The Galilean relativistic principle and nonlinear partial differential equation

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The second-order partial differential equations invariant under transformations of Galilei, rotation, scale and projection are described.

1. Introduction

From the mathematical point of view the Galilean relativistic principle (in a restricted sense) is nothing other than the requirement of the equations of motion to be invariant under the linear transformations

$$t \to t' = t, \qquad x_a \to x'_a = x_a + v_a t, \quad a = 1, 2, 3,$$

v_a being transformation parameters (the inertial reference system velocity $\boldsymbol{v}$ component). These transformations form a three-parameter Lie group. In order to construct linear and nonlinear partial differential equations (PDE)

$$\mathcal{L}U(t, \boldsymbol{x}) = 0, \qquad \boldsymbol{x} = (x_1, \dots, x_n)$$

(where $\mathcal{L}$ is a linear or nonlinear operator, which is invariant under the Galilean transformations) it is also necessary to give the law of transformation for the dependent variable of $U(t, \boldsymbol{x})$. Under different transformation laws of the function $U(t, \boldsymbol{x})$ we obtain different classes of PDE.

As is well known, the linear heat equation in the $(n+1)$-dimensional space

$$\begin{aligned} &\Delta U = \lambda U_0, \qquad \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2, \qquad U = U(t, \boldsymbol{x}), \\ &U_0 = U_t = \partial U/\partial t, \qquad \lambda = \text{const} \end{aligned} \tag{1.1}$$

is invariant under the following transformations:

$$t \to t' = t, \qquad x_a \to x'_a = x_a + v_a t, \quad a = \overline{1, n}, \tag{1.2}$$

$$U \to U' = \exp\left[-\frac{1}{2} v_a \left(x_a + \frac{1}{2} v_a t\right)\right], \tag{1.3}$$

v_a being the transformation parameters.

(1.3) defines the transformation law for the dependent function $U(t, \boldsymbol{x})$ under the Galilean transformations (1.2).

The $\frac{1}{2}(n^2 + 3n + 6)$-dimensional algebra with basic elements

$$G_a = t\partial_a - \frac{1}{2}\lambda x_a U \partial_U, \qquad \partial_a = \partial/\partial x_a, \qquad \partial_U = \partial/\partial U, \qquad a = \overline{1, n}, \tag{1.4a}$$

$$J_{ab} = x_a \partial_b - x_b \partial_a, \qquad a \neq b, \qquad a, b \neq \overline{1, n}, \tag{1.4b}$$

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$$\Pi = t^2\partial_t + tx_a\partial_a - \left(\frac{1}{4}\lambda|x|^2 + \frac{1}{2}nt\right)U\partial_U, \qquad |x|^2 = x_ax_a, \tag{1.4c}$$

$$D = 2t\partial_t + x_a\partial_a + kU\partial_U, \qquad k = \text{constant}, \tag{1.4d}$$

$$P_0 = \partial_t, \qquad P_a = \partial_a \tag{1.4e}$$

(where the repeated indices imply summation) is maximal in the Lie restriction invariance algebra (IA) of (1.1).

The set of operators (1.4) forms a Lie algebra, which will be noted by the symbol $SLi(1,n)$, i.e. the special Lie algebra. This name is natural because in the previous century Lie [10] (see also Ovsyannikov [13]) was the first to calculate the maximal IA of the two-dimensional $U(t,x_a)$ heat equation. The maximal IA of the $(3+1)$-dimensional Schrödinger equation, which coincides with (1.1) (differing only by constant coefficients), was calculated by Niederer [11]. For some more details on this, see, for example, Fushchych and Nikitin [6, 7].

From the group-theoretical point of view (1.3) defines the projective representation of the group (1.2). Apart from the projective representation (1.3) the group (1.2) has another representation, the infinitesimal operator of which

$$\widetilde{G}_a = t\partial_a, \qquad a = \overline{1,n} \tag{1.5}$$

being different from the G_a operators (1.4a).

The operators (1.5) generate the following transformations:

$$t \to t' = t, \qquad x_a \to x'_a = x_a + v_at, \qquad U \to U' = U. \tag{1.6}$$

We call (1.2) and (1.3) the projective Galilean transformations (PGT) and (1.6) the Galilean transformations (GT).

Equation (1.1) admits operators (1.4a) but does not admit operators (1.5).

In § 2 we describe the nonlinear second-order PDE

$$F(t,x,U,U_0,\underset{I}{U},\underset{II}{U}) \equiv -\Delta U + A(t,x,U)U_t + B(t,x,U,\underset{I}{U}) = 0, \tag{1.7}$$

where

$$\underset{I}{U} = (U_1,\dots,U_n), \qquad \underset{II}{U} = (U_{11},U_{12},\dots,U_{nn}),$$
$$U_a = \partial U/\partial x_a, \qquad U_{ab} = \partial^2 U/\partial x_a\partial x_b, \qquad a,b = \overline{1,n},$$

F, A, B being arbitrary differentiable functions, invariant under the PGT (1.2) and (1.3) as well as projective and scale transformations generated by operators (1.4c) and (1.4d).

In § 3 we construct the most general nonlinear PDE of the form

$$F(t,x,U,U_0,\underset{I}{U},U_{00},U_{01},\dots,U_{0n},\underset{II}{U}) = 0, \qquad \partial^2 U/\partial t\partial x_a = U_{0a} \tag{1.8}$$

which are invariant under the GT (1.6) and the translation group generated by the operators (1.4e). In particular, it is established that a set of equations of the form (1.8) does not contain linear equations (except, obviously $U_0 = 0$, $U_{00} = 0$) invariant under the GT (1.6) and the group of time and space translations.

In the final part of § 3 we give several examples of Galilei invariant equations in independent variables (t, x_1) space, for which general solutions are constructed.

It is to be noted that equations of the class (1.7) are widely used to describe nonlinear diffusion, heat and other processes. In particular, this class includes diffusion equation of the form

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x_a}\left(C(U)\frac{\partial U}{\partial x_a}\right) \tag{1.9}$$

as well as nonlinear Schödinger equations (if U is a complex function) and Hamilton–Jacobi equations. The group classification of (1.9) for the one-dimensional case was carried out by Ovsyannikov [12] and for the three-dimensional case by Dorodnitsyn et al [3] and Fushchych [4].

2. Equation invariant under the projective Galilean transformations

First of all in this section we are going to find the conditions to be imposed on the functions A and B under which (1.7) is invariant under the PGT (1.2) and (1.3). The complete solution of this problem is given by the following theorem.

Theorem 1. *Equation (1.7) is invariant under the PGT if and only if*

$$A(t, x, U) = f(t, w), \tag{2.1}$$

$$B(t, x, U, \underset{I}{U}) = Ug(t, w, w_1, \ldots, w_n) + (f(t, w) - \lambda)\left(\frac{x_a U_a}{t} + \frac{\lambda |x|^2}{4t^2}U\right), \tag{2.2}$$

where

$$w = U\exp\left(\frac{\lambda|x|^2}{4t}\right), \qquad |x|^2 = x_a x_a, \tag{2.3}$$

$$w_a = \left(U_a + \frac{1}{2}\lambda\frac{x_a}{t}U\right)\exp\left(\frac{\lambda|x|^2}{4t}\right), \qquad a = \overline{1, n}, \tag{2.4}$$

and f, g are arbitrary differentiable functions.

Proof. To prove the theorem let us use the Lie method (for a modern account, see Bluman and Cole [1] and Ibragimov [13]). According to Lie's approach, (1.7) is considered as a manifold in the space of the following variables: t, x, U, $\underset{I}{U}$, $\underset{II}{U}$. (1.7) is invariant under the transformations generated by an infinitesimal operator

$$X = \xi^\mu(t, x, U)\frac{\partial}{\partial x_\mu} + \eta(t, x, U)\frac{\partial}{\partial U}, \qquad \mu = \overline{0, n}$$

when the following invariance condition is fulfilled:

$$\overset{2}{X}F = \overset{2}{X}(-\Delta U + AU_t + B)|_{F=0} = 0, \tag{2.5}$$

where $\overset{2}{X}$ is the second prolongation of the infinitesimal operator X, i.e.

$$\overset{2}{X} = X + \rho^\mu(t, x, U)\frac{\partial}{\partial U_\mu} + \sigma^{\mu\nu}(t, x, U)\frac{\partial}{\partial U_{\mu\nu}}, \qquad \mu, \nu = \overline{0, n}, \tag{2.6}$$

$$\rho^{\mu} = \eta_{\mu} + U_{\mu}\eta_U - U_i(\xi^i_{\mu} + U_{\mu}\xi^i_U), \qquad i = \overline{0,n},$$

$$\sigma^{\mu\nu} = \eta_{\mu\nu} + U_{\nu}\eta_{\mu U} + U_{\mu}\eta_{\nu U} + U_{\mu}U_{\nu}\eta_{UU} + U_{\mu\nu}\eta_U - U_i(\xi^i_{\mu\nu} + U_{\nu}\xi^i_{\mu U}) -$$
$$- U_{\mu}U_i(\xi^i_{\nu U} + U_{\nu}\xi^i_{UU}) - U_{\mu i}(\xi^i_{\nu} + U_{\nu}\xi^i_U) -$$
$$- U_{i\nu}(\xi^i_{\mu} + U_{\mu}\xi^i_U) - U_{\mu\nu}U_i\xi^i_U, \qquad i = \overline{0,n}.$$

Substituting (2.6) into (2.5), we obtain

$$\left[-(\sigma^{11} + \ldots + \sigma^{nn}) + \xi^{\mu}\left(\frac{\partial A}{\partial x_{\mu}}U_0 + \frac{\partial B}{\partial x_{\mu}}\right) + \eta\left(\frac{\partial A}{\partial U}U_0 + \frac{\partial B}{\partial U}\right) + \right.$$
$$\left. + \rho^0 A + \rho^a \frac{\partial B}{\partial x_a}\right]\Bigg|_{F=0} = 0, \qquad a = \overline{1,n}. \tag{2.7}$$

After explicit expressions for ρ^{μ}, $\sigma^{\mu\nu}$ have been substituted into (2.7) and the obtained relation being split into separate parts for coefficients at U_{0a} and U_{ab}, $a \neq b$, the conditions for ξ^{μ} are found:

$$\xi^0_a \equiv \partial\xi^0/\partial x_a = 0, \qquad \xi^{\mu}_U \equiv \partial\xi^{\mu}/\partial U = 0, \qquad \xi^a_b + \xi^b_a = 0,$$
$$a \neq b, \qquad a, b = \overline{1,n}, \qquad \mu = \overline{0,n}. \tag{2.8}$$

After taking into account (2.8) the invariance condition, written in its complete form, is given by

$$\left[\xi^{\mu}\left(\frac{\partial A}{\partial x_{\mu}}U_0 + \frac{\partial B}{\partial x_{\mu}}\right) + \eta\left(\frac{\partial A}{\partial U}U_0 + \frac{\partial B}{\partial U}\right) + (\eta_0 + \eta_U U_0 - U_{\mu}\xi^{\mu}_0)A + \right.$$
$$+ (\eta_a + \eta_U U_a - U_b\xi^b_a)\frac{\partial U}{\partial U_a} - \Delta\eta - U_aU_a\eta_{UU} - \tag{2.9}$$
$$\left. - 2U_a\eta_{aU} - \eta_u\Delta U + 2U_{aa}\xi^a_a + U_a\Delta\xi^{\mu}\right]\Bigg|_{F=0} = 0.$$

In our case, taking into consideration the explicit form of the operators (1.4a) the coefficient functions ξ^{μ}, η of the operator X are written in the form

$$\xi^0 = 0, \qquad \xi^a = g_a t, \qquad \eta = -\frac{1}{2}\lambda g_a x_a U,$$

where g_a, $a = \overline{1,n}$ are arbitrary parameters.

Having used the explicit form of ξ^{μ} and η as well as the arbitrary nature and independence of the parameters g_a (2.9) is reduced to the following linear differential equation system, which enables one to find the functions $A(t, x, U)$ and $B(t, x, U, \underset{I}{U})$:

$$t\frac{\partial A}{\partial x_a} - \frac{1}{2}\lambda x_a U\frac{\partial A}{\partial U} = 0, \qquad a = \overline{1,n}, \tag{2.10}$$

$$\frac{2}{\lambda}t\frac{\partial B}{\partial x_a} - x_a U\frac{\partial B}{\partial U} - U\frac{\partial B}{\partial U_a} - x_a U_1\frac{\partial B}{\partial U_1} - \cdots - x_a U_n\frac{\partial B}{\partial U_n} +$$
$$+ x_a B - \frac{2}{\lambda}U_a(A - \lambda) = 0, \qquad a = \overline{1,n}. \tag{2.11}$$

Thus, the proof of the theorem is reduced to the construction of the general solution of the strongly overdetermined system (2.10) and (2.11) consisting of $2n$ equations for the functions A and B.

Now let us proceed in using the standard method to find the solutions of the first-order PDE (see, e.g., Courant and Hilbert [2]).

Let us write the system of characteristic ordinary differential equations (ODE) corresponding to the system (2.10)

$$\frac{dx_a}{t} = \frac{dU}{-\frac{1}{2}\lambda x_a U}, \qquad a = \overline{1,n}. \tag{2.12}$$

From (2.12) we obtain two invariants necessary for the construction of the general solution of the system (2.10):

$$w = U \exp\left(\frac{\lambda |x|^2}{4t}\right), \qquad w_0 = t. \tag{2.13}$$

Consequently, the general solution of (2.10) is determined by invariants (2.13) and has the form

$$A(t, x, U) = f(w, w_0), \tag{2.14}$$

where f is an arbitrary differentiable function.

Now let us write the characteristic system of ODE (2.11):

$$\begin{gathered} -\frac{dx_a}{(2/\lambda)t} = \frac{dU}{x_a U} = \frac{dU_a}{U + x_a U_a} = \frac{dU_1}{x_a U_1} = \cdots = \frac{dU_{a-1}}{x_a U_{a-1}} = \\ = \frac{dU_{a+1}}{x_a U_{a+1}} = \cdots = \frac{dU_n}{x_a U_n} = \frac{dB}{x_a B + (2/\lambda)(\lambda - f(w, w_0))}, \quad a = \overline{1,n}. \end{gathered} \tag{2.15}$$

In (2.15), contrary to all the previous ones, the repeated indices do not mean summation.

Having solved the system (2.15) we obtain the following system of invariants necessary for the determination of the function B:

$$\begin{gathered} w = U \exp\left(\frac{\lambda |x|^2}{4t}\right), \qquad w_0 = t, \\ w_a = \left(U_a + \frac{\lambda x_a}{2t} U\right) \exp\left(\frac{\lambda |x|^2}{4t}\right), \qquad a = \overline{1,n}, \\ I = \left[B + (\lambda - f(w, w_0)) \left(\frac{x_a U_a}{t} + \frac{\lambda |x|^2}{4t^2} U\right)\right] \exp\left(\frac{\lambda |x|^2}{4t}\right). \end{gathered} \tag{2.16}$$

The function B is, consequently, determined from the functional equation

$$\phi(w, w_0, w_1, \ldots, w_n, I) = 0 \tag{2.17}$$

which gives us the general solution of (2.11):

$$B = U g(w, w_0, w_1, \ldots, w_n) + (f(w, w_0) - \lambda) \left(\frac{x_a U_a}{t} + \frac{\lambda |x|^2}{4t^2} U\right), \tag{2.18}$$

where g is an arbitrary differentiable function.

Thus, we are able to construct all the equations of the form (1.7), which are invariant under pot, completing by this the proof of the theorem.

Consequence 1. If one supposes the coefficient B in (1.7) to be independent of the derivatives $\underset{I}{U}$, then

$$\Delta U = \lambda U_0 + Ug(w,t) \tag{2.19}$$

is the most general equation, invariant under the PGT, g being here an arbitrary differentiable function.

A class of equations (1.7) with coefficients (2.1) and (2.2) contains as a subclass a set of equations which are invariant under the operators (1.4b) of the rotation group. The complete description of (1.7) which admits both operators (1.4a) and (1.4b) is given by the following theorem.

Theorem 2. *Equations from the class (1.7) are invariant under the operators (1.4a) and (1.4b) if and only if they have the form*

$$\Delta U = f(w,t)U_t + Ug(w, w_a w_a, t) + (f(w,t) - \lambda)\left(\frac{x_a U_a}{t} + \frac{\lambda |x|^2}{4t^2} U\right), \tag{2.20}$$

where

$$w_a w_a = \left[U_a U_a + \lambda x_a U_a \frac{U}{t} + \left(\frac{\lambda |x| U}{2t}\right)^2\right] \exp\left(\frac{\lambda |x|^2}{2t}\right).$$

This theorem is proved in the same way as the first one. The only difference is that one should substitute into the invariance condition (2.9) the coefficients A and B from (2.1) and (2.2) and the values of ξ^μ, η from (1.4b).

It should be noted that equations of the form (2.19) are obtained as a particular case of (2.20), i.e. when the function B in (1.7) is independent on the derivatives $\underset{I}{U}$. Invariance under PGT automatically implies invariance under the rotation group.

The further restriction of the class of equations (2.19) is achieved by the requirement for the equations to be invariant under the projective operator Π (1.4c) and the operator of scale transformations D (1.4d). The two following theorems are proved in quite a similar way to the ones above.

Theorem 3. *Among equations (2.19) only equations*

$$\Delta U = \lambda U_t + \frac{U}{t^2} g\left(t^{n/2} w\right), \tag{2.21}$$

where g is an arbitrary differentiable function, admit the operator Π (1.4c).

Theorem 4. *Among equations (2.19) only equations*

$$\begin{gathered} \Delta U = \lambda U_1 + \lambda_1 \frac{U}{t^2}\left(\frac{U}{\varepsilon(t,x)}\right)^\beta, \qquad t^{n/2} w = \frac{U}{\varepsilon} \times \text{constant}, \\ \lambda_1 = \text{constant}, \qquad \beta = \text{constant}, \end{gathered} \tag{2.22}$$

where

$$\varepsilon(t,x) = \left[\frac{1}{2}\left(\frac{\lambda}{\pi t}\right)^{1/2}\right]^n \exp\left(-\frac{\lambda |x|^2}{4t}\right) \tag{2.23}$$

is a fundamental solution of (1.1), admit the operator Π *(1.4c) and the operator*

$$D = 2t\partial_t + x_a\partial_{x_a} + (2/\beta - n)U\partial_U. \tag{2.24}$$

Note 1. If one implies $\beta = 0$ in (2.22), the obtained equation has the form

$$\Delta U = \lambda U_t + \lambda_1 U/t^2 \tag{2.25}$$

which may be reduced to (1.1) by means of the local substitution

$$U = W(t,x)\exp(\lambda_1/\lambda t) \qquad \lambda \neq 0.$$

Note 2. The coefficients of all classes of equations constructed above contain (explicitly or implicitly) the fundamental solution $\varepsilon(t,x)$ of (1.1). This is apparently due to the fact that $\varepsilon(t,x)$ (with an approximation to an arbitrary constant) is the complete solution of the system

$$\begin{aligned} &\Delta = \lambda U_0, \\ &G_a(U) \equiv tU_a + \tfrac{1}{2}\lambda x_a U = 0, \qquad a = \overline{1,n}. \end{aligned} \tag{2.26}$$

Note 3. The above theorems may be generalised for the systems of equations of the form

$$\begin{aligned} \Delta U^{(k)} = {} & A^{(k)}\left(t, x, U^{(1)}, \ldots, U^{(m)}\right) + \\ & + B^{(k)}\left(t, x, U^{(1)}, \ldots, U^{(m)}\right), \qquad k = 1, 2, \ldots, m. \end{aligned} \tag{2.27}$$

In particular, amongst the equations (2.27) only equations

$$\Delta U^{(k)} = \lambda U_0^{(k)} + U^{(k)} g^{(k)}\left(t, w^{(1)}, \ldots, w^{(m)}\right), \qquad k = 1, 2, \ldots, m,$$

where $w^{(k)} = U^{(k)}\exp\left(\lambda|x|^2/4t\right)$, $g^{(k)}$ are arbitrary differentiable functions, are invariant under the Galilean transformations with the infinitesimal operators

$$G_a = t\frac{\partial}{\partial x_a} - \frac{1}{2}\lambda x_a\left(U^{(1)}\frac{\partial}{\partial U^{(1)}} + \cdots + U^{(m)}\frac{\partial}{\partial U^{(m)}}\right), \qquad a = \overline{1,n}.$$

3. The second-order equations, invariant under the Galilean transformations

In this section we shall construct all the equations of the form

$$U_t = C(t,x,U)\Delta U + K(t,x,U,\underset{I}{U}), \tag{3.1}$$

where $C(t,x,U)$, $K(t,x,U,\underset{I}{U})$ are arbitrary differentiable functions, invariant under the operators $\widetilde{G}_a$ (1.5), generating the GT (1.6). Also we shall distinguish all the second-order equations of the form (1.8) which admit the following operators:

$$\widetilde{G}_a = tP_a, \qquad P_a = \partial_a, \qquad P_0 = \partial_t, \qquad a = \overline{1,n}. \tag{3.2}$$

These operators satisfy the commutational relations

$$[\widetilde{G}_a, P_b] = 0, \qquad [P_\mu, P_\nu] = 0, \qquad [\widetilde{G}_a, P_0] = -P_a. \tag{3.3}$$

It turns out that the class of such equations is rather broad. In particular, it contains the many-dimensional Monge–Ampère equation (see Fushchych and Serov [8]) and the non-relativistic analogue of the latter. All these equations are considerably nonlinear, and as a rule they cannot be reduced to the form containing a linear plus a nonlinear term.

The following statement gives the solution of the first problem, which was posed at the beginning of this section.

Theorem 5. *(3.1) is invariant under the GT (1.6) if and only if*

$$C(t,x,U) = f(t,U), \tag{3.4}$$

$$K(t,x,U,\underset{I}{U}) = g(t,U,\underset{I}{U}) - x_a U_a/t, \tag{3.5}$$

where f, g are arbitrary differentiable functions.

To prove this theorem one should repeat the same procedures used in proving theorem 1, with the only obvious difference that the coefficient functions of the $\widetilde{G}_a$ operator, i.e.

$$\xi^0 = 0, \qquad \xi^a = g_a t, \qquad a = \overline{1,n}, \qquad \eta = 0$$

should be substituted into (2.9).

Now let us formulate several more statements, giving the complete description of the equations of class (3.1), invariant under $\widetilde{G}_a$, J_{ab} and the operators

$$\widetilde{\Pi} = t^2\partial_t + t x_a \partial_{x_a}, \tag{3.6}$$

$$\widetilde{D} = 2t\partial_t + x_a\partial_{x_a}. \tag{3.7}$$

Theorem 6. *Among the set of equations (3.1) only the equations given by*

$$\begin{aligned} &U_t = f(t,U)\Delta U + g(t,U,w_{n+1}) - x_a U_a/t, \\ &w_{n+1} = U_a U_a, \qquad U_a = \partial U/\partial x_a \end{aligned} \tag{3.8}$$

are invariant under the operators $\widetilde{G}_a$ and J_{ab}, $a,b=\overline{1,n}$.

Theorem 7. *(3.8) is invariant under the projective transformations generated by the operator (3.6) if and only if*

$$f(t,U) = \widetilde{f}(U), \qquad g(t,U,w_{n+1}) = t^{-2}\widetilde{g}(U, t^2 w_{n+1}), \tag{3.9}$$

where $\widetilde{f}$, $\widetilde{g}$ are arbitrary differentiable functions.

Theorem 8. *Amongst equations of the form (3.8) only equations*

$$U_t = \widetilde{f}(U)\Delta U + U_a U_a \widetilde{g}(U) - x_a U_a/t \tag{3.10}$$

are invariant under the projective and scale transformations generated by the operators (3.6) and (3.7).

Theorem 9. *The maximal IA of the simplest linear equation from the class (3.10):*

$$U_t = \lambda\Delta U - x_a U_a/t, \qquad \lambda = \text{constant} \tag{3.11}$$

is an algebra $SLi(1,n)$ with basic operators:

$$\widetilde{G}_a = t\partial_a, \qquad J_{ab} = x_a\partial_b - x_b\partial_a, \qquad \widetilde{\Pi} = t^2\partial_t + tx_a\partial_{x_a}, \qquad I = U\partial_U,$$
$$\widetilde{D} = 2t\partial_t + x_a\partial_{x_a}, \qquad \widetilde{P}_a = \partial_{x_a} + \frac{x_a}{2\lambda t}I, \qquad \widetilde{P}_t = \partial_t + \left(\frac{n}{2t} - \frac{|x|^2}{4\lambda t^2}\right)I.$$

Note 4. (3.11), by means of the local substitution

$$U = W(t,x)t^{n/2}\exp\left(\frac{\lambda|x|^2}{4t}\right)$$

or, in the equivalent notation,

$$U = \frac{W(t,x)}{\varepsilon(t,x)}, \qquad \varepsilon(t,x) = \left[\frac{1}{2}\left(\frac{\lambda}{\pi t}\right)^{1/2}\right]^n \exp\left(-\frac{\lambda|x|^2}{4t}\right)$$

may be reduced to (1.1) for the function $W(t,x)$.

Note 5. The classes of equations given in theorems 5 and 6 can be obtained from the equations given in theorems 1 and 2. For this purpose it would be enough to apply the above substitution from note 4.

Note 6. Equations invariant under GT (1.6) (see theorem 5) can be transformed by means of the substitution of the independent variables

$$t = \theta(t'),$$
$$x_a = \theta(t')x_a + \theta^{(a)}(t'), \qquad a = \overline{1,n},$$

where $\theta(t') \neq \text{constant}$, $\theta^{(a)}$, $a = \overline{1,n}$ being arbitrary differentiable functions, to the equations given by

$$U'_{t'} = f'(t',U')\Delta U' + g'(t',U',\underset{I}{U'}),$$

where

$$U'(t',x') = U(t,x),$$
$$f'(t',U') = \frac{d\theta}{dt'}(\theta(t'))^{-2}f(\theta(t'),U'),$$
$$g'(t,,U',\underset{I}{U'}) = \frac{d\theta}{dt'}g(\theta(t'),U',\underset{I}{U'}(\theta(t'))^{-1})+$$
$$+\left(\frac{d\theta^{(a)}(t')}{dt'}(\theta(t'))^{-1} - \frac{d\theta}{dt'}\theta^{(a)}(t')(\theta(t'))^{-2}\right)U'_a.$$

In particular if

$$\theta(t') = t', \qquad \theta^{(a)}(t') = 0, \qquad a = \overline{1,n}$$

one obtains the equations

$$U'_{t'} = t'^{-2}f(t',U')\Delta U' + g(t',U',\underset{I}{U'}t^{-1}).$$

Consequence 2. It follows from the theorems given in §§ 2 and 3 that the nonlinear diffusion equation (1.9) is invariant neither under PGT (1.2) and (1.3), nor under GT (1.6). It means that the Galilean principle of invariance is not satisfied by (1.9). Nonlinear equations, invariant under PGT and x and t translations, are obtained by Fushchych [5].

Now let us proceed in solving the second problem: to describe all the second-order equations

$$F(x_0, x_1, U, U_0, U_1, U_{00}, U_{01}, U_{11}) = 0 \tag{3.12}$$

in the two-dimensional space (x_0, x_1), which are invariant under GT and translations generated by operators (3.2).

Theorem 10. *Amongst the set of equations (3.12) only the equations given by*

$$F_1(w^{(I)}, w^{(II)}, U, U_1, U_{11}) = 0 \tag{3.13}$$

are invariant under ot (1.6) and translations.

(3.13) contains the following notation:

$$w^{(I)} = \det \begin{pmatrix} U_0 & U_1 \\ U_{01} & U_{11} \end{pmatrix}, \qquad w^{(II)} = \det \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix} \tag{3.14}$$

of the determinant of matrices, the elements of which are the first- and second-order derivatives of the function U. Here F_1 is an arbitrary differentiable function.

Proof. The invariance of (3.12) under translations, i.e. operators P_0, P_1, is equivalent to the requirement

$$\frac{\partial F}{\partial x_0} = \frac{\partial F}{\partial x_1} = 0. \tag{3.15}$$

Taking into account (3.15) we obtain the following expression for the action of the twice prolonged operator $\overset{2}{X}$ on the manifold (3.12) (see (2.6))

$$\left(\eta \frac{\partial F}{\partial U} + \rho^\mu \frac{\partial F}{\partial U_\mu} + \sigma^{\mu\nu} \frac{\partial F}{\partial U_{\mu\nu}} \right)\Bigg|_{F=0} = 0, \qquad \mu, \nu = 0, 1. \tag{3.16}$$

The coefficient functions of operators $\{G_a\}$ are given by

$$\xi^0 = \eta = 0, \qquad \xi^1 = t. \tag{3.17}$$

The coefficient functions $\{\rho^\mu\} = \{\rho^0, \rho^1\}$, $\{\sigma^{\mu\nu}\} = \{\sigma^{00}, \sigma^{01}, \sigma^{10}, \sigma^{11}\}$ are determined from the formulae given in § 2. Taking into account (3.17) we obtain

$$\begin{aligned} &\rho^0 = -U_1, \qquad \rho^1 = 0, \\ &\sigma^{00} = -2U_{01}, \qquad \sigma^{01} = \sigma^{10} = -U_{11}, \qquad \sigma^{11} = 0. \end{aligned} \tag{3.18}$$

With the help of formulae (3.17) and (3.18) the invariance condition (3.16) can easily be reduced to the following linear pde for the function F:

$$U_1 \frac{\partial F}{\partial U_0} + 2U_{01} \frac{\partial F}{\partial U_{00}} + U_{11} \frac{\partial F}{\partial U_{01}} = 0 \tag{3.19}$$

which can be readily solved. The general solution of (3.19) is an arbitrary differentiable function

$$F = F_1(w^{(I)}, w^{(II)}, U, U_1, U_{11})$$

which depends on five variables. The theorem is proved.

Theorem 10, without any substantial complications, is generalised for the case of $(n+1)$-dimensional space

$$\begin{aligned} &F(x_0, x_1, \dots, x_n, U, U_0, \underset{I}{U}, U_{00}, U_{01}, \dots, U_{0n}, \underset{II}{U}) = 0, \\ &\underset{I}{U} = (U_1, \dots, U_n), \qquad \underset{II}{U} = (U_{11}, U_{12}, \dots, U_{nn}) \end{aligned} \tag{3.20}$$

i.e. we have the following theorem.

Theorem 11. *Amongst equations of the class (3.20) only equations given by*

$$F_1(w^{(I)}, w^{(II)}, U, \underset{I}{U}, \underset{II}{U}) = 0 \tag{3.21}$$

are invariant under GT (1.6) and $x_0, x_1, \dots, x_n$ *coordinate translations, where*

$$\begin{aligned} w^{(I)} &= \det \begin{pmatrix} U_0 & U_1 & \cdots & U_n \\ U_{10} & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ U_{n0} & U_{n1} & \cdots & U_{nn} \end{pmatrix}, \\ w^{(II)} &= \det \begin{pmatrix} U_{00} & U_{01} & \cdots & U_{0n} \\ U_{10} & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ U_{n0} & U_{n1} & \cdots & U_{nn} \end{pmatrix}. \end{aligned} \tag{3.22}$$

Note 7. In the specific case when

$$F_1 \equiv w^{(II)} = \det(U_{\mu\nu}) = 0, \qquad U_{\mu\nu} = \partial^2 U/\partial x_\mu \partial x_\nu$$

a many-dimensional Monge–Ampère equation is obtained, the group properties of which have been studied by Fushchych and Serov [8].

Note 8. In the case

$$F_1 = w^{(I)} - \lambda = 0, \qquad \lambda = \text{constant} \tag{3.23}$$

the maximal IA of this equation is generated by an operator

$$\begin{aligned} &X = \xi^\mu \frac{\partial}{\partial x_\mu} + \eta \frac{\partial}{\partial U}, \\ &\xi^0 = C_{00} t + d_0, \qquad \xi^a = C_{ab} x_b + f_a(t), \qquad a, b = \overline{1, n}, \\ &\eta = CU + d, \qquad C = \frac{C_{00} + 2(C_{11} + \cdots + C_{nn})}{n+1}, \end{aligned} \tag{3.24}$$

where C_{00}, C_{ab}, d_0, d are arbitrary constants, and $f_a(t)$, $a = \overline{1, n}$ are arbitrary differentiable functions.

It means that the maximal IA of (3.23) is infinitely dimensional. In particular, this algebra contains operators of the form

$$\partial_{x_0},\ \partial_{x_a},\ \partial_U,\ x_b\partial_{x_a},\quad a\neq b,\quad a,b=\overline{1,n}, \tag{3.25a}$$

$$\begin{gathered}D_0=x_0\partial_{x_0}+\frac{U}{n+1}\partial_U,\\ D_1=x_1\partial_{x_1}+\frac{2U}{n+1}\partial_U,\quad \ldots,\quad D_n=x_n\partial_{x_n}+\frac{2U}{n+1}\partial_U,\end{gathered} \tag{3.25b}$$

$$X_1=f_1(t)\partial_{x_1},\quad \ldots,\quad X_n=f_n(t)\partial_{x_n}. \tag{3.25c}$$

Note 9. It is possible to construct a general solution for the two-dimensional equation

$$w^{(I)}=\det\begin{pmatrix} U_0 & U_1\\ U_{01} & U_{11}\end{pmatrix}=0. \tag{3.26}$$

To prove this, we represent (3.26) as follows:

$$\frac{\partial}{\partial x_1}\left(\frac{U_1}{U_0}\right)=0,\qquad U_1=\frac{\partial U}{\partial x_1},\qquad U_0=\frac{\partial U}{\partial x_0}$$

and then we obtain the general solution

$$U=F(x_1+G(x_0)),$$

where F and G are arbitrary differentiable functions. Direct verification shows that

$$U=F(\mathcal{L}_a x_a+G(x_0)),\qquad a=\overline{1,n},\qquad \mathcal{L}_a=\text{constant}$$

is a particular solution of $(n+1)$-dimensional equation (3.23) under $\lambda=0$.

Note 10. Equations

$$w^{(I)}=\begin{vmatrix} U_0 & \cdots & U_n\\ U_{10} & \cdots & U_{1n}\\ \cdots & \cdots & \cdots\\ U_{n0} & \cdots & U_{nn}\end{vmatrix}=F(U), \tag{3.27}$$

where $F(U)$ is an arbitrary twice differentiable function, can be reduced to (3.23) at $\lambda=1$ for the function $W(x_0,\ldots,x_n)$ by the substitution

$$W=\int [F(U)]^{-1/(n+1)}dU.$$

Note 11. Maximal IA of the equation

$$w^{(I)}=F(U_aU_a),\qquad U_aU_a=U_1^2+\cdots+U_n^2 \tag{3.28}$$

is generated by the basis operators (3.25c) and

$$\begin{gathered}\partial_{x_0},\ \partial_{x_a},\partial_U,\ x_a\partial_{x_b}-x_b\partial_{x_a},\quad a\neq b,\quad a,b=\overline{1,n},\\ D=(1-n)\partial_{x_0}+x_a\partial_{x_a}+U\partial_U.\end{gathered}$$

In particular, in the case of $n = 1$ for equations of the class (3.28)

$$\begin{vmatrix} U_0 & U_1 \\ U_{10} & U_{11} \end{vmatrix} = U_1^2, \tag{3.29}$$

$$\begin{vmatrix} U_0 & U_1 \\ U_{10} & U_{11} \end{vmatrix} = U_1^3 \tag{3.30}$$

one can obtain the general solutions, namely $U = F(x_1 e^{-x_0} + G(x_0))$ is the general solution of (3.29) and $\phi(U, x_0 U + G(x_0) - x_1) = 0$ is the general solution of (3.30) written in an implicit form, F, G, ϕ being arbitrary differentiable functions.

In conclusion, we note that among the Galilei invariant equations (3.21) one can distinguish a class of equations

$$U_0 = \lambda(U, \underset{I}{U})\Delta U + Q(U, \underset{I}{U}) - w^{(III)}/w^{(II)}, \tag{3.31}$$

$$w^{(III)} = \begin{vmatrix} 0 & U_1 & \cdots & U_n \\ U_{10} & U_{11} & \cdots & U_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ U_{n0} & U_{n1} & \cdots & U_{nn} \end{vmatrix}, \quad w^{(II)} = \begin{vmatrix} U_{11} & \cdots & U_{1n} \\ U_{21} & \cdots & U_{2n} \\ \cdots & \cdots & \cdots \\ U_{n1} & \cdots & U_{nn} \end{vmatrix},$$

λ, Q being arbitrary functions.

As to the structure, equations of the form (3.31) are diffusive type nonlinear equations with a strongly nonlinear addition. The properties of (3.31) will be studied by us in a further paper.

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