On nonlocal transformations

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A procedure of finding finite transformations generated by a linear arbitrary-order differential operators is presented. Dirac equation is shown to be Galilei invariant with the nonlocal law of transformation of the Ψ-function.

At the present time special interest in the study of the invariance properties of partial differential equations (PDE) excite nonlocal symmetries such as contact, Lie–Bäcklund [1] non-Lie [2, 3]. Recently it was shown [3] that many fundamental equations of theoretic physics possess an additional (non-Lie) invariance. The basis elements of such invariance algebras are arbitrary order differential operators even pseudo-differential, while the Lie symmetry is generated by first-order differential operators only. It will be noted that for systems of linear PDE non-Lie symmetry generated by finite-order differential operators can be obtained by the Lie–Bäcklund approach [1], but with more formidable calculations. In other words, the non-Lie method [2, 3] applicable to systems of linear PDE gives the same results as Lie–Bäcklund approach does, but more reliable and easy.

In this note we solve the problem of finding finite transformations generated by non-Lie operators, and show that any such operator leads to a one-parametrical group of transformations.

Formulae of finite transformations discussed here can be used for generating new solutions of equations in question by analogy with that done in the local case [4–8].

Any linear arbitrary-order differential operator Q acting in the space of r-component ψ-function ($\psi=\psi(x)$, $x=\{x_0,x_1,\ldots,x_n\}$) can be written down in the form

$$Q(x,\partial)=\xi^\mu\partial_\mu+\eta(x,\partial), \tag{1}$$

where $\xi^\mu(x)$ are scalar functions, $\mu=0,1,\ldots,n$; $\partial=\{\partial_\nu=\partial/\partial x_\nu\}$, $\eta(x,\partial)$ is a matrix ($r\times r$), the differential operator does not contain terms like $\xi^\mu(x)\partial_\mu$.

Definition. *A linear system of PDE*

$$L(x,\partial)\psi(x)=0 \tag{2}$$

is invariant under the transformations

$$x\to x'=f(x,\theta),\qquad \psi(x)\to\psi'(x')=R(x,\partial,\theta)\psi(x), \tag{3}$$

if

$$L(x',\partial')\psi'(x')=0. \tag{4}$$

Theorem. *Operator Q (1) will be an operator of symmetry of eq.(2), if on the manifold of solutions of eq.(1) the following condition holds true:*

$$LQ\psi=0\quad or\quad [L,Q]\psi\equiv(LQ-QL)\psi=0, \tag{5}$$

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the transformations generated by Q having the form

$$x'_\nu = \exp[\theta\xi\cdot\partial]x_\nu\exp[-\theta\xi\partial], \qquad \psi'(x') = \exp[\theta\xi\cdot\partial]\exp[-\theta Q]\psi(x). \tag{6}$$

Proof. As a result of transformations (6) operator $L(x,\partial)$ of eq.(2) will be rewritten in such a manner

$$L(x,\partial)\to L(x',\partial') = \exp[\theta\xi\cdot\partial]L(x,\partial)\exp[-\theta\xi\cdot\partial]. \tag{7}$$

Hence, we have

$$L(x',\partial')\psi'(x') = \exp[\theta\xi\cdot\partial]L(x',\partial)\exp[-\theta\xi\cdot\partial]\exp[\theta\xi\partial]\exp[-\theta Q]\psi(x) = \\ = \exp[\theta\xi\partial]L(x,\partial)\exp[-\theta Q]\psi(x) = 0,$$

since eq.(5) takes place. According to (4) it proves our theorem.

Remark 1. If Q is a first-order differential operator (case of Lie symmetry), that is $\eta(x,\partial) = \eta(x)$, then formulae (6) give the same result as does integration of corresponding Lie equations.

Remark 2. Transformations (6) form a one-dimensional group. Indeed,

$$x''_\nu = \exp[\beta\xi\cdot\partial]x'_\nu\exp[-\beta\xi\cdot\partial] = \exp[\beta\xi\cdot\partial]\exp[\theta\xi\cdot\partial]x_\nu\times \\ \times\exp[-\theta\xi\partial]\exp[-\beta\xi\cdot\partial] = \exp[(\theta+\beta)\xi\cdot\partial]x_\nu\exp[-(\theta+\beta)\xi\cdot\partial].$$

As far as the transformation of $\psi(x)$ is concerned, let us rewrite it in this way

$$\psi'(x') = \exp[\theta\xi\cdot\partial]\exp[-\theta Q]\psi(x) = \exp[\theta R]\psi(x).$$

To do it, we have used the Campbell–Baker–Hausdorff formula. $R = R(x,\partial)$ is an operator constructed from $\xi\cdot\partial$ and Q and their various commutators. So we have

$$\psi''(x'') = \exp[\beta R]\psi'(x') = \exp[\beta R]\exp[\theta R]\psi(x) = \exp[(\theta+\beta)R]\psi(x).$$

This proves our statement.

In ref. [2, 3] it is shown that Dirac equation is the sense of condition (5), under the set operators satisfying the commutation relations of the Poincaré algebra

$$P_0 = i\partial_0, \qquad P_a = -i\partial_a, \qquad a = 1,2,3,$$
$$J_{ab} = x_aP_b - x_bP_a - \frac{i}{2}\gamma_a\gamma_b,$$
$$J_{0a} = tP_a - \frac{1}{2}(Hx_a + x_aH), \qquad H = \gamma_0\gamma_aP_a + \gamma_0 m.$$

But now operator J_{0a} does not generate Lorentz transformations. In accordance with formulae (6) we get

$$x'_0 = x_0, \qquad x'_a = x_a + v_ax_0,$$

which are the well-known Galilei transformations;

$$\psi'(x') = \exp[ix_0v_ap_a]\exp\left[-ix_0v_ap_a + \frac{i}{2}(Hx_a + x_aH)\right]\psi(x)$$

and it is a nonlocal law of transformation.

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