that $\tau(\hat{F}) = F$. If L_1 is not conjugated to any subalgebra of the $AO(n-2)$ then by Proposition 2.2 the algebra $\hat{F}$ is splitting. If L_1 is conjugated to some subalgebra of $AO(n-2)$ then $F = \langle X\rangle \oplus F_1$ where $X \neq 0$, $\langle X\rangle$, and F_1 are subalgebras of the algebra $AO(n-2)\oplus\langle P_0\rangle$. The algebra

$$\hat{F} = \langle P_1,\dots,P_{n-1},G_1,\dots,G_{n-2},X+G_{n-1}\rangle \subset\!\!\!\!+ F_1$$

is not splitting by Lemma 2.1. The case $L_2 = 0$ can be treated similarly.

Let $P_0 \in F$. If $L_1 \subset AO(n-2)$ then algebra $\langle P_0 + G_{n-1}\rangle \subset\!\!\!\!+ L_1$ is nonsplitting. If L_1 is not conjugated to any subalgebra of the algebra $AO(n-2)$ then by way of complete reducibility of the algebra L_1 we get that $P_0 \in \hat{F}$ and whence algebra $\hat{F}$ is splitting. The proposition is proved.

Proposition 4.2. *The subalgebra F of the algebra $AO(n-1)\oplus\langle P_0\rangle$ has only splitting extensions in the $A\tilde{G}(n-1)$ if and only if F is a semisimple algebra.*

Lemma 4.1. *Let $W_1 = \langle Y_1,\dots,Y_m\rangle$, $W_2 = \langle Z_1,\dots,Z_m\rangle$ be Euclidean spaces over the field R, $O(W_i)$ the isometry group of W_i $(i = 1,2)$, $0 < \alpha_1 \le \alpha_2 \le \dots \le \alpha_t$, $S_0^t = 0$, $S_j^t = \langle Z_{t+1},\dots,Z_{t+j}\rangle$ $(j = 1,\dots,m-t)$. The subspaces of the space $W_1\oplus W_2$ are exhausted with respect to $O(W_1)\times O(W_2)$ conjugation by the following spaces:*

$$O,\ \langle Y_1,\dots,Y_r\rangle,\ \langle Z_1,\dots,Z_s\rangle,\ \langle Y_1,\dots,Y_r,Z_1,\dots,Z_s\rangle\ (r,s = 1,\dots,m),$$
$$\langle Y_1,\dots,Y_k,Y_{k+1}+\alpha_1 Z_1,\dots,Y_{k+t}+\alpha_t Z_t\rangle \oplus S_j^t$$
$$(k = 1,\dots,m-1,\ t = 1,\dots,m-k,\ j = 0,1,\dots,m-t),$$
$$\langle Y_1+\alpha_1 Z_1,\dots,Y_t+\alpha_t Z_t\rangle \oplus S_j^t\ (t = 1,\dots,m,\ j = 0,1,\dots,m-t).$$

Proof. Let N be a subspace of $W_1\oplus W_2$ and $N \neq W_1'\oplus W_2'$, where W_i' is a subspace of W_i $(i = 1,2)$. If $B_i = N\cap W_i$, N_i is a projection of N onto W_i $(i = 1,2)$ and then $N_1/B_1 \cong N_1/B_2$. Let dim $B_1 = k$. By Witt's theorem the space B_1 is conjugated to the space $\langle Y_1,\dots,Y_k\rangle$. If dim $(N_1|B_1) = t$ then N contains elements $Y_{k+j} + \alpha_{1j}Z_1 + \dots + \alpha_{tj}Z_t$ $(j = 1,\dots,t)$, and moreover the matrix $A = (\alpha_{ij})$ is nonsingular. The matrix A can be represented uniquely in the form CT, where C is an orthogonal matrix and T is a positively definite symmetric matrix.

The isometry $\mathrm{diag}\,[E_m, C^{-1}, E_{m-t}]$ maps N onto the space to which the matrix $C^{-1}(CT) = T$ corresponds. There exists such orthogonal matrix C_1 that $C_1TC_1^{-1} = \mathrm{diag}\,[\lambda_1, \ldots, \lambda_t]$. The isometry $\mathrm{diag}\,[E_k, C_1, E_{m-k-t}, C_1, E_{m-t}]$ maps N onto the space to which the matrix $C_1TC_1^{-1}$ corresponds. Therefore N is conjugated to the space

$$B_1 \oplus \langle Y_{k+1} + \alpha_1 Z_1, \ldots, Y_{k+t} + \alpha_t Z_t\rangle \oplus B_2,$$

where $0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_t$. The lemma is proved.

Let K be the primary subalgebra of the algebra $AO(n-1)$ which is a subdirect sum of irreducible subalgebras of the algebras $AO(V_1(1,q))$, $AO(V_1(q+1,2q))$, …, $AO(V_1((r-1)q+1,rq))$, respectively, and W nonzero subspace of the space $\mathfrak{M}$ with the property $[K,W] = W$. If $\tau_1(W) = 0$ then by way of Theorem 3.1 W is conjugated to the space $V_2(1,iq)$ $(1 \le i \le r)$. If $\tau_2(W) = 0$ then W is conjugated to $V_1(1,iq)$ $(1 \le i \le r)$. Let us suppose that $\tau_1(W) \ne 0$, $\tau_2(W) \ne 0$. Then W is a subdirect sum of $\tau_1(W)$, $\tau_2(W)$, where $\tau_1(W) = V_1(1,m)$ and $\tau_2(W)$ coincides with $V_2(1,k)$ or $V_2(m+1,m+l)$ or a subdirect sum of $V_2(1,k)$ and $V_2(m+1,m+l)$ $(k \le m)$. Every number of k, m, and l is divisible by q. Let us consider the case when $\tau_2(W)$ is a subdirect sum of $V_2(1,k)$ and $V_2(m+1,m+l)$. In the space W we choose the basis in the following form:

$$\begin{aligned}&G_a + \alpha_a^i P_i, \quad \beta_c^i P_i\\ &(a = 1,\ldots,m,\ c = m+1,\ldots,m+t,\ i = 1,\ldots,k,m+1,\ldots,m+l).\end{aligned} \tag{4.1}$$

The coefficients of the decomposition we write down as the corresponding columns of the matrix

$$\Gamma = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix},$$

having $m+t$ columns and $k+l$ lines. We call the matrix Γ a coupling matrix of elementary spaces in the space W. With the coupling matrix we shall carry out the transformations corresponding to definite $O(n-1)$ automorphisms and transformations to new bases of the form (4.1). Let $C_1 \in O(k)$, $C_2 \in O(m-k)$, $C_3 \in O(l)$, $S = \mathrm{diag}\,[C_1, C_2]$, T be a $t \times m$ matrix, and T_2 a nonsingular matrix of degree t. The most general admissible transformations of the coupling matrix have the form

$$\begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \end{pmatrix} \to \begin{pmatrix} C_1A_1S^{-1} + C_1B_1T_1 & C_1B_1T_2 \\ C_3A_2S^{-1} + C_3B_2T_1 & C_3B_2T_2 \end{pmatrix}.$$

If $B_2 \ne 0$ then according to Theorem 3.1 for some matrices C_3, T_2, the following equality is correct:

$$C_3B_2T_2 = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_1 \end{pmatrix},$$

where $\Delta_1 = \mathrm{diag}\,(\mu_1E_q, \ldots, \mu_aE_q]$, $\mu_1 = \cdots = \mu_a = 1$. By this transformation algebra K is left invariant. Applying Theorem 3.1 again we get that with $k = m$ the matrix A_2 can be transformed into matrix

$$\begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where Δ_2 is a square matrix of degree bq. For simplicity we shall assume that Δ_2 is a coupling matrix of elementary spaces in the subdirect sum of the spaces $V_1(1,bq)$ and $V_2(bq+1,2bq)$. One can admit that

$$K = \operatorname{diag}[A^b, A^b] = \{\operatorname{diag}[\underbrace{X,\ldots,X}_{2b}|X \in A\},$$

where A is an irreducible subalgebra of the algebra $AO(q)$. Since for every matrix $Y \in A^b$ the equality $\Delta_2 Y = Y\Delta_2$ takes place then $\Delta_2 = QS$, where S is a symmetric matrix, Q is an orthogonal matrix, and $Y \cdot Q = Q \cdot Y$. Applying the automorphism $\operatorname{diag}[E, Q^{-1}]$ we transform the coupling matrix Δ_2 into S. There exists such matrix $C \in O(bq)$ that

$$CSC^{-1} = \operatorname{diag}[\lambda_1 E_{(1)}, \lambda_2 E_{(2)}, \ldots, \lambda_t E_{(t)}],$$

where $\lambda_i \neq \lambda_j$ when $i \neq j$, and $E_{(i)}$ is the unit matrix $(i,j = 1,\ldots,t)$. The automorphism $\operatorname{diag}[C,C]$ transforms K into $\operatorname{diag}[CA^bC^{-1}, CA^bC^{-1}]$ and the coupling matrix S into CSC^{-1}. If $Y \in CA^bC^{-1}$ then $Y(CSC^{-1}) = (CSC^{-1})Y$. Whence $Y = \operatorname{diag}[Y_1, Y_2, \ldots, Y_t]$, where $\deg Y_i = \deg E_{(i)}$. The further decomposition of the blocks Y_i by $O(2bq)$ automorphisms $\operatorname{diag}[\bar{C}, \bar{C}]$, where $\bar{C} = \operatorname{diag}[C_1, \ldots, C_t]$, $\deg C_i = \deg E_{(i)}$ does not change the coupling matrix. Since irreducible parts of an algebra are defined uniquely then by the considered transformations of the coupling matrix the algebra K is left invariant. That is why one can suppose that with $k = m$

$$C_3A_2S^{-1} + C_3B_2T_1 = \begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\Delta_2 = \operatorname{diag}[\lambda_1 E_q, \ldots, \lambda_b E_q]$, $0 < \lambda_1 \leq \cdots \leq \lambda_b$, and $(a+b)q = l$ or $\lambda_1 = \cdots = \lambda_b = 0$ and $aq = l$. If $B_1 \neq 0$ then for some C_1, T_2 we have

$$C_1B_1T_2 = \begin{pmatrix} 0 & 0 \\ 0 & \Delta_3 \end{pmatrix},$$

where $\Delta_3 = \operatorname{diag}[E_q, \ldots, E_q]$.

The complete classification of coupling matrices one can get for large n.

Further we shall use the following notation:

$$\mathfrak{M} = \langle P_0, M, P_1, \ldots, P_{n-1}, G_1, \ldots, G_{n-1}\rangle, \qquad m = [(n-1)/2],$$

$$\Gamma(n-1) = \left\{ \sum_{i=1}^{m} \gamma_i J_{2i-1,2i} | \gamma_i = 0, 1 \right\},$$

$X_a \cap X_b = 0$ if $X_a, X_b \in \Gamma(n-1)$ and have no common summand.

Lemma 4.2. *Let $T = \alpha_1 X_1 + \cdots + \alpha_k X_k + Z$, $Z = \beta J_{0n} + \gamma \mathbb{D} + \delta P_0$, where $X_i \in \Gamma(n-1)$, $\alpha_i \neq 0$, $\alpha_i^2 \neq \alpha_j^2$, $X_i \neq X_j$ when $i \neq j$ $(i,j = 1,\ldots,k)$. If W is a subspace of the space $\mathfrak{M}$ and $[T,W] \subset W$ then $W = W_1 \oplus \cdots \oplus W_k \oplus \tilde{W}$, where $W_i = [X_i, W] = [X_i, W_i]$, $[Z, W_i] \subset W_i$, $[X_j, W_i] = 0$ when $j \neq i$, $[X_i, \tilde{W}] = 0$, $[Z, \tilde{W}] \subset \tilde{W}$.*

Proof. Let $X = T - Z$, $\mathfrak{M}' = [X, \mathfrak{M}]$, $\tilde{\mathfrak{M}} = \{Y \in \mathfrak{M} | [X,Y] = 0\}$, W' be a projection of W onto $\mathfrak{M}'$, and $\tilde{W}$ a projection of W onto $\tilde{\mathfrak{M}}$. Evidently, $\mathfrak{M} = \mathfrak{M}' \oplus \tilde{\mathfrak{M}}$ (as spaces). Since composition factors of the $\langle Z\rangle$ module $\mathfrak{M}$ are one dimensional, then the composition factors of the $\langle Z\rangle$ module $\tilde{W}$ are one dimensional, too. Let $\mathfrak{M}(P)=\{P_a\in\mathfrak{M}|[X,P_a]\neq 0\}$. It is easy to see that $\langle\mathfrak{M}(P)\rangle$ and $\mathfrak{M}'/\langle\mathfrak{M}(P)\rangle$ can be represented as direct sums of two-dimensional irreducible $\langle T\rangle$ submodules. Whence the dimensions of composition factors of the $\langle T\rangle$ module W' are equal to 2, too. When we now apply Lemma 3.1 we conclude that $W=W'\oplus\tilde{W}$.

Let $\mathfrak{M}_i=[X_i,\mathfrak{M}]$ and W_i be a projection of W' onto $\mathfrak{M}_i$. Clearly $\mathfrak{M}'=\mathfrak{M}_1\oplus\cdots\oplus\mathfrak{M}_k$. At first let us establish that $[Z,W_i]\subset W_i$. Since for any $Y_i\in W_i$ we have $[J_{0n}-\mathbb{D},Y_i]=-Y_i$, then we may assume that $\beta=0$. Obviously

$$[T,[T,Y_i]]=-\alpha_i^2Y_i+2\alpha_i[X_i,[Z,Y_i]]+\gamma[Z,Y_i].$$

Let

$$Y_i'=2\alpha_i[X_i,[Z,Y_i]]+\gamma[Z,Y_i],$$
$$Y_i''=2\alpha_i[X_i,[Z,Y_i']]+\gamma[Z,Y_i'].$$

The space W_i contains Y_i', Y_i''. It is easy to check that

$$Y_i''=4\alpha_i\gamma^2[X_i,[Z,Y_i]]+\gamma(\gamma^2-4\alpha_i^2)[Z,Y_i].$$

The determinant constructed by the coefficients of $[X_i,[Z,Y_i]]$, $[Z,Y_i]$ in Y_i', Y_i'' is equal to $-2\alpha_i\gamma(\gamma^2+4\alpha_i^2)$. If $\gamma\neq 0$ then $[Z,Y_i]\in W_i$. If $\gamma=0$ then W_i contains $Y_i'=[X_i,[\delta P_0,Y_i]]$ and $Y_i''=[T,Y_i']=-\alpha_i[\delta P_0,Y_i]$.

In the composition factors of the $\langle T\rangle$ module $\mathfrak{M}_i$ one can choose the basis so that the matrix of the operator T is one of the matrices

$$\begin{pmatrix}\gamma & -\alpha_i\\ \alpha_i & \gamma\end{pmatrix},\quad \begin{pmatrix}-\beta & -\alpha_i\\ \alpha_i & -\beta\end{pmatrix}.$$

If for $i\neq j$ the modules $\mathfrak{M}_i$ and $\mathfrak{M}_j$ are possessed by isomorphic composition factors then one of the following conditions is satisfied: $\alpha_i^2=\alpha_j^2$; $2\gamma=-2\beta$, $\gamma^2+\alpha_i^2=\beta^2+\alpha_j^2$. Since it is impossible then on the basis of Lemma 3.1 we conclude that $W'=W_1\oplus\cdots\oplus W_k$. The lemma is proved.

Proposition 4.3. *Let L_1 be a subalgebra of the $AO(n-1)$, $L_2=\langle\beta J_{0n}+\gamma\mathbb{D}+\delta P_0\rangle$, and F a subdirect sum of L_1 and L_2. If W is a subspace of $\mathfrak{M}$ and $[F,W]\subset W$ then $[L_j,W]\subset W$ $(j=1,2)$.*

This is proved by virtue of Lemma 4.2.

Theorem 4.1. *Let $V_1=\langle G_1,\ldots,G_{n-1}\rangle$, $V_2=[P_0,V_1]$, $V_{1,a}$ be a subspace of V_1, $V_{2,a}=[P_0,V_{1,a}]$; $K_1,K_2,\ldots,K_q$ be primary parts of nonzero subalgebra L_1 of the algebra $AO(n-1)$; $\mathfrak{R}$ be the maximal subalgebra of algebra $\mathfrak{M}$, annulled by L_1; and L_2 be a subalgebra of the algebra $\mathfrak{R}\Subset\langle J_{0n},\mathbb{D}\rangle$. If F is the subdirect sum of L_1 and L_2, and W is a subspace of $\mathfrak{M}$ invariant under F, then $W=W_1\oplus\cdots\oplus W_q\oplus\tilde{W}$, where $W_i=[K_i,W]=[K_i,W_i]$, $[L_2,W_i]\subset W_i$, $[K_j,W_i]=0$ when $j\neq i$, $[K_i,\tilde{W}]=0$, $[L_2,\tilde{W}]\subset\tilde{W}$ $(i,j=1,\ldots,q)$.*

If a primary algebra K is a subdirect sum of irreducible subalgebras of the algebras $AO(V_{1,1},\ldots,AO(V_{1,r})$, respectively, then nonzero subspaces W of the space $\mathfrak{M}$ with the property $[K,W]=W$ are conjugated to

$$\sum_{i=1}^{a}V_{1,i},\quad \sum_{i=1}^{a}V_{2,i}\quad (a=1,\ldots,r)$$

or to subdirect sums of such spaces

$$\sum_{i=1}^{\tilde{a}} V_{1,i} \quad \textit{and} \quad \sum_{i=1}^{\tilde{b}} V_{2,i}; \quad \sum_{i=1}^{a} V_{1,i} \quad \textit{and} \quad \sum_{i=a+1}^{c} V_{2,i};$$

$$\sum_{i=1}^{a} V_{1,i}, \quad \sum_{i=1}^{b} V_{2,i}, \quad \textit{and} \quad \sum_{i=a+1}^{c} V_{2,i}$$

$$(\tilde{a} = 1,\ldots,r,\ \tilde{b} = 1,\ldots,\tilde{a},\ a = 1,\ldots,r-1,\ b = 1,\ldots,a,\ c = a+1,\ldots,r).$$

The subdirect sums of the spaces

$$\sum_{i=1}^{a} V_{1,i}, \quad \sum_{i=a+1}^{c} V_{2,i}$$

are exhausted with respect to $O(n-1)$ conjugation by the following spaces:

$$\sum_{i=1}^{a} V_{1,i} \oplus \sum_{j=a+1}^{c} V_{2,j};$$

$$\sum_{i=1}^{b} (V_{1,i}, V_{2,a+1}, \lambda_i I(V_{1,i}, V_{2,a+1})) \oplus \sum_{j=b+1}^{a} V_{1,j} \oplus \sum_{k=a+b+1}^{c} V_{2,k}$$

$$(0 < \lambda_1 \leq \cdots \leq \lambda_b,\ b = 1,\ldots,\min\{a, c-a\}).$$

The written spaces are mutually nonconjugated.

Proof. Let $Q = [L_1, W]$, S be a projection of W onto $\mathfrak{R}$. It is easy to see that W is the subdirect sum of Q and S. Since the composition factors of the L_2 module $\mathfrak{R}$ are one dimensional and the composition factors of the L_1 module $[L_1, \mathfrak{M}]$ have dimension not less than 2 then in view of Lemma 3.1 $W = Q + S$. In virtue of Proposition 4.3 $[L_2, Q] \subset Q$. We can show, as in Theorem 3.1, that $Q = W_1 \oplus \cdots \oplus W_q$, where $W_i = [K_i, Q]$, $W_i = [K_i, W_i]$ $(i = 1,\ldots,q)$. The truthfulness of the further statements is established earlier when considering the transformations of the coupling matrix of elementary spaces in the space W. The theorem is proved.

Theorem 4.2. *Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$, $\alpha_1 = 0$, and $\alpha_s \in \{0,1\}$, $AH(0) = 0$, $AH(2d) = \langle J_{12}, J_{34}, \ldots, J_{2d-1,2d}\rangle$, and L be a nonzero Abelian subalgebra of the algebra $A\tilde{G}(n-1)$. If the projection $\tau_0(L)$ of the algebra L onto $\langle P_0\rangle$ is equal to 0 then L is conjugated to the subdirect sum of the algebras L_1, L_2, L_3, and L_4, where $L_1 \subset AH(2d)$ $(0 \leq d \leq m)$, $L_2 = 0$ or $L_2 = \langle G_{2d+1} + \alpha_1 P_{2d+1}, G_{2d+2} + \alpha_2 P_{2d+2}, \ldots, G_{2d+s} + \alpha_s P_{2d+s}\rangle$, $L_3 = 0$ or $L_3 = \langle P_{2d+s+1}, \ldots, P_l\rangle$, $L_4 = 0$ or $L_4 = \langle M\rangle$. If $\tau_0(L) \neq 0$ then L is conjugated to the subdirect sum of the algebras L_1, L_2, L_3, and L_4, where $L_1 \subset AH(2d)$, $L_2 = \langle P_0 + \alpha G_{2d+1}\rangle$ $(\alpha \in \{0,1\})$, $L_3 = 0$ or $L_3 = \langle P_r, \ldots, P_t\rangle$, $L_4 = 0$ or $L_4 = \langle M\rangle$ $(0 \leq d \leq m;\ r = 2d+1$ when $\alpha = 0$; $r = 2d+2$ when $\alpha = 1)$.*

Proof. Let

$$X_i = G_i + \sum_{j=2d+1}^{2d+s} \beta_{ji} P_j, \quad L = \langle X_{2d+1}, \ldots, X_{2d+s}\rangle.$$

Obviously, $[X_i, X_k] = (\beta_{ki} - \beta_{ik})M$. Since L is an Abelian algebra then $\beta_{ik} = \beta_{ki}$ and therefore $B = (\beta_{ik})$ $(i, k = 2d+1, \dots, 2d+s)$ is a symmetric matrix. Hence, there exists a matrix $C \in O(s)$ such that $CBC^{-1} = \operatorname{diag}[\lambda_1, \dots, \lambda_s]$. Whence we can assume up to conjugacy under $O(n-1)$ that $X_{2d+j} = G_{2d+j} + \lambda_j P_{2d+j}$ $(j = 1, \dots, s)$. $O(n-1)$ automorphisms permit us to change the numeration of generators $G_{2d+1}, \dots, G_{2d+s}$. That is why we can suppose that $\lambda_1 \le \dots \le \lambda_s$. Applying the automorphism $\exp(-\lambda_1 P_0)$ we get generators $G_{2d+j} + \mu_j P_{2d+j}$ $(j = 1, \dots, s)$, where $\mu_1 = 0$, $0 \le \mu_2 \le \dots \le \mu_s$. If $\mu_s > 0$ then $\mu_s = \exp\theta$ $(\theta \in R)$. Evidently,

$$\exp(-\theta J_{0n})(G_{2d+j} + \mu_j P_{2d+j})\exp(\theta J_{0n}) = \exp\theta \cdot (G_{2d+j} + \mu_j \exp(-\theta) P_{2d+j}).$$

Therefore when $\mu_s > 0$ we can assume that $\mu_s = 1$.

The rest of the assertion of the theorem follows from Proposition 4.1. The theorem is proved.

Corollary 1. *Let*

$$A(r,t) = \langle G_r + \alpha_r P_r, G_{r+1} + \alpha_{r+1} P_{r+1}, \dots, G_t + \alpha_t P_t, M\rangle,$$

where $\alpha_r \le \alpha_{r+1} \le \dots \le \alpha_t$, $a_r = 0$ *and* $\alpha_t = 1$ *when* $\alpha_t \ne 0$. *The maximal Abelian subalgebras of the algebra* $A\tilde{G}(n-1)$ *are exhausted up to conjugacy under* $\tilde{P}(1,n)$ *by the following algebras:*

$$U;\ A(1,n-1);\ A(1,s) \oplus V_2(s+1,n-1)\ (s = 1, \dots, n-2);$$

$$\langle G_1 + P_0, M\rangle \oplus V_2(2,n-1);\ AH(n-2) \oplus \langle G_{n-1} + P_0, M\rangle\ [n \equiv 0\ (\mathrm{mod}\ 2)];$$

$$AH(2d) \oplus \langle P_0\rangle \oplus V_2(2d+1,n)\ (d = 1, \dots, [(n-1)/2]);$$

$$AH(2d) \oplus A(2d+1,n-1)\ (d = 1, \dots, [(n-2)/2]);$$

$$AH(2d) \oplus A(2d+1,s) \oplus V_2(s+1,n-1)\ (d = 1, \dots, [(n-3)/2]);$$

$$AH(2d) \oplus \langle G_{2d+1} + P_0, M\rangle \oplus V_2(2d+2,n-1)\ (d = 1, \dots, [(n-3)/2]).$$

The written algebras are not mutually conjugated.

Corollary 2. *Let* $n \ge 3$, $X_t = \alpha_1 J_{12} + \alpha_2 J_{34} + \dots + \alpha_t X_{2t-1,2t}$; $\alpha_1 = 1$, $0 < \alpha_2 \le \dots \le \alpha_t \le 1$; $t = 1, \dots, [(n-1)/2]$; $s = 1, \dots, [(n-2)/2]$.

The one-dimensional subalgebras of the algebra $A\tilde{G}(n-1)$ *are exhausted with respect to* $\tilde{P}(1,n)$ *conjugation by the following algebras:* $\langle P_0\rangle$; $\langle M\rangle$; $\langle P_1\rangle$; $\langle G_1\rangle$; $\langle G_1 + P_2\rangle$; $\langle G_1 + P_0\rangle$; $\langle X_t\rangle$; $\langle X_t + P_0\rangle$; $\langle X_t + M\rangle$; $\langle X_t + P_{2t+1}\rangle$; $\langle X_s + G_{2s+1}\rangle$; $\langle X_s + G_{2s+1} + P_0\rangle$; $\langle X_r + G_{2r+1} + P_{2r+2}\rangle$ $(r = 1, \dots, [(n-3)/2])$.

The written algebras are not mutually conjugated.

Let

$$\begin{aligned}
&\Phi(0) = \langle M\rangle,\ \Phi(i) = \langle M, P_1, \dots, P_i\rangle,\ \Omega(0) = \langle M, P_0\rangle,\\
&\Omega(i) = \langle M, P_0, P_1, \dots, P_i\rangle,\ V_2(s,t) = \langle P_s, \dots, P_t\rangle\ (s \le t),\\
&\Lambda_{r+1,k+1}(j) = \langle P_{r+d} + \lambda_d P_{k+d} | d = 1, 2, \dots, j\rangle,
\end{aligned} \tag{4.2}$$

where $0 < \lambda_1 \le \dots \le \lambda_j$ $(1 \le j \le k-r)$.

Proposition 4.4. *Let $L = \langle G_1, \dots, G_k \rangle$. The subspaces of the space $U = \langle P_0, P_1, \dots, P_n \rangle$, which are invariant under L, are exhausted with respect to $\tilde{O}(1,n)$ conjugation by the following spaces:*

$$0,\ \Phi(i),\ \Omega(k),\ V_2(k+1,t),\ \Phi(i) \oplus V_2(k+1,t),\ \Omega(k) \oplus V_2(k+1,t),$$
$$\Phi(r) \oplus \Lambda_{r+1,k+1}(j),\ \Phi(r) \oplus \Lambda_{r+1,k+1}(j) \oplus V_2(k+j+1,s),$$

where $i = 0, 1, \dots, k$, $t = k+1, \dots, n-1$, $r = 0, 1, \dots, k-1$, $j = 1, \dots, k-r$, $s = k+j+1, \dots, n-1$.

Proof. Let W be a subspace of the space $\Omega(k)$ invariant under L. Since $[P_a, G_a] = M$ then with $W \neq 0$ we have $M \in W$. The normalizer of the algebra L in $O(n-1)$ contains $O(k)$. It follows from this and Witt's theorem that if $W \neq \langle M \rangle$ and $P_0 \notin W$ then $W = \Phi(i)$ $(1 \le i \le k)$. If $P_0 \in W$ then $W = \Omega(k)$.

For a description of all subspaces of the space U which are invariant under L we shall use the Goursat twist method [25]. Since by Witt's theorem the nonzero subspaces of the space $V_2(k+1, n-1)$ are exhausted with respect to $O(n-1)$ conjugation by the spaces $V_2(k+1,t)$ $(t = k+1, \dots, n-1)$ we need to classify the subdirect sums of the following pairs of spaces $\Omega(k)$, $V_2(k+1,t)$; $\Phi(i)$, $V_2(k+1,t)$ $(i = 0, 1, \dots, k,\ t = k+1, \dots, n-1)$.

Let N be the subdirect sum of $\Omega(k)$ and $V_2(k+1,t)$. If $P_0 + \lambda P_{k+1} \in N$ $(\lambda \neq 0)$ then N contains P_1, $P_1 = -[G_1, P_0 + \lambda P_{k+1}]$, and whence it contains M, too. Let

$$N' = \exp(\theta G_{k+1}) \cdot N \cdot \exp(-\theta G_{k+1}).$$

The space N' contains $P_0 + (\lambda - \theta) P_{k+1} + (\theta^2/2 - \lambda\theta) M$. Since $M \in N'$ then $P_0 + (\lambda - \theta) P_{k+1} \in N'$. Putting $\theta = \lambda$ we get that $P_0 \in N'$ and whence $\Omega(k) \subset N'$. Therefore $N' = \Omega \oplus V_2(k+1, t')$.

Let N be the subdirect sum of $\Phi(i)$ and $V_2(k+1,t)$. If $i = 0$, $M + \lambda P_{k+1} \in N$ $(\lambda \neq 0)$ then N' contains $(1 - \theta\lambda) M + \lambda P_{k+1}$. Putting $1 - \theta\lambda = 0$ we get that $N' = V_2(k+1,t)$. If $i \neq 0$ then $M \in N$. Let us assume that $N \neq \Phi(i) \oplus V_2(k+1,t)$. Then $\Phi(i)/S_1 \cong V_2(k+1,t)/S_2$, where $S_1 = N \cap \Phi(i)$, $S_2 = N \cap V_2(k+1,t)$. Let $\dim(\Phi(i)/S_1) = i - r = j$. Within the conjugation we can assume that $S_1 = \Phi(r)$ and $S_2 = 0$ or $S_2 = V_2(k+j+1,s)$ and that is why by means of Lemma 4.1 N is conjugated to one of the spaces,

$$\Phi(r) \oplus \Lambda_{r+1,k+1}(j);\ \Phi(r) \oplus \Lambda_{r+1,k+1}(j) \oplus V_2(k+j+1,s).$$

The proposition is proved.

5. On subalgebras of the normalizer of isotropic space

In virtue of Theorem 2.1 the normalizer of the isotropic space $\langle P_0 + P_n \rangle$ in $A\tilde{P}(1,n)$ coincides with the algebra $K = A\tilde{G}(n-1) \mathbin{\supset\!\!\!\!+} \langle J_{0n}, \mathbb{D} \rangle$. In this section we shall establish a number of assertions on subalgebras of the algebra K possessing nonzero projection onto $\langle J_{0n}, \mathbb{D} \rangle$. On the grounds of these results in Theorem 5.1 we describe all Abelian subalgebras of the algebra K that are nonconjugate to the subalgebras of $A\tilde{G}(n-1)$. As a corollary, we obtain the list of maximal Abelian subalgebras and one-dimensional subalgebras of the algebra K as well as one-dimensional subalgebras of the algebra $A\tilde{P}(1,n)$.

Further ε denotes the projection of K onto $\langle J_{0n}, \mathbb{D} \rangle$ and ξ denotes the projection of K onto $AO(n-1) \oplus \langle J_{0n}, \mathbb{D} \rangle$.

Proposition 5.1. *Let $L = \langle G_1, \ldots, G_k\rangle$ $(1 \le k \le n-1)$, and F be a subdirect sum of L and $\langle\mathbb{D}\rangle$. The algebra F has only splitting extensions in $A\tilde{P}(1,n)$.*

Proof. Let $\hat{F}$ be a subalgebra of $A\tilde{P}(1,n)$ such that $\pi(\hat{F}) = F$. Up to an $O(n-1)$ automorphism one can assume that $\hat{F}$ contains the generator

$$X_1 = G_1 + \sum_{\nu=0}^{n} \alpha_\nu P_\nu + \gamma\mathbb{D} \quad (\gamma \ne 0).$$

Clearly,

$$\exp\left(\sum_{\mu=0}^{n} b_\mu P_\mu\right)\cdot X_1 \cdot \exp\left(-\sum_{\mu=0}^{n} b_\mu P_\mu\right) = G_1 + \gamma\mathbb{D} + (\alpha_0 - \gamma b_0 + b_1)P_0 +$$
$$+(\alpha_1 + b_0 - b_n - \gamma b_1)P_1 + (\alpha_n + b_1 - \gamma b_n)P_n + \sum_{i=2}^{n-1}(\alpha_i - \gamma b_i)P_i.$$

We put

$$\begin{array}{l} \alpha_0 - \gamma b_0 + b_1 = 0, \quad \alpha_1 + b_0 - b_n - \gamma b_1 = 0, \\ \alpha_n + b_1 - \gamma b_n = 0, \quad \alpha_i - \gamma b_i = 0 \quad (i = 2, \ldots, n-1). \end{array} \tag{5.1}$$

The determinant of coefficients by b_0, b_1, and b_n is equal to $-\gamma^3$. Since $\gamma \ne 0$ then the systems (5.1) has a solution. Therefore one can assume that $X_1 = G_1 + \gamma\mathbb{D}$. Let $a \ne 1$,

$$X_a = G_a + \sum_{\mu=0}^{n} \alpha_\mu P_\mu + \delta\mathbb{D}.$$

Since

$$[X_1, X_a] = -(\alpha_0 - \alpha_n)P_1 - \alpha_1 M + \gamma \sum \alpha_\mu P_\mu,$$
$$[X_1, X_a] - \gamma X_a = -\gamma G_a - \gamma\delta\mathbb{D} - (\alpha_0 - \alpha_n)P_1 - \alpha_1 M,$$

we shall assume that

$$X_a = G_a + \alpha M + \beta P_1 + \delta\mathbb{D}.$$

Then

$$[X_1, X_a] = (\gamma\alpha - \beta)M + \gamma\beta P_1 \quad (2 \le a \le k).$$

If $\gamma\alpha - \beta \ne 0$ then we shall consider that $\alpha = 0$, $\beta \ne 0$. Since

$$[X_1, [X_1, X_a]] = -2\gamma\beta M + \gamma^2\beta P_1,$$

then $\hat{F}$ containts $M - \gamma P_1$, $-2M + \gamma P_1$ and whence $M, P_1 \in \hat{F}$. That is why $G_a + \delta\mathbb{D} \in \hat{F}$.

Let $\gamma\alpha - \beta = 0$. If $\beta \ne 0$ then $P_1 \in \hat{F}$. Since $[X_1, P_1] = [G_1 + \gamma\mathbb{D}, P_1] = -M + \gamma P_1$ then $M \in \hat{F}$ and therefore $G_a + \delta\mathbb{D} \in \hat{F}$. If $\beta = 0$ then $\alpha = 0$. It proves that F is a splitting algebra. The proposition is proved.

The record $F : W_1, \dots, W_s$ means that we deal with the subalgebras $W_1 \subset\!\!\!\!+ F$, …, $W_s \subset\!\!\!\!+ F$.

In virtue of Propositions 4.4 and 5.1 we conclude that the subalgebras of the algebra $\mathfrak{M} \subset\!\!\!\!+ \mathbb{D}$ possessing a nonzero projection onto $\langle \mathbb{D} \rangle$ are exhausted with respect to $\tilde{P}(1,n)$ conjugation by the following algebras [see notations (4.2)]:

$$\begin{aligned}
&\langle \mathbb{D} \rangle :\ 0,\ \Phi(i),\ V_2(s,t)\ (i = 0, 1, \dots, n-1,\ s = 0, 1,\ t = s, s+1, \dots, n);\\
&\langle G_1 + \alpha_1 \mathbb{D}, \dots, G_k + \alpha_k \mathbb{D}, \beta \mathbb{D} \rangle :\ 0,\ \Phi(i),\ \Omega(k),\ V_2(k+1,t),\\
&\Phi(i) \oplus V_2(k+1,t),\ \Omega(k) \oplus V_2(k+1,t),\ \Phi(r) \oplus \Lambda_{r+1,k+1}(j),\\
&\Phi(r) \oplus \Lambda_{r+1,k+1}(j) \oplus V_2(k+j+1,s)\ (k = 1, \dots, n-1,\ i = 0, 1, \dots, k,\\
&t = k+1, \dots, n-1,\ r = 0, 1, \dots, k-1,\ j = 1, \dots, k-r,\\
&s = k+j+1, \dots, n-1).
\end{aligned}$$

These algebras must then be simplified using transformations contained in the normalizer of each algebra in the group of $O(1,n)$ automorphisms. If, for example, the normalizer contains $\exp(\theta J_{12})$ then instead of $\langle G_1 + \alpha_1 \mathbb{D}, G_2 + \alpha_2 \mathbb{D} \rangle$ we can take $\langle G_1 + \alpha_1 \mathbb{D}, G_2 \rangle$.

Proposition 5.2. *Let L be a subalgebra of $AO(n)$, and F be the subdirect sum of L and $\langle \mathbb{D} \rangle$. The algebra F possesses only the splitting extensions in $A\tilde{P}(1,n)$.*

Proposition 5.2 is proved by virtue of Propositions 2.1 and 3.2.

Proposition 5.3. *Let L_1 be a subalgebra of $AO(n-1)$, $L_2 = \langle \mathbb{D}, J_{0n} \rangle$ or $L_2 = \langle \mathbb{D} + \gamma J_{0n} \rangle$, where $\gamma = 0$, $\gamma^2 \neq 0$, $2\gamma + 1 \neq 0$. If F is a subdirect sum of the algebras L_1 and L_2 then every subalgebra $\hat{F}$ of the algebra K with the property $\xi(\hat{F}) = F$ is conjugated to the algebra $(W_1 + W_2) \subset\!\!\!\!+ F$, where $W_1 \subset U$, $W_2 \subset V_1 = \langle G_1, \dots, G_{n-1} \rangle$.*

Proof. Let $L_2 = \langle \mathbb{D}, J_{0n} \rangle$. On the basis of Propositions 2.2 and 5.1 algebra $\hat{F}$ contains the elements

$$X_1 = J_{0n} + \sum_{i=0}^{n} \alpha_i P_i, \quad X_2 = \mathbb{D} + \sum_{j=1}^{n-1} \beta_j G_j.$$

Since $[X_1, X_2] = \sum \gamma_i P_i - \sum \beta_j G_j$ then $\mathbb{D} + \sum \gamma_i P_i \in \hat{F}$. Therefore one can suppose that $\mathbb{D} \in \hat{F}$. Whence $J_{0n} \in \hat{F}$ and $F \subset \hat{F}$.

Let $L_2 = \langle \mathbb{D} + \gamma J_{0n} \rangle$. Since $[\mathbb{D} + \gamma J_{0n}, P_a] = P_a$, $[\mathbb{D} + \gamma J_{0n}, G_a] = -\gamma G_a$ ($a = 1, \dots, n-1$), then by virtue of Proposition 5.2 one can admit that $\hat{F}$ contains the subdirect sum of F and subalgebra of the algebra $\langle P_0, P_n \rangle$. Evidently

$$\begin{aligned}
&\exp(\theta_0 P_0 + \theta_n P_n) \cdot (\mathbb{D} + \gamma J_{0n} + \alpha_0 P_0 + \alpha_n P_n) \cdot \exp(-\theta_0 P_0 - \theta_n P_n) =\\
&\quad = \mathbb{D} + \gamma J_{0n} + (\alpha_0 - \theta_0 + \gamma \theta_n) P_0 + (\alpha_n + \gamma \theta_0 - \theta_n) P_n.
\end{aligned}$$

Since $\gamma^2 \neq 1$, then coefficients by P_0, P_n can be transformed into zero. On the basis of the conditions $\gamma^2 \neq 1$, $[\mathbb{D} + \gamma J_{0n}, \hat{F} \cap \mathfrak{M}] \subset \hat{F} \cap \mathfrak{M}$ it is not difficult to get that $F \subset \hat{F}$.

Let $W = \hat{F} \cap \mathfrak{M}$, $Y = \sum \delta_a G_a + \sum \rho_i P_i \in W$. Since

$$[\mathbb{D} + \gamma J_{0n}, Y] = -\gamma \sum \delta_a G_a - \gamma(\rho_0 P_n + \rho_n P_0) + \sum \rho_i P_i$$

and $\gamma^2 \neq 1$ then one can assume that $Y = \sum \delta_a G_a + \rho_0 P_0 + \rho_n P_n$. By the direct calculations we find that

$$[\mathbb{D} + \gamma J_{0n}, Y] = -\gamma \sum \delta_a G_a + (\rho_0 - \gamma\rho_n) P_0 + (\rho_n - \gamma\rho_0) P_n,$$

$$[\mathbb{D} + \gamma J_{0n}, [\mathbb{D} + \gamma J_{0n}, Y]] = \\ = \gamma^2 \sum \delta_a G_a + (\gamma^2 \rho - 2\gamma\rho_n + \rho_0) P_0 + (\gamma^2 \rho_n - 2\gamma\rho_0 + \rho_n) P_n.$$

The determinant Δ constructed by the coefficients of $\sum \delta_a G_a$, P_0, P_n in Y and the vectors received is equal to $\gamma(2\gamma + 1)(\rho_n^2 - \rho_0^2)$. If $\Delta \neq 0$ then $\sum \delta_a G_a, P_0, P_n \in W$. If $\Delta = 0$ then $\rho_n = \pm\rho_0$. When $\rho_n = \rho_0$ we get that

$$[\mathbb{D} + \gamma J_{0n}, Y] - (1 - \gamma) Y = -\sum \delta_a G_a.$$

If $\rho_n = -\rho_0$ then

$$[\mathbb{D} + \gamma J_{0n}, Y] - (1 + \gamma) Y = (-2\gamma - 1) \sum \delta_a G_a.$$

The proposition is proved.

Proposition 5.4. *The subalgebras of the algebra $\mathfrak{M} \mathbin{\supset\!\!\!\!+} \langle J_{0n}, \mathbb{D}\rangle$ containing J_{0n} or having the property that their projection F onto $\langle J_{0n}, \mathbb{D}\rangle$ coincides with $\langle \mathbb{D} + \gamma J_{0n}\rangle$, where $\gamma \neq 0$, $\gamma^2 \neq 1$, $2\gamma + 1 \neq 0$, are exhausted with respect to $\tilde{P}(1,n)$ conjugation by the following algebras [see notation (4.2)]:*

$$F:\ 0,\ \Phi(a),\ \Omega(a),\ V_2(1,d)\ (a = 0,1,\ldots,n-1,\ d = 1,\ldots,n-1);$$
$$(G_1,\ldots,G_k) \mathbin{\supset\!\!\!\!+} F:\ 0,\ \Phi(i),\ \Omega(k),\ V_2(k+1,t),\ \Phi(i) \oplus V_2(k+1,t),$$
$$\Omega(k) \oplus V_2(k+1,t),\ \Phi(r) \oplus \Lambda_{r+1,k+1}(j),\ \Phi(r) \oplus \Lambda_{r+1,k+1}(j) \oplus V_2(k+j+1,s)$$
$$(i = 0,1,\ldots,k,\ t = k+1,\ldots,n-1,\ r = 0,1,\ldots,k-1,\ j = 1,\ldots,k-r,$$
$$s = k+j+1,\ldots,n-1,\ k = 1,\ldots,n-1).$$

The proof of Proposition 5.4 is based on Proposition 5.3.

Proposition 5.5. *Let L_1 be a subalgebra of $AO(n-1)$, $L_2 = \langle 2\mathbb{D} - J_{0n}\rangle$, F a subdirect sum of L_1 and L_2, and $\hat{F}$ such subalgebra of K that $\xi(\hat{F}) = F$. The algebra $\hat{F}$ is conjugated to the algebra $W \mathbin{\supset\!\!\!\!+} F$, where $W \subset \mathfrak{M}$ and satisfies the following condition: if $Y \in W$ and projection of Y onto $V_1 = \langle G_1,\ldots,G_{n-1}\rangle$ is equal to $\sum \delta_a G_a$ then W contains $\sum \delta_a G_a + \rho P_0$ and ρM or $\sum \delta_a G_a + \rho(P_0 - P_n)$.*

Proposition 5.6. *Let L_1 be a subalgebra of $AO(n-1)$, $L_2 = \langle \mathbb{D} + J_{0n} + \gamma M\rangle$ ($\gamma \in \{0,1\}$), and F the subdirect sum of L_1 and L_2. If a subspace W of the space $\mathfrak{M}$ is invariant under F then $W = W_1 + W_2$, where $W_1 \subset U$, $W_2 \subset V_1$.*

The proof of Propositions 5.5 and 5.6 is similar to that of Proposition 5.3.

Let $\theta = (\gamma_0 - \gamma_n)/2$. Since

$$\exp(\theta P_0) \cdot (\mathbb{D} + J_{0n} + \gamma_0 P_0 + \gamma_n P_n) \cdot \exp(-\theta P_0) = \mathbb{D} + J_{0n} + \frac{1}{2}(\gamma_0 + \gamma_n) M,$$

then further we shall suppose that the projection of the algebra $\hat{F} \subset A\tilde{P}(1,n)$ onto $\langle \mathbb{D} + J_{0n}, P_0, P_n\rangle$ contains $\mathbb{D} + J_{0n} + \alpha M$, where $\alpha \in \{0,1\}$. Proposition 5.6 gives the considerable information on the structure of such algebras.

Proposition 5.7. *Let L_1 be a subalgebra of $AO(n-1)$, $L_2 = \langle \mathbb{D} - J_{0n} + \gamma P_0 \rangle$ ($\gamma \in \{0,1\}$), and F the subdirect sum of the algebras L_1 and L_2. If a subspace W of the space $\mathfrak{M}$ is invariant under F, then W contains its own projection onto $\langle P_0, P_n \rangle$ and $[L_1, W] \subset W$, $[\gamma P_0, W] \subset W$.*

Proof. On the basis of Proposition 4.3 $[L_i, W] \subset W$ ($i = 1,2$). Let $\tilde{\mathfrak{M}} = \{Y \in \mathfrak{M} | [L_1, Y] = 0\}$, and $\tilde{W}$ be a projection of W onto $\tilde{\mathfrak{M}}$. It is easy to see that the matrix $\operatorname{diag}[2,0]$ is the matrix of the operator $\mathbb{D} - J_{0n}$ in the basis $P_0 + P_n$, $P_0 - P_n$ of the space $\langle P_0, P_n \rangle$ and in the basis of the space $\mathfrak{M} | \langle P_0, P_n \rangle$ the matrix of the same operator is the unit one. Whence by Lemma 3.1 we conclude that $\tilde{W}$ contains its own projection onto $\langle P_0, P_n \rangle$. It remains for us to note that for arbitrary

$$Y = \sum_{j=1}^{n-1} (\alpha_j P_j + \beta_j G_j)$$

we have $[\mathbb{D} - J_{0n} + \gamma P_0, Y] = Y + [\gamma P_0, Y]$. The proposition is proved.

Proposition 5.8. *Let F be a subalgebra of the algebra $AO(1,n)$ generated by J_{0n} and G_a, where a runs through some subset I of the set $\{1, 2, \ldots, n-1\}$. If $\hat{F}$ is a subalgebra of $AP(1,n)$ with $\pi(\hat{F}) = F$, then within the conjugation with respect to the group of translations the algebra $\hat{F}$ contains elements G_a ($a \in I$) and $J_{0n} + \sum \delta_i P_i$ ($i = 1, \ldots, n-1$).*

Proposition 5.9. *Let L be a subalgebra of the algebra $AP(1,n)$, $X = J_{ab} + \delta J_{0n} + \beta P_c$, $Y = G_c + \sum \gamma_i P_i$ ($i = 1, \ldots, n$), where $\beta \neq 0$, $\delta \neq 0$, and a, b, and c are different numbers of $\{1, 2, \ldots, n-1\}$. If $X, Y \in L$ then L contains G_c.*

Theorem 5.1. *Let L be an Abelian subalgebra of the algebra K and $\varepsilon(L) \neq 0$. If $\varepsilon(L) = \langle J_{0n} \rangle$ then L is $\tilde{P}(1,n)$ conjugated to the subdirect sum of algebras L_1, L_2, $\langle J_{0n} \rangle$, where $L_1 \subset AH(2d)$, $L_2 = 0$, or $L_2 = \langle P_{2d+1}, \ldots, P_{2d+s} \rangle$. If $\varepsilon(L) = \langle \mathbb{D} \rangle$ then L is $\tilde{P}(1,n)$ conjugated to the subdirect sum of L_1, L_2, $\langle \mathbb{D} \rangle$, where $L_1 \subset AH(2d)$, $L_2 = 0$ or $L_2 = \langle G_{2d+1}, \ldots, G_{2d+s} \rangle$. If $\varepsilon(L) = \langle \mathbb{D}, J_{0n} \rangle$ or $\varepsilon(L) = \langle \mathbb{D} + \gamma J_{0n} \rangle$, where $\gamma \neq 0$, $\gamma^2 \neq 1$ then L is $\tilde{P}(1,n)$ conjugated to the subdirect sum of algebras $\varepsilon(L)$ and $L_1 \subset AH(2d)$. If $\varepsilon(L) = \langle \mathbb{D} + J_{0n} \rangle$, then L is conjugated to the subdirect sum of the algebras L_1, L_2, L_3, where $L_1 \subset AH(2d)$, $L_2 \subset \langle M \rangle$, $L_3 = \langle J_{0n} + \mathbb{D} \rangle$.*

Proof. If $\varepsilon(L) = \langle J_{0n} \rangle$ then in view of Propositions 2.2 and 4.3 the algebra L contains its own projection onto $\langle M, P_0 - P_n, G_1, \ldots, G_{n-1} \rangle$. Since $[J_{0n}, G_a] = -G_a$, $[J_{0n}, M] = -M$, $[J_{0n}, P_0 - P_n] = P_0 - P_n$ then this projection is equal to zero. Therefore L is the subdirect sum of $L_1 \subset AH(2d)$ and $L_2 \subset \langle P_{2d+1}, \ldots, P_{n-1} \rangle$. If $L_2 \neq 0$ then by Witt's theorem L_2 is conjugated to $\langle P_{2d+1}, \ldots, P_{2d+s} \rangle$.

If $\varepsilon(L) = \langle \mathbb{D} \rangle$ then in virtue of Propositions 4.3 and 5.2 the projection of L onto U is equal to 0.

If $\varepsilon(L) = \langle \mathbb{D}, J_{0n} \rangle$ or $\varepsilon(L) = \langle \mathbb{D} + \gamma J_{0n} \rangle$, where $\gamma \neq 0$, $\gamma^2 \neq 1$, $2\gamma + 1 \neq 0$, then by Proposition 5.3 the algebra L is conjugated to the subdirect sum of the algebras $\varepsilon(L)$ and $L_1 \subset AH(2d)$. With $\varepsilon(L) = \langle 2\mathbb{D} - J_{0n} \rangle$ Proposition 5.5 is applicable.

Let $\varepsilon(L) = \langle \mathbb{D} - J_{0n} \rangle$. On the basis of Propositions 2.2 and 4.3 the projection of L onto $\langle G_1, \ldots, G_{n-1} \rangle$ is equal to 0. Applying the $O(1,n)$ automorphism corresponding to the matrix $\operatorname{diag}\,[1, \ldots, 1, -1]$ we get that $\varepsilon(L) = \langle \mathbb{D} + J_{0n} \rangle$. According to Proposition 5.2 the projection of L onto $\langle P_1, \ldots, P_{n-1} \rangle$ is equal to 0. Since $[J_{0n} + \mathbb{D}, P_0 + P_n] = 0$, $[J_{0n} + \mathbb{D}, P_0 - P_n] = 2(P_0 - P_n)$ then by Propositions 2.1 and 4.3 the projection of L onto $\langle P_0, P_n \rangle$ belongs to $\langle P_0 + P_n \rangle$. The theorem is proved.

Corollary 1. *The maximal Abelian subalgebras of the algebra K with the condition $\varepsilon(K) \neq 0$ are exhausted with respect to $\tilde{P}(1,n)$ conjugation by the following algebras:*

$$AH(n-1) \oplus \langle J_{0n}, \mathbb{D}\rangle,\ AH(n-1) \oplus \langle M, J_{0n}, \mathbb{D}\rangle,$$

$$AH(2d) \oplus \langle P_{2d+1}, \ldots, P_{n-1}, J_{0n}\rangle,$$

$$AH(2d) \oplus \langle G_{2d+1}, \ldots, G_{n-1}, \mathbb{D}\rangle\ (d = 0, 1, \ldots, [(n-2)/2]).$$

The written algebras are not conjugated mutually.

Corollary 2. *Let $n \geq 3$, $X_t = \alpha_1 J_{12} + \alpha_2 J_{34} + \cdots + \alpha_t J_{2t-1,2t}$; $\alpha_1 = 1$, $0 < \alpha_2 \leq \cdots \leq \alpha_t \leq 1$; $t = 1, \ldots, [(n-1)/2]$; $s = 1, \ldots, [(n-2)/2]$; $\alpha > 0$. The one-dimensional subalgebras of the algebra K with the condition $\varepsilon(K) \neq 0$ are exhausted with respect to $\tilde{P}(1,n)$ conjugation by the following algebras:*

$$\langle J_{0n}\rangle;\ \langle \mathbb{D}\rangle;\ \langle \mathbb{D} + \alpha J_{0n}\rangle;\ \langle J_{0n} + P_1\rangle;\ \langle \mathbb{D} + G_1\rangle;\ \langle \mathbb{D} + J_{0n} + M\rangle;$$

$$\langle X_t + \alpha\mathbb{D} + \beta J_{0n}\rangle\ (\beta \geq 0);\ \langle X_t + \alpha J_{0n}\rangle;\ \langle X_t + \alpha(\mathbb{D} + J_{0n} + M)\rangle;$$

$$\langle X_t + G_{2s+1} + \alpha\mathbb{D}\rangle;\ \langle X_s + P_{2s+1} + \alpha J_{0n}\rangle.$$

The written algebras are not conjugated mutually.

Proposition 5.10. *The one-dimensional subalgebras of the algebra $\tilde{P}(1,n)$ are exhausted wtih respect to the $\tilde{P}(1,n)$ conjugation by the one-dimensional subalgebras of the algebra K and the following algebras:*

$$\langle J_{12} + \beta_1 J_{34} + \cdots + \beta_{n/2-1} J_{n-1,n} + \gamma\mathbb{D}\rangle,$$

$$\langle J_{12} + \beta_1 J_{34} + \cdots + \beta_{n/2-1} J_{n-1,n} + P_0\rangle,$$

where $n \equiv 0 \pmod 2$, $\gamma \geq 0$, $0 < \beta_1 \leq \cdots \leq \beta_{n/2-1} \leq 1$.

6. Subalgebras of the algebras $A\tilde{P}(1,4)$

In this section we make use of the previous results to provide a classification of all subalgebras of $A\tilde{P}(1,4)$ with respect to $\tilde{P}(1,4)$ conjugation.

Let $\hat{F}$ be an subalgebra of $A\tilde{P}(1,4)$ such that $\pi(\hat{F}) = F$. An expression $\hat{F} + W$ means that W is a subspace of U, $[F, W] \subset W$, and $\hat{F} \cap U \subset W$. As concerns the algebras $\hat{F} + W_1, \ldots, \hat{F} + W_s$ we will use the notation $\hat{F} : W_1, \ldots, W_s$.

Lemma 6.1. *Let $\alpha, \beta, \gamma \in R$, $\alpha > 0$, $\beta \geq 0$, $\gamma \neq 0$, and F run through the full system of representatives of the classes of $O(1,4)$-conjugated subalgebras of the algebra $AO(1,4)$ [4]. The subalgebras of the algebra $AO(1,4) \oplus \langle \mathbb{D}\rangle$ are exhausted with respect to $\tilde{O}(1,4)$ conjugation by the algebras F, $F \oplus \langle \mathbb{D}\rangle$ and the following algebras:*

$\langle J_{12} + \alpha\mathbb{D}\rangle$; $\langle J_{12} + cJ_{34} + \alpha\mathbb{D}\rangle$ $(0 < c \leq 1)$; $\langle J_{04} + \alpha\mathbb{D}\rangle$; $\langle J_{12} + cJ_{04} + \alpha\mathbb{D}\rangle$ $(c > 0)$;
$\langle G_3 + \mathbb{D}\rangle$; $\langle G_3 - J_{12} + \alpha\mathbb{D}\rangle$; $\langle J_{12} + \alpha\mathbb{D}, J_{34} + \beta\mathbb{D}\rangle$; $\langle J_{04} + \alpha\mathbb{D}, J_{12} + \beta\mathbb{D}\rangle$; $\langle J_{04}, J_{12} + \alpha\mathbb{D}\rangle$;
$\langle G_3 + \mathbb{D}, J_{12} + \beta\mathbb{D}\rangle$; $\langle G_3, J_{12} + \alpha\mathbb{D}\rangle$; $\langle G_1 + \mathbb{D}, G_2$; $\langle G_3, J_{04} + \gamma\mathbb{D}\rangle$;
$\langle G_3, J_{12} + cJ_{04} + \gamma\mathbb{D}\rangle$ $(c > 0)$; $\langle G_3, J_{04} + \gamma\mathbb{D}, J_{12} + \beta\mathbb{D}\rangle$; $\langle G_3, J_{04}, J_{12} + \alpha\mathbb{D}\rangle$;
$\langle G_1, G_2, J_{12} + \alpha\mathbb{D}\rangle$; $\langle G_1, G_2, J_{04} + \gamma\mathbb{D}\rangle$; $\langle G_1, G_2, J_{12} + cJ_{04} + \gamma\mathbb{D}\rangle$ $(c > 0)$;

$\langle G_1+\mathbb{D}, G_2, G_3\rangle$; $\langle G_1, G_2, G_3-J_{12}+\alpha\mathbb{D}\rangle$; $\langle J_{03}, J_{04}, J_{34}, J_{12}+\alpha\mathbb{D}\rangle$;
$\langle J_{12}+J_{34}, J_{13}-J_{24}, J_{23}+J_{14}, J_{34}+\gamma\mathbb{D}\rangle$; $\langle G_1, G_2, J_{12}+\alpha\mathbb{D}, J_{04}+\delta\mathbb{D}\rangle$;
$\langle G_1, G_2, J_{12}, J_{04}+\gamma\mathbb{D}\rangle$; $\langle G_1, G_2, G_3+\mathbb{D}, J_{12}+\beta\mathbb{D}\rangle$; $\langle G_1, G_2, G_3, J_{12}+\alpha\mathbb{D}\rangle$;
$\langle G_1, G_2, G_3, J_{04}+\gamma\mathbb{D}\rangle$; $\langle G_1, G_2, G_3, J_{12}+cJ_{04}+\gamma\mathbb{D}\rangle$ $(c>0)$;
$\langle J_{12}, J_{13}, J_{23}, J_{04}+\alpha\mathbb{D}\rangle$; $\langle G_1, G_2, G_3, J_{12}+\alpha\mathbb{D}, J_{04}+\delta\mathbb{D}\rangle$;
$\langle G_1, G_2, G_3, J_{12}, J_{13}, J_{23}, J_{04}+\gamma\mathbb{D}\rangle$.

Lemma 6.1 is proved with the Goursat method [25] and the result on the classification of subalgebras of the algebra $AO(1,4)$ [4].

Theorem 6.1. *Let $\Delta(\Gamma)$ be the system of representatives of the classes of conjugated subalgebras of the algebra $A\tilde{O}(1,4)$ (respectively, $AO(1,4)$) found in Lemma 6.1. The splitting subalgebras of the algebra $A\tilde{P}(1,4)$ are exhausted with respect to $\tilde{P}(1,4)$ conjugation by the following algebras:*

(1) $W \subset\!\!\!+ F$, where $F\in\Gamma$, $W\subset U$, and $[F,W]\subset W$;

(2) $W \subset\!\!\!+ \hat{F}$, where $\hat{F}\in\Delta$ and the projection of $\hat{F}$ onto $AO(1,4)$ coincides with F, $F\in\Gamma$;

(3) $\langle J_{12}, J_{34}+\alpha\mathbb{D}\rangle$: $\langle P_1,P_2\rangle$, $\langle P_0,P_1,P_2\rangle$ $(\alpha>0)$;

(4) $\langle G_1+\alpha\mathbb{D}, G_2+\beta\mathbb{D}\rangle$: $\langle M,P_1\rangle$, $\langle M,P_1+\omega P_3\rangle$, $\langle M,P_1,P_3\rangle$, $\langle M,P_1+\omega P_3,P_2\rangle$ $(\omega>0,\ \alpha\geq 0,\ \beta\geq 0,\ \alpha^2+\beta^2\neq 0)$;

(5) $\langle G_1+\alpha\mathbb{D}, G_2+\beta\mathbb{D}, G_3, M, P_1\rangle$ $(\alpha\geq 0,\ \beta\geq 0,\ \alpha^2+\beta^2\neq 0)$;

(6) $\langle G_1+\alpha\mathbb{D}, G_2, G_3+\beta\mathbb{D}, M, P_1, P_2\rangle$ $(\alpha\geq 0,\ \beta\geq 0,\ \alpha^2+\beta^2\neq 0)$.

Proof. Let $\hat{F}$ be the subdirect sum of $F\in\Gamma$ and $\mathbb{D}$, and W a subspace of U invariant under $\hat{F}$. Then $[F,W]\subset W$ and on the contrary, if $[F,W]\subset W$ then $[\hat{F},W]\subset W$. Therefore we can use the results on the classification of the splitting subalgebras of $AP(1,4)$ [9]. Only the cases of the algebras $\hat{F}\in\Delta$ simplified by $O(1,4)$ automorphisms demand an additional consideration. Such algebras correspond to the algebra F coinciding with $\langle J_{12}, J_{34}\rangle$, $\langle G_1, G_2\rangle$, or $\langle G_1, G_2, G_3\rangle$. If, for example, $\hat{F}=\langle G_1+\alpha_1\mathbb{D}, G_2+\alpha_2\mathbb{D}, G_3+\alpha_3\mathbb{D}\rangle$ then this algebra must be simplified using transformations contained in the normalizer of $\langle M,P_1\rangle$, $\langle M,P_1,P_2\rangle$, respectively, in the group of $O(1,4)$ automorphisms. The theorem is proved.

We conceive the classification of nonsplitting subalgebras of $A\tilde{P}(1,4)$ with respect to $\tilde{P}(1,4)$ conjugation by virtue of the known classification of the nonsplitting subalgebras of $AP(1,4)$ with respect to $P(1,4)$ conjugation [11]. The application of the automorphism $\exp(\theta\mathbb{D})$ allows us to substitute one of the continuous parameters by the translation generators onto 1.

Let $(i_1,\ldots,i_q)=\langle P_{i_1},\ldots,P_{i_q}\rangle$; $(a\omega b)=\langle P_a+\omega P_b\rangle$ $(\omega>0)$; $(04)=\langle M\rangle$.

Theorem 6.2. *The nonsplitting subalgebras of the algebra $A\tilde{P}(1,4)$ are exhausted with respect to $\tilde{P}(1,4)$ conjugation by the nonsplitting subalgebras of the algebra $AP(1,4)$ and the following algebras:*

$\langle J_{04}-\mathbb{D}+P_0\rangle$: 0, (1), (04), (1,2), (04,1), (1,2,3), (04,1,2), (04,1,2,3);

$\langle J_{12}+c(J_{04}-\mathbb{D}+P_0)\rangle$: 0, (04), (3), (04,3), (1,2), (1,2,3), (04,1,2), (04,1,2,3) $(c>0)$;

$\langle J_{04}+\mathbb{D}+M, J_{12}+\alpha M\rangle$: 0, (3), (1,2), (1,2,3) $(\alpha>0)$;

$\langle J_{04}+\mathbb{D}, J_{12}+M\rangle$: 0, (3), (1,2), (1,2,3); $\langle J_{04}+\mathbb{D}+M, J_{12}\rangle$: 0, (3), (1,2), (1,2,3);

$\langle J_{04}-\mathbb{D}+P_0, J_{12}+\alpha P_0\rangle$: (04), (04,3), (04,1,2), (04,1,2,3) $(\alpha\geq 0)$;

$\langle J_{04}-\mathbb{D}, J_{12}+P_0\rangle$: (04), (04,3), (04,1,2), (04,1,2,3);

$\langle J_{04}-2\mathbb{D}, G_3+P_0\rangle$: (04), (04,1), (04,1$\omega$3), (04,3), (04,1$\omega$3,2), (04,1,2), (04,1,3), (04,1,2,3);

$\langle J_{04}-2\mathbb{D}, G_3+P_0-P_4\rangle$: 0, (1), (1,2); $\langle J_{04}-\mathbb{D}, G_3+P_1\rangle$: 0, (04), (04,3), (0,3,4);

$\langle J_{04}-\mathbb{D}, G_3+P_2\rangle$: (1), (04,1), (04,1$\omega$3), (04,1,3), (0,1,3,4);

$\langle G_3+\alpha P_1, J_{04}-\mathbb{D}+P_0, M, P_3\rangle$ $(\alpha>0)$; $\langle J_{04}-\mathbb{D}+P_0, G_3+\alpha P_2, M, P_1, P_3\rangle$ $(\alpha>0)$;

$\langle G_3, J_{04}-\mathbb{D}+P_0\rangle$: (04,3), (04,1,3), (04,1,2,3); $\langle G_3, J_{04}+\mathbb{D}+M\rangle$: 0, (1), (1,2);

$\langle G_3+P_0, J_{12}+c(J_{04}-2\mathbb{D})\rangle$: (04), (04,3), (04,1,2), (04,1,2,3) $(c>0)$;

$\langle G_3+P_0-P_4, J_{12}+c(J_{04}-2\mathbb{D})\rangle$: 0, (1,2) $(c>0)$;

$\langle G_3, J_{12}+c(J_{04}-\mathbb{D}+P_0)\rangle$: (04,3), (04,1,2,3); $\langle G_3, J_{12}+c(J_{04}+\mathbb{D}+M)\rangle$: 0, (1,2);

$\langle G_3+P_0, J_{12}, J_{04}-2\mathbb{D}\rangle$: (04), (04,3), (04,1,2), (04,1,2,3);

$\langle G_3+P_0-P_4, J_{12}, J_{04}-2\mathbb{D}\rangle$: 0, (1,2);

$\langle G_3, J_{12}+\alpha P_0, J_{04}-\mathbb{D}+P_0\rangle$: (04,3), (04,1,2,3) $(\alpha\geq 0)$;

$\langle G_3, J_{12}+P_0, J_{04}-\mathbb{D}\rangle$: (04,3), (04,1,2,3);

$\langle G_3, J_{12}+\alpha M, J_{04}+\mathbb{D}+M\rangle$: 0, (1,2) $(\alpha\geq 0)$; $\langle G_3, J_{12}+M, J_{04}+\mathbb{D}\rangle$: 0, (1,2);

$\langle G_1, G_2+P_0, J_{04}-2\mathbb{D}\rangle$: (04,1), (04,1,2), (04,1,2ω3), (04,1,3), (04,1,2,3);

$\langle G_1+P_3, G_2+\mu P_2+\delta P_3, J_{04}-\mathbb{D}\rangle$ $(\mu>0,\ \delta\geq 0)$; $\langle G_1+P_3, G_2, J_{04}-\mathbb{D}\rangle$;

$\langle G_1, G_2+P_2+\delta P_3, J_{04}-\mathbb{D}\rangle$ $(\delta\geq 0)$; $\langle G_1, G_2+P_2, J_{04}-\mathbb{D}, P_3\rangle$;

$\langle G_1+P_2+\lambda P_3, G_2-P_1+\mu P_2+\delta P_3, J_{04}-\mathbb{D}, M\rangle$ $(\mu>0,\ \lambda>0\ \vee\ \lambda=0,\ \delta\geq 0)$;

$\langle G_1+P_2+\lambda P_3, G_2-P_1, J_{04}-\mathbb{D}, M\rangle$ $(\lambda\geq 0)$; $\langle G_1+P_3, G_2, J_{04}-\mathbb{D}, M\rangle$;

$\langle G_1+\lambda P_3, G_2+P_2+\delta P_3, J_{04}-\mathbb{D}, M\rangle$ $(\lambda>0\ \vee\ \lambda=0,\ \delta\geq 0)$;

$\langle G_1+P_2, G_2-P_1+\mu P_2, J_{04}-\mathbb{D}, M, P_3\rangle$ $(\mu\geq 0)$; $\langle G_1, G_2+P_2, J_{04}-\mathbb{D}, M, P_3\rangle$;

$\langle G_1+P_2, G_2-P_1+\mu P_2, J_{04}-\mathbb{D}, M, P_3\rangle$ $(\mu\geq 0)$;

$\langle G_1+\alpha P_2+\beta P_3, G_2+P_3, J_{04}-\mathbb{D}, M, P_1\rangle$ $(\alpha>0\ \vee\ \alpha=0,\ \beta\geq 0)$;

$\langle G_1+P_2+\beta P_3, G_2, J_{04}-\mathbb{D}, M, P_1\rangle$ $(\beta\geq 0)$; $\langle G_1+P_3, G_2, J_{04}-\mathbb{D}, M, P_1\rangle$;

$\langle G_1+\alpha P_2+\beta P_3, G_2+P_3, J_{04}-\mathbb{D}, M, P_1+\omega P_3\rangle$ $(\omega>0)$;

$\langle G_1+P_2+\beta P_3, G_2, J_{04}-\mathbb{D}, M, P_1+\omega P_3\rangle$ $(\omega>0)$;

$\langle G_1+P_3, G_2, J_{04}-\mathbb{D}, M, P_1+\omega P_3\rangle$ $(\omega>0)$; $\langle G_1+P_3, G_2, J_{04}-\mathbb{D}, M, P_1, P_2\rangle$;

$\langle G_1+P_2, G_2, J_{04}-\mathbb{D}, M, P_1, P_3\rangle$; $\langle G_1, G_2+P_3, J_{04}-\mathbb{D}, M, P_1+\omega P_3, P_2\rangle$ $(\omega>0)$;

$\langle G_1+P_3, G_2, J_{04}-\mathbb{D}, P_0, P_1, P_2, P_4\rangle$; $\langle G_1+\beta P_3, G_2, J_{04}-\mathbb{D}+P_0, M, P_1, P_2\rangle$ $(\beta\geq 0)$;

$\langle G_1, G_2, J_{04}-\mathbb{D}+P_0, M, P_1, P_2, P_3\rangle$; $\langle G_1, G_2, J_{04}+\mathbb{D}+M\rangle$; $\langle G_1, G_2, J_{04}+\mathbb{D}+M, P_3\rangle$;

$\langle G_1+P_2, G_2-P_1, J_{12}+c(J_{04}-\mathbb{D})\rangle$: (04), (04,3) $(c>0)$;

$\langle G_1, G_2, J_{12}+c(J_{04}-\mathbb{D}+P_0), M, P_1, P_2, sP_3\rangle$ $(c>0,\ s=0,1)$;

$\langle G_1, G_2, J_{12}+c(J_{04}+\mathbb{D}+M)\rangle$: 0, (3) $(c>0)$;

$\langle G_1, G_2, J_{12}+P_0, J_{04}-\mathbb{D}, M, P_1, P_2, sP_3\rangle$ $(s=0,1)$;

$\langle G_1, G_2, J_{12}+M, J_{04}+\mathbb{D}\rangle$: 0, (3);

$\langle G_1, G_2, J_{12}+\delta P_0, J_{04}-\mathbb{D}+P_0, M, P_1, P_2, sP_3\rangle$ $(\delta\geq 0,\ s=0,1)$;

$\langle G_1+P_2, G_2-P_1, J_{12}, J_{04}-\mathbb{D}, M, sP_3\rangle$ $(s=0,1)$;

$\langle G_1, G_2, J_{12}+\alpha M, J_{04}+\mathbb{D}+M\rangle$: 0, (3) $(\alpha\geq 0)$;

$\langle G_1, G_2, G_3+P_0, J_{04}-2\mathbb{D}, M, P_1, P_2, sP_3\rangle$ $(s=0,1)$;

$\langle G_1, G_2+P_2, G_3+\alpha P_3, J_{04}-\mathbb{D}\rangle$; $\langle G_1, G_2+P_2, G_3+\alpha P_3, J_{04}-\mathbb{D}, M\rangle$;

$\langle G_1+P_2+\beta P_3, G_2-P_1+\mu P_2+\gamma P_3, G_3+\beta P_1+\gamma P_2+\delta P_3, J_{04}-\mathbb{D}, M\rangle$ $(\mu\geq 0,\ \beta>0\ \vee\ \beta=0,\ \gamma\geq 0)$;

$\langle G_1+P_2+\beta P_3, G_2-P_1, G_3+\beta P_1+\delta P_3, J_{04}-\mathbb{D}, M\rangle$ $(\beta\geq 0)$;

$\langle G_1+\beta P_2, G_2+P_3, G_3-P_2, J_{04}-\mathbb{D}, M, P_1\rangle$ $(\beta\geq 0)$;

$\langle G_1+\beta P_2+\gamma P_3, G_2+P_3, G_3-P_2+\mu P_3, J_{04}-\mathbb{D}, M, P_1\rangle$ ($\mu>0$, $\beta>0 \vee \beta=0$, $\gamma\geq 0$);

$\langle G_1+\beta P_2+\gamma P_3, G_2, G_3+P_3, J_{04}-\mathbb{D}, M, P_1\rangle$ ($\beta>0 \vee \beta=0$, $\gamma\geq 0$);

$\langle G_1+P_2, G_2, G_3, J_{04}-\mathbb{D}, M, P_1\rangle$; $\langle G_1+P_3, G_2, G_3, J_{04}-\mathbb{D}, M, P_1, P_2\rangle$;

$\langle G_1, G_2, G_3, J_{04}-\mathbb{D}+P_0, M, P_1, P_2, P_3\rangle$; $\langle G_1, G_2, G_3, J_{04}+\mathbb{D}+M\rangle$;

$\langle G_1, G_2, G_3+P_0, J_{12}+c(J_{04}-2\mathbb{D}), M, P_1, P_2, sP_3\rangle$ ($c>0$, $s=0,1$);

$\langle G_1+P_2, G_2-P_1, G_3+\beta P_3, J_{12}+c(J_{04}-\mathbb{D}), M\rangle$ ($c>0$);

$\langle G_1+P_2, G_2-P_1, G_3, J_{12}+c(J_{04}-\mathbb{D}), M, P_3\rangle$ ($c>0$);

$\langle G_1, G_2, G_3+P_3, J_{12}+c(J_{04}-\mathbb{D})\rangle$: 0, (04);

$\langle G_1, G_2, G_3, J_{12}+c(J_{04}+\mathbb{D}+M)\rangle$ ($c>0$);

$\langle J_{12}, J_{13}, J_{23}, J_{04}-\mathbb{D}+P_0\rangle$: 0, (04), (1,2,3), (04,1,2,3);

$\langle G_1, G_2, G_3+P_0, J_{12}, J_{04}-2\mathbb{D}, M, P_1, P_2, sP_3\rangle$ ($s=0,1$);

$\langle G_1, G_2, G_3, J_{12}+P_0, J_{04}-\mathbb{D}, M, P_1, P_2, P_3\rangle$;

$\langle G_1, G_2, G_3, J_{12}+\delta P_0, J_{04}-\mathbb{D}+P_0, M, P_1, P_2, P_3\rangle$ ($\delta\geq 0$);

$\langle G_1+P_2, G_2-P_1, G_3+\beta P_3, J_{12}, J_{04}-\mathbb{D}, M\rangle$;

$\langle G_1+P_2, G_2-P_1, G_3, J_{12}, J_{04}-\mathbb{D}, M, P_3\rangle$; $\langle G_1, G_2, G_3+P_3, J_{12}, J_{04}-\mathbb{D}\rangle$: 0, (04);

$\langle G_1, G_2, G_3, J_{12}+M, J_{04}+\mathbb{D}\rangle$; $\langle G_1, G_2, G_3, J_{12}+\delta M, J_{04}+\mathbb{D}+M\rangle$ ($\delta\geq 0$);

$\langle G_1, G_2, G_3, J_{12}, J_{13}, J_{23}, J_{04}-\mathbb{D}+P_0, M, P_1, P_2, P_3\rangle$;

$\langle G_1, G_2, G_3, J_{12}, J_{13}, J_{23}, J_{04}+\mathbb{D}+M\rangle$.

7. Conclusions

The results of the present paper may be summarized in the following way.

(1) The maximal Abelian subalgebras of the algebra $A\tilde{P}(1,n)$ have been explicitly found in Corollary 1 to Theorem 4.2 and Corollary 1 to Theorem 5.1.

(2) The full classification of one-dimensional subalgebras of algebra $A\tilde{P}(1,n)$ is contained in Corollary 2 to Theorem 4.2, Corollary 2 to Theorem 5.1 and Proposition 5.10.

(3) The completely reducible subalgebras of $A\tilde{O}(1,n)$ which possess only splitting extensions in the algebra $A\tilde{P}(1,n)$ have been picked out. We have established in Theorem 3.1 that the description of the splitting subalgebras $\hat{F}$ of $A\tilde{P}(1,n)$ whose projection F onto $A\tilde{O}(1,n)$ does not have any invariant isotropic subspaces in the space of translations or annul such subspaces, could be reduced to the description of the irreducible parts of the algebra F.

(4) A number of assertions on the subalgebras of the algebra $U \subset\!\!\!\!+ K'$ has been proved where K' is the normalizer of $\langle P_0+P_n\rangle$ in $A\tilde{O}(1,n)$. These assertions concern the following matters: The splittability of all extensions of the subalgebra $L\subset K'$ in $A\tilde{P}(1,n)$ or in some other algebras (Propositions 4.1, 4.2, 5.1, and 5.2); the decomposition of invariant subspaces into a direct sum of its projections onto certain subspaces (Propositions 5.3, 5.5, 5.6, 5.7, and 5.8); the explicit description of some classes of the conjugated subalgebras of the algebra $A\tilde{P}(1,n)$ (Theorem 4.1, Propositions 4.4 and 5.4).

(5) The full classification with respect to $\tilde{P}(1,4)$ conjugation of the nonsplitting subalgebras of $A\tilde{P}(1,4)$ which are nonconjugate to the subalgebras of $AP(1,4)$ has been carried out.

Note added in proof: In Refs. [26–28] the subalgebras of the algebra $AP(1,n)$ were used to construct the exact solutions of many-dimensional nonlinear d'Alembert and Dirac equations. The invariants of subgroups of the generalized Poincaré group

$P(1,n)$ were constructed in Ref. [29]. A number of general results on continuous subgroups of pseudoorthogonal pseudounitary groups had been obtained [30].

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1. Patera J., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1975, **16**, 1597.
2. Patera J., Winternitz P., Sharp R.T., Zassenhaus H., *Can. J. Phys.*, 1976, **54**, 950.
3. Patera J., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1975, **16**, 1615.
4. Patera J., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1976, **17**, 717.
5. Patera J., Sharp R.T., Winternitz P., Zassenhaus H., *J. Math. Phys.*, 1977, **18**, 2259.
6. Burdet G., Patera P., Perrin M., Winternitz P., *J. Math. Phys.*, 1978, **19**, 1758.
7. Beckers J., Patera J., Perroud M., Winternitz P., *J. Math. Phys.*, 1977, **18**, 72.
8. Burdet G., Patera J., Perrin M., Winternitz P., *Ann. Sci. Math. Quebec*, 1978, **2**, 81.
9. Fedorchuk V.M., *Ukrainian Math. J.*, 1979, **31**, 717.
10. Fedorchuk V.M., *Ukrainian Math. J.*, 1981, **33**, 696.
11. Fushchych W.I., Barannik A.F., Barannik L.F., Fedorchuk V.M., *J. Phys. A: Math. Gen.*, 1985, **18**, 2893.
12. Fushchych W.I., Barannik A.F., Barannik L.F., The continuous subgroups of the generalized Galilei group. I, Preprint 85.19, Mathematical Institute, Kiev, 1985.
13. Fushchych W.I., Barannik A.F., Barannik L.F., *Ukrainian Math. J.*, 1986, **38**, 67.
14. Lassner W., *Acta Phys. Slovaca*, 1973, **23**, 193.
15. Bacry H., Combe Ph., Sorba P., *Rep. Math. Phys.*, 1974, **5**, 145.
16. Bacry H., Combe Ph., Sorba P., *Rep. Math. Phys.*, 1974, **5**, 361.
17. Sorba P., *J. Math. Phys.*, 1976, **17**, 941.
18. Fushchych W.I., The symmetry of mathematical physics problems, in Algebraic-Theoretical Studies in Mathematical Physics, Mathematical Institute, Kiev, 1981 (in Russian).
19. Fushchych W.I., On symmetry and particular solutions of some multidimensional equations of mathematical physics, in Algebraic-Theoretical Methods in Mathematical Physics Problems, Mathematical Institute, Kiev, 1983 (in Russian).
20. Grundland A.M., Harnad J., Wintemitz P., *J. Math. Phys.*, 1984, **25**, 791.
21. Nikitin A.G., Fushchych W.I., Jurik I.I., *Theor. Math. Phys.*, 1976, **26**, 206.
22. Fushchych W.I., Nikitin A.G., *J. Phys. A: Math. Gen.*, 1980, **13**, 2319.
23. Jacobson N., Lie algebras, Dover, New York, 1962.
24. Goto M., Grosshans F.D., Semisimple Lie algebras, Dekker, New York, 1978.
25. Coursat E., *Ann. Sci. Ecole Norm. Sup.*, 1889, **3**, 69.
26. Fushchych W.I., Nikitin A.G., Symmetries of Maxwell's equations, Reidel, Dordrecht, 1987.
27. Fushchych W.I., Serov N.I., *J. Phys. A: Math. Gen.*, 1983, **16**, 3645.
28. Fushchych W.I., Shtelen W.M., *J. Phys. A: Math. Gen.*, 1985, **18**, 271.
29. Barannik L.F., Fushchych W.I., The invariants of subgroups of the generalized Poincaré group $P(1,n)$, Preprint 86.86, Mathematical Institute, Kiev,1986.
30. Barannik A.F., Fushchych W.I., On continuous subgroups of pseudoorthogonal and pseudounitary groups, Preprint 86.87, Mathematical Institute, Kiev,1986.