Conformal invariance of relativistic equations for arbitrary spin particles

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We show that any Poincaré-invariant equation for particles of zero mass and of discrete spin provide a unitary representation of the conformal group, and find an explicit expression of the conformal group generators in terms of Poincaré group generators.

It is well-known that the relativistic equations for massless particles are invariant under the conformal transformations. This was first established for the Maxwell equations [1] and then for the equations describing the massless particles of spin 1/2 [2] and of any spin [3].

L. Gross [4] has demonstrated that the solutions of the Maxwell and of the Rarita–Schwinger (with mass $m = 0$) equations provide a unitary representation of the conformal group C_4. The proof given in [4] is rather tedious and in some sense non-constructive, since it does not give an algorithm to obtain an explicit form of Hermitian generators of the group C_4 for any conformal invariant equation.

In this note, we shall formulate a theorem, which generalizes the results [1–4] and give a simple and constructive proof of it. Without restricting ourselves by any concrete form of equations for massless particles we show that any (generally speaking, reducible) representation of the Lie algebra of Poincaré group $P(1,3)$, which corresponds to zero mass and discrete spin, can be extended to provide a representation of the conformal group Lie algebra, and find the explicit expression of the generators of the group C_4 through the generators of its subgroup $P(1,3)$.

Theorem 1. *Any Poincaré-invariant equation for particles of zero mass and of discrete spin is invariant under the conformal algebra C_4[1], basis elements of which are given by the operators P_μ, $J_{\mu\nu}$ and*

$$\begin{aligned}
&D = \frac{1}{2}[P_0P_a/P^2, J_{0a}]_+, \qquad a,b = 1,2,3,\\
&K_0 = \frac{1}{2}[P_0/P^2, J_{0a}J_{0a} + \Lambda^2 - (1/2)]_+,\\
&K_a = \frac{1}{2}\left([P_0/P^2, [J_{0b}, J_{ab}]_+]_+ - [P_a/P^2, J_{0b}J_{0b} + \Lambda^2 - (1/2)]_+\right),
\end{aligned} \tag{1}$$

where P_μ and $J_{\mu\nu}$ are the basis elements of the Poincaré algebra $P(1,3)$, $\mu,\nu = 0,1,2,3$, $\Lambda = \frac{1}{2}\varepsilon_{abc}J_{ab}P_cP_0^{-1}$; $P^2 = P_1^2 + P_2^2 + P_3^3$; $[A,B]_+ = AB + BA$ and D, K_μ are the operators, which extend the algebra $P(1,3)$ to the algebra C_4.

Proof. Inasmuch as the operators P_μ and $J_{\mu\nu}$ satisfy, by definition, the algebra

$$\begin{aligned}
&[P_\mu, P_\nu]_- = 0, \qquad [J_{\mu\nu}, P_\lambda]_- = i(g_{\nu\lambda}P_\mu - g_{\mu\lambda}P_\nu),\\
&[J_{\mu\nu}, J_{\lambda\sigma}]_- = i(g_{\nu\lambda}J_{\mu\sigma} + g_{\mu\sigma}J_{\nu\lambda} - g_{\mu\lambda}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\lambda})
\end{aligned} \tag{2}$$

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[1] We use the same notation for the groups and for the corresponding Lie algebras.

the theorem proof is reduced to the verification of the correctness of the following commutation relations

$$
\begin{gathered}
[J_{\mu\nu}, K_\lambda]_- = i(g_{\nu\lambda}K_\mu - g_{\mu\lambda}K_\nu), \qquad [K_\mu, P_\nu]_- = 2i(g_{\mu\nu}D - J_{\mu\nu}), \\
[D, P_\mu]_- = iP_\mu, \quad [D, K_\mu]_- = -iK_\mu, \quad [K_\mu, K_\nu]_- = 0, \quad [J_{\mu\nu}, D]_- = 0,
\end{gathered} \tag{3}
$$

which determine together with (2) the algebra C_4 (see e.g. [5]). It is not difficult to realize such a verification, bearing in mind that, on the set of the solutions of any relativistic equation for a particle of zero mass and of discrete spin, the following relations are satisfied:

$$
P_\mu P^\mu = 0, \qquad W_\mu W^\mu = 0, \qquad W_\mu = \Lambda P_\mu,
$$

where W_μ is the Lubanski–Pauli vector

$$
W_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J_{\nu\rho}P_\sigma.
$$

So the formulas (1) determine the explicit form of the conformal group generators by the given generators P_μ, $J_{\mu\nu}$ of the group $P(1,3)$. Let us note that the generators K_μ and D are written in a transparently Hermitian form, from which follows that they generate together with P_μ, $J_{\mu\nu}$ the unitary representation of the conformal group. The theorem is proved.

Let us demonstrate the constructive character of Theorem 1 by some examples. First consider the Weyl equation

$$
\sigma_\mu p^\mu \varphi(x_0, \vec{x}) = 0, \qquad p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}. \tag{4}
$$

On the set of solutions of equation (4) the Poincaré group generators have the form

$$
\begin{gathered}
P_0 = \sigma_a p_a, \qquad P_a = p_a = -i\frac{\partial}{\partial x_a}, \\
J_{ab} = x_a p_b - x_b p_a + \frac{i}{4}[\sigma_a, \sigma_b]_-, \qquad J_{0a} = x_0 p_a - \frac{1}{2}[x_a, P_0]_+,
\end{gathered} \tag{5}
$$

where σ_a are the Pauli matrices. Substituting (5) into (1), one obtains the remaining generators of the conformal group in the form

$$
D = \frac{1}{2}[x_\mu, P^\mu]_+, \qquad K_\mu = -[J_{\mu\nu}, x^\nu]_+ + \frac{1}{2}[P_\mu, x_\nu x^\nu]_+. \tag{6}
$$

On the set of solutions of Equation (4), the generators (5) and (6) may be written also in the usual differential form (see e.g. [5])

$$
\begin{gathered}
P_\mu = p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad D = x_\mu p^\mu + \frac{3}{2}i, \\
J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{4}[\sigma_\mu, \sigma_\nu]_-, \qquad K_\nu = 2x_\nu D - x_\mu x^\mu p_\nu - \frac{1}{2}x^\mu[\sigma_\mu, \sigma_\nu]_-,
\end{gathered}
$$

which however is not manifestly Hermitian.

Taking P_μ, $J_{\mu\nu}$ in the Foldy–Shirokov form [6]

$$\begin{aligned} &P_0 = p = \left(p_1^2 + p_2^2 + p_3^2\right)^{1/2}, \qquad P_a = p_a, \\ &J_{ab} = x_a p_b - x_b p_a + S_{ab}, \qquad J_{0a} = x_0 p_a - \frac{1}{2p}[x_a, P_0]_+ - (S_{ab} p_b / p), \end{aligned} \tag{7}$$

we obtain from (1)

$$D = \frac{1}{2}[x_\mu, P^\mu]_+, \qquad K_\mu = -[J_{\mu\nu}, X^\nu]_+ + \frac{1}{2}\left[P_\mu, X_\nu X^\nu + \frac{1}{2p}\left(\Lambda^2 + \frac{1}{4}\right)\right]_+, \tag{8}$$

where

$$X_0 = x_0, \qquad X_a = x_a + (S_{ab} p_b)/p^2.$$

Using (1), it is not difficult to be convinced that (8) is a universal form of the generators K_μ, D for any representation of the conformal group, in which P_a and J_{ab} have the structure (7).

Lastly, if P_μ and $J_{\mu\nu}$ are the generators of the irreducible representation of the Poincaré group in Lomont–Moses form [7], then the formulas (1) give the conformal group generators in the form of Bose and Parker [8].

In connection with the above results, the following question arises naturally: Do there exist Poincaré invariant equations, for particles with nonzero mass, which would be invariant under the conformal group? A positive answer to this question may be given only for equations describing particles with variable mass. As an example, one may consider the relativistic equations with proper time, conformal invariance of which has been established in [9].

It has been proposed in [10] to use the group of rotations and translations in five-dimensional Minkowski space for the description of physical systems with variable mass and spin. This group, which will be further denoted by the symbol $P(1,4)$, contains as subgroups both the Poincaré group $P(1,3)$ and the Galilei group $G(3)$.

The main property of $P(1,4)$-invariant equations is that they are constant also under the conformal algebra C_4. More precisely, the following statement is valid:

Theorem 2. *Any $P(1,4)$-invariant equation is invariant under the Lie algebra of the group $SO(1,5)$.*

Proof. Using the method proposed in [11], we consider the operator

$$J_{\mu 5} = \frac{1}{2}(P_\mu P^\mu)^{-1/2}(P^\nu J_{\mu\nu} + J_{\mu\nu} P^\nu), \qquad P_\mu P^\mu \neq 0,$$

where P_μ and $J_{\mu\nu}$ are the generators of the group $P(1,4)$, $\mu, \nu = 0, 1, 2, 3, 4$. The set of the operators $J_{\mu\nu}$ and $J_{\mu 5}$ satisfy the commutation relations of the Lie algebra of the group $SO(1,5)$ (which is locally isomorphic to the Euclidean conformal group)

$$\begin{aligned} &[J_{\mu 5}, J_{\nu 5}]_- = i J_{\mu\nu}, \qquad [J_{\mu 5}, J_{\nu\lambda}]_- = i(g_{\mu\nu} J_{\lambda 5} - g_{\mu\lambda} J_{\nu 5}), \\ &[J_{\mu\nu}, J_{\lambda\sigma}]_- = i(g_{\nu\lambda} J_{\mu\sigma} + g_{\mu\sigma} J_{\nu\lambda} - g_{\mu\lambda} J_{\nu\sigma} - g_{\nu\sigma} J_{\mu\lambda}). \end{aligned}$$

In the case in which the Casimir operator $P_\mu P^\mu = 0$, the proof is reduced to that of Theorem 1.

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