Continuous subgroups of the Poincaré group $P(1,4)$

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An exhaustive description of the non-splitting subalgebras of the $LP(1,4)$ algebra with respect to $P(1,4)$ conjugation is presented.

1. Introduction

The generalised Poincaré group $P(1,4)$ is the group of inhomogeneous pseudoorthogonal transformations of the five-dimensional pseudo-Euclidean space with the scalar product $(\boldsymbol{X},\boldsymbol{Y}) = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4$. The $P(1,4)$ group is the simplest one which contains the Poincaré group $P(1,3)$ as a subgroup. Fushchych and Krivsky [11, 12] and Fushchych [10] have used the $P(1,4)$ group and its unitary representations to describe particles with variable mass and spin. An arbitrary partial differential equation which is invariant under the $P(1,4)$ group is also invariant under the $P(1,3)$ group as well as under the extended Galilei group $\widetilde{G}(1,3)$ since $\widetilde{G}(1,3) \subset P(1,4)$ (Fushchych and Nikitin [13]). The papers of Aghassi et al [1, 2] deal with irreducible representations of $P(1,4)$ and $C(1,4)$, using the latter in the theory of elementary particles. Kadyshevsky [16] proposed using the $P(1,4)$ group in field theory with the fundamental length. The $P(1,4)$ group is the invariance group of the relativistic Hamilton–Jacobi equation (Fushchych and Serov [14]) and the Monge–Ampere equation (Fushchych and Serov [15]). These nonlinear equations are invariant under transformations of the $P(1,4)$ group with the fifth coordinate as $x_4 \equiv u$, where $u = u(x_0, x_1, x_2, x_3)$. So it is important to investigate the subgroup structure of the $P(1,4)$ group. In particular, these results can be used in the separation of variables of many important partial differential equations.

The splitting subalgebras of $LP(1,4)$ were described by Fedorchuk [6, 7]. Some high-dimension non-splitting subalgebras of $LP(1,4)$ were listed by Fedorchuk and Fushchych [9] and Fedorchuk [8]. In this paper we list all the non-splitting subalgebras of the $LP(1,4)$ algebra with respect to $P(1,4)$ conjugation. In the papers of Lassner [17], Bacry et al [3, 4, 5] and Patera et al [18] all the subalgebras of $LP(1,3)$ are classified with respect to $P(1,3)$ conjugation, so we consider such subalgebras of $LP(1,4)$ which are non-conjugate to the subalgebras of $LP(1,3)$. In our paper we use the method due to Patera et al [16].

2. Some auxiliary remarks

The $LP(1,4)$ algebra is defined by the following computation relations:

$$[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta}J_{\beta\gamma} + g_{\beta\gamma}J_{\alpha\delta} - g_{\alpha\gamma}J_{\beta\delta} - g_{\beta\delta}J_{\alpha\gamma},$$
$$[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta, \qquad J_{\beta\alpha} = -J_{\alpha\beta}, \qquad [P_\alpha, P_\beta] = 0,$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\alpha\beta} = 0$ if $\alpha \neq \beta$ $(\alpha, \beta = 0, 1, 2, 3, 4)$.

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Below we shall use the following notation: $K_a = J_{0a} - J_{a4}$ $(a = 1, 2, 3)$; $W = \langle X_1, \ldots, X_s\rangle$ is a space or Lie algebra over the real number field R with the generating elements $X_1, \ldots, X_s$; $V = \langle P_0, P_1, P_2, P_3, P_4\rangle$; π is a projection $LP(1,4)$ on $LO(1,4)$; $\pi_{a,\ldots,q}$ is a projection $LP(1,4)$ on $\langle P_a, \ldots, P_q\rangle$.

Lemma 1. *Let W be a subspace of V invariant under* $\mathrm{Ad}\, J_{ab}$ $(1 \le a < b \le 4)$. *If $\pi_{a,b}(W) \ne 0$ then $P_a, P_b \in W$.*

Proof. Let $X = \sum x_\alpha P_\alpha \in W$ and $\pi_{ab}(X) \ne 0$. Obviously,

$$[J_{ab}, X] = x_a P_b - x_b P_a, \qquad [J_{ab}, [J_{ab}, X]] = -x_a P_a - x_b P_b.$$

Since the vectors obtained are linearly independent, so $P_a, P_b \in W$ and this proves the lemma.

Lemma 2. *If $W \subset V$ and $[J_{0a}, W] \subset W$ and if $\pi_{0,a}(W) \ne 0$, then the subspace W contains $P_0 + P_a$ or $P_0 - P_a$.*

Corollary. *Let $W \subset V$ and $[J_{0a}, W] \subset W$. If $\pi_{0,a}(W) \ne 0$, then within the conjugation corresponding to the element*

$$\mathit{diag}\,(\underbrace{1, \ldots, -1}_{a+1}, \ldots, 1)$$

from $O(1,4)$ group W contains $P_0 + P_a$.

Lemma 3. *Let W be a subspace of V invariant under* $\mathrm{Ad}\,(J_{0a} + \gamma J_{cd})$, *where $\gamma \in R$, $\gamma \ne 0$, $0, a, c, d$ are mutually different. Then $W = \pi_{0,a}(W) \oplus s\langle P_b\rangle$, where $s \in \{0, 1\}$, $b \notin \{0, a, c, d\}$.*

Proof. If

$$X = \sum_0^4 \alpha_j P_j \in W$$

then W contains the elements

$$X_1 = [J_{0a} + \gamma J_{cd}, X] = -\alpha_0 P_a - \alpha_a P_0 + \gamma(\alpha_c P_d - \alpha_d P_c),$$
$$X_2 = [J_{0a} + \gamma J_{cd}, X_1] = \alpha_0 P_a + \alpha_a P_0 + \gamma^2(-\alpha_c P_c - \alpha_d P_d),$$
$$X_3 = [J_{0a} + \gamma J_{cd}, X_2] = -\alpha_0 P_a - \alpha_a P_0 + \gamma^3(-\alpha_c P_d + \alpha_d P_c).$$

Since $X_1 - X_3 = (\gamma + \gamma^3)(\alpha_c P_d - \alpha_d P_c)$ and $\gamma \ne 0$, then $\alpha_c P_d - \alpha_d P_c \in W$ whence $\pi_{c,d}(X), \pi_{0,a}(X) \in W$. Thus, this lemma is proved.

Lemma 4. *Let W be a subspace of V invariant under* $\mathrm{Ad}\, K_a$. *If $\pi_{0,4}(W) \not\subset \langle P_0 + P_4\rangle$ then $P_0 + P_4, P_a \in W$. If $\pi_a(W) \ne 0$ then $P_0 + P_4 \in W$.*

Proof. Let W contains the vector $X = \sum \alpha_j P_j$, then W also contains $X_1 = [X, K_a] = \alpha_a(P_0 + P_4) + (\alpha_0 - \alpha_4)P_a$, $X_2 = [X_1, K_a] = (\alpha_0 - \alpha_4)(P_0 + P_4)$. If $\alpha_0 - \alpha_4 \ne 0$ then $P_0 + P_4, P_a \in W$. If $\alpha_0 - \alpha_4 = 0$, $\alpha_a \ne 0$ then $P_0 + P_4 \in W$. Thus this lemma is proved.

Lemma 5. *Let W be a subspace of V invariant under* $\mathrm{Ad}\,(K_a - J_{bc})$, *where $\{a, b, c\} = \{1, 2, 3\}$. Then W is invariant under* $\mathrm{Ad}\, K_a$ *and* $\mathrm{Ad}\, J_{bc}$.

Proof. Let $X = K_a - J_{bc}$, $Y \in W$. Since $[X, [X, [X, Y]]] = [J_{bc}, Y]$, then $[J_{bc}, W] \subset W$, $[K_a, W] \subset W$. Thus, the lemma is proved.

Lemma 6. *Let F be a subalgebra of $LO(1,4)$ with the generators J_{04} and K_a, where a covers a subset I of the set $\{1,2,3\}$. If A is a subalgebra of $LP(1,4)$ and $\pi(A)=F$, then within the conjugation with respect to the group of translations A contains elements K_a $(a\in I)$ and $J_{04}+\delta_1P_1+\delta_2P_2+\delta_3P_3$.*

Proof. Let $X_a=K_a+\sum\alpha_iP_i$, $Y=J_{04}+\sum\delta_iP_i$ $(i=0,1,2,3,4)$. By the automorphism $\exp(t_1P_0+t_2P_4)$ the coefficients δ_0, δ_4 can be made zero. Since $[Y,X_a]=-K_a+\delta_a(P_0+P_4)-\alpha_0P_4-\alpha_4P_0$, one can therefore consider $X_a=K_a+\gamma P_0$ within the automorphism $\exp(tP_a)$. Evidently $[Y,X_a]+X_a=(\delta_a+\gamma)P_0+(\delta_a-\gamma)P_4$. If $\gamma\neq 0$ then $P_0+P_4\in A$ by lemma 4. Therefore we have $P_0,P_4\in A$ and hence $\gamma=0$ within the conjugation. Thus, this lemma is proved.

Lemma 7. *Let A be a subalgebra of $LP(1,4)$, $X=J_{12}+cJ_{04}+\beta P_3$, $Y=K_3+\sum\gamma_iP_i$ $(i=1,2,3,4;\ c>0)$. If $X,Y\in A$, then A contains K_3.*

Proof. It is easy to obtain

$$cY-[X,Y]=(\beta-c\gamma_4)P_0+(c\gamma_1-\gamma_2)P_1+(c\gamma_2+\gamma_1)P_2+c\gamma_3P_3+(c\gamma_4+\beta)P_4.$$

According to lemma 3 $(\beta-c\gamma_4)P_0+(c\gamma_4+\beta)P_4,(c\gamma_1-\gamma_2)P_1+(c\gamma_2+\gamma_1)P_2\in A$. If $\gamma_4\neq 0$ then lemma 4 yields $P_0,P_4\in A$. If $c\gamma_1-\gamma_2=0$, $c\gamma_2+\gamma_1=0$ then $\gamma_1=\gamma_2=0$. Thereafter using lemma 1 we can put $\gamma_1=\gamma_2=0$. Since $c\gamma_3P_3\in A$ one can admit that $\gamma_3=0$. Thus the lemma is proved.

Lemma 8. *Let A be a subalgebra of $LP(1,4)$, $\varphi=\exp(-\omega K_b)$ $(\omega\in R,\ \omega\neq 0)$. If $P_0+P_4,P_b+\omega^{-1}P_4\in A$ $(1\le b\le 3)$ then the algebra $\varphi(A)$ contains P_0 and P_4.*

Proof. According to the Campbell–Hausdorff formula we have

$$\varphi(P_0+P_4)=P_0+P_4,\qquad \varphi(P_0+\omega^{-1}P_4)=\omega^{-1}P_4+\frac{1}{2}\omega(P_0+P_4).$$

This gives that $P_0+P_4,P_4\in\varphi(A)$, therefore $P_0,P_4\in\varphi(A)$. Thus this lemma is proved.

3. The non-splitting subalgebras of the $LP(1,4)$ algebra

Let $\widetilde{F}$ be an subalgebra of $LP(1,4)$ such that $\pi(\widetilde{F})=F$. An expression $\widetilde{F}+W$ means that $[F,W]\subset W$ and $\widetilde{F}\cap V\subset W$. As concerns the non-splitting algebras $\widetilde{F}+W_1,\ldots,\widetilde{F}+W_s$ we will use the notation $\widetilde{F}:W_1,\ldots,W_s$.

Theorem. *Let $\alpha,\beta,\delta,\mu,\omega\in R$, $\alpha>0$, $\omega>0$, $\mu\ge 0$ and this takes place for all labelling variables. The non-splitting subalgebras of the $LP(1,4)$ algebra are exhausted by the non-splitting subalgebras of the $LP(1,3)$ algebra and the following subalgebras:*

$\langle J_{12}+\alpha P_0\rangle$: $\langle P_3,P_4\rangle$, $\langle P_1,P_2,P_3,P_4\rangle$;

$\langle J_{12}+P_0+P_3\rangle$: $\langle P_4\rangle$, $\langle P_1,P_2,P_4\rangle$;

$\langle J_{12}+\alpha P_3\rangle$: $\langle P_4\rangle$, $\langle P_0+P_4\rangle$, $\langle P_0,P_4\rangle$, $\langle P_1,P_2,P_4\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0,P_1,P_2,P_4\rangle$;

$\langle J_{12}+P_0\rangle$: $\langle P_0+P_4,P_3\rangle$, $\langle P_0+P_4,P_1,P_2,P_3\rangle$;

$\langle J_{12}+J_{34}+\alpha P_0\rangle$: 0, $\langle P_1,P_2\rangle$, $\langle P_1,P_2,P_3,P_4\rangle$;

$\langle J_{12}+cJ_{34}+\alpha P_0\rangle$: 0, $\langle P_1,P_2\rangle$, $\langle P_3,P_4\rangle$, $\langle P_1,P_2,P_3,P_4\rangle$ $(0<c<1)$;

$\langle J_{04}+\alpha P_3\rangle$: $\langle P_1,P_2\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0,P_1,P_2,P_4\rangle$;

$\langle J_{12}+cJ_{04}+\alpha P_3\rangle$: 0, $\langle P_0+P_2\rangle$, $\langle P_0,P_4\rangle$, $\langle P_1,P_2\rangle$, $\langle P_0+P_4,P_1,P_4\rangle$, $\langle P_0,P_1,P_2,P_4\rangle$ $(c>0)$;

$\langle K_3+P_2\rangle$: $\langle P_1\rangle$, $\langle P_0+P_4,P_1\rangle$, $\langle P_0+P_4,P_1+\omega P_3\rangle$, $\langle P_0+P_4,P_1,P_3\rangle$, $\langle P_0,P_1,P_3,P_4\rangle$;
$\langle K_3+P_4\rangle$: $\langle P_1,P_2\rangle$, $\langle P_0+P_4,P_1+\omega P_3,P_2\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0+P_4,P_1,P_2,P_3\rangle$;
$\langle K_3-J_{12}+\alpha P_4\rangle$: 0, $\langle P_0+P_4\rangle$, $\langle P_0+P_4,P_3\rangle$, $\langle P_1,P_2\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0+P_4,P_1,P_2,P_3\rangle$;
$\langle J_{12}+\alpha P_0,J_{34}+\mu P_0\rangle$: 0, $\langle P_1,P_2\rangle$, $\langle P_1,P_2,P_3,P_4\rangle$; $\langle J_{12},J_{34}+\alpha P_0,P_1,P_2\rangle$;
$\langle J_{04}+\alpha P_3,J_{12}+\mu P_3\rangle$: 0, $\langle P_0+P_4\rangle$, $\langle P_0,P_4\rangle$, $\langle P_1,P_2\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0,P_1,P_2,P_4\rangle$;
$\langle J_{04},J_{12}+\alpha P_3\rangle$: 0, $\langle P_0+P_4\rangle$, $\langle P_0,P_4\rangle$, $\langle P_1,P_2\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0,P_1,P_2,P_4\rangle$;
$\langle J_{12}+P_0+P_4,K_3+\mu P_4\rangle$; $\langle J_{12},K_3+P_4\rangle$;
$\langle J_{12}+\mu P_3,K_3+P_4,P_0+P_4\rangle$; $\langle J_{12}+\alpha P_3,K_3,P_0+P_4\rangle$;
$\langle J_{12}+P_0+P_4,K_3+\mu P_4,P_1,P_2\rangle$; $\langle J_{12},K_3+P_4,P_1,P_2\rangle$;
$\langle J_{12}+\mu P_4,K_3+P_4,P_0+P_4,P_3\rangle$; $\langle J_{12}+P_4,K_3,P_0+P_4,P_3\rangle$;
$\langle J_{12}+\mu P_3,K_3+P_4,P_0+P_4,P_1,P_2\rangle$; $\langle J_{12}+\alpha P_3,K_3,P_0+P_4,P_1,P_2\rangle$;
$\langle J_{12}+\mu P_4,K_3+P_4,P_0+P_4,P_1,P_2,P_3\rangle$; $\langle J_{12}+P_4,K_3,P_0+P_4,P_1,P_2,P_3\rangle$;
$\langle K_1+\mu P_2+P_3,K_2+\mu P_1+\beta P_2\rangle$; $\langle K_1,K_1\pm P_2,P_3\rangle$;
$\langle K_1+P_2,K_2+P_1+\beta P_2,P_3\rangle$; $\langle K_1+\alpha P_2+P_3,K_2+\beta_1P_1+\beta_2P_2,P_0+P_4\rangle$;
$\langle K_1+P_3,K_2+\mu P_1+\beta P_2,P_0+P_4\rangle$; $\langle K_1+\mu_2P_2+\mu_3P_3,K_2+P_4,P_0+P_4,P_1\rangle$;
$\langle K_1+P_2+\alpha P_3,K_2+\beta P_3,P_0+P_4,P_1\rangle$; $\langle K_1+P_2,K_2+\alpha P_3,P_0+P_4,P_1\rangle$;
$\langle K_1+P_3,K_2+\mu P_3,P_0+P_4,P_1\rangle$; $\langle K_1,K_2+P_3,P_0+P_4,P_1\rangle$;
$\langle K_1+P_2,K_2+\beta_1P_1+\beta_2P_2,P_0+P_4,P_3\rangle$; $\langle K_1,K_2\pm P_2,P_0+P_4,P_3\rangle$;
$\langle K_1+P_2+\beta P_3,K_2+\delta P_3,P_0+P_4,P_1+\omega P_3\rangle$;
$\langle K_1+P_3,K_2+\mu P_3,P_0+P_4,P_1+\omega P_3\rangle$; $\langle K_1,K_2+P_3,P_0+P_4,P_1+\omega P_3\rangle$;
$\langle K_1+P_3,K_2,P_0+P_4,P_1,P_2\rangle$; $\langle K_1+P_4,K_2+\alpha P_3,P_0+P_4,P_1,P_2\rangle$;
$\langle K_1+P_2,K_2,P_0+P_4,P_1,P_3\rangle$; $\langle K_1+P_2,K_2+\alpha P_4,P_0+P_4,P_1,P_3\rangle$;
$\langle K_1,K_2+P_4,P_0+P_4,P_1,P_3\rangle$; $\langle K_1,K_2+P_3,P_0+P_4,P_1+\omega P_3,P_2\rangle$;
$\langle K_1+P_4,K_2+\mu P_3,P_0+P_4,P_1+\omega P_3,P_2\rangle$; $\langle K_1+P_3,K_2,P_0,P_1,P_2,P_4\rangle$;
$\langle K_1+P_4,K_2,P_0+P_4,P_1,P_2,P_3\rangle$; $\langle K_3,J_{04}+\alpha P_1,P_0+P_4,P_1+\omega P_3,P_2\rangle$;
$\langle K_3,J_{04}+\alpha P_2\rangle$: $\langle P_1\rangle$, $\langle P_0+P_4,P_1\rangle$, $\langle P_0+P_4,P_1+\omega P_3\rangle$, $\langle P_0+P_4,P_1,P_3\rangle$, $\langle P_0,P_1,P_3,P_4\rangle$;
$\langle K_3,J_{04}+\alpha P_3,P_0+P_4,P_1,P_2\rangle$; $\langle K_3,J_{04}+\alpha_1P_1+\alpha_2P_2,P_0+P_4,P_1+\omega P_3\rangle$;
$\langle K_3,J_{04}+\alpha_2P_2+\alpha_3P_3,P_0+P_4,P_1\rangle$;
$\langle K_3,J_{12}+cJ_{04}+\alpha P_3\rangle$: $\langle P_0+P_4\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$ $(c>0)$;
$\langle K_3,J_{04}+\mu_1P_3,J_{12}+\mu_2P_3\rangle$: $\langle P_0+P_4\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$ $(\mu_1^2+\mu_2^2>0)$;
$\langle K_1,K_2,J_{12}+\alpha P_3\rangle$; $\langle K_1,K_2,J_{12}+P_0+P_4,P_3\rangle$; $\langle K_1,K_2,J_{12}+\alpha P_3,P_0+P_4\rangle$;
$\langle K_1+P_2,K_2-P_1,J_{12}+\alpha P_3,P_0+P_4\rangle$; $\langle K_1+P_2,K_2-P_1,J_{12},P_0+P_4,P_3\rangle$;
$\langle K_1,K_2,J_{12}+\alpha P_3,P_0+P_4,P_1,P_2\rangle$; $\langle K_1,K_2,J_{12}+\alpha P_3,P_0,P_1,P_2,P_4\rangle$;
$\langle K_1,K_2,J_{12}+P_4,P_0+P_4,P_1,P_2,P_3\rangle$;
$\langle K_1,K_2,J_{04}+\alpha P_1\rangle$: $\langle P_0+P_4,P_3\rangle$, $\langle P_0+P_4,P_1+\omega P_3\rangle$, $\langle P_0+P_4,P_1+\omega P_3,P_2\rangle$;
$\langle K_1,K_2,J_{04}+\alpha P_2\rangle$: $\langle P_0+P_4,P_1+\omega P_3\rangle$, $\langle P_0+P_4,P_1,P_3\rangle$;
$\langle K_1,K_2,J_{04}+\alpha P_3\rangle$: 0, $\langle P_0+P_4\rangle$, $\langle P_0+P_4,P_1\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0,P_1,P_2,P_4\rangle$;
$\langle K_1,K_2,J_{04}+\alpha_1P_1+\alpha_2P_2,P_0+P_4,P_1+\omega P_3\rangle$;
$\langle K_1,K_2,J_{04}+\alpha_1P_1+\alpha_3P_3,P_0+P_4\rangle$; $\langle K_1,K_2,J_{04}+\alpha_2P_2+\alpha_3P_3,P_0+P_4,P_1\rangle$;
$\langle K_1,K_2,J_{12}+cJ_{04}+\alpha P_3\rangle$: 0, $\langle P_0+P_4\rangle$, $\langle P_0+P_4,P_1,P_2\rangle$, $\langle P_0,P_1,P_2,P_4\rangle$ $(c>0)$;
$\langle K_1+P_2,K_2+P_1+\beta P_2+\mu P_3,K_3+\mu P_2+\delta P_3\rangle$; $\langle K_1,K_2\pm P_2,K_3+\beta P_3\rangle$;
$\langle K_1+P_2,K_2+\beta_1P_1+\beta_2P_2+\alpha P_3,K_3+\delta_1P_1+\delta_2P_2+\delta_3P_3,P_0+P_4\rangle$;
$\langle K_1+P_2,K_2+\beta_1P_1+\beta_2P_2,K_3+\alpha P_1+\delta_2P_2+\delta_3P_3,P_0+P_4\rangle$;
$\langle K_1+P_2,K_2+\beta_1P_1+\beta_2P_2,K_2+\mu P_2+\delta P_2,P_0+P_4\rangle$;

$\langle K_1, K_2 \pm P_2, K_3 + \beta P_3, P_0 + P_4\rangle$; $\langle K_1 + P_2, K_2 + \alpha P_3, K_3 + \beta P_2 + \delta P_3, P_0 + P_4, P_1\rangle$;
$\langle K_1 + P_2, K_2, K_3 + \mu P_2 + \beta P_3, P_0 + P_4, P_1\rangle$;
$\langle K_1, K_2 + P_3, K_3 + \beta P_2 + \delta P_3, P_0 + P_4, P_1\rangle$; $\langle K_1, K_2, K_3 \pm P_3, P_0 + P_4, P_1\rangle$;
$\langle K_1 + P_3, K_2, K_3, P_0 + P_4, P_1, P_2\rangle$; $\langle K_1, K_2, K_3 + P_4, P_0 + P_4, P_1, P_2\rangle$;
$\langle K_1 + \alpha P_3, K_2, K_3 + P_4, P_0 + P_4, P_1, P_2\rangle$; $\langle K_1 + P_4, K_2, K_3, P_0 + P_4, P_1, P_2, P_3\rangle$;
$\langle K_1 \pm \alpha P_1, K_2 \pm \alpha P_2, J_{12} - K_3\rangle$;
$\langle K_1 + \beta P_1 + \mu P_2, K_2 - \mu P_1 + \beta P_2, J_{12} - K_3, P_0 + P_4\rangle$ $(\beta^2 + \mu^2 > 0)$;
$\langle K_1 + \alpha P_2, K_2 - \alpha P_1, J_{12} - K_3, P_0 + P_4, P_3\rangle$;
$\langle K_1, K_2, J_{12} - K_3 + \alpha P_4, P_0 + P_4, P_1, P_2, sP_3\rangle$ $(s = 0, 1)$;
$\langle J_{12} + J_{34}, J_{12} - J_{24}, J_{23} + J_{14}, J_{34} + \alpha P_0\rangle$: 0, $\langle P_1, P_2, P_3, P_4\rangle$;
$\langle K_1, K_2, J_{04} + \alpha P_3, J_{12} + \mu P_3\rangle$: 0, $\langle P_0 + P_4\rangle$, $\langle P_0 + P_4, P_1, P_2\rangle$, $\langle P_0, P_1, P_2, P_4\rangle$;
$\langle K_1, K_2, J_{04}, J_{12} + \alpha P_3\rangle$: 0, $\langle P_0 + P_4\rangle$, $\langle P_0 + P_4, P_1, P_2\rangle$, $\langle P_0, P_1, P_2, P_4\rangle$;
$\langle K_1, K_2, K_3 \pm P_3, J_{12}\rangle$; $\langle K_1, K_2, K_3 + \beta P_3, J_{12} + P_0 + P_4\rangle$;
$\langle K_1 + P_2, K_2 - P_1, K_3 + \beta P_3, J_{12} + \mu P_3, P_0 + P_4\rangle$;
$\langle K_1, K_2, K_3 \pm P_3, J_{12} + \mu P_3, P_0 + P_4\rangle$; $\langle K_1, K_2, K_3, J_{12} + \alpha P_3, P_0 + P_4\rangle$;
$\langle K_1 + P_2, K_2 - P_1, K_3, J_{12}, P_0 + P_4, P_3\rangle$; $\langle K_1, K_2, K_3 + P_4, J_{12} + \mu P_3, P_0 + P_4, P_1, P_2\rangle$;
$\langle K_1, K_2, K_3, J_{12} + \alpha P_3, P_0 + P_4, P_1, P_2\rangle$;
$\langle K_1, K_2, K_3 + P_4, J_{12} + \mu P_4, P_0 + P_4, P_1, P_2, P_3\rangle$;
$\langle K_1, K_2, K_3, J_{12} + P_4, P_0 + P_4, P_1, P_2, P_3\rangle$; $\langle K_1, K_2, K_3, J_{04} + \alpha P_1, P_0 + P_4\rangle$;
$\langle K_1, K_2, K_3, J_{04} + \alpha P_2, P_0 + P_4, P_1\rangle$; $\langle K_1, K_2, K_3, J_{04} + \alpha P_3, P_0 + P_4, P_1, P_2\rangle$;
$\langle K_1, K_2, K_3, J_{12} + cJ_{01} + \alpha P_3\rangle$: $\langle P_0 + P_4\rangle$, $\langle P_0 + P_4, P_1, P_2\rangle$ $(c > 0)$;
$\langle K_1, K_2, K_3, J_{04} + \mu_1 P_3, J_{12} + \mu_2 P_3\rangle$: $\langle P_0 + P_4\rangle$, $\langle P_0 + P_4, P_1, P_2\rangle$ $(\mu_1^2 + \mu_2^2 > 0)$.

Proof. The subalgebras of $LO(1,4)$ are classified by Patera et at [19]. For every algebra Fedorchuk [6, 7] has found invariant subspaces of the space V. Using these results together with lemmas 1–8, we will find the non-splitting subalgebras of the $LP(1,4)$ algebra. Below we consider some examples in detail.

Let A be a subalgebra $LP(1,4)$, $W = A \cap V$.

Suppose that $\pi(A) = \langle J_{12}\rangle$. Within the automorphism $\exp(t_1 P_1 + t_2 P_2)$ the algebra A contains the element $X = J_{12} + \lambda P_0 + \rho P_3 + \sigma P_4$ $(\lambda, \rho, \sigma \in R)$. Since

$$\exp(tJ_{04})(\lambda P_0 \sigma P_4) = (\lambda \cosh t - \sigma \sinh t)P_0 + (\sigma \cosh t - \lambda \sinh t)P_4$$

then if $P_0 + P_4 \in W$ one can write $X = J_{12} + e^t(\lambda - \sigma)P_0 + \rho P_3$. Since $\exp(\pi J_{13})(X) = -J_{12} + e^t(\lambda - \sigma)P_0 - \rho P_3$, we consider $\lambda - \sigma \geq 0$. If $\lambda - \sigma > 0$ then putting $t = -\ln(\lambda - \sigma)$, we obtain the algebra $W \supset\!\!\!\!+ \langle J_{12} + P_0 + \rho P_3\rangle$. Applying the automorphism $\exp(tK_3)$, one can put $\rho = 0$. If $\lambda - \sigma = 0$ then $A = W \supset\!\!\!\!+ \langle J_{12} + \rho P_3\rangle$, $\rho \neq 0$.

Let $P_0 + P_4 \notin W$. If $P_3, P_4 \in W$ then $\lambda > 0$, $\rho = \sigma = 0$. If $W = \langle P_4\rangle$ or $W = \langle P_1, P_2, P_4\rangle$ then $\sigma = 0$. Applying the automorphism $\exp(tJ_{03})$ we reduce this case to the following ones $\lambda = \rho = 1$ or $\lambda = 0$, $\rho > 0$.

Suppose that $\pi(A) = \langle K_1, K_2, J_{12} + cJ_{04}\rangle$ $(c > 0)$ one can suppose that A contains the elements

$$X_1 = K_1 + \sum_0^4 \lambda_i P_i, \qquad X_2 = K_2 + \sum_0^4 \rho_i P_i, \qquad X_3 = J_{12} + cJ_{04} + \sigma P_3.$$

Obviously, $[X_1, X_2] = (\lambda_2 - \rho_1)(P_0 + P_4) + (\lambda_0 - \lambda_4)P_2 - (\rho_0 - \rho_4)P_1$. If $\lambda_0 - \lambda_4 \neq 0$ or $\rho_0 - \rho_4 \neq 0$ then using lemma 1, we obtain $P_1, P_2, \in A$. Therefore $P_0 + P_4 \in A$ and one can put $\lambda_i = \rho_i = 0$ for $i = 0, 1, 2$. Later, $[X_3, X_1] = K_2 - cK_1 - c\lambda_4 P_0$,

$[X_3, X_2] = -K_1 - cK_2 - c\rho_4 P_0$. Therefore $\lambda_3 = \rho_3 = 0$, $\lambda_4 P_4 + c\rho_4(P_4 - P_0), -\rho_4 P_4 + c\lambda_4(P_4 - P_0) \in A$. The determinant constructed by the coefficients of P_4, $P_4 - P_0$ is equal to $c(\lambda_4^2 + \rho_4^2)$. If $\lambda_4^2 + \rho_4^2 \neq 0$ then $P_4, P_4 - P_0 \in A$. So we have the algebra $\langle K_1, K_2, J_{12} + cJ_{04} + \sigma P_3, P_0 + P_4, P_1, P_2, sP_0\rangle$ $(s = 0, 1)$.

Let $\lambda_0 - \lambda_4 = 0$, $\rho_0 - \rho_4 = 0$, $\lambda_3 = \rho_3 = 0$. Obviously,

$$[X_3, X_1] = K_2 - cK_1 + \lambda_1 P_2 + \lambda_2 P_1 - c\lambda_0(P_0 + P_4),$$
$$[X_3, X_2] = -K_1 - cK_2 + \rho_1 P_2 - \rho_2 P_1 - c\rho_0(P_0 + P_4),$$
$$[X_3, X_1] + cX_1 - X_2 = (c\lambda_1 - \lambda_2 - \rho_1)P_1 + (c\lambda_2 + \lambda_1 - \rho_2)P_2 - \rho_0(P_0 + P_4),$$
$$[X_3, X_2] + X_1 + cX_2 = (\lambda_1 + c\rho_1 - \rho_2)P_1 + (\lambda_2 + c\rho_2 + \rho_1)P_2 + \lambda_0(P_0 + P_4).$$

If on the right-hand side of one of the last two equalities some coefficients of P_1, P_2 are non-zero, so by lemmas 1 and 3 $P_1, P_2, P_0 + P_4 \in A$. Let $c\lambda_1 - \lambda_2 - \rho_1 = 0$, $c\lambda_2 + \lambda_1 - \rho_2 = 0$, $\lambda_1 + c\rho_1 - \rho_2 = 0$, $\lambda_2 + c\rho_2 + \rho_1 = 0$. The determinant formed by the coefficients of λ_1, λ_2, ρ_1, ρ_2 is equal $c^2(4 + c^2)$. We obtain $\lambda_1 = \lambda_2 = 0$, $\rho_1 = \rho_2 = 0$, $\lambda_0(P_0 + P_4), \rho_0(P_0 + P_4) \in A$ and therefore

$$A = W + \langle K_1, K_2, J_{12} + cJ_{04} + \sigma P_3\rangle, \qquad W \subset V.$$

Let $\pi(A) = \langle J_{12}, J_{13}, J_{23}, J_{04}\rangle$. Because of the simplicity of the algebra $\langle J_{12}, J_{13}, J_{23}\rangle$ one can assume that A contains the elements J_{12}, J_{13}, J_{23}, $X = J_{04} + \sum \gamma_i P_i$ $(i = 1, 2, 3)$. Applying lemma 1 to $[J_{12}, X]$, $[J_{13}, X]$, we conclude that $\sum \gamma_i P_i \in A$, i.e. A is a splitting algebra.

When the algebra $\pi(A)$ coincides with one of the following algebras: $\langle K_3, J_{04}\rangle$, $\langle K_1, K_2, J_{04}\rangle$, $\langle K_1, K_2, K_3, J_{04}\rangle$, one has to apply lemma 6. If $\pi(A)$ contains $J_{12} + cJ_{04}$, K_a, where $a \in I \subset \{1, 2, 3\}$, then we apply lemma 7. Thus, this theorem is proved.

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