On some exact solutions of the nonlinear Schrödinger equation in three spatial dimensions

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1. In 1881 Lie introduced the study of the solutions of partial differential equations, based, on the infinitesimal transformations of the continuous groups. Afterwards these methods have been used for finding exact and approximate solutions of the nonlinear partial differential equations by various authors [1]. In the main these solutions were obtained in one spatial dimension. In this paper some exact similarity solutions of nonlinear parabolic partial differential equations, possessing high symmetry, are obtained, in three spatial dimensions.

Consider the nonlinear equation

$$i\frac{\partial u}{\partial x_0}+\frac{1}{2M}\Delta u=F(u), \tag{1}$$

where

$$u=u(x_0,\boldsymbol{x}),\qquad \boldsymbol{x}\in\mathbf{R}^n,\qquad M=\text{const},\qquad \Delta\equiv\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}.$$

Lemma 1.* *Equation (1) is invariant under a* $(n^2+3n+8)/2$*-parameter Schrödinger group if*

$$F(u)=\lambda u|u|^{4/n}, \tag{2}$$

where λ *is an arbitrary constant,* n *is a number of the spatial variables in eq. (1).*

Lemma 2.* *Equation (1) is invariant under a* $(n^2+3n+6)/2$*-parameter transformation group if*

$$F(u)=\lambda u|u|^{m}, \tag{3}$$

where λ, m *are arbitrary constants.*

This group consists of the Galilean group and the one-parameter group of scale transformations.

Lemma 1 and lemma 2 can be proved by using the finite or infinitesimal transformations of the Schrödinger group [2]. By means of the Lie–Ovsjannikov method [3] one can show that the above-mentioned groups are the maximal ones in the sense of Lie which leave eq. (1) with the nonlinearities (2) and (3) invariant.

In the sequel we restrict ourselves to $\mathbf{R}^3$ and consider the infinitesimal transformations of the Schrödinger group

$$x_0'=x_0+\varepsilon A_0+O\left(\varepsilon^2\right),\qquad x_i'=x_i+\varepsilon A_i+O\left(\varepsilon^2\right),$$

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* Misprints in formulations of Lemmas 1 and 2 are corrected.

where

$$A_0 = -cx_0^2 + 2qx_0 + b, \qquad A_i = (-cx_0 + q)x_i + \varepsilon_{ijk}r_jx_k + v_ix_0 + a_i.$$

The parameters a_i, b, c, q, v_i, r_i are arbitrary real constants. The parameter a_i, represents spatial translations, b represents time translations, c represents invariance under the one-parameter group of projective transformations, q represents dilatations, v_i signifies Galilean invariance and r_i denotes rotation invariance.

2. We need the invariants ω^1, ω^2, ω^3 of the Schrödinger group for finding the solutions of eq. (1). These invariants are obtained by solving the Lagrange equations

$$\frac{dx_0}{A_0} = \frac{dx_1}{A_1} = \frac{dx_2}{A_2} = \frac{dx_3}{A_3}.$$

We will give the explicit form of these invariants.

Case I. $cb + q^2 \neq 0$:

$$\omega^1 = \sqrt{\frac{z_iz_i}{-cx_0^2 + 2qx_0 + b}}, \qquad \omega^3 = \frac{r_iz_i}{\sqrt{r_lr_l\left(-cx_0^2 + 2qx_0 + b\right)}},$$
$$\omega^2 = \left|\arcsin\frac{z_2\Big/\sqrt{-cx_0^2 + 2qx_0 + b} - r_2\omega^3\Big/\sqrt{r_lr_l}}{\sqrt{\left(\left(r_1^2 + r_3^2\right)/r_lr_l\right)\left[(\omega^1)^2 - (\omega^3)^2\right]}}\right| - \sqrt{r_lr_l}\int^{x_0}\frac{dt}{-ct^2 + 2qt + b}, \tag{4}$$

where $z_i = x_i - \alpha_ix_0 - \beta_i$,

$$\alpha_i = \frac{qv_i + ca_i - \varepsilon_{ijk}v_jr_k}{bc + q^2 + r_lr_l} + r_i\frac{c(r_ka_k) + q(r_kr_k)}{\left(bc + q^2\right)\left(bc + q^2 + r_lr_l\right)},$$
$$\beta_i = \frac{bv_i - qa_i + \varepsilon_{ijk}r_ja_k}{bc + q^2 + r_lr_l} + r_i\frac{r_k(bv_k - qa_k)}{\left(bc + q^2\right)\left(bc + q^2 + r_lr_l\right)}.$$

Here and in the sequel the summation convention is being employed.

Case II. $cb + q^2 = 0$, $c \neq 0$:

1) $r_3 \neq 0$:

$$\omega^1 = \frac{\sqrt{y_iy_i}}{\tau}, \qquad \omega^3 = \frac{r_iy_i}{\tau\sqrt{r_lr_l}},$$
$$\omega^2 = \left|\arcsin\frac{y_2/\tau - r_2\omega^3/\sqrt{r_lr_l}}{\left(\left(r_1^2 + r_3^2\right)/r_lr_l\right)\left[(\omega^1)^2 - (\omega^3)^2\right]}\right| + \frac{\sqrt{r_lr_l}}{\tau}, \tag{5}$$

where

$$y_i = x_i - \gamma_i\tau - \delta_i + \frac{\eta_i}{\tau}, \qquad \tau = -cx_0 + q,$$
$$\gamma_i = \frac{1}{c(r_lr_l)}\left[\varepsilon_{ijk}v_jr_k - (v_iq + ca_i) + r_ir_k(v_kq + ca_k)\right],$$
$$\delta_i = \frac{1}{c(r_lr_l)}\left[\varepsilon_{ijk}r_j(qv_k + ca_k) + r_i(r_kv_k)\right], \qquad \eta_i = \frac{1}{2}\frac{r_ir_k}{r_lr_l}\left(\frac{v_kq}{c} + a_k\right).$$

2) $r_1 = r_2 = r_3 = 0$:

$$\omega^i = \frac{1}{\tau}\left[x_i - \frac{v_i}{c} + \frac{1}{\tau}\left(\frac{v_iq}{c} + a_i\right)\right]. \tag{6}$$

Case III. $c = 0,\ q = 0,\ b \neq 0$:
1) $r_3 \neq 0$:

$$\begin{gathered} \omega^1 = \sqrt{S_i S_i}, \qquad \omega^3 = \frac{r_i S_i}{\tau\sqrt{r_l r_l}}, \\ \omega^2 = \left| \arcsin \frac{S_2 - (r_2\omega^3)/\sqrt{r_l r_l}}{\sqrt{\left(\left(r_1^2 + r_3^2\right)/r_l r_l\right)\left[(\omega^1)^2 - (\omega^3)^2\right]}} \right| + \frac{\sqrt{r_l r_l}}{b} x_0, \end{gathered} \tag{7}$$

where

$$S_i = x_i - \frac{x_0^2}{2b r_l r_l} r_i (r_k v_k) - \frac{x_0}{b(r_l r_l)}[r_i(a_k r_k) + b\varepsilon_{ijk} r_j v_k] - \\ - \frac{1}{r_l r_l}\{\varepsilon_{ijk} r_j a_k + b[v_i - (z_k v_k)\tau_i]\}.$$

2) $r_1 = r_2 = r_3 = 0$:

$$\omega^i = x_i - \frac{v_i}{2b} x_0^2 - \frac{a_i}{b} x_0. \tag{8}$$

Case IV. $c = 0,\ q = 0,\ b = 0,\ v_i x_0 + a_i \neq 0$:
1) $r_3 \neq 0,\ r_2 \neq 0,\ r_1 \neq 0$:

$$\omega^1 = x_0, \qquad \omega^2 = \sqrt{w_i w_i}, \qquad \omega^3 = \frac{r_i w_i}{\sqrt{r_l r_l}}, \tag{9}$$

where

$$w_i = x_i + \frac{\varepsilon_{ijk} r_j (v_k x_0 + a_k)}{(r_i r_l/(v_i x_0 + a_i))(v_l x_0 - a_l) - r_k r_k}$$

(there is no summation over i).
2) $r_3 \neq 0,\ r_2 = 0,\ r_1 = 0$:

$$\begin{gathered} \omega^1 = x_0, \qquad \omega^2 = x_1^2 + x_2^2 + \frac{2}{r_3}[(v_2 x_0 + a_2)x_1 - (v_1 x_0 + a_1)x_2], \\ \omega^3 = (v_3 x_0 + a_3) \arcsin \left| \frac{r_3 x_1 + v_2 x_0 + a_2}{\sqrt{(v_1 x_0 + a_1)^2 + (v_2 x_0 + a_2) + r_3^2 \omega^2}} \right| - x_3. \end{gathered} \tag{10}$$

3) $r_1 = r_2 = r_3 = 0$:

$$\begin{gathered} \omega^1 = x_0, \qquad \omega^2 = x_1(v_2 x_0 + a_2) - x_2(v_1 x_0 + a_1), \\ \omega^3 = x_1(v_3 x_0 + a_3) - x_3(v_1 x_0 + a_1). \end{gathered} \tag{11}$$

Case V. $c = q = b = 0,\ v_i x_0 + a_i = 0,\ r_i \neq 0$:

$$\omega^1 = x_0, \qquad \omega^2 = \sqrt{x_i x_i}, \qquad \omega^3 = \frac{r_i x_i}{\sqrt{r_l r_l}}. \tag{12}$$

Now we construct the solutions of the eq. (1) of the form

$$u = \varphi\left(\omega^1, \omega^2, \omega^3\right) f(x_0, \boldsymbol{x}), \qquad F(u) = \lambda u |u|^m. \tag{13}$$

Substituting (13) into eq. (1), we require the functions φ and f to satisfy the non-coupled equations. In accordance with this requirement the following equations are obtained:

$$\frac{1}{2M}\varphi_{\omega^l\omega^k}\psi^{lk}(\omega^1,\omega^2,\omega^3)+\varphi_{\omega^k}\psi^k(\omega^1,\omega^2,\omega^3)+\varphi\psi(\omega^1,\omega^2,\omega^3)=\lambda\varphi|\varphi|^m, \quad (14)$$

$$i\frac{\partial f}{\partial x_0}+\frac{1}{2M}\Delta f=f|f|^m\psi(\omega^1,\omega^2,\omega^3), \quad (15)$$

$$\omega^l_{x_i}\omega^k_{x_i}=|f|^m\psi^{lk}(\omega^1,\omega^2,\omega^3), \quad (16)$$

$$\left(i\omega^k_{x_0}+\frac{1}{2M}\Delta\omega^k\right)f+\frac{1}{M}\omega^k_{x_i}f_{x_i}=f|f|^m\psi^k(\omega^1,\omega^2,\omega^3), \quad (17)$$

where

$$\varphi_{\omega^k}\equiv\frac{\partial\varphi}{\partial\omega^k},\qquad \varphi_{\omega^k\omega^l}\equiv\frac{\partial^2}{\partial\omega^k\partial\omega^l},\qquad \omega^k_{x_0}\equiv\frac{\partial\omega^k}{x_0}.$$

In fact, the nonlinear equation (15) with the additional conditions (16) and (17) is the inhomogeneous linear Schrödinger equation which one can easily integrate. Substituting the solution of eq. (15) into eq. (14), we obtain for φ a nonlinear partial differential equation. Thus the solution of eq. (14) is a function of only three variables ω^1, ω^2, ω^3.

3. Let us consider some exact solutions of eqs. (14)–(17).

For the invariants (4) and the nonlinearity $F=\lambda u|u|^{4/3}$, the functions f and φ take the form ($c\neq 0$)

$$f=\left(-cx_0^2+2qx_0+b\right)^{-3/4}\left[\frac{-cx_0+q-\sqrt{bc+q^2}}{-cx_0+q+\sqrt{bc+q^2}}\right]^{i\rho}\times$$
$$\times\exp\left[iM(\alpha_i z_i)+\frac{iM}{2}\frac{-cx_0+q-\sqrt{cb+q^2}}{-cx_0^2+2qx_0+b}z_iz_i+\frac{iM}{2}(\alpha_i\alpha_i)x_0\right],$$
$$\varphi=\left[\frac{1}{2\lambda M}\left(\frac{9}{4}-B^2\right)\right]^{3/4}\left[(\omega^1)^2-(\omega^3)^2\right]^{-3/4}\exp\left[iB\omega^2\right],$$

where

$$B=\frac{2\sqrt{bc+q^2}}{\sqrt{r_lr_l}}\rho,\qquad \rho=\frac{M}{4\sqrt{cb+q^2}}[2(v_i\beta_i)+b(\alpha_i\alpha_i)-c(\beta_i\beta_i)].$$

For the invariants (4) and the nonlinearity $F=\lambda\varphi|\varphi|^m$ the functions f and φ take the form ($c=0$, $q\neq 0$)

$$f=(2qx_0+b)^{-1/m+i\rho}\exp\left\{iM(\alpha_iz_i)+\frac{iM}{2}(\alpha_i\alpha_i)x_0\right\},$$
$$\varphi=\left[\frac{1}{2\lambda M}\left(\frac{4}{m^2}-B^2\right)\right]^{1/m}\left[(\omega^1)^2-(\omega^3)^2\right]^{-1/m}\exp\left[iB\omega^2\right],$$

where

$$B = \frac{2q}{\sqrt{r_l r_l}}\rho, \qquad \rho = \frac{M}{4q}[2(v_i\beta_i) + b(\alpha_i\alpha_i)].$$

For the invariants (12) and the nonlinearity $F = \lambda\varphi|\varphi|^m$ the functions f and φ take the form $u_1 = f\varphi_1$, $u_2 = f\varphi_2$:

$$f = \exp[ic_1], \qquad \varphi_1 = \left[\frac{c_2}{2\lambda M}\left(\frac{2}{m}\right)^2\right]^{1/m}\left[x_i x_i - \frac{(r_i x_i)^2}{r_l r_l}\right]^{-1/m},$$

$$\varphi_2 = c_3 x_0^{-1}\exp\left[i\frac{\lambda c_3^m}{m-1}x_0^{-m+1} + i\frac{m}{2x_0}\left[x_i x_i - \frac{(r_i x_i)^2}{r_l r_l}\right]\right],$$

where $c_1 = \mathrm{const}$, $c_2 = \mathrm{const}$, $c_3 = \mathrm{const}$.

Remark. We have considered only a part of the exact solutions obtained by the same method for the invariants (4)–(12) and the nonlinearities (2) and (3). Three-spatial-dimension exact solutions of Liouville, eikonal, Hamilton–Jacobi and Navier–Stokes equations are obtained by this method too [4, 5].

1. Ames W.F., Nonlinear Partial Differential Equations in Engineering, Vol. I and II, New York, N. Y., 1965, 1972.
2. Niederer V., *Helv. Phys. Acta*, 1972, **45**, 802.
3. Ovsjannikov L.V., The Group Analysis of Differential Equations, Moscow, 1978 (in Russian).
4. Fushchych W.I., Serov N.I., *Ukr. Math. J.*, 1981, **33**, № 6, 780–784 (in Russian).
5. Fushchych W.I. (Editor), Algebraic-Theoretical Investigations in Mathematical Physics, Kiev, 1981 (in Russian).