The symmetry and some exact solutions of the relativistic eikonal equation

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We discuss the symmetry group of the relativistic eikonal equation. It is found to be the conformal group $C_{1,4}$ of the $(4+1)$-dimensional Poincaré–Minkowski space. Some exact multiparametrical solutions of the equation are obtained.

Introduction

The relativistic eikonal or the relativistic Hamilton–Jacobi equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = m^2 \tag{1}$$

is a fundamental one in theoretical physics.

Without loss of generality we shall put $m = 1$ and consider the equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \tag{2}$$

In the study of partial differential equations one often gains deep insight by studying the symmetry of the equation both from the point of view of its physical interpretation and from being able to find exact solutions and to generate new solutions from known ones.

In this note we have shown that the maximally extensive local (in sense of Lie) invariance group of eq. (2) is the conformal group $C_{1,4}$ of the $(4+1)$-dimensional Poincaré-Minkowski space with the metric

$$s^2 = x^A x_A = g^{AB} x_A x_B = x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2, \tag{3}$$

where $A, B = 0, 1, \ldots, 4$; $x_4 = u$; $g^{AB} = g_{AB} = \{1, -1, -1, -1, -1\}\delta_{AB}$, δ_{AB} is the Kronecker delta.

Some exact multiparametrical solutions of eq. (2) are obtained with the help of the method recently proposed [1]. A procedure of generating new exact solutions from known ones is presented.

The symmetry group

Theorem. *The maximally extensive local invariance group of eq. (2) is the 21-parametrical Lie group, basis elements of its Lie algebra having the form*

$$\begin{aligned}
&P_0 = \frac{\partial}{\partial x_0}, \qquad P_a = -\frac{\partial}{\partial x_a}, \qquad P_4 = -\frac{\partial}{\partial u}, \qquad a, b = 1, 2, 3,\\
&J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \qquad \mu, \nu = 0, 1, 2, 3,\\
&J_{04} = x_0 P_4 - u P_0, \qquad J_{a4} = x_a P_4 - u P_a,\\
&D = x^A P_A \equiv x_0 P_0 - x_1 P_1 - x_2 P_2 - x_3 P_3 - u P_4,\\
&K_\mu = 2x_\mu D - x^B x_B P_\mu, \qquad x^B x_B \equiv x_0^2 - x_1^2 - x_3^3 - u^2,\\
&K_4 = 2uD - x^B x_B P_4,
\end{aligned} \tag{4}$$

Lettere al Nuovo Cimento, 1982, **34**, № 16, P. 498–502.

and satisfying commutation rules of the conformal algebra $C_{1,4}$

$$\begin{gathered}
[P_A,P_B]=0, \qquad [P_A,J_{BC}]=g_{AB}P_C-g_{AC}P_A,\\
[J_{AB},J_{CD}]=g_{BC}J_{AD}+g_{AD}J_{BC}-g_{AC}J_{BD}-g_{BD}J_{AC}, \qquad [P_A,D]=P_A,\\
[J_{AB},D]=0, \qquad [P_A,K_B]=2(g_{AB}D-J_{AB}), \qquad [D,K_A]=K_A,\\
[J_{AB},K_C]=g_{BC}K_A-g_{AC}K_B, \qquad [K_A,K_B]=0, \qquad A,B,C,D=0,1,\ldots,4.
\end{gathered} \tag{5}$$

One can get the proof of this theorem living Lie's method [2]. This being the case, one has to solve the set of first-order coupled partial differential equations:

$$\begin{gathered}
\frac{\partial\xi^a}{\partial x_b}+\frac{\partial\xi^b}{\partial x_a}=0, \qquad a\neq b, \quad a,b=1,2,3,\\
\frac{\partial\xi^0}{\partial x_a}-\frac{\partial\xi^a}{\partial x_0}=0, \qquad \frac{\partial\xi^0}{\partial x_0}=\frac{\partial\xi^1}{\partial x_1}=\frac{\partial\xi^2}{\partial x_2}=\frac{\partial\xi^3}{\partial x_3}=\frac{\partial\eta}{\partial u},\\
\frac{\partial\xi^a}{\partial u}+\frac{\partial\eta}{\partial x_a}=0, \qquad \frac{\partial\xi^0}{\partial u}-\frac{\partial\eta}{\partial x_0}=0,
\end{gathered} \tag{6}$$

which can be integrated in a straightforward manner to obtain the infinitesimal transformations ξ^μ and η and then, from the formula

$$Q=\xi^\mu(x,u)\frac{\partial}{\partial x_\mu}+\eta(x,u)\frac{\partial}{\partial u}, \tag{7}$$

the vector fields (4).

It is emphasized that because u is a dependent variable, the nonlinear representation of the conformal group $C_{1,4}$ is realized on the manifold of the solutions of eq. (2). Let us remind that the Klein–Gordon equation with $m\neq 0$ is invariant under the Poincaré group $A_{1,3}\subset C_{1,3}\subset C_{1,4}$ only; the massless Klein–Gordon equation is invariant under the conformal group $C_{1,3}$. In both cases one has usual (linear) group representations as contrasted with the case of eq. (2).

The finite group transformations

Below we present the finite transformations generated by the operators (4), which can be obtained by direct integration of corresponding Lie equations:

$$\begin{aligned}
P_\mu: &\quad x'_\mu=x_\mu+a_\mu, \qquad u'(x')=u(x), \qquad \mu=0,1,2,3,\\
P_4: &\quad x'_\mu=x_\mu, \qquad u'(x')=u(x)+a_4,
\end{aligned} \tag{8}$$

$$\begin{aligned}
J_{ab}: &\quad x'_0=x_0, \qquad u'(x')=u(x),\\
&\quad \boldsymbol{x}'=\boldsymbol{x}\cos\alpha+\frac{\boldsymbol{x}\times\boldsymbol{\alpha}}{\alpha}\sin\alpha+\frac{\boldsymbol{\alpha}\cdot(\boldsymbol{\alpha}\cdot\boldsymbol{x})}{\alpha^2}(1-\cos\alpha),\\
&\quad \boldsymbol{x}\equiv(x_1,x_2,x_3), \qquad \boldsymbol{\alpha}\equiv(\alpha_1,\alpha_2,\alpha_3), \qquad \alpha\equiv\left(\alpha_1^2+\alpha_2^2+\alpha_3^2\right)^{1/2},
\end{aligned} \tag{9}$$

$$\begin{aligned}
J_{a4}: &\quad x'_0=x_0, \qquad x'_a=x_a\cos\beta_a-u(x)\sin\beta_a,\\
&\quad x'_b=x_b, \qquad x'_c=x_c, \qquad a\neq b\neq c, \quad a,b,c=1,2,3,\\
&\quad u'(x')=u(x)\cos\beta_a+x_a\sin\beta_a,
\end{aligned} \tag{10}$$

$$\begin{aligned}
J_{0a}: &\quad x'_0=x_0\cosh\theta_a+x_a\sinh\theta_a, \qquad x'_a=x_a\cosh\theta_a+x_0\sinh\theta_a,\\
&\quad x'_b=x_b, \qquad x'_c=x_c, \qquad a\neq b\neq c, \quad a,b,c=1,2,3,\\
&\quad u'(x')=u(x),
\end{aligned} \tag{11}$$

$$J_{04}: \quad \begin{aligned} & x_0' = x_0 \cosh\theta_4 + u(x)\sinh\theta_4, \qquad x_a' = x_a, \quad a = 1,2,3, \\ & u'(x') = u(x)\cosh\theta_4 + x_0 \sinh\theta_4, \end{aligned} \tag{12}$$

$$D: \quad x_\mu' = e^{\varkappa} x_\mu, \qquad u'(x') = e^{\varkappa} u(x), \qquad \mu = 0,1,2,3, \tag{13}$$

$$K_A: \quad \begin{aligned} & x_\mu' = \frac{x_\mu - C_\mu\left(x^2 - u^2(x)\right)}{1 - 2C_\nu x^\nu + 2C_4 u(x) + C^A C_A\left(x^2 - u^2(x)\right)}, \\ & u'(x') = \frac{u(x) - C_4\left(x^2 - u^2(x)\right)}{1 - 2C_\nu x^\nu + 2C_4 u(x) + C^A C_A\left(x^2 - u^2(x)\right)}, \\ & x^2 \equiv x_\mu x^\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2, \\ & C_A C^A \equiv C_0^2 - C_1^2 - C_2^2 - C_3^2 - C_4^2, \quad A = 0,1,2,3,4, \end{aligned} \tag{14}$$

where a_μ, a_4, α_a, β_a, θ_a, θ_4, $\varkappa$, C_A are arbitrary real constants.

Contrary to the usual (linear with respect to the dependent function) transformations hare we have the nonlinear ones. Hence it is *the nonlinear representation* of the conformal group $C_{1,4}$, previously mentioned.

Some exact solutions of the equation

One can make sure by the straightforward calculations that the following functions satisfy eq. (2):

$$u(x) = F(\alpha^\nu x_\nu) + \beta^\nu x_\nu, \qquad \alpha^\nu x_\nu = \alpha^\nu \beta_\nu = 0, \qquad \beta^\nu \beta_\nu = 1, \tag{15}$$

where F is an arbitrary differentiable function;

$$u(x) = \left[(\alpha^\nu x_\nu)^2 + x^\nu x_\nu\right]^{1/2}, \qquad \alpha^\nu \alpha_\nu = -1, \tag{16}$$

$$u(x) = \left[x_0^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{x})^2\right]^{1/2}, \qquad \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = 1, \tag{17}$$

$$u(x) = (x_\nu x^\nu)^{1/2} \equiv \left(x_0^2 - \boldsymbol{x}^2\right)^{1/2}, \tag{18}$$

where α_ν, β_ν are arbitrary real constants satisfying the mentioned conditions.

Equation (2) is invariant under the transformations

$$x \to x' = f(x, u(x), \{\theta\}), \qquad u(x) \to u'(x') = g(x, u(x), \{\theta\}), \tag{19}$$

where $\{\theta\}$ are parameters of transformations; the functions f and g are defined by (8)–(14). It is obvious that if $u(x) = \varphi(x)$ is a solution of eq. (2), then the new solutions can be obtained from the functional equation

$$g(x, u_{\text{new}}(x), \{\theta\}) = \varphi\left(x' = f(x, u_{\text{new}}(x), \{\theta\})\right). \tag{20}$$

For example, the functions

$$\begin{aligned} & u(x) = \frac{-1 \pm [1 + 4A(Ax^\nu x_\nu + \beta^\nu x_\nu)]^{1/2}}{2A}, \\ & \quad A \equiv C_4 - \beta^\nu C_\nu \neq 0, \quad \beta^\nu \beta_\nu = 1, \end{aligned} \tag{21}$$

$$u(x) = \frac{C_4 \pm \left[C_4^2 + C^A C_A (1 - 2C^\nu x_\nu + C^A C_A x_\nu x^\nu)\right]^{1/2}}{C_A C^A}, \qquad (22)$$
$$C_A C^A = C_0^2 - C_1^2 - C_2^2 - C_3^2 - C_4^2 \neq 0$$

are obtained from (16) with $F = 0$ and (18), respectively, by means of eqs. (14) and (20).

Formulae (21), (7) imply that the function

$$u(x) = (2A)^{-1}[1 + 4A(Ax_\nu x^\nu + \beta^\nu x_\nu)]^{1/2}, \qquad (23)$$

where $\beta^\nu \beta_\nu = 1$, $A \neq 0$ and are arbitrary real constants, satisfies eq. 2.

Upon application of (20) and (12) to (23), we have another solution of eq. (2):

$$u(x) = (2A)^{-1}\beta_0 \sinh\theta_4 \pm \left[\left(\frac{\beta_0 \sinh\theta_4}{2A}\right)^2 + \left(\frac{1}{2A}\right)^2 + x_\nu x^\nu + \right.$$
$$\left. + \frac{1}{A}(\beta_0 x_0 \cosh\theta_4 - \boldsymbol{\beta}\cdot\boldsymbol{x})\right], \qquad A \neq 0, \qquad \beta^\nu \beta_\nu = 1. \qquad (24)$$

It is obvious that one can use the rest of finite group transformations to generate more exact solutions of eq. (2).

Remarks

Firstly, it id important to note that what has been said about the symmetry of eq. (2) holds true for the equation

$$\frac{\partial u}{\partial x_0} \pm \left(\frac{\partial u}{\partial x_a}\frac{\partial u}{\partial x_a} + m^2\right)^{1/2} = 0$$

recently proposed [1] as the most natural relativistic generalization of the Hamilton–Jacobi equation.

Secondly, the symmetry group of eq. (1) with $m = 0$ turns out to be an infinite-dimensional one because of the arbitrary dependence of ξ^μ and η from (7) on u. The arbitrary dependence of η on u implies that the arbitrary differentiable function $F(u(x))$ will be a solution of eq. (1) with $m = 0$ as well as $u(x)$. The contrary is the case $m \neq 0$.

1. Fushchych W.I., in Algebraic-theoretical studies in mathematical physics, Kiev, 1981, 6–28 (in Russian).
2. Ovsyannikov L.V., The group analysis of differential equation, Moscow, 1978 (in Russian).