New symmetries and conservation laws for electromagnetic fields

A.G. NIKITIN, W.I. FUSHCHYCH, V.A. VLADIMIROV

It is well known that classical conservation laws of energy, momentum, angular momentum and center-of-energy movement of the electromagnetic field are the consequences of the Maxwell equations invariance with respect to Poincare transformations. However, the relativistic invariance does not exhaust all symmetry properties of these equations. A natural question arises whether there exist any other conservation laws for electromagnetic fields different from those above. One could expect a positive answer to this question to be obtained provided that Maxwell equations possess an additional symmetry different from the relativistic and conformal invariances, because the symmetry under the proper conformal transformations does not lead to any new conserved quantities [1]. We will show in this paper that electromagnetic field equations do possess an additional (nongeometric) symmetry with respect to the $GL(2)\otimes GL(2)$ group, which gives rise to new conservation laws.

1. It is well known [2] that the maximal symmetry group of Maxwell equations

$$\frac{\partial \boldsymbol{E}}{\partial t} = \text{rot}\ \boldsymbol{H}, \qquad \frac{\partial \boldsymbol{H}}{\partial t} = -\text{rot}\ \boldsymbol{E}, \qquad \text{div}\ \boldsymbol{E} = \text{div}\ \boldsymbol{H} = 0 \tag{1}$$

in the class of local transformations is the $C(1,3)\otimes\mathcal{H}$ Lie group where $C(1,3)$ is a 15-parameter conformal group [3, 4] and $\mathcal{H}$ is one-parameter Larmore–Heaviside–Rainich transformation group [5–7]:

$$\begin{aligned} &\boldsymbol{E}\to \boldsymbol{E}\cos\varphi + \boldsymbol{H}\sin\varphi,\\ &\boldsymbol{H}\to \boldsymbol{H}\cos\varphi - \boldsymbol{E}\sin\varphi. \end{aligned} \tag{2}$$

In 1970 a method was proposed (hereafter cited as the non-Lie method) in which no restrictions are imposed on the order of operators available by systems of differential equations under consideration [8, 9]. By means of this method the existence of additional invariances was established for many important equations of relativistic and nonrelativistic physics [10–16]. As for the electromagnetic field equations, the results of the investigations of their symmetry properties obtained within the framework of the non-Lie method are formulated below in Theorems 1 and 2.

Let us rewrite Eqs.(1) in matrix form:

$$\begin{aligned} &L_1\psi = 0, \qquad L_1 = i\frac{\partial}{\partial t} + \sigma_2 \boldsymbol{S}\cdot\boldsymbol{p},\\ &L_2 = 0, \qquad L_2 = p_1 - \boldsymbol{S}\cdot\boldsymbol{p}S_1, \qquad \psi = \begin{pmatrix} \boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}, \end{aligned} \tag{3}$$

where

$$S_a = \begin{pmatrix} \hat{S}_a & \hat{0}\\ \hat{0} & \hat{S}_a\end{pmatrix}, \qquad \sigma_2 = i\begin{pmatrix} \hat{0} & -\hat{1}\\ \hat{1} & \hat{0}\end{pmatrix}, \qquad a = 1,2,3, \tag{4}$$

in Group-Theoretical Methods in Physics, Harwood, Harwood Academic Publishers, 1985, P. 497–505.

$\hat{1}$ and $\hat{0}$ are three-dimensional units and zero matrices, respectively, S_a are spin matrices which correspond to spin $s = 1$, $(\hat{S}_a)_{bc} = i\varepsilon_{abc}$. Let us denote the set of basic elements of a finite-dimensional Lie algebra by $\{Q_A\}$, $A = 1, 2, \ldots, j$. The $\{Q_A\}$ form the invariance algebra (IA) of Maxwell equations if for every $A = 1, \ldots, j$ operator Q_A is defined on the set of solutions of Eq.(3) and transforms this set into itself, i.e., the following equations hold:

$$L_1 Q_A \Psi = 0, \qquad L_2 Q_A \Psi = 0, \tag{5}$$

where Ψ is any solution of system (3). As an example of symmetry algebra of Eq.(3) we have the well-known 16-dimensional Lie algebra of the $C(1,3) \otimes \mathcal{H}$ group. Yet the Maxwell equations possess certain additional symmetry stated by the following theorem.

Theorem 1. *The Maxwell equations are invariant under the nine-dimensional Lie algebra A_8, basic elements of which have the form*

$$\begin{aligned} &Q_1 = \sigma_3 \boldsymbol{S} \cdot \hat{\boldsymbol{p}} D, \qquad Q_2 = i\sigma_2, \qquad Q_3 = \sigma_1 \boldsymbol{S} \cdot \hat{\boldsymbol{p}} D, \\ &Q_{3+a} = i\sigma_2 \boldsymbol{S} \cdot \hat{\boldsymbol{p}} Q_a, \qquad Q_7 = 1, \qquad Q_8 = i\sigma_2 \boldsymbol{S} \cdot \hat{\boldsymbol{p}}, \qquad a = 1, 2, 3, \end{aligned} \tag{6}$$

where

$$D = \sum_{a \neq b \neq c} \left[\left(p_a^2 p_b^2 + p_a^2 p_c^2 - p_b^2 p_c^2 \right) (1 - S_a) + p_1 p_2 p_3 S_a S_b p_c \right] \varphi^{-1}, \tag{7}$$

$$\varphi = \frac{1}{\sqrt{2}} \left[p_1^4 \left(p_2^2 - p_3^2 \right) + p_2^4 \left(p_1^2 - p_3^2 \right) + p_3^4 \left(p_1^2 - p_2^2 \right) \right]^{1/2}, \tag{8}$$

and σ_a are the Pauli matrices commuting with $\hat{S}_a$, $\hat{p}_a = p_a/p$, $p = \sqrt{\boldsymbol{p}^2}$. Operators (6) satisfy the following relations:

$$\begin{aligned} &[Q_a, Q_b] = -[Q_{3+a}, Q_{3+b}] = -\varepsilon_{abc} Q_c, \qquad a, b, c = 1, 2, 3, \\ &[Q_{3+a}, Q_b] = \varepsilon_{abc} Q_{3+c}, \qquad [Q_7, Q_A] = [Q_8, Q_A] = 0, \qquad A = 1, 2, \ldots, 8 \end{aligned} \tag{9}$$

forming an algebra isomorphic to Lie algebra of the $GL(2) \otimes GL(2)$ group.

Proof. One can convince oneself that the statements of Theorem 1 are true by straightforward calculation making use of the following relations:

$$\begin{aligned} &D\sigma_a = \sigma_a D, \qquad D\boldsymbol{S} \cdot \hat{\boldsymbol{p}} = -\boldsymbol{S} \cdot \hat{\boldsymbol{p}} D, \\ &D(\boldsymbol{S} \cdot \hat{\boldsymbol{p}})^2 = D - f(p_1 + ip_2 S_3 - ip_3 S_2) S_2, \qquad f = p_1^2 p_2^2 + p_1^2 p_3^2 + p_3^2 p_2^2, \\ &D^2 \boldsymbol{S} \cdot \hat{\boldsymbol{p}} = \boldsymbol{S} \cdot \hat{\boldsymbol{p}}, \qquad L_2 \boldsymbol{S} \cdot \hat{\boldsymbol{p}} = 0, \qquad [D, L_2] = -p_2^2 p_3^2 L_2. \end{aligned} \tag{10}$$

It is obvious that the additional symmetry algebra of Eq.(3) could not be obtained within the framework of the classical Lie method, which is based on the infinitesimal approach.

Since Q_A in (6) are integro-differential operators, we give the corresponding finite transformations for the Fourier components of $\boldsymbol{E}$ and $\boldsymbol{H}$. From the relation

$$\tilde{\psi} \to \tilde{\psi}' = \exp(\theta_A Q_A) \tilde{\psi}, \qquad \tilde{\psi} = (2\pi)^{-3/2} \int \psi(x) \exp(-i\boldsymbol{p} \cdot \boldsymbol{x}) d^3 x \tag{11}$$

we have

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\cos\theta_1+i\varepsilon_{abc}\hat p_bD_{cd}\tilde E_d\sin\theta_1,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\cos\theta_1-i\varepsilon_{abc}\hat p_bD_{cd}\tilde H_d\sin\theta_1,\end{aligned}\tag{12a}$$

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\cos\theta_2+\tilde H_a\sin\theta_2,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\cos\theta_2-\tilde E_a\sin\theta_2,\end{aligned}\tag{12b}$$

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\cos\theta_3-i\varepsilon_{abc}\hat p_bD_{cd}\tilde H_d\sin\theta_3,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\cos\theta_3-i\varepsilon_{abc}\hat p_bD_{cd}\tilde E_d\sin\theta_3,\end{aligned}\tag{12c}$$

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\cosh\theta_4-D_{ab}\tilde H_b\sinh\theta_4,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\cosh\theta_4-D_{ab}\tilde E_b\sinh\theta_4,\end{aligned}\tag{12d}$$

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\cosh\theta_5+i\varepsilon_{abc}\hat p_b\tilde E_c\sinh\theta_5,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\cosh\theta_5+i\varepsilon_{abc}\hat p_b\tilde H_c\sinh\theta_5,\end{aligned}\tag{12e}$$

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\cosh\theta_6-D_{ab}\tilde E_b\sinh\theta_6,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\cosh\theta_6+D_{ab}\tilde H_b\sinh\theta_6,\end{aligned}\tag{12f}$$

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\exp\theta_7,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\exp\theta_7,\end{aligned}\tag{12g}$$

$$\begin{aligned}&\tilde E_a\to\tilde E'_a=\tilde E_a\cos\theta_8+i\varepsilon_{abc}\hat p_b\tilde H_c\sin\theta_8,\\&\tilde H_a\to\tilde H'_a=\tilde H_a\cos\theta_8-i\varepsilon_{abc}\hat p_b\tilde E_c\sin\theta_8,\end{aligned}\tag{12h}$$

where θ_A $(A=1,2,\ldots,8)$ are real parameters,

$$\begin{aligned}&D_{ab}=\left[\delta_{ab}\left(p_a^2p_d^2+p_a^2p_e^2-p_d^2p_e^2\right)+p_1p_2p_3p_c\right]\varphi^{-1},\\&c\neq d\neq e,\qquad c\neq e,\qquad c\neq a,b.\end{aligned}\tag{13}$$

Using the inverse Fourier transformation one can obtain the finite transformations generated by (6) in the basic representation:

$$\begin{aligned}&H'_a(t,\boldsymbol{x})=(2\pi)^{-3/2}\int\tilde H'_a\exp(i\boldsymbol{px})d^3p,\\&E'_a(t,\boldsymbol{x})=(2\pi)^{-3/2}\int\tilde E'_a\exp(i\boldsymbol{px})d^3p.\end{aligned}\tag{14}$$

Transformations (12a)–(12h) form the representation of the $GL(2)\otimes GL(2)$ group which includes the one-parameter HLR group (2).

2. Recently [16] within the framework of the non-Lie approach, group properties of the equations for vector-potential of the electromagnetic field,

$$\begin{aligned}&\Box A_\mu=0,\\&\partial_\mu A^\mu=0,\qquad\mu=0,1,2,3,\end{aligned}\tag{15}$$

were investigated. The additional symmetry of Eqs.(15) proved to be even higher than that of the Maxwell equations.

Theorem 2. *Equations (15) are invariant under the Lie algebra of the $GL(3)$ group. Basic elements of this symmetry algebra on the set of solutions of Eqs.(15) have the form*

$$(F_{ab}A)^{\mu} = \frac{1}{p^2}\left(g_0^{\mu}p_0p_a - g_a^{\mu}p_0^2\right)A_b, \qquad a,b = 1,2,3, \tag{16}$$

where g_{ν}^{μ} is the metric tensor of the Minowski space and $g_{\nu\nu} = (1,-1,-1,-1)$; $1/p^2$ is the integral operator defined as

$$\frac{1}{p^2}f(t,\boldsymbol{x}) = \int \frac{f(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x'. \tag{17}$$

The proof of this theorem is given in Ref. [16]. Obviously, the additional symmetry algebra of Eqs.(15) generated by nonlocal operators (16) cannot be obtained in the classical Lie approach.

3. What conservation laws correspond to the symmetries stated by Theorems 1 and 2? Since basic elements of the additional symmetry algebras are nonlocal operators the traditional method for construction of conserved quantities based on the Noether theorem is of no use. Another possibility of building up the conserved quantities is to put every element of the invariance algebra of Maxwell equations into correspondence with a four-vector:

$$J_0^A = \psi^+ MQ^A\psi, \qquad J_a^A = -\psi^+ M\sigma_2 S_a Q^A\psi \tag{18}$$

satisfying the continuity equation

$$\partial_{\mu}(J^A)^{\mu} = 0, \tag{19}$$

where ψ is vector-function from (3), σ_2, S_a are matrices introduced in (4), M is an operator which satisfies the following equation

$$\left[i\frac{\partial}{\partial t} + \sigma_2\boldsymbol{S}\cdot\boldsymbol{p}, M\right]\psi = 0. \tag{20}$$

Employing the Gauss–Ostrogradsky theorem we can conclude from (19) and (20) that the integrals

$$\langle Q_A\rangle = \int d^3xJ_A^0 = \int d^3x\psi^+ MQ_A\psi \tag{21}$$

are independent of time. In this way it is possible to obtain all classical conserved quantities as well as new conserved quantities which correspond to the non-Lie symmetry of Maxwell equations. Operator M must be chosen in accordance with the demand for integrals (21) to have a clear physical interpretation. The following operator does satisfy this requirement:

$$M = \frac{p_0}{p} = -\frac{\sigma_2\boldsymbol{S}\cdot\boldsymbol{p}}{p^2}, \tag{22}$$

where $1/p^2$ is the integral operator defined in (17). As a matter of fact, substituting (22) into (21) and choosing $Q_A = \{P_{\mu}, J_{\mu\nu}\}$, where P_{μ} and $J_{\mu\nu}$ are basic elements of the Poincaré algebra, we obtain classical expressions for energy, momentum, angular momentum and center-of-energy of the electromagnetic field. Inserting (6) into (21) one obtains

$$\langle Q_1\rangle = \int \frac{d^3p}{\varphi p}\left\{ f\tilde{\boldsymbol{E}}(t,-\boldsymbol{p})\cdot\tilde{\boldsymbol{H}}(t,\boldsymbol{p}) + \sum_a p_a^2 \dot{\tilde{E}}_a(t,-\boldsymbol{p})\dot{\tilde{H}}_a(t,\boldsymbol{p})\right\}, \tag{23a}$$

$$\langle Q_2\rangle = \int \frac{d^3p}{2p^2}\left\{ \boldsymbol{p}\cdot\left[\tilde{\boldsymbol{E}}(t,-\boldsymbol{p})\times\tilde{\boldsymbol{E}}(t,\boldsymbol{p}) + \tilde{\boldsymbol{H}}(t,\boldsymbol{p})\times\tilde{\boldsymbol{H}}(t,\boldsymbol{p})\right]\right\}, \tag{23b}$$

$$\begin{aligned}\langle Q_3\rangle = \int \frac{d^3p}{2\varphi p}&\left\{ \sum_a f\left[\tilde{H}_a(t,\boldsymbol{p})\tilde{H}_a(t,-\boldsymbol{p}) - \tilde{E}_a(t,\boldsymbol{p})\tilde{E}_a(t,-\boldsymbol{p})\right] + \right.\\ &\left. + \sum_a p_a^2\left[\dot{\tilde{H}}_a(t,\boldsymbol{p})\dot{\tilde{H}}_a(t,-\boldsymbol{p}) - \dot{\tilde{E}}_a(t,\boldsymbol{p})\dot{\tilde{E}}_a(t,-\boldsymbol{p})\right]\right\},\end{aligned} \tag{23c}$$

$$\langle Q_8\rangle = \int \frac{d^3p}{2p}\left[\tilde{\boldsymbol{E}}(t,\boldsymbol{p})\cdot\tilde{\boldsymbol{E}}(t,-\boldsymbol{p}) + \tilde{\boldsymbol{H}}(t,\boldsymbol{p})\cdot\tilde{\boldsymbol{H}}(t,-\boldsymbol{p})\right], \tag{23d}$$

$$\langle Q_4\rangle = \langle Q_5\rangle = \langle Q_6\rangle = \langle Q_7\rangle = 0, \qquad \dot{A} = \frac{\partial A}{\partial t}. \tag{23e}$$

Thus, the existence of additional symmetry algebras for the electromagnetic field equations gives rise to the new conserved quantities independent of classical ones.

In a similar way we can show that the additional symmetry (16) of Eqs.(15) leads us to the following conserved quantities:

$$\tilde{S}_a = \frac{i}{2}\varepsilon_{abc}\int A_b(t,\boldsymbol{x})\overleftrightarrow{p_0}A_c(t,\boldsymbol{x})d^3x, \quad a,b,c = 1,2,3, \tag{24a}$$

$$\tilde{\Sigma}_{ab} = \frac{1}{2}\int\left\{A_a(t,\boldsymbol{x})\overleftrightarrow{p_0}\left[\frac{p_0}{p}A_b(t,\boldsymbol{x})\right] + A_b(t,\boldsymbol{x})\overleftrightarrow{p_0}\left[\frac{p_0}{p}A_a(t,\boldsymbol{x})\right]\right\}d^3x. \tag{24b}$$

Formulas (24a) express the spin of the vector field [17]. The time independence of spin components (24a) was originally derived by consideration of properties of energy-momentum tensor of the vector fields having nothing to do with their symmetry properties. Now we see that conservation of (24a) as well as the existence of six new conserved quantities (24b) are the consequences of the non-Lie symmetry of Eqs.(15).

In conclusion we discuss briefly a physical meaning of the new conserved quantities (23) and (24). It is readily shown that if a monochromatic wave solution of Eq.(1) is substituted into the following expression

$$K_a = \frac{\langle Q_a\rangle}{\langle Q_8\rangle}, \qquad a = 1,2,3, \tag{25}$$

the Stokes parameters describing polarization of this wave are obtained. In general integrals (23a)–(23d) can be regarded as a generalization of these parameters for arbitrary solutions of Maxwell equations. Equations (24) can be reduced to matrix elements of the polarization density matrix for the field with nonzero spin provided that (A_μ) is the solution of (15) corresponding to a monochromatic wave.

Thus, the non-Lie symmetry of the equations of motion can be employed to describe the polarization properties of the electromagnetic field. The analogous statement holds in the case of any relativistic equation for particles with non-zero mass and arbitrary spin, e.g., the additional symmetry of the Dirac equation [8, 9] was used in Ref. [18] to describe polarization of the electron. More extended discussion of non-Lie symmetry of Maxwell equations is given in Refs. [19, 20].

1. Plybon D., *Am. J. Phys.*, 1974, **42**, 998.
2. Ibragimov N.H., *Dokl. Acad. Sci. USSR*, 1968, **178**, 566.
3. Bateman H., *Proc. London Math. Soc.*, 1909, **8**, 223.
4. Cunningham E., *Proc. London Math. Soc.*, 1909, **8**, 77.
5. Heaviside O., *Phil. Trans. Roy. Soc. A*, 1893, **183**, 423.
6. Larmor I., Collected Papers, London, 1928.
7. Rainich G.I., *Trans. Am. Math. Soc.*, 1925, **27**, 106.
8. Fushchych W.I., Preprint, ITP-70-32E, Kyiv, 1970.
9. Fushchych W.I., *Teor. Mat. Fiz.*, 1971, **7**, 3.
10. Fushchych W.I., *Lett. Nuovo Cim.*, 1974, 508.
11. Nikitin A.G., Segeda Ju.N., Fushchych W.I., *Teor. Mat. Fiz.*, 1976, **29**, 82.
12. Fushchych W.I., Nikitin A.G., *J. Phys. A*, 1979, **12**, 747.
13. Fushchych W.I., Nikitin A.G., *Lett. Nuovo Cim.*, 1979, **24**, 220.
14. Fushchych W.I., Nakonechny V.V., *Ukr. Mat. Zh.*, 1980, **32**, 267.
15. Fushchych W.I., Nikitin A.G., *Czech. J. Phys. B*, 1982, **32**, 476.
16. Fushchych W.I., Vladimirov V.A., *Dokl. Acad. Sci. USSR*, 1981, **257**, 1105.
17. Bogoliubov N.N., Shirkov D.V., Introduction to quantum field theory, Moscow, Nauka, 1973.
18. Strazhev V.I., *Vesti Akad. Sci. Byelorus. SSR*, 1981, **5**, 75.
19. Fushchych W.I., Nikitin A.G., *Fiz. Elem. Chast. i Atom. Jadra*, 1983, **14**, 5.
20. Fushchych W.I., Nikitin A.G., Symmetry of Maxwell equations, Kiev, Naukova Dumka, 1983.