On the new conformally invariant equations for spinor fields and their exact solutions

W.I. FUSHCHYCH, W.M. SHTELEN, R.Z. ZHDANOV

The Poincaré and conformally invariant nonlinear generalizations of the Dirac equation are discussed and, in particular, the conformally invariant version of the Dirac–Heisenberg equation is obtained. For the latter equation some exact solutions are found and among them there is a family which is invariant under the full 15-parameter conformal group.

Consider the following Poincaré invariant nonlinear generalization of the Dirac equation

$$\begin{gathered}\gamma^\mu[i\partial_\mu + F_1\bar\psi\gamma_\mu\psi + F_2\bar\psi\gamma_4\gamma_\mu\psi + F_3(\bar\psi\gamma_\mu\psi)\gamma_4 + F_4(\bar\psi\gamma_4\gamma_\mu\psi)\gamma_4]\psi + \\ +F_5(\bar\psi\sigma_{\mu\nu}\psi)\sigma^{\mu\nu}\psi + F_6(\bar\psi\sigma_{\mu\nu}\psi)\gamma_4\sigma^{\mu\nu}\psi = (F_7 + F_8\gamma_4)\psi,\end{gathered} \tag{1}$$

where $F_1, \ldots, F_8$ are arbitrary functions of $\bar\psi\psi$ and $\bar\psi\gamma_4\psi$,

$$\gamma_4 = i\gamma_0\gamma_1\gamma_2\gamma_3, \qquad \sigma_{\mu\nu} = \frac{1}{4}i(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu).$$

The well-known Dirac–Heisenberg [1] and Dirac–Gürsey [2] equations belong to this class.

We shall choose from (1) such equations which are invariant under the scale transformation

$$x'_\mu = e^\theta x_\mu, \qquad \psi'(x') = e^{k\theta}\psi(x), \qquad k, \theta = \text{const}. \tag{2}$$

and under the conformal ones (see e.g. ref. [3])

$$\begin{gathered}x'_\mu = \frac{x_\mu - c_\mu x^2}{\sigma(x)}, \qquad \psi'(x') = \sigma(x)(1 - \gamma c \gamma x)\psi(x), \\ \sigma(x) = 1 - 2cx + c^2x^2, \qquad cx \equiv c^\nu x_\nu, \qquad c^2 \equiv c^\nu c_\nu, \qquad \nu = 0,1,2,3.\end{gathered} \tag{3}$$

Theorem 1. *Eq.(1) is invariant under the scale transformation (2) if and only if*

$$\begin{aligned}
&F_1 = \phi_1\left[(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi)\right]^{-(1+2k)/4k}, && F_2 = \phi_2\left[(\bar\psi\gamma_4\gamma_\mu\psi)(\bar\psi\gamma_4\gamma^\mu\psi)\right]^{-(1+2k)/4k}, \\
&F_3 = \phi_3\left[(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi)\right]^{-(1+2k)/4k}, && F_4 = \phi_4\left[(\bar\psi\gamma_4\gamma_\mu\psi)\bar\psi\gamma_4\gamma^\mu\psi\right]^{-(1+2k)/4k}, \\
&F_5 = \phi_5\left[(\bar\psi\sigma_{\mu\nu}\psi)\bar\psi\sigma^{\mu\nu}\psi\right]^{-(1+2k)/4k}, && F_6 = \phi_6\left[(\bar\psi\sigma_{\mu\nu}\psi)\bar\psi\sigma^{\mu\nu}\psi\right]^{-(1+2k)/4k}, \\
&F_7 = \phi_7(\bar\psi\psi)^{-1/2k}, && F_8 = \phi_8(\bar\psi\psi)^{-1/2k},
\end{aligned} \tag{4}$$

where $\phi_1, \ldots, \phi_8$ are arbitrary functions of $\bar\psi\psi/\bar\psi\gamma_4\psi$.

Physics Letters B, 1985, **159**, № 2/3, P. 189–191.

Proof. It is easy to see that transformations (2) leave eq.(1) invariant if

$$\begin{aligned} & e^{\theta(2k+1)}F_B\left(\bar\psi\psi e^{2k\theta},\bar\psi\gamma_4\psi e^{2k\theta}\right)=F_B\left(\bar\psi\psi,\bar\psi\gamma_4\psi\right), \qquad B=1,2,\ldots,6,\\ & e^{\theta(k+1)}F_C\left(\bar\psi\psi e^{2k\theta},\bar\psi\gamma_4\psi e^{2k\theta}\right)=F_C\left(\bar\psi\psi,\bar\psi\gamma_4\psi\right), \qquad C=7,8. \end{aligned} \tag{5}$$

Taking into account the well-known identities [4]

$$\begin{aligned} & (\bar\psi\psi)^2+(\bar\psi\gamma_4\psi)^2-(\bar\psi\sigma_{\mu\nu}\psi)\bar\psi\sigma^{\mu\nu}\psi=0,\\ & (\bar\psi\psi)^2-(\bar\psi\gamma_4\psi)^2-(\bar\psi\gamma_4\gamma_\mu\psi)\bar\psi\gamma_4\gamma_\mu\psi=0,\\ & (\bar\psi\gamma_\mu\psi)\bar\psi\gamma^\mu\psi-(\bar\psi\gamma_4\gamma_\mu\psi)\bar\psi\gamma_4\gamma^\mu\psi=0, \end{aligned} \tag{6}$$

the general solution of (5) can be written as (4). One can directly verify that eq.(1) with functions (4) is invariant under transformations (2).

Theorem 2. *Eq.(1) is invariant under the conformal group $C(1,3)$ if and only if functions $F_1,\ldots,F_8$ have the form (4) with $k=-3/2$.*

Proof. Since the conformal group $C(1,3)$ contains the extended Poincaré group $\widetilde{P}(1,3)=\{P(1,3),D\}$, we can use the result of theorem 1. Then one can make sure that transformations (3) leave eq.(1) with functions (4) invariant when $k=-3/2$ and this proves the theorem.

Corollary. If $F_7=\lambda(\bar\psi\psi)^{1/3}$ and $F_A=0$, $A=1,\ldots,6,8$ then eq.(1) coincides with the Dirac–Gürsey (2) one:

$$\left[i\gamma\partial-\lambda(\bar\psi\psi)^{1/3}\right]\psi=0, \qquad \lambda=\mathrm{const}. \tag{7}$$

In another case when $F_4=\lambda[(\bar\psi\gamma_\mu\psi)\bar\psi\gamma^\mu\psi]^{-1/3}$, $F_B=0$, $B=1,2,3,5,\ldots,8$, we obtain a conformally invariant version of the Dirac–Heisenberg equation

$$\left\{i\gamma\partial+\lambda(\bar\psi\gamma_\mu\psi)\gamma^\mu/[(\bar\psi\gamma_\nu\psi)\bar\psi\gamma^\nu\psi]^{1/3}\right\}\psi=0. \tag{8}$$

As is well known the original Dirac–Heisenberger equation (1) is not invariant under the conformal transformations.

Now we use the symmetry properties of eq.(8) to construct its exact solutions. Following refs. [5, 3] we take the anzatze

$$\psi=\varphi(\beta x), \qquad \beta x\equiv\beta^\nu x_\nu, \qquad \beta^\nu=\mathrm{const}, \tag{9}$$

$$\psi=\left[\gamma x/(x^\nu x_\nu)^2\right]\phi(\beta x/x^\nu x_\nu), \tag{10}$$

which are translationally and conformally invariant respectively. The substitution of (9), (10) into (8) gives rise to the following system of ordinary differential equations

$$i\gamma\beta du/d\omega+\nu(\bar u\gamma_\mu u)\gamma^\mu u/[(\bar u\gamma_\nu u)\bar u\gamma^\nu u]^{1/3}=0, \tag{11}$$

where $u=\{\varphi(\omega),\omega=\beta x$ or $\phi(\omega)=\beta x/x^\nu x_\nu\}$, $\nu=\lambda$ for φ and $\nu=-\lambda$ for ϕ. Depending on ν, there are three different cases (χ is a constant spinor, $\beta_\mu=\bar\chi\gamma_\mu\chi/[(\bar\chi\gamma_\nu\chi)\bar\chi\gamma^\nu\chi]^{1/3}$)

(a) $\quad \mathrm{Im}\, v=0, \quad u=e^{iv\omega}\chi,$

(b) $\quad \mathrm{Re}\, v=0, \quad u=\left(c+\frac{2}{3}z\omega\right)^{-3/2}\chi, \quad z=\mathrm{Im}\, v,$

$$
\begin{aligned}
(c)\quad & \operatorname{Im} v \operatorname{Re} v \neq 0, \quad u = (f_1 + if_2)\chi, \quad v = v_1 + iv_2, \\
& f_1 = \pm\left[(w-2v)^{1/2} + (w+2v)^{1/2}\right], \\
& f_2 = \mp\left[(w-2v)^{1/2} - (w+2v)^{1/2}\right], \\
& \int \frac{dv}{\left[c_1 - 2(v_2/v_1)v^2\right]^{2/3}} = 2v_1\omega + c_2, \quad w = \left[c_1 - 2(v_2/v_1)v^2\right]^{1/2}.
\end{aligned}
\tag{12}
$$

Remark. Let us show that the conformally invariant ansatz (10) can be obtained from (9) by applying the procedure of generation of solutions if one uses the conformal transformations (3). As is shown in ref. [3] the formula of generating solutions in this case has the form

$$
\begin{aligned}
& \psi_{\text{new}}(x) = \left[(1 - \gamma x\gamma c)/\sigma^2(x)\right]\psi_{\text{old}}(x'), \\
& x'_\mu = (x_\mu - c_\mu x^2)/\sigma(x), \qquad \sigma(x) = 1 - 2cx + c^2x^2.
\end{aligned}
\tag{13}
$$

Applying (13) with $c_0 = 1$, $c_1 = c_2 = c_3 = 0$ to (9) and then changing x_0 in $x_0 + 1$ at the expense of translation invariance we obtain the ansatz (10).

Now let us use the procedure of generating solutions to the conformally invariant one (10), for the case (12a),

$$
\begin{aligned}
& \psi(x) = \left[\gamma x/(x^\nu x_\nu)^2\right]\exp(-i\lambda\beta x/x^\nu x_\nu)\chi, \\
& \beta_\mu = \bar\chi\gamma_\mu\chi/[(\bar\chi\gamma_\nu\chi)\bar\chi\gamma^\nu\chi]^{1/3}.
\end{aligned}
\tag{14}
$$

Having done transformations of translations we obtain from (14) another family of solutions of eq.(8)

$$
\begin{aligned}
& \psi(x) = \left[(\gamma x + \gamma a)/\left(x^2 + 2ax + a^2\right)\right]\exp\left[-i\lambda(\beta x + \beta a)/\left(x^2 + 2ax + a^2\right)\right], \\
& \beta_\mu = \bar\chi\gamma_\mu\chi/[(\bar\chi\gamma_\nu\chi)\bar\chi\gamma^\nu\chi]^{1/3}.
\end{aligned}
\tag{15}
$$

It is a remarkable family of solutions, because it is invariant within the transformations of the parameters under the full 15-parameter conformal group. Indeed, it is obvious that (15) is invariant under displacements. Let us also show that it cannot be generated by the procedure (13). Applying (13) to the solution (15) we obtain

$$
\begin{aligned}
\psi(x) = {} & \frac{1 - \gamma x\gamma c}{\sigma^2(x)}\frac{\left(\gamma x - \gamma c x^2\right)/\sigma(x) + \gamma a}{\left[a^2 + 2\left(ax - acx^2 + x^2\right)/\sigma(x)\right]^2}\times \\
& \times\exp\left[-i\lambda\frac{\left(\beta x - \beta c x^2\right)/\sigma(x) + \beta a}{a^2 + 2\left(ax - acx^2 + x^2\right)/\sigma(x)}\right]\chi.
\end{aligned}
\tag{16}
$$

One can make sure that (16) can be rewritten in the form (15) only with the new parameters

$$
\begin{aligned}
& a_\mu \to \tilde a_\mu = -\left(a_\mu - c_\mu a^2\right)/\sigma(a,c), \qquad \chi \to \tilde\chi = (1 - \gamma c\gamma a)/\sigma^2(a,c), \\
& \beta_\mu \to \tilde\beta_\mu = \bar{\tilde\chi}\gamma_\mu\tilde\chi\Big/[(\bar{\tilde\chi}\gamma_\nu\tilde\chi)(\bar{\tilde\chi}\gamma^\nu\tilde\chi)]^{1/3}.
\end{aligned}
\tag{17}
$$

It is also clear that (15) cannot be generated by the remaining transformations of the conformal group.

In conclusion let us note that we have used symmetry to obtain exact solutions of nonlinear Dirac equation [3], nonlinear equations of quantum electrodynamics [6], Yang–Mills equations [7] and some scalar nonlinear equations [8, 9].

1. Heisenberg W., Introduction to the unified field theory of elementary particles, London, Interscience, 1966.
2. Gürsey F., *Nuovo Cimento*, 1956, **3**, 988.
3. Fushchych W.I., Shtelen W.M., *J. Phys. A*, 1983, **16**, 271.
4. Finkelstein R., Fronsdal C., Kaust P., *Phys. Rev.*, 1956, **103**, 5.
5. Fushchych W.I., in: Algebraic-theoretical studies in mathematical physics, Kiev, Mathematical Institute, 1981, 6.
6. Fushchych W.I., Shtelen W.M., *Phys. Lett. B*, 1983, **128**, 215.
7. Fushchych W.I., Shtelen W.M., *Lett. Nuovo Cimento*, 1983, **38**, 37.
8. Fushchych W.I., Shtelen W.M., *Lett. Nuovo Cimento*, 1982, **34**, 498.
9. Fushchych W.I., Serov N.I., *J. Phys. A*, 1983, **16**, 3645.