The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d'Alembert and eikonal equations

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Multiparametrical exact solutions of the many-dimensional nonlinear d'Alembert, Liouville, sine-Gordon and eikonal equations arc obtained. The maximally extensive local invariance groups of the equations are determined and invariants of the extended Poincaré group are found.

1. Introduction

In 1881 Sophus Lie propounded to use the groups of continuous transformations for finding the exact solutions of partial differential equations (PDE). Later on many authors exploited Lie's ideas to study PDE of mechanics and physics (see Ames [2], Bluman and Cole [5], where a vast bibliography is cited and the historical aspects are discussed).

The classical work of Birkhoff [4] is devoted to the construction of the exact solutions of nonlinear hydrodynamics equations with the help of Lie's methods. Birkhoff [4] was the first to formulate the group method to obtain similarity (automodel) solutions of PDE. Many of the exact solutions have been obtained mainly for two-dimensional PDE. Lately Ovsyannikov's book [16] has dealt with the modern development of Lie's theory. Ovsyannikov formulated the method of finding the partly invariant solutions of PDE. To find such solutions one has to enumerate all the non-equivalent subgroups of the PDE invariance group. It is a very complicated problem. For example, the five-dimensional d'Alembert equation invariance group has more than 500 subgroups. Hence it is natural to seek more effective approaches for obtaining the exact solutions of many-dimensional PDE admitting a wide invariance group.

The main ideas we use in our work arc closely connected with those of Birkhoff [4] and Morgan [15]. The aim of our paper is to find the exact solutions of the following nonlinear pde widely used in mathematical and theoretical physics:

$$p_\mu p^\mu u + \lambda \exp u = 0, \tag{1.1}$$

$$\Box u + \lambda u^k = 0, \tag{1.2}$$

$$p_\mu u p^\mu u = 0, \tag{1.3}$$

where $p_\mu = ig^{\mu\nu}\partial/\partial x_\nu$, $g_{\mu\nu}$ is the metric tensor with the signature $(+1, -1, \ldots, -1)$, $p_\mu p^\mu = -\partial^2/\partial x_0^2 + \Delta \equiv -\Box$, $u = u(x)$, $x = (x_0, x_1, \ldots, x_{n-1})$, λ, k are arbitrary real constants. We use the summation convention for the repeated indices.

Equation (1.2) plays a special role in the quantum field theory when $k = 3$ and $x = (x_0, \ldots, x_3)$: its solutions may be used to construct some solutions of the Yang–Mills equation by virtue of the 'tHooft–Corrigan–Wilczek ansatz (see Actor [1]).

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For the solutions of (1.1)–(1.3) we adopt the ansatz suggested by Fushchych [6]:

$$u(x) = \varphi(\omega) f(x) + g(x), \tag{1.4}$$

where $\varphi(\omega)$ is an unknown function of the new variables $\omega = \omega(x) = \{\omega_1(x), \dots, \omega_{n-1}(x)\}$, the number of which is one less than the number of variables $x = (x_0, \dots, x_{n-1})$. The new variables $\omega(x)$ and the functions $f(x)$, $g(x)$ are determined from the Lagrange equations

$$\frac{dx_0}{\xi^0} = \frac{dx_1}{\xi^1} = \cdots = \frac{dx_{n-1}}{\xi^{n-1}} = \frac{du}{\eta}, \tag{1.5}$$

where ξ^μ and η are the functions from the infinitesimal invariance transformations

$$x'_\mu = x_\mu + \varepsilon \xi^\mu(x,u) + O\left(\varepsilon^2\right), \qquad u' = u + \varepsilon \eta(x,u) + O\left(\varepsilon^2\right). \tag{1.6}$$

If ξ^μ and η have the form

$$\xi^\mu = \xi^\mu(x), \qquad \eta = a(x)u + b(x), \tag{1.7}$$

it implies (1.4).

Having substituted (1.4) into (1.1)–(1.3) one obtains equations for $\varphi(\omega)$ which are often rather easy to solve.

2. The group properties of (1.1)–(1.3)

It is evident from the above, that to find the new variables $\omega(x)$ and the functions $f(x)$ and $g(x)$ it is necessary to know the functions $\xi^\mu(x)$ and $\eta(x,u)$ explicitly. Hence we shall study the group properties of (1.1)–(1.3).

Theorem 1. *Equation (1.1) is invariant under the Poincaré group $P(1,n-1)$ and under the scale transformation group $D(1)$. The basis elements of the corresponding Lie algebra $\tilde{P}(1,n-1) = \{P(1,n-1), D(1)\}$ have to form*

$$p_\mu = ig^{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \qquad D = x_\nu p^\nu - 2i\frac{\partial}{\partial u}. \tag{2.1}$$

Theorem 2. *Equation (1.2) is invariant under the extended Poincaré group $\tilde{P}(1,n-1)$, with basis elements of its Lie algebra having the form*

$$p_\mu = ig^{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \qquad D = x_\nu p^\nu + \frac{2i}{1-k}u\frac{\partial}{\partial u}. \tag{2.2}$$

Theorem 3. *Equation (1.3) admits the infinite-dimensional invariance group. The infinitesimal operator of this group is as follows (we use Ovsyannikov's [16] notations):*

$$\begin{aligned} &X = \xi^\mu(x,u)\partial/\partial x_\mu + \eta(x,u)\partial/\partial u, \\ &\xi^\mu = -b_\mu(u)x_\nu x^\nu + 2x_\mu b_\nu(u)x^\nu + c_{\mu\nu}x^\nu + d_\mu(u), \qquad \eta = \eta(u), \end{aligned} \tag{2.3}$$

where b_ν, $c_{\mu\nu}$, d_μ, η are arbitrary real functions of u and $c_{0a} = c_{a0}$, $c_{ab} = -c_{ba}$, $c_{00} = c_{11} = \cdots = c_{n-1\,n-1}$, $a, b = \overline{1, n-1}$.

Theorem 4. *The equation*

$$\Box u + F(x,u) = 0 \tag{2.4}$$

is invariant under the extended Poincaré group if and only if

$$F(x,u) = \lambda_1 \exp u, \tag{2.5}$$

or

$$F(x,u) = \lambda_2 u^k. \tag{2.6}$$

where λ_1, λ_2, k are arbitrary constants, $k \neq 1$, the infinitesimal generators are given in (2.1) and (2.2) respectively.

To prove these theorems one can use the Lie algorithm following e.g. Ovsyannikov [16]. One can make sure that (2.4) with nonlinearities (2.5), (2.6) is invariant under the group $\tilde{P}(1,n-1)$ using final invariance transformations.

Note 1. Theorem 4 implies that there is only one equation of the form (2.4) with non-polynomial nonlinearity invariant under $\tilde{P}(1,n-1)$, and it is the Liouville equation.

Note 2. If $n=2$, equation (1.1) admits the infinite-dimensional Lie group with the generator $X = \xi^0\partial/\partial x_0 + \xi^1\partial/\partial x_1 + \eta\partial/\partial u$, where

$$\begin{aligned} &\xi^0 = f(x_0+x_1) + g(x_0-x_1), \qquad \xi^1 = f(x_0+x_1) - g(x_0-x_1) + c_1, \\ &\eta = c_2 - \partial\xi^0/\partial x_0, \end{aligned} \tag{2.7}$$

f and g are arbitrary differentiable functions, c_1, c_2 are constants.

Note 3. Equation (1.2) with $n=2$ and $\lambda=0$ is invariant under the infinite-dimensional Lie group, as it takes place for the Liouville equation. The two-dimensional equation of gas dynamics has the same properties (see Fushchych and Serova [10]). Apparently this property gives the possibility of finding the general solution of the equations mentioned above.

3. The group $\tilde{P}(1,2)$ invariants

The question of finding of the invariants $\omega(x)$ is connected with the integration of the Lagrange system (1,5). Generally speaking, equations (1.5) have infinitely many solutions according to the various functions ξ^μ. Ovsyannikov [16] has proposed to enumerate all the non-conjugate subgroups of the equations invariance group and to integrate the system (1.5) for each subgroup. This way, as was previously mentioned, is connected with the algebraic difficulties.

In this section we shall show the particular case of the group (1.6) for which the system (1.5) is usually integrated.

Many fundamental equations of mathematical and theoretical physics are invariant under the group $IGL(n,R)$ of inhomogeneous linear transformations of n-dimensional Minkowski space or under its subgroups, e.g. the Lorentz group, the Poincaré group, the Galilei group etc.

It is well known that the functions ξ^μ for this group have the form

$$\xi^\mu = c_{\mu\nu}x^\nu + d_\mu, \qquad \mu = \overline{0,n-1}, \tag{3.1}$$

where $c_{\mu\nu}$ and d_μ are arbitrary constants.

Let us introduce the notations

$$\frac{dx_0}{\xi^0} = \frac{dx_1}{\xi^1} = \cdots = \frac{dx_{n-1}}{\xi^{n-1}} = dt. \tag{3.2}$$

Using (3.1) and (3.2) one can write down the system (1.5) in the form

$$\dot{x}_\mu = c_{\mu\nu}x^\nu + d_\mu, \qquad \mu = \overline{0, n-1}. \tag{3.3}$$

Equation (3.3) is a system of ordinary differential equations with constant coefficients and it is well known how to find its general solution. After doing this one has to eliinate the parameter to obtain the invariants $\omega(x)$.

If (1.1)–(1.3) are invariant under the group $\tilde{P}(1, n-1)$, which is a subgroup of $IGL(n, R)$, we shall consider the determination of $\tilde{P}(1, n-1)$ invariants in detail. For simplicity we put $n = 3$. According to the conditions between the coefficients $c_{\mu\nu}$ and d_μ in (3.1) we have obtained the following independent solutions of the system (1.5).

$$(1) \quad \omega_1 = \alpha_\nu y^\nu (\beta_\nu y^\nu)^a, \qquad \omega_2 = y_\nu y^\nu (\beta_\nu y^\nu)^{-2},$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0$, $\beta_\nu\beta^\nu = b \neq 0$.

$$(2) \quad \omega_1 = \beta_\nu y^\nu (\alpha_\nu y^\nu)^{-1} + \ln \alpha_\nu y^\nu, \qquad \omega_2 = y_\nu y^\nu (\alpha_\nu y^\nu)^{-2},$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0$, $\beta_\nu\beta^\nu = b \neq 0$.

$$(3) \quad \omega_1 = \ln \alpha_\nu y^\nu + b_1 \tan^{-1}[\gamma_\nu\gamma^\nu (\beta_\nu y^\nu)^{-1}], \qquad \omega_2 = y_\nu y^\nu (\alpha_\nu y^\nu)^{-2},$$

where $\alpha_\nu\alpha^\nu = b_2 \neq 0$, $\beta_\nu\beta^\nu = \gamma_\nu\gamma^\nu = b_3 \neq 0$, $\alpha_\nu\beta^\nu = \alpha_\nu\gamma^\nu = \beta_\nu\gamma^\nu = 0$, $(\beta_\nu y^\nu)^2 + (\gamma_\nu y^\nu)^2 = b_3(\alpha_\nu x^\nu)^{-2}(1 - b_2\omega_2)$.

$$(4) \quad \omega_1 = \alpha_\nu y^\nu + \ln \beta_\nu y^\nu, \qquad \omega_2 = \gamma_\nu z^\nu (\beta_\nu y^\nu)^{-2},$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = \beta_\nu\gamma_\nu = \gamma_\nu\gamma^\nu = 0$, $\alpha_\nu\gamma^\nu = b_1 \neq 0$, $\beta_\nu\beta^\nu = b_2 \neq 0$.

$$(5) \quad \omega_1 = (\beta_\nu y^\nu)^2 + y_\nu y^\nu, \qquad \omega_2 = \beta_\nu y^\nu + a \ln \alpha_\nu y^\nu,$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0$, $\beta_\nu\beta^\nu = -1$.

$$(6) \quad \omega_1 = (\beta_\nu y^\nu)^2 - y_\nu y^\nu, \qquad \omega_2 = \beta_\nu y^\nu + a \tan^{-1}[\gamma_\nu y^\nu (\alpha_\nu y^\nu)^{-1}],$$

where $\alpha_\nu\alpha^\nu = \gamma_\nu\gamma^\nu = b \neq 0$, $\beta_\nu\beta^\nu = 1$, $\alpha_\nu\beta^\nu = \alpha_\nu\gamma^\nu = \beta_\nu\gamma^\nu = 0$, $(\alpha_\nu y^\nu)^2 + (\gamma_\nu y^\nu)^2 = b\omega_1$.

$$(7) \quad \omega_1 = \frac{1}{2}(\alpha_\nu y^\nu)^2 + a\beta_\nu y^\nu, \qquad \omega_2 = \frac{1}{2}(\alpha_\nu y^\nu)^3 + a\alpha_\nu y^\nu \beta_\nu y^\nu + a^2\gamma_\nu y^\nu,$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = \beta_\nu\gamma^\nu = 0$, $\alpha_\nu\gamma^\nu = -\beta_\nu\beta^\nu = \gamma_\nu\gamma^\nu = b \neq 0$.

$$(8) \quad \omega_1 = \alpha_\nu x^\nu, \qquad \omega_2 = x_\nu x^\nu, \qquad \alpha_\nu\alpha^\nu = b \neq 0.$$

$$(9) \quad \omega_1 = (\beta_\nu y^\nu)(\alpha_\nu y^\nu)^{-1}, \qquad \omega_2 = \gamma_\nu y^\nu (\alpha_\nu y^\nu)^{-1},$$

where $\alpha_\nu\alpha^\nu = a_{11}$, $\alpha_\nu\beta^\nu = a_{12}$, $\ldots$, $\gamma_\nu\gamma^\nu = a_{33}$.

$$(10) \quad \omega_1 = \alpha_\nu x^\nu, \qquad \omega_2 = \beta_\nu x^\nu,$$

where $\alpha_\nu\alpha^\nu = -\beta_\nu\beta^\nu = 1$, $\alpha_\nu\beta^\nu = 0$.

In these formulae $y_\nu = x_\nu + a_\nu$, $z_\nu = x_\nu + \frac{1}{2}a_\nu$, a_ν, α_ν, β_ν, γ_ν, a, b, b_k, a_{ik} are constants connected with the group parameters $c_{\mu\nu}$ and d_μ.

To find f and g from (1.4) it is sufficient to integrate the equation

$$(du)/\eta = dt. \tag{3.4}$$

(3.4) yields

$$u(x) = \varphi(\omega) + g(x), \tag{3.5}$$

$$u(x) = \varphi(\omega) f(x), \tag{3.6}$$

$$u(x) = \Phi(\varphi(\omega) + g(x)), \tag{3.7}$$

for (1.1), (1.2), (1.3) respectively. Here Φ is an arbitrary differentiable function.

The formulae (2.1)–(2.3) yield

$$\begin{array}{ll} g(x) = -2\ln\psi(x) & \text{for (1.1)}, \\ f(x) = [\psi(x)]^{2/(1-k)} & \text{for (1.2)}, \\ g(x) = \ln\psi(x) & \text{for (1.3)}. \end{array} \tag{3.8}$$

Below we present the explicit form of $\psi(x)$:

(1) $\psi(x) = \beta_\nu y^\nu$, (2) $\psi(x) = \alpha_\nu y^\nu$, (3) $\psi(x) = \alpha_\nu y^\nu$,
(4) $\psi(x) = \beta_\nu y^\nu$, (9) $\psi(x) = \alpha_\nu y^\nu$.

In the other cases $\psi(x) = 1$.

4. The exact solutions of the Liouville equation

Substituting (3.5) into (1.1) and using (1)–(10) and (3.8) one obtain the following PDE:

$$\begin{aligned} (1)\quad & a^2\omega_1^2\varphi_{11} + 4\omega_1(\omega_2 + a + 1)\varphi_{12} + 4\omega_2(\omega_2 - 1)\varphi_{22} + a(a-1)\omega_1\varphi_1 + \\ & +2(3\omega_2 - 1)\varphi_2 + 2 + \lambda\exp\varphi = 0, \end{aligned} \tag{4.1}$$

where $\lambda_1 = \lambda/b$, $\varphi_{ik} = \partial^2/\partial\omega_i\partial\omega_k$, $i,k = 1,2$.

$$(2)\quad b\varphi_{11} + 4\varphi_{12} - 4\omega_2\varphi_{22} - 2\varphi_2 + \lambda\exp\varphi = 0. \tag{4.2}$$

$$\begin{aligned} (3)\quad & [b_2 - 1/(b_2\omega_2 - 1)]\varphi_{11} - 4(b_2\omega_2 - 1)\varphi_{12} + 4\omega_2(b\omega_2 - 1)\varphi_{22} - \\ & -b_2\varphi_1 + 2(3b_2\omega_2 - 1)\varphi_2 + 2b_2 + \lambda\exp\varphi = 0. \end{aligned} \tag{4.3}$$

$$(4)\quad \varphi_{11} + 2(2\omega_2 + b_1)\varphi_{12} + 4\omega_2^2\varphi_{22} - \varphi_1 + b\omega_2\varphi_2 + 2 + (\lambda/b_2)\exp\varphi = 0. \tag{4.4}$$

$$(5)\quad 4\omega_1\varphi_{11} - 4a\varphi_{12} - \varphi_{22} + 4\varphi_1 + \lambda\exp\varphi = 0. \tag{4.5}$$

$$(6)\quad -4\omega_1\varphi_{11} + (a^2\omega_1^{-1} + 1)\varphi_{22} + 4\varphi_1 + \lambda\exp\varphi = 0. \tag{4.6}$$

$$(7)\quad -\varphi_{11} + 2(\omega_1 + a^2)\varphi_{22} + (\lambda/a^2 b)\exp\varphi = 0. \tag{4.7}$$

$$(8)\quad b\varphi_{11} + 4\omega_1\varphi_{12} + 4\omega_2\varphi_{22} + b\varphi_2 + \lambda\exp\varphi = 0. \tag{4.8}$$

$$\begin{aligned} (9)\quad & (a_{11}\omega_1^2 - 2a_{12}\omega_1 + a_{22})\varphi_{11} + 2(a_{11}\omega_1\omega_2 - a_{13}\omega_1 - a_{12}\omega_2 + a_{23})\varphi_{12} + \\ & +(a_{11}\omega_2^2 - a_{13}\omega_2 + a_{33})\varphi_{22} + 2(a_{11}\omega_1 - a_{12})\varphi_1 + \\ & +2(a_{11}\omega_2 - a_{13})\varphi_2 + 2a_{11} + \lambda\exp\varphi = 0. \end{aligned} \tag{4.9}$$

(10) $\varphi_{11} - \varphi_{22} + \lambda \exp\varphi = 0.$ (4.10)

If one obtains at least one particular solution of any of equations (4.1)–(4.10) then (3.5) gives a solution of (1.1). Let us consider, (4.1) and (4.10) as an example. If one supposes that $\partial\varphi/\partial\omega_2 = 0$ then (4.1) is reduced to the ordinary differential equation for the function φ:

$$a^2\omega_1^2\varphi_{11} + a(a-1)\omega_1\varphi_1 + 2 + \lambda_1 \exp\varphi = 0, \tag{4.11}$$

the general solution of which has the form

$$\varphi(\omega_1) = \begin{cases} -2\ln\left[\left(-\lambda/2bc_1^2\right)^{1/2}\omega_1^{-1/a}\sinh\left(c_1\omega_1^{1/a} + c_2\right)\right], & \lambda b < 0, \\ -2\ln\left[\left(\lambda/2bc_1^2\right)^{1/2}\omega_1^{-1/a}\cosh\left(c_1\omega_1^{1/a} + c_2\right)\right], & \lambda b > 0, \\ -2\ln\left[\left(-\lambda/2bc_1^2\right)^{1/2}\omega_1^{-1/a}\cos\left(c_1\omega_1^{1/a} + c_2\right)\right], & \lambda b < 0, \\ -2\ln\left[\left(\lambda/2bc_1^2\right)^{1/2}\omega_1^{-1/a}\left(\omega_1^{1/a} + c_2\right)\right], & \lambda b > 0. \end{cases} \tag{4.12}$$

Hence from (3.5) and (4.12) one obtains the solution of (1.1)

$$\begin{aligned} &u = -2\ln[\gamma P(x)\sinh(c_1Q(x) + c_2)], \quad u = -2\ln[\delta P(x)\cosh(c_1Q(x) + c_2)], \\ &u = -2\ln[\gamma P(x)\cos(c_1Q(x) + c_2)], \quad u = -2\ln[\gamma P(x)(Q(x) + c_2)], \end{aligned} \tag{4.13}$$

where $P(x) = (\alpha_\nu y^\nu)^{-1/a}$, $Q(x) = \beta_\nu y^\nu(\alpha_\nu y^\nu)^{1/2}$, $\gamma^2 = -\delta^2 = -\lambda/2bc_1^2$.

Equation (4.10) is the two-dimensional Liouville equation. Its general solution was found by Liouville [14]:

$$\varphi(\omega_1,\omega_2) = \ln\left(-\frac{8}{\lambda}\frac{f_1'(\omega_1+\omega_2)f_2'(\omega_1-\omega_2)}{[f_1(\omega_1+\omega_2) + f_2(\omega_1-\omega_2)]^2}\right), \tag{4.14}$$

where f_1 and f_2 are arbitrary differentiable functions.

Note. The two-dimensional Liouville equation can be solved in other ways, e.g. with the help of the theory of complex variables. But we believe the simplest way is to linearise the Liouville equation. Fushchych and Tychinin [12], using non-local substitutions

$$u = \ln\left[W_\xi W_\eta\left(1 - \tanh^2\frac{c_1 - W}{\sqrt{2}}\right)\right], \qquad \xi = x_0 + x_1, \qquad \eta = x_0 - x_1,$$

or

$$u = \ln\left[2W_\xi W_\eta/(W + c_2)^2\right],$$

or

$$u = \ln\left[W_\xi W_\eta\left(1 + \tan^2\frac{W + c_3}{\sqrt{2}}\right)\right]$$

reduce the Liouville equation to $\Box W = 0$, the general solution of which was obtained by d'Alembert. Using those formulae we obtain the Liouville solution (4.14).

From (3.5) and (4.14) one obtains a solution of (1.1):

$$u = \ln\left(-\frac{8}{\lambda}\frac{f_1'(\gamma_\nu x^\nu)f_2'(\delta_\nu x^\nu)}{[f_1(\gamma_\nu x^\nu) + f_2(\delta_\nu x^\nu)]^2}\right), \tag{4.15}$$

where $\gamma_\nu\gamma^\nu = \delta_\nu\delta^\nu = 0$, $\gamma_\nu\delta^\nu = 2$.

The other solutions of (1.1) we have obtained have the form (4.13) with

$$
\begin{array}{lll}
(a) & P(x) = F^{-1}(\alpha_\nu y^\nu), & Q(x) = \beta_\nu y^\nu F(\alpha_\nu y^\nu), \\
(b) & P(x) = F^{-1}(\alpha_\nu y^\nu), & Q(x) = \beta_\nu y^\nu F(\alpha_\nu y^\nu) - \ln F(\alpha_\nu y^\nu), \\
(c) & P(x) = \alpha_\nu y^\nu, & Q(x) = (y_\nu y^\nu)^{1/2}(\alpha_\nu y^\nu)^{-1}, \\
(d) & P(x) = \omega_1, & Q(x) = \ln \omega_1, \quad \omega_1 = (\beta_\nu y^\nu)^2 + y_\nu y^\nu, \\
(e) & P(x) = 1, & Q(x) = \beta_\nu y^\nu + a \ln \alpha_\nu y^\nu, \\
(f) & P(x) = 1, & Q(x) = \beta_\nu y^\nu, \\
(g) & P(x) = 1, & Q(x) = \beta_\nu y^\nu + F(\alpha_\nu y^\nu),
\end{array}
$$

where F is an arbitrary differentiable function, $\alpha_\nu \alpha^\nu = \alpha_\mu \beta^\nu = 0$, $\beta_\nu \beta^\nu = b \neq 0$.

Besides, from (4.8) we have the particular solution of (1.1) in the form

$$u(x) = -\ln\left(\frac{1}{2}\lambda x_\nu x^\nu\right). \tag{4.16}$$

We have obtained the solutions of (1.1) when $n = 3$ and they are easily generalised to more general cases $n \geq 4$. For $n \geq 4$ some solutions of (1.1) may be obtained in an analogous way, integrating (1.5) and determining the invariants $\omega(x)$.

5. The exact solutions of the nonlinear d'Alembert equation

Using (3.6) arid the explicit form of invariants $\omega(x)$ and function $f(x)$ (1)–(10) one obtains the tollowing PDE:

$$
\begin{aligned}
(1)\quad & a^2\omega_1^2\varphi_{11} - 4a\omega_1(\omega_2 - a)\varphi_{12} + 4\omega_2(\omega_2 - b)\varphi_{22} + \\
& +a[a - 1 + 4/(1-k)]\omega_1\varphi + 2(k-1)^{-1}[(3k+1)\omega_2 - 2b(k+1)]\varphi_2 + \\
& +2(k+1)(k-1)^{-2}\varphi + (\lambda/b)\varphi^k = 0.
\end{aligned} \tag{5.1}
$$

$$(2)\quad b\varphi_{11} + 4\varphi_{12} - 4\omega_2\varphi_{22} + 2(3+k)(1-k)^{-1}\varphi_2 + \lambda\varphi^k = 0. \tag{5.2}$$

$$
\begin{aligned}
(3)\quad & [b_2 - 1/(b_2\omega_2 - 1)]\varphi_{11} - 4(b_2\omega_2 - 1)\varphi_{12} + 4\omega_2(b_2\omega_2 - 1)\varphi_{22} + \\
& +b_2(3+k)(1-k)^{-1}\varphi_1 - 4(1-k)^{-1}[\omega_2 b(3-k) - (1+k)]\varphi_2 + \\
& +2b_2(1+k)(1-k)^{-2}\varphi + \lambda\varphi^k = 0.
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
(4)\quad & \omega_1^2\varphi_{11} + 2\omega_1\omega_2[1 - (b_1/2b_2)\omega_2^2]\varphi_{12} + \omega_2^2\varphi_{22} + 4(1-k)^{-1}(\omega_1\varphi_1 + \\
& +\omega_2\varphi_2) + 2(1+k)(1-k)^{-2}\varphi + (\lambda/b_2)\varphi^k = 0.
\end{aligned} \tag{5.4}
$$

$$(5)\quad 4\omega_1\varphi_{11} - 4a\varphi_{12} - \varphi_{22} + 4\varphi_1 + \lambda\varphi^k = 0. \tag{5.5}$$

$$(6)\quad -4\omega_1\varphi_{11} + (1 + a^2\omega_1^{-1})\varphi_{22} + 4\varphi_1 + \lambda\varphi^k = 0. \tag{5.6}$$

$$(7)\quad -\varphi_{11} + (2\omega_1 + a^2)\varphi_{22} + (\lambda_1/a^2)\varphi^k = 0. \tag{5.7}$$

$$(8)\quad b\varphi_{11} + 4\omega_1\varphi_{12} + 4\omega_2\varphi_{22} + b\varphi_2 + \lambda\varphi^k = 0. \tag{5.8}$$

$$
\begin{aligned}
(9)\quad & (a_{11}\omega_1^2 - 2a_{12}\omega_1 + a_{22})\varphi_{11} + 2(a_{11}\omega_1\omega_2 - a_{13}\omega_1 - a_{12}\omega_2 + a_{23})\varphi_{12} + \\
& +(a_{11}\omega_2^2 - 2a_{13}\omega_2 + a_{33})\varphi_{22} + 2(k+1)(k-1)^{-1}[(a_{11}\omega_1 - a_{12})\varphi_1 + \\
& +(a_{11}\omega_2 - a_{13})\varphi_2] + 2a_{11}(k+1)(k-1)^{-2}\varphi + \lambda\varphi^k = 0.
\end{aligned} \tag{5.9}
$$

$$(10) \quad \varphi_{11} - \varphi_{22} + \lambda\varphi^k = 0. \tag{5.10}$$

Equation (5.1) when $\partial\varphi/\partial\omega_2 = 0$ becomes the Emden–Fowler one

$$\xi^2 V_{\xi\xi} + 2\xi V_\xi + (\lambda/b)\xi^{k+1}V^k = 0 \tag{5.11}$$

via the substitution

$$\varphi = \xi^{(k+1)/(k-1)}V(\xi), \qquad \xi = \omega_1^{1/a}. \tag{5.12}$$

We have found some particular solutions of (5.2)–(5.10) and then we have the following solutions of (1.2):

$$u = [\beta_\nu y^\nu + \alpha_\nu y^\nu\,(c_2 + \ln a_\nu y^\nu)]^{2/(1-k)}, \tag{5.13}$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0$, $\beta_\nu\beta^\nu = b = -\frac{1}{2}\lambda(1-k)^2/(1+k)$.

$$u = \left\{\frac{1}{2}\left[(k-1)^2/(k-3)\right] y_\nu y^\nu\right\}^{1/(1-k)}, \tag{5.14}$$

$$u = \left\{-\frac{1}{2}\lambda\left(1-k^2\right)\left[(\beta_\nu y^\nu)^2 + y_\nu y^\nu\right]\right\}^{1/(1-k)}, \tag{5.15}$$

where $\beta_\nu\beta^\nu = -1$.

$$u = \left\{c_2 \pm (1-k(\left[\frac{1}{2}\lambda(1+k)^{-1}\right]^{1/2}(\beta_\nu y^\nu + a\ln\alpha_\nu y^\nu)\right\}^{2/(1-k)}, \tag{5.16}$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0$, $\beta_\nu\beta^\nu = -1$.

$$u = \left[c_2 + (2a)^{-1}(\alpha_\nu y^\nu)^2 + \beta_\nu y^\nu\right]^{2/(1-k)}, \tag{5.17}$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0$, $\beta_\nu\beta^\nu = -\frac{1}{2}\lambda(1-k)^2(1+k)$.

From (5.13)–(5.17) one can see that all the solutions of (1.2) obtained have the form

$$u = [F(y) + G(z)]^\alpha, \tag{5.18}$$

where α takes the values $1/(1-k)$ and $2/(1-k)$, and

$$y = \left(y^1, \ldots, y^{n-1}\right), \qquad z = \left(z^1, \ldots, z^{n-1}\right),$$
$$y^a = y^a(x), \qquad z^a = z^a(x), \qquad a = \overline{1, n-1}, \qquad x \in R_n.$$

If one searches for the solutions of (1.2) in the form (5.18), then the substitution of (5.18) in (1.2) leads to the equation for the functions F, G, y^a, z^a:

$$\begin{aligned}&(\alpha-1)A_\mu A^\mu + (F+G)\left(F_{ab}y^a_\mu y^{b\mu} + G_{ab}z^a_\mu z^{b\mu} + F_a\Box y^a + G_a\Box z^a\right) + \\ &\quad + (\lambda/\alpha)(F+G)^{\alpha(k-1)+2} = 0,\end{aligned} \tag{5.19}$$

where

$$\begin{aligned}&A_\mu = F_a y^a_\mu + G_a z^a_\mu, \qquad F_a \equiv \partial F/\partial y^a, \qquad F_{ab} = \partial^2 F/\partial y^a\partial y^b,\\ &G_a \equiv \partial G/\partial z^a, \qquad G_{ab} = \partial^2 G/\partial z^a\partial z^b, \qquad y^a_\mu \equiv \partial y^a/\partial x_\mu,\\ &z^a_\mu \equiv \partial z^a/\partial x_\mu, \qquad \mu = \overline{0, n-1}, \qquad a, b = \overline{1, n-1}.\end{aligned} \tag{5.20}$$

Below we list some particular solutions of (5.19):

$$(a)\quad F(y) = \left(y^1 + c\right)^2, \qquad G(z) = z^1 z^2, \qquad y^1 = \alpha_\nu x^\nu, \qquad z^1 = \beta_\nu x^\nu,$$
$$z^2 = \gamma_\nu x^\nu, \qquad \alpha = 1/(1-k), \qquad \alpha_\nu\beta^\nu = \alpha_\nu\gamma^\nu = \beta_\nu\beta^\nu = \gamma_\nu\gamma^\nu = 0,$$
$$2\alpha_\nu\alpha^\nu = \beta_\nu\gamma^\nu = \lambda(k-1)^2(k-3)^{-1}, \qquad x = (x_0, \ldots, x_{n-1}), \qquad n \geq 3.$$

$$(b)\quad F(y) = y^2\varphi\left(y^1\right), \qquad G(z) = z^2\psi\left(z^1\right), \qquad y^1 = z^1 = \alpha_\nu x^\nu,$$
$$y^2 = \beta_\nu x^\nu, \qquad z^2 = \gamma_\nu x^\nu, \qquad \alpha = 2/(1-k),$$

where φ and ψ are arbitrary differentiable functions, satisfying the condition

$$\begin{gathered} \varphi^2 + \psi^2 = \frac{1}{2}\lambda(k-1)^2/(k+1), \\ \alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = \alpha_\nu\gamma^\nu = \beta_\nu\gamma^\nu = 0, \qquad \beta_\nu\beta^\nu = \gamma_\nu\gamma^\nu = -1, \qquad n \geq 3. \end{gathered} \tag{5.21}$$

(c) $F(y) = F\left(y^1\right)$ is an arbitrary differentiable function,

$$G(z) = z^1, \qquad y^1 = \alpha_\nu x^\nu, \qquad z^1 = \beta_\nu x^\nu,$$
$$\alpha_\nu\alpha^\nu = \alpha_\nu\beta_\nu = 0, \qquad \beta_\nu\beta^\nu = -\frac{1}{2}(k-1)^2/(k+1).$$

So according to (5.18) we have the following solutions of (1.2)

$$u = \left[(\alpha_\nu x^\nu + c)^2 + \beta_\nu x^\nu \gamma_\nu x^\nu\right]^{1/(k-1)}, \tag{5.22}$$

where $\alpha_\nu\beta^\nu = \alpha_\nu\gamma^\nu = \beta_\nu\beta^\nu = \gamma_\nu\gamma^\nu = 0$, $2\alpha_\nu\alpha^\nu = \beta_\nu\gamma^\nu = \lambda(k-1)^2/(k-3)$, $k \neq 3$.

$$u = [\beta_\nu x^\nu \varphi(\alpha_\nu x^\nu) + \gamma_\nu x^\nu \psi(\alpha_\nu x^\nu)]^{2/(1-k)}, \tag{5.23}$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = \alpha_\nu\gamma^\nu = \beta_\nu\gamma^\nu = 0$, $\beta_\nu\beta^\nu = \gamma_\nu\gamma^\nu = -1$, $\varphi^2 + \psi^2 = \frac{1}{2}\lambda(k-1)^2/(k+1)$, $k \neq -1$.

$$u = [F(\alpha_\nu x^\nu) + \beta_\nu x^\nu]^{2/(1-k)}, \tag{5.24}$$

where $\alpha_\nu\alpha^\nu = \alpha_\nu\beta_\nu = 0$, $\beta_\nu\beta^\nu = -\frac{1}{2}(k-1)^2/(k+1)$, $k \neq -1$. If in (5.23)–(5.24) φ, ψ, F are arbitrary functions we have the wide class of exact solutions of (1.2).

Ibragimov [13] established that if $k = (n+2)/(n-2)$, $n \geq 3$ (1.2) is conformally invariant. It is well known that the conformal transformations have the form (see e.g. Fushchych and Nikitin [8])

$$x'_\mu = \sigma^{-1}(x_\mu + c_\mu x_\nu x^\nu), \qquad u' = \sigma^{(n-2)/2}u, \tag{5.25}$$

where $\sigma = (1 + 2c_\nu x^\nu + c_\lambda c^\lambda x_\nu x^\nu)$, c_μ are constants. Using (5.25) one can produce new solutions of the equation

$$\Box u + \lambda u^{(n+2)/(n-2)} = 0. \tag{5.26}$$

in such a way. Let $u = F(x)$ be a solution of (5.26) for $n \geq 3$, then

$$u = \sigma^{(2-n)/n} F((x + c x_\nu x^\nu)/\sigma), \tag{5.27}$$

where $c = (c_0, \ldots, c_{n-1})$, will be another solution of (5.26) and

$$\Box u + \lambda u^{(n+2)/(n-2)} = \sigma^{-(n+2)/2}\left(\Box F + \lambda F^{(n+2)/(n-2)}\right). \tag{5.28}$$

When $n = 4$, equation (5.26) has the form

$$\Box u + \lambda u^3 = 0. \tag{5.29}$$

Its particular solutions are given in (5.15)–(5.17), (5.23)–(5.24). These expressions give the solutions of Yang–Mills equations after using the 'tHooft–Corrigan–Wilczek ansatz.

In the conclusion of this section we consider another nonlinear d'Alembert equation

$$\Box u + \lambda \sin u = 0, \qquad \lambda = 1, \tag{5.30}$$

which is known as a sine-Gordon equation. Below we present some exact solutions of this equation

$$u = 4\tan^{-1}\left\{\exp\left[f(\alpha_\nu x^\nu) + \beta_\nu x^\nu\right]\right\}, \tag{5.31}$$

$$k\int_0^{u/2} \frac{d\psi}{\left(1 - k^2\sin^2\psi\right)^{1/2}} = f(\alpha_\nu x^\nu) + \beta_\nu x^\nu + c_0, \tag{5.32}$$

where f is an arbitary differentiable function, α_ν, β_ν, k are constants, $\alpha_\nu\alpha^\nu = \alpha_\nu\beta^\nu = 0$, $\beta_\nu\beta^\nu = -1$.

6. The exact solutions of the eikonal equation

The eikonal equation (1.3) is one of the main equations of geometrical optics and it is the characteristic equation for the linear d'Alembert one. In this section we shall find some exact solutions of (1.3) by analogy with that done in the previous sections and show how to generate new solutions using the conformal transformations. Upon substituting (3.7) into (1.3) we obtain some PDE for the function $\varphi(\omega)$ and we have solved some of them.

Below we present the final result:

$$u = \Phi(\alpha_\nu x^\nu), \tag{6.1}$$

$$u = \Phi\left(\beta_\nu y^\nu \pm \left[(\beta_\nu y^\nu)^2 - a y_\nu y^\nu\right]^{1/2}\right), \tag{6.2}$$

$$u = \Phi(y_\nu y^\nu / \alpha_\nu y^\nu), \tag{6.3}$$

where Φ is an arbitrary differentiable function, $y_\nu = x_\nu + a_\nu$, α_ν, β_ν, a_ν, a are constants, $\alpha_\nu\alpha^\nu = 0$, $\beta_\nu\beta^\nu = a \neq 0$.

One can see from (2.3) that (1.3) is conformally invariant, the conformal transformations being as follows:

$$x'_\mu = \sigma^{-1}(x_\mu + c_\mu x_\nu x^\nu), \qquad u' = u, \tag{6.4}$$

where σ is from (5.25). The new solutions u_{new} have the form

$$u_{\text{new}}^{(x)} = u_{\text{old}}((x + c x_\nu x^\nu)/\sigma). \tag{6.5}$$

In conclusion we formulate the following statement.

Theorem 5. *The equation*

$$p_\mu u p^\mu u = F(u), \qquad \mu = \overline{0, n-1}, \tag{6.6}$$

is reduced to the form

$$p_\mu V p^\mu V = 1, \qquad \mu = \overline{0, n-1} \tag{6.7}$$

by the substitution

$$V = \int \frac{du}{(F(u))^{1/2}}. \tag{6.8}$$

Note. Equation (6.7) upon substituting

$$W(x, V) = 0 \tag{6.9}$$

takes the form

$$p_\nu W p^\nu W = 0, \qquad \nu = \overline{0, n}, \tag{6.10}$$

where $W = W(\tilde{x})$, $\tilde{x} = (x_0, \ldots, x_{n-1}, x_n \equiv V)$. Equation (6.10) is the eikonal equation (1.3) in $(n+1)$-dimensional space. With the help of ansatz (1.4), we have obtained multiparametrical exact solutions of many-dimensional nonlinear Schrödinger (Fushchych and Moskaliuk [7], Born–Infeld (Fushchych and Serov [9]) and Dirac equations (Fushchych and Shtelen [11]).

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Appendix. The reduction of (1.1)–(1.2) to ordinary differential equations

If the function φ from the ansatz (1.4) depends on one variable ω only it means that (1.1)–(1.2) are ordinary differential ones.

The Liouville equation (1.1) is reduced to the equation

$$\omega_\nu \omega^\nu \varphi'' + \Box\omega\varphi' + \Box g + \lambda \exp g \exp \varphi = 0,$$

via the substitution (3.5) if the conditions

$$\omega_\nu \omega^\nu = \psi_1(\omega) \exp g, \qquad \Box\omega = \psi_2(\omega) \exp g, \qquad \Box g = \psi_3(\omega) \exp g,$$

are satisfied.

The equation (1.2) will be reduced to the ordinary differential equation

$$\omega_\nu \omega^\nu f \varphi'' + (\Box\omega f + 2\omega_\nu f^\nu)\varphi' + \Box f \cdot \varphi + \lambda f^k \varphi^k = 0$$

under the conditions

$$\omega_\nu \omega^\nu = \psi_1(\omega) f^{k-1}, \qquad \Box\omega f + 2\omega_\nu f^\nu = \psi_2(\omega) f^k, \qquad \Box f = \psi_3(\omega) f^k.$$

The ansatz (1.4) in this case has the form (3.6).

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