On the new conservation laws for vector field equations

W.I. FUSHCHYCH, V.A. VLADIMIROV

The new conservation laws corresponding to the non-Lie symmetry of vector field equations are obtained.

1. Introduction

The classical Lie method (see e.g. Ovsyannikov [11]) which is commonly used to investigate the group theoretical properties of differential equations has one essential disadvantage. Based on the infinitesimal approach, it does not permit one to find out the maximal Lie algebra available by a given system of differential equations if among its basic elements there are operators of higher order. A method was proposed (Fushchych [2]), hereafter quoted as the non-Lie method, in which no restriction is imposed on the order of operators available, by the systems of differential equation under consideration.

By means of the non-Lie method additional invariances were established: Dirac (Fushchych [2]); Maxwell (Fushchych [3]); Kemmer–Duffin–Petiau (Fushchych and Nikitin [4]) and many other theoretical and mathematical physics equations.

Recently (Fushchych and Vladimirov [5]) within the framework of the non-Lie (approach, group properties of the equations for the potential of an electromagnetic field have been investigated:

$$\begin{aligned} &p_\mu p^\mu \boldsymbol{A}^\nu(x) = 0, \qquad x \in \boldsymbol{R}^4, \\ &p_\mu \boldsymbol{A}^\mu(x) = 0, \qquad \mu,\nu = 0,1,2,3, \end{aligned} \tag{1}$$

where $p_\mu = i\partial/\partial x^\mu = ig_{\mu\nu}\partial/\partial x_\nu$ and $g_{\mu\nu} = g^{\mu\nu}$ the metric tensor of Minkowski space, $g_{00} = -g_{11} = -g_{22} = -g_{33} = 1$.

The maximal invariance algebra of equation (1) generated by first-order differential operators is the eleven-dimensional Weyl algebra which includes the Poincaré algebra $P(1,3)$ and operator $D = x_\mu p^\mu + 2i$. It has been shown recently (Fushchych and Vladimirov [5]) that equations (1) are additionally invariant under the nine-dimensional $GL(3)$ algebra with basic elements being integro-differential operators; defined on the set of solutions by the following formula

$$\begin{aligned} &(D_{ab}\boldsymbol{A})^\mu = p_0/|\boldsymbol{p}|^2(g_0^\mu p_a - g_a^\mu p_0)\boldsymbol{A}_b, \\ &|\boldsymbol{p}|^2 = p_1^2 + p_2^2 + p_3^2, \qquad a,b = 1,2,3. \end{aligned} \tag{2}$$

The operators (2) are non-local so there is no such point transformation of independent variables (x^μ): $(x'^\mu) = (Tx)^\mu$, which would give rise to a continuous group representation generated by D_{ab} on the set of solutions of (1).

The existence of additional symmetry for systems of differential equations which describe elementary particles is strictly connected with their polarisation properties.

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Thus, there is no additional symmetry in the case of the Klein–Gordon equation which describes a spin-zero relativistic particle, the additional symmetry algebra of the Dirac equation is $GL(2) \oplus GL(2)$, and generally the greater the spin, the greater the dimension of the additional symmetry algebra.

One of the most important consequences of the invariance of the evolution equation is the existence of integral quantities conserved in time. The purpose of this paper is to construct new conserved quantities which correspond to the non-Lie symmetry of vector field equations.

For the equations obtained from variational principles, the correspondence between local transformation groups which preserve the action integral and conservation laws is established by the well known Noether theorem. It is obvious that, because of non-locality of the transformation group generated by the operators (2), the Noether theorem is of no use in our case. However, there is another method of building up conserved quantities. Good [6] succeeded in obtaining all classical conserved quantities for the Maxwell equations without reference to the Noether theorem. Later O'Connell and Tompkins [8, 9, 10] and several other authors extended this result on some other Poincaré-invariant equations. Employing the same techniques as in the above mentioned papers it is possible to construct conserved quantities corresponding to the non-local additional symmetry of the vector field equations.

In § 2 we perform such a construction for the four-vector potential of the electromagnetic field equation. In § 3 analogous conserved quantities are obtained for the Proca equation. In § 4 we discuss the results obtained.

2. The new conserved quantities for the equations (1)

Theorem 1. *Integrals*

$$\boldsymbol{S}_a = \frac{i}{2}\varepsilon_{abc}\int\{\boldsymbol{A}_b(t,\boldsymbol{x})p_0\boldsymbol{A}_c(t,\boldsymbol{x}) - [p_0\boldsymbol{A}_b(t,\boldsymbol{x})]\boldsymbol{A}_c(t,\boldsymbol{x})\}\,d^3\boldsymbol{x}, \quad a,b,c = 1,2,3, \tag{3}$$

$$\begin{aligned}\Sigma_{jk} = \frac{1}{2}\int\Bigg[&\boldsymbol{A}_j(t,\boldsymbol{x})p_0\left(\frac{p_0}{|p_0|}\boldsymbol{A}_k\right)(t,\boldsymbol{x}) - [p_0\boldsymbol{A}_j(t,\boldsymbol{x})]\left(\frac{p_0}{|p_0|}\boldsymbol{A}_k\right)(t,\boldsymbol{x}) + \\ &+\boldsymbol{A}_k(t,\boldsymbol{x})p_0\left(\frac{p_0}{|p_0|}\boldsymbol{A}_j\right)(t,\boldsymbol{x}) - [p_0\boldsymbol{A}_k(t,\boldsymbol{x})]\left(\frac{p_0}{|p_0|}\boldsymbol{A}_j\right)(t,\boldsymbol{x})\Bigg]\,d^3\boldsymbol{x}, \quad j,k = 1,2,3,\end{aligned} \tag{4}$$

are conserved in time.

Proof. Let us consider the following operator:

$$W = \exp\left\{(\ln\sqrt{2})\left[1 + \left(\varepsilon_{kjl}i\frac{p_k\boldsymbol{S}_{lj}}{2|\boldsymbol{p}|}\right)^2\right] + \frac{\pi}{4}\frac{p_0}{2|\boldsymbol{p}|^2}p_n(\boldsymbol{S}_{0j}\boldsymbol{S}_{jn} + \boldsymbol{S}_{jn}\boldsymbol{S}_{0j})\right\}, \tag{5}$$

where $j,k,l,n = 1,2,3$, $\boldsymbol{S}_{\mu\nu}$, $\mu,\nu = 0,1,2,3$ are the matrices of the $D\left(\frac{1}{2},\frac{1}{2}\right)$ representation of the Lie algebra of the $O(1,3)$ group*. It can be easily shown that matrix elements of symbols of this operator and the inverse one are

$$\begin{aligned}&[\boldsymbol{W}(p)]^\mu_\nu = \left(p_0/|\boldsymbol{p}|^2\right)\left(p_0 g^\mu_\nu - p^\mu g_{\nu 0} - g^\mu_0 p_\nu + 2g^\mu_0 g_{\nu 0} p_0\right), \\ &\left[\boldsymbol{W}^{-1}(p)\right]^\mu_\nu = \left(1/2|\boldsymbol{p}|^2\right)\left[2p_0^2(g^\mu_\nu - g^\mu_0 g_{\nu 0}) + p^\mu p_\nu\right]\end{aligned} \tag{6}$$

*Matrix elements of $S_{\mu\nu}$ have the form

$$(\boldsymbol{S}_{\mu\nu})^\alpha_\beta = i(g^\alpha_\mu g_{\nu\beta} - g^\alpha_\nu g_{\mu\beta}), \qquad \alpha,\beta = 0,1,2,3, \quad \mu,\nu = 0,1,2,3.$$

(for symbols see e.g. Shubin [12]).

Using the operators (5) we are allowed to transform (1) into the equivalent diagonal form

$$p_\mu p^\mu \tilde{\boldsymbol{A}}^\nu = 0, \qquad p_0 \tilde{\boldsymbol{A}}^0 = 0, \qquad \mu,\nu = 0,1,2,3, \tag{7}$$

where $\tilde{\boldsymbol{A}}^\nu = (\boldsymbol{W}^{-1}\boldsymbol{A})^\nu$. It is not difficult to show that for every $\tilde{\boldsymbol{A}}'$, $\tilde{\boldsymbol{A}}''$ satisfying (7) the following equation holds

$$p_\mu \left[\tilde{\boldsymbol{A}}'_\nu p^\mu \tilde{\boldsymbol{A}}''^\nu - (p^\mu \tilde{\boldsymbol{A}}'_\nu)\tilde{\boldsymbol{A}}''^\nu\right] = 0. \tag{8}$$

If we restrict ourselves to those solutions $\tilde{\boldsymbol{A}}'$, $\tilde{\boldsymbol{A}}''$ which tend to zero quickly enough with their first derivatives when $|x| \to \infty$, then by the Green–Gauss–Ostrogradsky theorem

$$\int \left[\tilde{\boldsymbol{A}}'_\mu p_0 \tilde{\boldsymbol{A}}''^\mu - (p_0 \tilde{\boldsymbol{A}}'_\mu)\tilde{\boldsymbol{A}}''^\mu\right] d^3\boldsymbol{x} = \text{constant}. \tag{9}$$

In canonical representation (equation (7)) basic elements of the symmetry algebra can be chosen as

$$(\tilde{\boldsymbol{S}}_\alpha)^\mu_\nu = -i\varepsilon_{abc} g^\mu_b g_{\nu c}, \qquad a,b,c = 1,2,3, \quad \mu,\nu = 0,1,2,3, \tag{10}$$

$$(\tilde{\boldsymbol{\Sigma}}_{jk})^\mu_\nu = -(g^\mu_j g_{\nu k} + g^\mu_k g_{\nu j}), \qquad j,k = 1,2,3, \quad \mu,\nu = 0,1,2,3. \tag{11}$$

Setting $\tilde{\boldsymbol{A}}' = \boldsymbol{W}^{-1}\boldsymbol{A}$, $\tilde{\boldsymbol{A}}'' = f(p)\tilde{\boldsymbol{Q}}_\alpha \boldsymbol{W}^{-1}\boldsymbol{A}$ in (9) where $f(p)$ is a scalar function and $\tilde{\boldsymbol{Q}}_\alpha$ belongs to the symmetry algebra of equation (7), we can get

$$\int\{(\boldsymbol{W}^{-1}\boldsymbol{A})_\mu p_0 [f(p)\tilde{\boldsymbol{Q}}_\alpha \boldsymbol{W}^{-1}\boldsymbol{A}]^\mu - (p_0 \boldsymbol{W}^{-1}\boldsymbol{A})_\mu [f(p)\tilde{\boldsymbol{Q}}_\alpha \boldsymbol{W}^{-1}\boldsymbol{A}]^\mu\} d^3\boldsymbol{x} = \text{const.} \tag{12}$$

Inserting into the above integral $\tilde{\boldsymbol{Q}}_\alpha = \tilde{\boldsymbol{S}}_a$, $a = 1,2,3$, $f(p) = -\frac{1}{2}$ one obtains formula (3). Substitution $f(p) = -p_0/2|p_0|$, $\tilde{\boldsymbol{Q}}_\alpha = \tilde{\boldsymbol{\Sigma}}_{jk}$, $j,k = 1,2,3$ gives us expression (4).

Now we see that besides such well known conserved quantities for the $\tilde{\boldsymbol{A}}_\mu$ as energy, momentum etc, integrals (3) and (4) are also independent of time.

Remark 1. Expression (3) represents three components of spin of the real vector field (see e.g. Bogoliubov and Shirkov [1]). Formula (4) gives us six new conserved quantities for equation (1) corresponding to the non-Lie (additional) symmetry.

3. The new conserved quantities for the Proca equation

In the paper of Fushchych and Vladimirov [5] the non-Lie symmetry of the Proca equation was also investigated

$$\begin{aligned} &\left(p_\mu p^\mu - m^2\right)\psi^\nu(t,\boldsymbol{x}) = 0, \qquad p_\mu \psi^\mu(t,\boldsymbol{x}) = 0, \\ &(t,\boldsymbol{x}) \in \boldsymbol{R}^4, \qquad m > 0. \end{aligned} \tag{13}$$

It was shown that equation (13) is also invariant under the nine-dimensional Lie algebra of the $GL(3)$ group.

Theorem 2. *Integrals*

$$\boldsymbol{S}_a = i\varepsilon_{abc}\int [\psi^*_b p_0 \psi_c - (p_0\psi^*_b)\psi_c]\, d^3\boldsymbol{x}, \qquad a,b,c = 1,2,3, \tag{14}$$

$$\mathbf{\Sigma}_{jk} = \int \Big[\psi_j^*(t,\boldsymbol{x}) p_0 \left(\frac{p_0}{|p_0|}\psi_k\right)(t,\boldsymbol{x}) - [p_0 \psi_j^*(t,\boldsymbol{x})] \left(\frac{p_0}{|p_0|}\psi_k\right)(t,\boldsymbol{x}) + \\ + \psi_k^*(t,\boldsymbol{x}) p_0 \left(\frac{p_0}{|p_0|}\psi_j\right)(t,\boldsymbol{x}) - [p_0 \psi_k^*(t,\boldsymbol{x})] \left(\frac{p_0}{|p_0|}\psi_j\right)(t,\boldsymbol{x})\Big] d^3\boldsymbol{x}, \quad j,k = 1,2,3, \tag{15}$$

are conserved in time.

The proof of this theorem is not different from that of the previous one. To diagonalise (13) we can use operators U, U^{-1} with symbols

$$[U(p)]^\mu_\nu = (1/p_0)\left(p_0 g^\mu_\nu - g^\mu_0 p_\nu - p^\mu g_{0\nu} + 2 g^\mu_0 g_{\nu 0} p_0\right), \\ \left[U^{-1}(p)\right]^\mu_\nu = \left[1/\left(p_0^2 + |\boldsymbol{p}|^2\right)\right]\left[\left(p_0^2 + |\boldsymbol{p}|^2\right)\left(g^\mu_\nu - g^\mu_0 g_{\nu 0}\right) + p^\mu p_\nu\right]. \tag{16}$$

Remark 2. Integrals (14) express spin components of the complex vector field (Bogoliubov and Shirkov [1]).

4. Conclusions

We have obtained the conserved quantities corresponding to non-Lie symmetry of equations (1) and (13) without reference to the Noether theorem. It is worth noting that classical conserved quantities for vector fields such as energy, momentum etc can also be obtained in this way by substitution of generators of Weyl (Poincaré) symmetry algebras into (9).

As has already been mentioned, integrals (3) and (14) are attributed to the spin of the classical vector fields. Conservation of (3) and (14) along with total angular momentum was obtained as a consequence of symmetry of the energy-momentum tensor for vector fields, namely $T_{\mu\nu} = T_{\nu\mu}$ (see e.g. Bogoliubov and Shirkov [1]). Generally this is not true and such conserved quantities connected in fact with non-Lie symmetry could not be obtained using the Noether theorem.

The natural question is what physical interpretation can be proposed for the new conserved quantities. It is well known that by substitution

$$\boldsymbol{E}_k = -\left[\left(\partial \boldsymbol{A}_0/\partial x^k\right) + \left(\partial \boldsymbol{A}_k/\partial x^0\right)\right], \quad \boldsymbol{H}_k = \varepsilon_{kjr}(\partial/\partial x^j)\boldsymbol{A}_r, \quad k,j,r = 1,2,3,$$

the energy and momentum integrals for $\boldsymbol{A}^\mu$ can be expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$ which satisfy the Maxwell equations. As for the integrals (3) and (4), their explicit dependence on the $\boldsymbol{A}^\mu$ could not be eliminated by similar substitution, therefore any interpretation in terms of measurable classical quantities is hardly possible. Nevertheless the interpretation of (3), (4) and (14) and (15) is possible in terms of quantum field theory.

In conclusion we want to say a few words about the independence of the integrals obtained. There exist several non-equivalent definitions of the independence of conserved quantities. According to Ibragirnov [7] a set of conserved quantities $(F_i)_{i=1}^N$,

$$F_i = \int f_i(t, x_1, \ldots, x_k)\, d^k x$$

is independent if functions f_i are linearly independent. Following this definition it is not difficult to show that classical conserved quantities p_μ, $J_{\mu\nu}(D)$ and new conserved quantities (3), (4), (14) and (15) are independent.

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