Reduction of the representations of the generalised Poincaré algebra by the Galilei algebra

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The realisations of all classes of unitary irreducible representations of the generalised Poincaré group $P(1,4)$ have been found in a basis in which the Casimir operators of its important subgroup, i.e. the Galilei group, are of diagonal form. The exact form of the unitary operator which connects the canonical basis of the $P(1,4)$ group and the Galilei basis has been established.

1. Introduction

Some years ago it was proposed to use the generalised Poincaré group $P(1,4)$ the group of displacements and rotations in five-dimensional Minkovsky space, for the description of particles with variable masses and spins (Fushchych and Krivsky [9, 10], Fushchych [8]). This and other generalised groups $P(1,n)$, $P(2,3)$ etc were considered and used successively by Castell [4], Aghassi et al [1], Barrabes and Henry [3], Elizalde and Gomish [5] and many others.

The main property of the $P(1,4)$ group is that it contains the Poincaré group $P(1,3)$ as well as the Galilei group $G(3)$ as its subgroups[1]. So the $P(1,4)$ group unified the groups of motion of relativistic and non-relativistic quantum mechanics.

For the elucidation of the physical grounds of the generalised quantum mechanics based on the $P(1,4)$ group (Fushchych and Krivsky [9, 10, 11] the important problem is the reduction of the irreducible representations IR of the $P(1,4)$ group, or the Lie algebra of the $P(1,4)$ group, by the IR of its subgroups, or its subalgebras[2]. The problem of the reduction of IR of the $P(1,4)$ algebra corresponding to the time-like five-momenta by its subalgebra $P(1,3)$ has been solved (Fushchich et al [12], Nikitin et al [15]), i.e. the type of representations of the $P(1,3)$ algebra contained in the IR of the $P(1,4)$ algebra has been investigated and the unitary operator was found which connects the canonical basis of the $P(1,4)$ group representation with the $P(1,3)$ basis, in which the Casimir operators of the Poincaré group have the diagonal form (the spectrum of these operators is nondegenerate).

In this paper we find the realisation of the IR of the $P(1,4)$ algebra in the "Galilei basis" namely, in the basis in which the invariant operators of the Galilei subalgebra are diagonal ones. We also obtain the explicit form of the unitary operator, which carries out the reduction $P(1,3) \to G(2)$ which plays an important role in the null-plane approach (see e.g. Leutwyler and Stern [13]).

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[1]The paper of Fedorchuck [6] is devoted to the classification and the description of all subgroups of the $P(1,4)$ group.

[2]We will indicate the groups and the corresponding Lie algebras by the same indices.

2. Statement of the problem

The Lie algebra of the $P(1,4)$ group is specified by the fifteen generators P_μ, $J_{\mu\nu}$ $(\mu,\nu=0,1,2,3,4)$ which satisfy the commutation relations

$$
\begin{aligned}
&[P_\mu,P_\nu]=0, \qquad [P_\mu,J_{\nu\sigma}]=i(g_{\mu\nu}P_\sigma-g_{\mu\sigma}P_\nu),\\
&[J_{\mu\nu},J_{\rho\sigma}]=i(g_{\mu\sigma}J_{\nu\rho}+g_{\nu\rho}J_{\mu\sigma}-g_{\mu\rho}J_{\nu\sigma}-g_{\nu\sigma}J_{\mu\rho}).
\end{aligned}
\tag{2.1}
$$

The algebra (2.1) has three main invariant (Casimir) operators (Fushchych and Krivsky [9, 10])

$$
P^2=P_\mu P^\mu=P_0^2-\boldsymbol{P}^2-P_4^2, \qquad V_1=\frac{1}{2}\omega_{\mu\nu}\omega_{\mu\nu}, \qquad V_2=-\frac{1}{4}J_{\mu\nu}\omega_{\mu\nu}, \tag{2.2}
$$

where

$$
\omega_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma\lambda}J^{\rho\sigma}P^\lambda.
$$

As in the case of the Poincaré group, one can specify four different classes of the representations of the algebra (2.1), corresponding to $P^2>0$, $P^2=0$, $P^2<0$ and $P_\mu\equiv 0$ (in the last case one arrives at the representations of the homogeneous group $SO(1,4)$, which are not considered here).

Algebra (2.1) contains the Lie algebras of the Poincaré and of the Galilei groups as subalgebras. In order to select the subalgebra $P(1,3)$ it is enough to consider the relations (2.1) for $\mu,\nu\neq 4$. The subalgebra $G(3)$ may be obtained by the transition to the new basis

$$
\begin{aligned}
&\hat P_0=\frac{1}{2}(P_0-P_4), \qquad M=P_0+P_4, \qquad \hat P_a=P_a, \qquad K=J_{04},\\
&J_a=\frac{1}{2}\varepsilon_{abc}J_{bc}, \qquad G_a^+=J_{0a}+J_{4a}, \qquad G_a^-=\frac{1}{2}(J_{0a}-J_{4a}).
\end{aligned}
\tag{2.3}
$$

The operators (2.3) satisfy the commutation relations

$$
\begin{aligned}
&[\hat P_0,\hat P_a]=[\hat P_0,M]=[\hat P_a,M]=[\hat P_a,\hat P_b]=0,\\
&[\hat P_0,J_a]=[M,J_a]=[G_a^+,G_b^+]=[M,G_a^+]=0,\\
&[\hat P_a,J_b]=i\varepsilon_{abc}\hat P_c, \qquad [G_a^+,\hat P_b]=i\delta_{ab}M,\\
&[J_a,J_b]=i\varepsilon_{abc}J_c, \qquad [\hat P_0,G_b^+]=i\hat P_b,
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
&[\hat P_0,G_a^-]=[G_a^-,G_b^-]=0, \qquad [G_a^-,M]=-i\hat P_a, \qquad [G_a^-,J_b]=i\varepsilon_{abc}G_c^-,\\
&[G_a^-,\hat P_b]=-i\delta_{ab}\hat P_0, \qquad [G_a^+,G_b^-]=i(\varepsilon_{abc}J_c+\delta_{ab}K), \qquad [\hat P_0,K]=-i\hat P_0,\\
&[\hat P_a,K]=[J_a,K]=0, \qquad [M,K]=iM, \qquad [G_a^\pm,K]=\pm G_a^\pm.
\end{aligned}
\tag{2.5}
$$

The commutation relations (2.4) specify the Lie algebra of the extended Galilei group (Bargman [2]). The invariant operators of this algebra are given by the formulae

$$
C_1=2M\hat P_0-\boldsymbol{P}^2, \qquad C_2=(M\boldsymbol{J}-\hat{\boldsymbol{P}}\times\boldsymbol{G}^+)^2, \qquad C_3=M. \tag{2.6}
$$

Our aim is to find the realisations of the generators (2.3) for any class of IR of the $P(1,4)$ algebra, in a basis where the Casimir operators (2.6) have a diagonal form. This enables us to answer the question what IR of the $G(3)$ algebra are contained in the given representation of the $P(1,4)$ algebra and to establish the connection between the vectors in the Poincaré and in the Galilei bases.

The realisations of all IR of the $P(1,4)$ algebra have already been found (Fushchych and Krivsky [9, 10, 11], Fushchych [8]). So the problem of the description of the IR of the $P(1,4)$ algebra in the Galilei basis reduces to transforming the known realisation to a form in which the operators (2.6) are diagonal.

3. The representations with $P^2 \geq 0$

Let us consider the IR of the $P(1,4)$ algebra, which corresponds to the positive values of the invariant operator $P^2 = \varkappa^2 > 0$. The generators P_μ, $J_{\mu\nu}$ in the canonical basis $|p_k, j_3, \tau_3; \varepsilon, j, \tau, \varkappa\rangle$ have the form (Fushchych and Krivsky [9, 10])

$$
\begin{aligned}
&P_0 = \varepsilon E \equiv \varepsilon\left(\boldsymbol{p}^2 + p_4^2 + \varkappa^2\right)^{1/2}, \qquad P_k = p_k, \\
&J_{kl} = i\left(p_l \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial p_l}\right) + S_{kl}, \qquad k, l = 1,2,3,4, \\
&J_{0k} = -i\varepsilon E \frac{\partial}{\partial p_k} - \varepsilon \frac{S_{kl} p_l}{E + \varkappa}, \qquad \varepsilon = \pm 1,
\end{aligned}
\tag{3.1}
$$

where S_{kl} $(k, l = 1,2,3,4)$ are the generators of the IR $D(j,\tau)$ of the $SO(4)$ group.

The basis of the realisation (3.1) is formed by the vectors $|p_k, j_3, \tau_3; \varepsilon, \tau, \varkappa\rangle$, which are the eigenfunctions of the complete set of the commuting operators

$$
T = P_k, \qquad J_3 = \frac{1}{2}(\omega_{12} + \omega_{43}), \qquad T_3 = \frac{1}{2}(\omega_{12} - \omega_{43}), \qquad \hat{\varepsilon} = P_0/|P_0|,
$$
$$
J^2 = \frac{1}{4\varkappa^2}(V_1 + 2\varepsilon\varkappa V_2), \qquad T^2 = \frac{1}{4\varkappa^2}(V_1 - 2\varepsilon\varkappa V_2), \qquad P^2,
$$

with the eigenvalues p_k, j_3, τ_3, ε, $j(j+1)$, $\tau(\tau+1)$ and $\varkappa^2$ correspondingly, where j and τ are the integers or half-integers labelling the IR of the $SO(4)$ group,

$$
j_3 = -j, -j+1, \ldots, j, \quad \tau_3 = -\tau, -\tau+1, \ldots, \tau, \quad \varepsilon = \pm 1, \quad -\infty < p_k < \infty.
$$

The basis vectors may be normalised according to

$$
\langle p_k, j_3, \tau_3; \varepsilon, j, \tau, \varkappa \mid p'_k, j'_3, \tau'_3; \varepsilon, j, \tau, \varkappa\rangle = 2E\delta(p_k - p'_k)\delta_{j_3 j'_3}\delta_{\tau_3\tau'_3},
$$

and the generators (3.1) are Hermitian with respect to the scalar product

$$
(\Psi_1, \Psi_2) = \int \left(d^4p/E\right) \Psi_1^\dagger(p_k, j_3, \tau_3)\Psi_2(p_k, j_3, \tau_3). \tag{3.2}
$$

The basis of the IR of the $P(1,4)$ algebra, in which the invariant operators (2.6) of the $G(3)$ algebra and the operators P_a $(a = 1,2,3)$ and $S_3 = J_3 - (1/m)(P_2 G_1^+ - P_1 G_2^+)$ have the diagonal form, will be called "Galilei basis" (or "$G(3)$ basis") and denoted by $|p_a, m, s, s_3; \varepsilon, j, \tau, \varkappa\rangle$.

We will normalise the basis vectors as

$$
\langle p_a, m, s, s_3; \varepsilon, j, \tau, \varkappa \mid p'_a, m', s', s'_3; \varepsilon, j, \tau, \varkappa\rangle = 2m\delta(m - m')\delta(p_a - p'_a)\delta_{ss'}\delta_{s_3 s'_3}.
$$

This will lead us to the scalar product

$$
(\phi_1, \phi_2) = \sum_{|j-\tau| \leq s \leq j+\tau} \int_{\varkappa}^{\infty} \frac{dm}{m} \int d^3p\, \phi_1^\dagger(s, s_3, m, \boldsymbol{p})\phi_2(s, s_3, m, \boldsymbol{p}). \tag{3.3}
$$

Our task is to establish the explicit form of the generators of the $P(1,4)$ group in the Galilei basis and to find the transition operator, which connects the canonical and Galilei bases. First we substitute (3.1) into (2.3) and (2.6) and obtain the Galilei generators $\hat{P}_\mu$, J_a, G_a^+ the invariant operators C_a and the remaining generators G_a^-, K in the canonical basis in a form

$$\begin{gathered}\hat{P}_0=\frac{1}{2}(\varepsilon E-p_4),\qquad M=\varepsilon E+p_4,\qquad J_a=-i(\boldsymbol{p}\times(\partial/\partial\boldsymbol{p}))_a+S_a,\\ G_a^+=x_4p_a-Mx_a-\frac{\varepsilon S_{ab}p_b-S_{4a}(E+\varkappa+\varepsilon p_4)}{E+\varkappa},\end{gathered}\tag{3.4}$$

$$\begin{gathered}C_1=\varkappa^2,\qquad C_3=M,\\ C_2=\Big\{\boldsymbol{S}^2\left[M(E+\varkappa)-\varepsilon\boldsymbol{p}^2\right]^2+\left[\boldsymbol{p}^2\boldsymbol{N}^2-(\boldsymbol{p}\cdot\boldsymbol{N})^2\right]\times\\ \times(E+\varkappa+\varepsilon p_4)^2+(\boldsymbol{p}\cdot\boldsymbol{S})^2\left[2\varepsilon M(E+\varkappa)-\boldsymbol{p}^2\right]\Big\}(E+\varkappa)^{-2},\end{gathered}\tag{3.5}$$

$$\begin{gathered}G_a^-=\frac{1}{2}\left[-x_4p_a-2\hat{P}_0x_a-\frac{\varepsilon S_{ab}p_b-S_{4a}(E+\varkappa-\varepsilon p_4)}{E+\varkappa}\right],\\ K=-\hat{P}_0x_4-\varepsilon\frac{S_{4a}p_a}{E+\varkappa},\end{gathered}\tag{3.6}$$

where

$$S_a=\frac{1}{2}\varepsilon_{abc}S_{bc},\qquad N_a=S_{4a},\qquad x_k=i(\partial/\partial p_k).\tag{3.7}$$

The Casimir operator C_2 (3.5) is in general the matrix which has elements depending on p_k. Our second step is to diagonalise this matrix with the help of some unitary transformation. We will look for the diagonalising operator in a form

$$U_1=\exp(iS_{4a}p_a\theta/p),\tag{3.8}$$

where $p=\left(p_1^2+p_2^2+p_3^2\right)^{1/2}$ and θ is an unknown function of p, p_4.

With the help of the operator (3.8) one may derive from (3.4) and (3.6) a new realisation:

$$\begin{gathered}\hat{P}_0'=U_1\hat{P}_0U_1^\dagger=\hat{P}_0,\qquad \hat{P}_a'=U_1\hat{P}_aU_1^\dagger=\hat{P}_a,\\ J_a'=U_1J_aU_1^\dagger=J_a,\qquad M'=U_1MU_1^\dagger=M,\end{gathered}\tag{3.9}$$

$$(G_a^+)'=U_1G_a^+U_1^\dagger=x_4'p_a-x_a'M-\frac{\varepsilon S_{ab}'p_b-S_{4a}'(E+\varkappa+\varepsilon p_4)}{E+\varkappa},\tag{3.10}$$

$$\begin{gathered}(G_a^-)'=U_1G_a^-U_1^\dagger=\frac{1}{2}\left(-x_4'p_a-2\hat{P}_0x_a'-\frac{\varepsilon S_{ab}'p_b-S_{4a}'(E+\varkappa-\varepsilon p_4)}{E+\varkappa}\right),\\ K'=U_1kU_1^\dagger=-\hat{P}_0x_4'-\varepsilon S_{4a}'p_a/(E+\varkappa),\end{gathered}\tag{3.11}$$

where

$$x_k'=U_1x_kU_1^\dagger,\qquad S_{kl}'=U_1S_{kl}U_1^\dagger.$$

Using the Hausdorf–Campbell formula

$$\exp(A)B\exp(-A)=\sum_{n=0}^{\infty}\frac{1}{n!}\{A,B\}^n,$$

$$\{A,B\}^n=[A,\{A,B\}^{n-1}],\qquad \{A,B\}^0=B$$

it is not difficult to calculate

$$\begin{aligned}
x'_a&=x_a+\left(p_aS_{4b}p_b/p^2\right)[\partial\theta/\partial p-(\sin\theta)/p]+\\
&\quad+\left(S_{ab}p_b/p^2\right)(1-\cos\theta)+(1/p)S_{4a}\sin\theta,\\
S'_{4a}&=S_{4a}\cos\theta+\left(p_aS_{4b}p_b/p^2\right)(1-\cos\theta)+S_{ab}p_b(\sin\theta)/p,\\
S'_{ab}p_b&=S_{ab}p_b\cos\theta+[(p_aS_{4b}p_b/p)-pS_{4a}]\sin\theta,\\
x'_4&=x_4+(S_{4b}p_b/p)(\partial\theta/\partial p_4).
\end{aligned}\tag{3.12}$$

Substituting (3.12) into (3.10), one obtains

$$\begin{aligned}
(G_a^+)'&=x_4p_a-Mx_a+\frac{p_aS_{4b}p_b}{p}\left[\frac{\partial\theta}{\partial p_4}-\frac{M}{p}\left(\frac{\partial\theta}{\partial p}-\frac{1}{p}\sin\theta\right)-\frac{\varepsilon}{E+\varkappa}\sin\theta+\right.\\
&\quad\left.+\frac{E+\varkappa+\varepsilon p_4}{(E+\varkappa)p}(1-\cos\theta)\right]+\frac{S_{ab}p_b}{p}\left[\left(\frac{M}{p}-\frac{\varepsilon p}{E+\varkappa}\right)-\frac{M}{p}+\right.\\
&\quad\left.+\frac{E+\varkappa+\varepsilon p_4}{E+\varkappa}\sin\theta\right]+S_{4a}\left[\left(\frac{\varepsilon p}{E+\varkappa}-\frac{M}{p}\right)\sin\theta+\frac{E+\varkappa+\varepsilon p_4}{E+\varkappa}\cos\theta\right].
\end{aligned}\tag{3.13}$$

The expression (3.13) for G_a^+ is much simplified, if one puts

$$\theta=2\tan^{-1}[p/(E+\varepsilon p_4+\varkappa)].\tag{3.14}$$

For such a value of the parameter θ, we have:

$$\sin\theta=\frac{p(E+\varkappa+\varepsilon p_4)}{(E+\varkappa)(E+\varepsilon p_4)},\qquad 1-\cos\theta=\left[p^2/(E+p)(E+\varepsilon p_4)\right],$$

$$\varepsilon\frac{\partial\theta}{\partial p_4}-\frac{E+\varepsilon p_4}{p}\frac{\partial\theta}{\partial p}=-\sin\theta\frac{E+\varepsilon p_4}{p^2}$$

and

$$(G_a^+)'=x_4p_a-Mx_a.\tag{3.15}$$

Substituting (3.9) and (3.15) into (2.6), we have

$$C'_2=M^2\boldsymbol{S}^2,\tag{3.16}$$

where the matrix $\boldsymbol{S}^2=S_1^2+S_2^2+S_3^2$ always may be chosen in the diagonal form,

$$\boldsymbol{S}^2\phi_s=s(s+1)\phi_s,\qquad |j-\tau|\le s\le j+\tau.$$

The operators (3.9)–(3.11) are defined in a Hilbert space of square integrable functions $\phi(p_1,p_2,p_3,p_4)$. In order to diagonalise the operator M and (3.5) we introduce in place of $\{p_1,p_2,p_3,p_4\}$ the new variables $\{p_1,p_2,p_3,m\}$, where $m=E+\varepsilon p_4$. Then

$$\frac{\partial}{\partial p_4}\to\left(\varepsilon+\frac{p_4}{E}\right)\frac{\partial}{\partial m},\qquad \frac{\partial}{\partial p_a}\to\frac{\partial}{\partial p_a}+\frac{p_a}{E}\frac{\partial}{\partial m}$$

and the operators (3.9)–(3.11) and (3.15) take the form

$$\hat{P}'_0 = m_0 + \varepsilon\frac{p^2}{2m}, \qquad \hat{P}'_a = p_a, \qquad M' = \varepsilon m,$$

$$J'_a = -i(\boldsymbol{p}\times(\partial/\partial\boldsymbol{p}))_a + S_a, \qquad (G_a^+)' = -i\varepsilon m(\partial/\partial p_a), \tag{3.17a}$$

$$C'_1 = \varkappa^2, \qquad C'_2 = m^2\boldsymbol{S}^2, \qquad C'_3 = \varepsilon m, \tag{3.17b}$$

$$\begin{aligned} &K' = -im(\partial/\partial m), \\ &(G_a^-)' = i[\varepsilon p_a(\partial/\partial m) - \hat{P}'_0(\partial/\partial p_a)] - \varepsilon(S_{ab}p_b + S_{4a}\varkappa)/m, \end{aligned} \tag{3.17c}$$

where

$$\varkappa \le m < \infty, \qquad m_0 = \varepsilon\left(\varkappa^2/2m\right).$$

The generators (3.17) are Hermitian with respect to the scalar product (3.3).

So we reach the following result:

Theorem. *The Hilbert space of the IR $D^\varepsilon(\varkappa, j, \tau)$ of the $P(1,4)$ algebra, corresponding to $P^2 = \varkappa^2 > 0$, is expanded into the direct integral of the subspaces, which correspond to the IR of the $G(3)$ algebra with the following values of the invariant operators: $C_1 = \varkappa^2$, $C_2 = m^2 s(s+1)$, $C_3 = \varepsilon m$, $|\varkappa| \le m < \infty$, $|j-\tau| \le s \le j+\tau$. The explicit form of the $P(1,4)$ group generators in the Galilei basis and that of the transition operator, which connects the canonical and the $G(3)$ bases, are given by the formulae (3.8), (3.14) and (3.17).*

To conclude this section we consider the IR of the $P(1,4)$ algebra, corresponding to $P^2 = 0$. The realisations of such an IR have been obtained in the form (Fushchych and Krivsky [9, 10]):

$$P_0 = \varepsilon E_0 \equiv \varepsilon\left(p^2 + p_4^2\right)^{1/2}, \qquad P_a = p_a, \qquad P_4 = p_4,$$

$$J_{0a} = -i\varepsilon E_0\frac{\partial}{\partial p_a} - \varepsilon\frac{S_{ab}p_b}{E_0 + p_4}, \qquad J_{04} = -i\varepsilon E_0\frac{\partial}{\partial p_4},$$

$$J_{4a} = i\left(p_a\frac{\partial}{\partial p_4} - p_4\frac{\partial}{\partial p_a}\right) + \varepsilon\frac{S_{ab}p_b}{E_0 + p_4},$$

where S_{ab} are the generators of the IR $D(s)$ of the $SO(3)$ group. Substituting (3.18) into (2.3), one obtains

$$\hat{P}_0 = \frac{1}{2}(\varepsilon E_0 - p_4), \qquad M = \varepsilon E_0 + p_4, \qquad J_1 = -i\left(\boldsymbol{p}\times\frac{\partial}{\partial\boldsymbol{p}}\right)_a + S_a,$$

$$G_a^+ = i\left(p_a\frac{\partial}{\partial p_4} - p_4\frac{\partial}{\partial p_a}\right) + i\varepsilon E_0\frac{\partial}{\partial p_a}, \qquad K = -i\varepsilon E_0\frac{\partial}{\partial p_4}, \tag{3.18}$$

$$G_a^- = \frac{1}{2}\left(-ip_a\frac{\partial}{\partial p_4} - i\hat{P}_0\frac{\partial}{\partial p_a}\right) - \varepsilon\frac{S_{ab}p_b}{E_0 + \varepsilon p_4}.$$

It is not difficult to see that replacement of the variables $\{\boldsymbol{p}, p_4\} \to \{\boldsymbol{p}, m\}$, where $m = E_0 + \varepsilon p_4$, reduces the generators (3.18) to the form (3.17), where, however, $\varkappa = 0$, $0 \le m < \infty$ and s has the fixed value, which characterises the IR of the $SO(3)$

group. So we have established the explicit form of the generators of the $P(1,4)$ group, corresponding to $P^2=0$, in the Galilei basis.

4. The representations with $P^2 < 0$

We now use the IR of the $P(1,4)$ group, which corresponds to $P^2=-\eta^2<0$. The generators of such representations have been obtained in the form (Fushchych and Krivsky [9, 10, 11])

$$\begin{gathered}
P_0=p_0, \qquad P_a=p_a, \qquad P_4=\varepsilon\left(p_0^2+\eta^2-p_a^2\right)^{1/2},\\
J_{\alpha\beta}=i\left(p_\beta\frac{\partial}{\partial p_\alpha}-p_\alpha\frac{\partial}{\partial p_\beta}\right)+S_{\alpha\beta}, \qquad \varepsilon=\pm1,\\
J_{4\alpha}=-iP_4\frac{\partial}{\partial p_\alpha}-\varepsilon\frac{S_{\alpha\beta}p^\beta}{|P_4|+\eta}, \qquad \alpha,\beta=0,1,2,3,
\end{gathered}\tag{4.1}$$

where $S_{\alpha\beta}$ are the matrices which realise IR of the Lie algebra of the $SO(1,4)$ group.

Reducing the representation (4.1) by the representations of the Lie algebra of the Galilei group, the mass operator $M=P_0+P_4$ may take the zero value. Let us impose the $G(3)$-invariant condition of turning into zero in the hyperspace, corresponding to zero eigenvalues of the operator M, on the functions from the space of the IR (4.1) (this hyperspace is the five-dimensional half-cylinder $p^2=\eta^2$, $\varepsilon p_0<0$).

Using the transformation operator on the generators (4.1)

$$U_2=\exp(iS_{0a}p_a\theta/p), \qquad \theta=2\tanh^{-1}[p/(\eta+|P_4|+\varepsilon p_0)]\tag{4.2}$$

and using the relations

$$\begin{gathered}
U_2x_0U_2^{-1}=x_0+S_{0a}p_a\frac{1}{p}\frac{\partial\theta}{\partial p_0}, \qquad x_\mu=i\frac{\partial}{\partial p_\mu},\\
U_2x_aU_2^{-1}=x_a+\frac{p_a}{p}\frac{S_{0b}p_b}{p}\left(\frac{\partial\theta}{\partial p}-\frac{1}{p}\sinh\theta\right)+\frac{1}{p}S_{0a}\sinh\theta+\frac{S_{ab}p_b}{p^2}(1-\cosh\theta),\\
U_2S_{0a}U_2^{-1}=S_{0a}\cosh\theta-(1/p)S_{ab}p_b\sinh\theta+(p_a/p)(S_{0b}p_b/p)(1-\cosh\theta),\\
U_2S_{ab}p_bU_2^{-1}=S_{ab}p_b\cosh\theta+[(p_aS_{0b}p_b/p)-pS_{0a}]\sinh\theta,\\
\sinh\theta=\frac{p(\varepsilon p_0+|P_4|+\eta)}{(\varepsilon p_0+|P_4|)(|P_4|+\eta)}, \qquad \frac{\partial\theta}{\partial p_0}=\frac{p}{|P_4|(|P_4|+\eta)},\\
1-\cosh\theta=\frac{-p^2}{(|P_4|+\eta)(\varepsilon p_0+|P_4|)}, \qquad \frac{\partial\theta}{\partial p}=\frac{|P_4|(\varepsilon p_0+\eta)+p_0^2+\eta^2}{|P_4|(|P_4|+\eta)(|P_4|+\varepsilon p_0)},
\end{gathered}$$

one comes to the realisation

$$\begin{gathered}
P_0''=p_0, \qquad P_a''=p_a, \qquad P_4''=\varepsilon\left(p_0^2+\eta^2-p^2\right)^{1/2},\\
J_{ab}''=i\left(p_b\frac{\partial}{\partial p_a}-p_a\frac{\partial}{\partial p_b}\right)+S_{ab},\\
J_{0a}''=i\left(p_a\frac{\partial}{\partial p_0}-p_0\frac{\partial}{\partial p_a}\right)-\frac{S_{ab}p_b+S_{a0}\eta}{|P_4''|+\varepsilon p_0},\\
J_{4a}''=-iP_4''\frac{\partial}{\partial p_a}+\frac{S_{ab}p_b+S_{0a}\eta}{|P_4''|+\varepsilon p_0}, \qquad J_{04}''=iP_4''\frac{\partial}{\partial p_0}.
\end{gathered}\tag{4.3}$$

Substituting (4.3) into (2.3) and going from $\{p_a, p_0\}$ to the new variables $\{p_a, m\}$, where $m = p_0 + \left(p_0^2 + \eta^2 - p_a^2\right)^{1/2}$, one obtains the Galilei group generators in the form (3.17a), and the remaining generators G_a^-, K in the form (3.17c), where, however, $m_0 = -\eta^2/2m$, $-\eta^2 < m < 0$, $0 < m < \infty$, and S_{ab} are the generators of the group $SO(3) \subset SO(1,3)$.

5. Covariant representation of the $P(1,4)$ group

Consider an arbitrary covariant representation of the Lie algebra of the $P(1,4)$ group. Such a representation is realised by the operators

$$P_\mu = p_\mu, \qquad J_{\mu\nu} = i\left(p_\nu \frac{\partial}{\partial p_\mu} - p_\mu \frac{\partial}{\partial p_\nu}\right) + S_{\mu\nu}, \tag{5.1}$$

where $S_{\mu\nu}$ are the generators of a representation of the $SO(1,4)$ group. Let us confine ourselves to the case where $P_\mu P^\mu \Psi > 0$.

Substituting (5.1) into (2.3), we obtain

$$\begin{aligned} &\hat{P}_0 = \frac{1}{2}(p_0 - p_4), \qquad \hat{P}_a = p_a, \qquad J_a = -i\left(\boldsymbol{p} \times \frac{\partial}{\partial \boldsymbol{p}}\right)_a + S_a, \\ &M = p_0 + p_4, \qquad G_a^+ = \tilde{x}_0 p_a - x_a M + \lambda_a^+, \\ &G_a^- = \tilde{x}_4 p_a - x_a \hat{P}_0 + \frac{1}{2}\lambda_a^-, \qquad K = \tilde{x}_4 M - \tilde{x}_0 \hat{P}_0 + S_{04}, \end{aligned} \tag{5.2}$$

where

$$\lambda^\pm = S_{0a} \pm S_{4a}, \qquad \tilde{x}_0 = 2i\left(\frac{\partial}{\partial p_0} - \frac{\partial}{\partial p_4}\right), \qquad \tilde{x}_4 = i\left(\frac{\partial}{\partial p_0} + \frac{\partial}{\partial p_4}\right).$$

For the transition of the realisation (5.2) into the Galilei basis we use the operator

$$U_3 = \exp[i\lambda^+ \boldsymbol{p}/M]. \tag{5.3}$$

With the help of the transformation

$$\begin{aligned} &\hat{P}_\mu \to \hat{P}_\mu''' = U_3 \hat{P}_\mu U_3^{-1}, \qquad J_a \to J_a''' = U_3 J_a U_3^{-1}, \\ &G_a^\pm \to (G_a^\pm)''' = U_3 G_a^\pm U_3^{-1}, \qquad K \to K''' = U_3 K U_3^{-1}, \end{aligned}$$

one comes to the realisation in which the invariant operators (2.6) of the $G(3)$ subalgebra are of diagonal form:

$$\begin{aligned} &\hat{P}_0''' = \frac{1}{2}(p_0 - p_4), \qquad \hat{P}_a''' = p_a, \qquad M''' = M = p_0 + p_4, \\ &J_a''' = -i(\boldsymbol{p} \times \partial/\partial \boldsymbol{p})_a + S_a, \qquad G_a^+ = \tilde{x}_0 p_a - x_a M, \\ &G_a^- = \tilde{x}_4 p_a - x_a \hat{P}_0''' - \frac{S_{ab} p_b + S_{40} p_a}{M} + \frac{1}{2}\lambda_a^- - \lambda^+ \frac{p_\mu p^\mu}{M^2}, \\ &K''' = \tilde{x}_4 M - \tilde{x}_0 \hat{P}_0''' + S_{04}, \end{aligned}$$

where $S_a = \frac{1}{2}\varepsilon_{abc} S_{bc}$. The operators C_a (2.6) take the form

$$C_1''' = p_\mu p^\mu, \qquad C_2''' = M^2 \boldsymbol{S}^2, \qquad C_3''' = M$$

i.e. the eigenvalues of the operator C_1 coincide with the values of P^2, the eigenvalues of the operator C_2 are characterised by the spectrum of the Casimir operator of the group $SO(3) \subset SO(1,4)$, and the eigenvalues of the operator C_3 lie in the interval $(C_1''')^{1/2} \leq C_3''' < \infty$.

The results of this section may be used for the diagonalisation of the wave equations, which are invariant under the $P(1,4)$ group. As an example we will consider the five-dimensional generalisation of the Dirac equation

$$(\gamma_\mu p^\mu + \varkappa)\Psi = 0, \qquad \mu = 0, 1, 2, 3, 4. \tag{5.4}$$

On the set of the solutions of the equation (5.4) the generators of the $P(1,4)$ group have the form (5.1) where $S_{\mu\nu} = \frac{1}{4} i[\gamma_\mu, \gamma_\nu]$. Using the operator (5.3) on equation (5.4), one obtains an equation, which is equivalent to (5.4) but is manifestly invariant under the Galilei group

$$\hat{P}_0''' \Phi_+ = \left(\varkappa/2m + p^2/2m\right) \Phi_+, \qquad \Phi_- = 0, \tag{5.5}$$

where

$$\Phi_\pm = \frac{1}{2}(1 \pm \gamma_0 \gamma_4)\Phi, \qquad \Phi = U_3 \Psi, \qquad \varkappa \leq m < \infty.$$

If one imposes the Galilean-invariant subsidiary condition $(p_0 + p_4)\Psi = m_0 \Psi$ and puts $\varkappa = 0$, then equation (5.4) is reduced to the Levi-Leblond equation for the non-relativistic particle of spin $s = \frac{1}{2}$ (Levi-Leblond [14]). In this case (5.3) coincides with the operator which diagonalises the Levi-Leblond equation (Nikitin and Salogub [16]).

6. IR of the Poincaré group in the $G(2)$ basis

The transition of the IR of the $P(1,3)$ group to the basis of a two-dimensional Galilei group $G(2)$ may be made by complete analogy with the reduction $P(1,4) \to G(3)$. Here we consider only the representations of the $P(1,3)$ group, which correspond to time-like four-momenta. The generators of such a representation in a Shirokov–Foldy realisation (Shirokov [17, 18], Foldy [7]) have the form (3.1) where $\mu, \nu = 0, 1, 2, 3$; $k, l = 1, 2, 3$. With the help of the transformation

$$P_\mu \to \tilde{P}_\mu = U P_\mu U^{-1}, \qquad J_{\mu\nu} \to \tilde{J}_{\mu\nu} = U J_{\mu\nu} U^{-1},$$

where

$$\begin{aligned} &U = \exp\left\{(iS_{3\alpha} p_\alpha / |p|) \tan^{-1}[|p|/(|P_0| + \varepsilon p_3 + \varkappa)]\right\}, \\ &|p| = \left(p_1^2 + p_2^2\right)^{1/2}, \qquad \alpha = 1, 2, \end{aligned}$$

and the following replacement of the variables $\{p_1, p_2, p_3\} \to \{p_1, p_2, m\}$, where $m = \varepsilon p_3 + \left(p_1^2 + p_2^2 + \varkappa^2\right)^{1/2}$, one obtains the generators of the Poincaré group in the $G(2)$ basis:

$$\begin{aligned} &\hat{P}_0 = \frac{1}{2}(\tilde{P}_0 + \tilde{P}_3) = \varkappa^2/2m + |p|^2/2m, \qquad \hat{P}_\alpha = p_\alpha, \\ &J_3 = i[p_2(\partial/\partial p_1) - p_1(\partial/\partial p_2)] + S_{12}, \qquad M = \varepsilon m, \end{aligned} \tag{6.1}$$

$$
\begin{aligned}
&G^+_\alpha = \tilde J_{0\alpha} + \tilde J_{3\alpha} = -i\varepsilon m\frac{\partial}{\partial p_\alpha}, \qquad |\varkappa| \le m < \infty,\\
&G^-_\alpha = \frac12(\tilde J_{0\alpha} - \tilde J_{3\alpha}) = i[p_\alpha(\partial/\partial m) - \hat P_0(\partial/\partial p_\alpha)] - \varepsilon(S_{\alpha\beta}p_\beta + S_{3\alpha}\varkappa)/m, \qquad (6.2)\\
&K = \tilde J_{03} = -im(\partial/\partial m).
\end{aligned}
$$

The operators (6.1) coincide with the "kinematical group generators", which are used in the null-plane formalism (see e.g. Leutwyler and Stern [13]).

Using the results of §§ 3–5, it is not difficult to make the transition into the $G(2)$ basis of the representations of the $P(1,3)$ algebra which corresponds to light-like and space-like four-momenta.

7. Connection between the Galilei and the Poincaré bases

We now consider the connection between the realisations of the generators of the $P(1,4)$ group (corresponding to time-like five-momenta) in both the Galilei and Poincaré bases.

The generators of the $P(1,4)$ group in the Poincaré basis (i.e. in the basis where the Casimir operators of the $P(1,3)$ group are of diagonal type) have the form (Fushchych et al [12], Nikitin et al [15])

$$
\begin{aligned}
&P_0 = E = \left(p^2 + \bar m^2\right)^{1/2}, \qquad P_a = p_a, \qquad P_4 = \varepsilon_4\left(\bar m^2 + \varkappa^2\right)^{1/2},\\
&J_{ab} = i[p_b(\partial/\partial p_a) - p_a(\partial/\partial p_b)], \qquad \varepsilon_4 = \pm 1,\\
&J_{0a} = -ip_0(\partial/\partial p_a) - S_{ab}p_b/(E+\bar m), \qquad a,b = 1,2,3,\\
&J_{04} = -iE\left\{\varepsilon_4\left(1 - \varkappa^2/\bar m^2\right)^{1/2}, \partial/\partial\bar m\right\} - (\varkappa/\bar m)(S_{4a}p_a/\bar m), \qquad (7.1)\\
&J_{4a} = ip_a\left\{\varepsilon_4\left(1 - \varkappa^2/\bar m^2\right)^{1/2}, \partial/\partial\bar m\right\} - i\varepsilon\bar m\left(1 - \varkappa^2/\bar m^2\right)^{1/2}\partial/\partial p_a +\\
&\qquad + \frac{\varkappa p_a S_{4b}p_b}{m^2(E+m)} + \varepsilon_4\left(1 - \varkappa^2/\bar m^2\right)^{1/2}[S_{ab}p_b/(E+\bar m)] + \frac{\varkappa S_{4a}}{\bar m},
\end{aligned}
$$

where

$$
\{A,B\} = AB + BA, \qquad |\varkappa| \le \bar m < \infty.
$$

The generators (7.1) are Hermitian with respect to the scalar product

$$
(\chi_1,\chi_2) = \sum_{s=|j-\tau|}^{j+\tau}\int_\varkappa^\infty d\bar m\int\frac{d^3p}{2E}\chi_1^\dagger(\boldsymbol p,\bar m,s,s_3)\chi_2(\boldsymbol p,\bar m,s,s_3).
$$

As soon as the operators (7.1) and (3.17) realise the same IR $D^+(\varkappa,j,\tau)$ of the $P(1,4)$ group, the equivalence transformation, which connects these two realisations, exists. In order to come from (7.1) to (3.17), we make the isometric transformation

$$
P_\mu \to WP_\mu W^{-1}, \qquad J_{\mu\nu} \to WJ_{\mu\nu}W^{-1} \tag{7.2}
$$

and the following replacement of variables

$$
p_a \to p_a, \qquad \bar m \to \bar m(m,\boldsymbol p), \tag{7.3}
$$

where

$$
\begin{aligned}
&W = \left(1 - \varkappa/\bar{m}^2\right)^{1/4} \exp[i(S_{4a}p_a/p)(\theta_1 - \theta_2)],\\
&\theta_1 = 2\tan^{-1}\left\{p/\left[E + \varepsilon_4\left(\bar{m}^2 - \varkappa^2\right)^{1/2} + \varkappa\right]\right\},\\
&\theta_2 = 2\tan^{-1}\left[\varepsilon_4 p\left(\bar{m}^2 - \varkappa^2\right)^{1/2}/(E+m)(m+\varkappa)\right],\\
&\bar{m} = (1/2m)\left[\left(m^2 - \varkappa^2 - p^2\right)^2 + 4m^2\varkappa^2\right]^{1/2}.
\end{aligned}
\tag{7.4}
$$

One can ensure by direct verification that the transformations (7.2)–(7.4) reduce the generators (7.1) into the Galilei basis (i.e. that the transformed generators coincide with (3.17) after substitution into (2.3)). We do not give the detailed calculations here because the transformations (7.2)–(7.4) may be represented as two consequent ones: namely, the transition from the Poincaré to the canonical basis (Nikitin et al [15])

$$
\begin{aligned}
&P_\mu \to VP_\mu V^{-1}, \qquad J_{\mu\nu} \to VJ_{\mu\nu}V^{-1},\\
&\bar{m} \to \bar{m}(p_4) = \varepsilon_4\left(p_4^2 + \varkappa^2\right)^{1/2},\\
&V = \left(1 - \varkappa^2/\bar{m}^2\right)^{1/4}\exp(iS_{0a}p_a\theta_2/p)
\end{aligned}
\tag{7.5}
$$

and then the transition from the canonical basis to the Galilei one (see § 3). So

$$
W = U_1 V,
$$

where V and U_1 are given by equations (7.5), (3.8), (3.14).

The transformation (7.2)–(7.4) may be used to establish the connection between the vectors in the Galilei and in the Poincaré bases. This connection is given by the equations:

$$
\begin{aligned}
&\phi(\boldsymbol{p}, m, s, s_3) = W\hat{P}_s\hat{P}_{s_3}P_{s'}P_{s_3'}\chi(\boldsymbol{p}, m(\bar{m}, \boldsymbol{p}), s, s_3),\\
&\chi(\boldsymbol{p}, m, s, s_3) = W^{-1}\tilde{P}_s\tilde{P}_{s_3}P_{s'}P_{s_3'}\phi(\boldsymbol{p}, m(\bar{m}, \boldsymbol{p}), s, s_3),\\
&m(\bar{m}, \boldsymbol{p}) = \varepsilon_4\left(\bar{m}^2 - \varkappa^2\right)^{1/2} + \left(p^2 + \bar{m}^2\right)^{1/2},\\
&|j - \tau| \le s, s' \le j + \tau, \qquad -s \le s_3 \le s, \qquad -s' \le s_3' \le s',
\end{aligned}
$$

where P_s, P_{s_3}, $\hat{P}_s$, $\hat{P}_{s_3}$, $\tilde{P}_s$, $\tilde{P}_{s_3}$ are the projectors into the subspace with the corresponding fixed value of s and s_3.

$$
P_s = \prod_{\tilde{s}\ne s}\frac{\boldsymbol{S}^2 - \tilde{s}(\tilde{s}+1)}{s(s+1) - \tilde{s}(\tilde{s}+1)}, \qquad P_{s_3} = \prod_{\tilde{s}_3\ne s_3}\frac{S_3 - \tilde{s}_3}{s_3 - \tilde{s}_3},
\tag{7.6}
$$

$$
\hat{P}_s = W^{-1}P_sW, \quad \hat{P}_{s_3} = W^{-1}P_{s_3}W, \quad \tilde{P}_{s_3} = WP_{s_3}W^{-1}, \quad \tilde{P}_s = WP_sW^{-1}.
$$

$\hat{P}_s$, $\hat{P}_{s_3}$, $\tilde{P}_s$, $\tilde{P}_{s_3}$ may be obtained from (7.6) by the substitution

$$
\begin{aligned}
&S_a \to \hat{S}_a = W^{-1}S_aW = S_a\cos\tilde{\theta} + \frac{p_aS_bp_b}{p^2}(1 - \cos\tilde{\theta}) + \frac{1}{p}\varepsilon_{abc}p_bS_{4c}\sin\tilde{\theta},\\
&S_a \to \tilde{S}_a = WS_aW^{-1} = S_a\cos\tilde{\theta} + \frac{p_aS_bp_b}{p^2}(1 - \cos\tilde{\theta}) - \frac{1}{p}\varepsilon_{abc}p_bS_{4c}\sin\tilde{\theta},\\
&\tilde{\theta} = \theta_1 - \theta_2.
\end{aligned}
$$

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1. Aghassi J.J., Roman P., Santilli R.M., *J. Math. Phys.*, 1970, **11**, 2297–2305.
2. Bargman V., *Ann. Math.*, 1954, **59**, 1–46.
3. Barrabes K., Henry J., *J. Phys. A: Math. Gen.*, 1976, **9**, 1428–1434.
4. Castell L., *Nuovo Cim.*, 1967, **49**, 285–294.
5. Elizalde E., Gomish J., *J. Math. Phys.*, 1978, **19**, 1790–1797.
6. Fedorchuk V.M., Preprint 78-18, Mathematical Institute, Kiev, 1978.
7. Foldy L.L., *Phys. Rev.*, 1956, **102**, 568–574.
8. Fushchych W.I., *Teor. Mat. Fiz.*, 1970, **4**, 360–367.
9. Fushchych W.I., Krivsky I.Yu., *Nucl. Phys. B*, 1968, **7**, 79–87.
10. Fushchych W.I., Krivsky I.Yu., Preprint 68-72, Institute for Theoretical Physics, Kiev, 1968.
11. Fushchych W.I., Krivsky I.Yu., *Nucl. Phys. B*, 1969, **14**, 537–544.
12. Fushchych W.I., Nikitin A.G., Yurik I.I., Preprint 75-85, Mathematical Institute, Kiev, 1975.
13. Leutwyler H., Stern J., *Ann. Phys.*, 1968, **112**, 94–164.
14. Levi-Leblond J.-M., *Comm. Math. Phys.*, 1967, **6**, 286–311.
15. Nikitin A.G., Fushchych W.I., Yurik I.I., *Teor. Mat. Fiz.*, 1976, **26**, 202–220 (transl. *Teor. and Math. Phys.*, 1976, **26**, 138–147).
16. Nikitin A.G., Salogub V.A., *Ukr. Fiz. J.*, 1975, **20**, 1730–1732.
17. Shirokov Yu.M., *Dokl. Akad. Nauk USSR*, 1954, **94**, 857–861.
18. Shirokov Yu.M., *Dokl. Akad. Nauk USSR*, 1954, **99**, 737–739.