Some exact solutions of the many-dimensional sine-Gordon equation

W.I. FUSHCHYCH, Yu.N. SEHEDA

In the present paper we construct the multiparametrical families of exact solutions of the many-dimensional nonlinear d'Alembert equation

$$\Box U = \sinh U, \tag{1}$$

where $\Box = \partial^2/\partial x_0^2 - \cdots - \partial^2/\partial x_n^2$. This equation is concerned with some problems of field theory [1]. In the case of $n = 1$ the analysis and the physical interpretation of solutions of this equations are given in [2].

Up to date the inverse-scattering method is applied for solving two-dimensional nonlinear equations (KdV, sine-Gordon, nonlinear Schrödinger and some others) mainly and the attempts to extend this method for solving many-dimensional equations are not so successful.

To construct some classes of exact solutions of the many-dimensional equation (1), we use group-theoretical ideas of Lie which were applied fruitfully by Birkhoff [3], Sedov [4] and Ovsyannikov [5] to nonlinear equations of hydrodynamics.

The maximal local invariance group of eq.(1) is the Poincaré group $P(1,n)$ of rotations and translations of the $(1+n)$-dimensional space $R^{1,n}$.

We look for the solutions to eq.(1) of the form

$$U(x) = \varphi(\omega), \tag{2}$$

where φ is a function of the invariant variable ω only (for more details see [6]). We use the following set of invariants which were presented in [7]. (Below the summation convention is employed. The parameters $\alpha_\nu, \beta_\nu, \ldots$ are arbitrary real constants.)

$$\omega = (x_\nu x^\nu)^{1/2}, \tag{3a}$$

$$\omega = \left[(\beta_\nu y^\nu)^2 + y_\nu y^\nu\right]^{1/2}, \qquad \beta_\nu\beta^\nu = -1, \tag{3b}$$

$$\omega = \left[(\beta_\nu y^\nu)^2 - y_\nu y^\nu\right]^{1/2}, \qquad \beta_\nu\beta^\nu = 1, \tag{3c}$$

$$\omega = \alpha_\nu x^\nu, \qquad \alpha_\nu\alpha^\nu = l = \pm 1,, \tag{3d}$$

$$\omega = \frac{1}{2}(\alpha_\nu y^\nu)^2 + a\beta_\nu y^\nu, \qquad \alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0, \qquad \beta_\nu\beta^\nu = l = \pm 1, \tag{3e}$$

$$\begin{gathered} \omega = \beta_\nu y^\nu + a\ln\alpha_\nu y^\nu, \qquad a \neq 0, \\ \alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0, \qquad \beta_\nu\beta^\nu = l = \pm 1, \qquad y^\nu = x^\nu + a^\nu, \end{gathered} \tag{3f}$$

a, a^ν are arbitrary constants.

Lettere al Nuovo Cimento, 1984, **41**, № 14, P. 462–464.

Substituting (2) into the many-dimensional partial differential eq.(1) we reduce it to the ordinary differential equations

$$U' + \frac{N_1}{\omega}U' = \sinh U \tag{4a}$$

(cases (3a), (3b))

$$-U' - \frac{N_2}{\omega}U' = \sinh U, \tag{4b}$$

(case (3c))

$$U' = l\sinh U, \qquad l = \pm 1 \tag{4c}$$

(cases (3d), (3e)).

Here N_1 and N_2 are natural numbers depending on the value of the space dimension n.

When $N_1 \neq 0$ and $N_2 \neq 0$ eqs.(4a) and (4b) cannot be solved in explicit form. Taking $l = 1$ we have from (4c)

$$w = \int \frac{du}{\sqrt{2\cosh U + C}},$$

C is an arbitrary constant. The solutions of eq.(1) are found by in inversion of elliptic integrals [8]:

$$U = 2\,\mathrm{tgh}^{-1}\{\mathrm{sn}\,(z,k)\}, \qquad z = \frac{\sqrt{C+2}}{2}\omega, \quad k^2 = \frac{C-2}{C+2}, \quad C > 2, \tag{5a}$$

$$U = 2\,\mathrm{tgh}^{-1}\{\sin z\}, \qquad z = \sqrt{2}\omega, \quad C = 2. \tag{5b}$$

Writing the integration constant in the form

$$w = \int \frac{du}{\sqrt{2\cosh U - C}}$$

we have analogously

$$U = \cosh^{-1}\frac{2 - C\,\mathrm{sn}^2(\omega,k)}{2\,\mathrm{cn}^2(\omega,k)}, \qquad k^2 = \frac{C+2}{4}, \quad 0 < C < 2, \tag{5c}$$

$$U = \cosh^{-1}\frac{C/2 - \mathrm{sn}^2(z,k)}{\mathrm{cn}^2(z,k)}, \qquad z = \frac{\sqrt{C+2}}{2}\omega, \quad k^2 = \frac{4}{C+2}, \quad C > 2, \tag{5d}$$

$$U = 4\,\mathrm{tgh}^{-1}\exp[\omega], \qquad C = 2, \tag{5e}$$

$$U = \cosh^{-1}\{\mathrm{cn}\,(\omega,k)\}^{-1}, \qquad C = 0, \quad k^2 = \frac{1}{2}. \tag{5f}$$

When $l = -1$ we have the solution

$$\begin{gathered} U = \cosh^{-1}\left\{\frac{C}{2}\,\mathrm{cn}^2(z,k) + \mathrm{sn}^2(z,k)\right\}, \\ z = \frac{\sqrt{C+2}}{2}\omega, \quad k^2 = \frac{C-2}{C+2}, \quad C > 2. \end{gathered} \tag{5g}$$

Here $\mathrm{sn}\,z$ and $\mathrm{cn}\,z$ are Jacobi elliptic functions.

We add that in ref. [6, 7] the same way some solutions of the many-dimensional equation $\Box U = \sin U$ are found. In [9] the solutions of this equation were obtained by symmetry reduction.

1. Newell A., in Solitons, eds. R.K. Bullough and P.J. Caudrey, 1980, p. 177.
2. Pogrebkov A.K., Polivanov M.K., *Part. Nucl.*, 1983, **14**, 1073.
3. Birkhoff G., Hydrodynamics, Princeton, 1950.
4. Sedov L.I., Metody podobiya i razmernosti v mekhanike, Moscov, 1967 (in Russian).
5. Ovsyannikov L.V., Group properties of differential equations, Novosibirsk, 1962 (in Russian).
6. Fushchych W.I., in Algebraic-theoretical studies in mathematical physics, Kiev, Institute of Mathematics, 1981, 6–24 (in Russian).
7. Fushchych W.I., Serov N.I., *J. Phys. A.*, 1983, **16**, 3645.
8. Gradshtein I.S., Ryzhik I.M., Tables of integrals, series, products, New York, 1965.
9. Grundland A.M., Harnad J., Winternitz P., *Kinam*, 1982, **4**, 333.