On the invariance groups of relativistic equations for the spinning particles interacting with external fields

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All relativistic free-particle motion equations, including the Dirac and Kemmer–Duffin–Petiau (KDP) ones, are invariant under the Poincaré group $P_{1,3}$. But such a group does not exhaust symmetry of the relativistic equations. It has been shown in [1] with help of non-Lie method, that any Poincaré-invariant equation for a free particle with spin $s \geq \frac{1}{2}$ has additional invariance under $SU_2 \otimes SU_2$ group. The same invariance group is possessed by Maxwell equations [2].

It has been shown in [3, 4], that the free equations of KDP (for $s = 1$) and of Rarita–Schwinger (for $s = \frac{3}{2}$) have more extensive symmetry group than the group $SU_2 \otimes SU_2$. It follows from the results of these papers, that any relativistic equation for a free particle of spin $s \geq 1$ possesses SU_3 symmetry.

In this note, which is an extention of the paper [4], the invariance groups of the Dirac and KDP equations for the particles, interacting with an external field have been established.

Theorem 1. *The Dirac equation with the Pauli-type interaction*

$$L\Psi = 0, \qquad L = \gamma_\mu \pi^\mu + \frac{i}{4m}(1 + i\gamma_4)\gamma_\mu\gamma_\nu F^{\mu\nu} + m, \tag{1}$$

where

$$\pi_\mu = p_\mu - eA_\mu, \qquad p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x^\nu},$$

A_μ is the vector potential of electromagnetic field, $F_{\mu\nu} = -i[\pi_\mu, \pi_\nu]_-$, is invariant under the Lie algebra of the $SU_2 \otimes SU_2$ group. This algebra basis elements $Q_{\mu\nu}$ have the form

$$Q_{\mu\nu} = i\gamma_\mu\gamma_\nu + \frac{i}{m}(1 + i\gamma_4)(\gamma_\mu\pi_\nu - \gamma_\nu\pi_\mu). \tag{2}$$

Proof may be carried out in a way, which has been described in [4]. The theorem validity, i.e. that the operators $Q_{\mu\nu}$ satisfy the invariance condition of eq.(1) [4]

$$[Q_{\mu\nu}, L]_- = \Gamma_{\mu\nu}L, \qquad \Gamma_{\mu\nu} = \frac{i}{m}(\gamma_\mu\pi_\nu - \gamma_\nu\pi_\mu)$$

and the commutation relations

$$[Q_{\mu\nu}, Q_{\lambda\sigma}]_- = 2i(g_{\mu\lambda}Q_{\nu\sigma} + g_{\nu\sigma}Q_{\mu\lambda} - g_{\mu\sigma}Q_{\nu\lambda} - g_{\nu\lambda}Q_{\mu\sigma})$$

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may be established by the direct verification. Putting in (1), (2) $A_\mu = 0$, one comes to the invariance algebra of the free Dirac equation, which has been obtained in [4].

Theorem 2. *The Dirac equation for a particle in a constant inhomogeneous magnetic field*

$$\pi_0\varphi = H\varphi, \qquad H = \gamma_0\gamma_a\pi_a + \gamma_0 m, \tag{3}$$

where

$$\pi_0 = p_0, \qquad \pi_3 = p_3, \qquad \pi_1 = p_1 - eA_1(x_1, x_2), \qquad \pi_2 = p_2 - eA_2(x_1, x_2)$$

is invariant under the Lie algebra of $SU_2 \otimes SU_2$ group. The basis elements Σ_{kl} of this algebra have the form

$$\begin{gathered}\Sigma_{12} = \frac{i\gamma_3\gamma_0\gamma_\alpha\pi_\alpha}{|\gamma_0\gamma_\alpha\pi_\alpha|}, \qquad \Sigma_{31} = \frac{i\gamma_4(\gamma_3 m + p_3)}{(p_3^2 + m^2)^{1/2}}, \\ \Sigma_{23} = i\Sigma_{12}\Sigma_{31}, \qquad \Sigma_{4a} = \frac{H}{|H|}\Sigma_{bc}, \qquad \alpha = 1, 2, \quad (a, b, c) \text{ is cykl } (1, 2, 3).\end{gathered} \tag{4}$$

Proof. Let us use the canonical transformation method. Passing to the new wave function Ψ':

$$\Psi \to \Psi' = W\Psi, \qquad H \to H' = WHW^{-1}, \tag{5}$$

where

$$\begin{gathered}W = V_1V_2V_3, \quad V_1 = \frac{\mathcal{E} + q_3 + i\gamma_1\gamma_2\gamma_0\gamma_\alpha\pi_\alpha}{\sqrt{2\mathcal{E}(\mathcal{E} + q_3)}}, \quad \mathcal{E} = \left(m^2 + \pi^2 - i\gamma_1\gamma_2\mathcal{H}\right)^{1/2}, \\ \pi^2 = \pi_1^2 + \pi_2^2 + \pi_3^2, \quad q_3 = \left(m^2 + p_3^2\right)^{1/2}, \quad \mathcal{H} = -i[\pi_1, \pi_2]_-, \\ V_2 = V_2^{-1} = \frac{1}{2}\left[1 + i\gamma_3\gamma_4 + (1 - i\gamma_3\gamma_4)\frac{\gamma_0\gamma_\alpha\pi_\alpha}{|\gamma_0\gamma_\alpha\pi_\alpha|}\right], \\ V_3 = \left(V_3^{-1}\right)^\dagger = (m + q_3 + \gamma_3 p_3)[2q_3(q_3 + m)]^{-1/2}\end{gathered}$$

one obtains the equation

$$i\frac{\partial}{\partial t}\Psi' = i\gamma_1\gamma_2\left(m^2 + \pi^2 - i\gamma_1\gamma_2\mathcal{H}\right)^{1/2}\Psi'. \tag{6}$$

Equation (6) is obviously invariant under the transformations $\Psi' \to \Sigma'_{kl}\Psi'$, where

$$\Sigma'_{12} = \frac{i}{2}\gamma_3, \qquad \Sigma'_{31} = \frac{i}{2}\gamma_4, \qquad \Sigma'_{23} = \frac{i}{2}\gamma_4\gamma_3, \qquad \Sigma'_{4a} = \frac{i}{2}\gamma_1\gamma_2\Sigma_{bc}. \tag{7}$$

Operators (7) satisfy commutation relations of the Lie algebra of the $O_4 \sim SU_2 \otimes SU_2$ group. The exact form (4) of these operators in the initial Ψ-representation one obtains by the inverse transformation, $\Sigma_{kl} = W^{-1}\Sigma'_{kl}W$. The theorem is proved.

Remark 1. An analogous theorem takes place also for the Dirac equation, which describes the particle in alternating the electric field with the fixed direction (say, in a field, which is directed along the third co-ordinate axis). Such an equation may be written in the form (3), where

$$\pi_0 = p_0 - eA_0(t, x_3), \qquad \pi_1 = p_1, \qquad \pi_2 = p_2, \qquad \pi_3 = p_3 - eA_3(t, x_3). \tag{8}$$

The exact form of the $SU_2 \times SU_2$-group generators is given by the following formulae:

$$\tilde{\Sigma}_{12} = \frac{i\gamma_2\gamma_1\gamma_\lambda\pi^\lambda}{|\gamma_1\gamma_\lambda\pi^\lambda|}, \qquad \tilde{\Sigma}_{31} = \frac{i\gamma_4(\gamma_2 m + p_2)}{(p_2^2+m^2)^{1/2}}, \qquad \lambda = 0, 3,$$

$$\tilde{\Sigma}_{32} = i\tilde{\Sigma}_{12}\tilde{\Sigma}_{31}, \qquad \tilde{\Sigma}_{4a} = \frac{i\gamma_1(\gamma_\lambda\pi^\lambda - \gamma_2\pi_2 - m)}{|i\gamma_1(\gamma_\lambda\pi^\lambda - \gamma_2\pi_2 - m)|}\tilde{\Sigma}_{bc}.$$

These operators as like as (4) ones, are integrodifferential operators, in contrast with (2), where $Q_{\mu\nu}$ are differential ones.

Let us consider the KDP equation for a particle of spin $s = 1$ charge e and the anomalous magnetic moment k, which interacts with the constant homogeneous magnetic field $\mathcal{H}$

$$\left(\beta_\mu\pi^\mu + m + \frac{ek}{4m}S_{\mu\nu}F^{\mu\nu}\right)\Psi = 0, \tag{9}$$

where

$$\begin{aligned} &\pi_0 = p_0, \qquad \pi_1 = p_1 - e\mathcal{H}x_2, \qquad \pi_2 = p_2, \qquad \pi_3 = p_3, \\ &S_{\mu\nu} = i(\beta_\mu\beta_\nu - \beta_\nu\beta_\mu), \qquad S_{\mu\nu}F^{\mu\nu} = 2S_{12}\mathcal{H}. \end{aligned} \tag{10}$$

Theorem 3. *Equation (9) and (10) have six independent constants of motion Q_A which form the Klein group. If $k = 1$, eqs.(9) and (10) are invariant under ten-dimensional Lie algebra A_{10}, which contains subalgebra O_4.*

Proof. Let us reduce eqs.(9) and (10) to the canonical diagonal form, for which the theorem statements become obvious. Multiplying (9) from the left by

$$\tilde{V}_1 = \exp\left[i\frac{S_{5\lambda}p^\lambda}{p_5}\frac{\pi}{2}\right], \qquad \lambda = 0, 3, \qquad p_5 = \left(p_0^2 - p_3^2\right)^{1/2}, \tag{11}$$

gives the equation

$$(i\beta_5 p_5 - \beta_\alpha\pi_\alpha + k\omega S_{12} + m)\Psi' = 0, \qquad \Psi' = \tilde{V}_1\Psi, \qquad \omega = \frac{e\mathcal{H}}{2m}. \tag{12}$$

This equation may be written in the equivalent form

$$\begin{aligned} &p_5\Psi' = \left[S_{5\alpha}\pi_\alpha + i\beta_5\hat{M} - \hat{M}^{-1}\beta_\alpha\pi_\alpha\beta_5\left(\beta_\alpha\pi_\alpha - \hat{M}\right)\right]\Psi', \\ &\left(1+\beta_5^2\right)\left(\beta_\alpha\pi_\alpha + \hat{M}\right)\Psi' = 0, \qquad \hat{M} = m + k\omega S_{12}, \qquad |k\omega| \neq m. \end{aligned} \tag{13}$$

With the help of the transformation $\Psi' \to \Phi = \tilde{V}_2\tilde{V}_3\Psi'$, where

$$\begin{aligned} &\tilde{V}_2 = \exp\left[-\hat{M}^{-1}\beta_\alpha\pi_\alpha\beta_5^2\right] = 1 - \hat{M}^{-1}\beta_\alpha\pi_\alpha\beta_5^2, \\ &\tilde{V}_3 = 1 + \beta_5^2\left(1 + \frac{E + \hat{H}\varkappa}{\sqrt{E\left(2E + [\hat{H}, \varkappa]_+\right)}}\right), \\ &\tilde{V}_3^{-1} = 1 + \beta_5^2\left(1 + \frac{E + \varkappa\hat{H}}{\sqrt{E\left(2E + [\hat{H}, \varkappa]_+\right)}}\right), \end{aligned} \tag{14}$$

$$\hat{H} = i\beta_5\left(\beta_\alpha\pi_\alpha M^{-1}\beta_\alpha\pi_\alpha - M\right), \qquad M = m + \omega S_{12},$$
$$E = |\hat{H}| = \sqrt{\hat{H}^2}, \qquad \varkappa = S_{12} + i\beta_5\left(1 - S_{12}^2\right), \qquad |\omega| \neq m,$$

one reduces eq.(13) to the diagonal form

$$\begin{gathered} p_5\Phi = H^c\Phi, \qquad H^c = S_{12}\left(m^2 + \pi_\alpha^2 - 2\omega S_{12} + \omega^2\right)^{1/2} + \\ + i\beta_5 S_{12}(k-1)\omega + i\beta_5\left(1 - S_{12}^2\right)\left(m^2 + \frac{m^2\pi_\alpha^2 + 2k^2\omega^2}{m^2 - k^2\omega^2}\right)^{1/2}, \\ (1+\beta_5^2)\Phi = 0, \qquad \pi_\alpha^2 = \pi_1^2 + \pi_2^2. \end{gathered} \tag{15}$$

Equations (15) are obviously invariant with respect to transformations $\Phi \to Q_A\Phi$, where Q_A are arbitrary matrices, which commute with β_5 and S_{12}. The complete set of such matrices may be chosen in the form

$$\begin{gathered} Q_1' = i\beta_5\left(1 + S_{12} + S_{12}^2\right), \qquad Q_2' = i\beta_5\left(1 - S_{12} - S_{12}^2\right), \\ Q_3' = i\beta_5\left(1 - 2S_{12}^2\right), \qquad Q_{3+a}' = i\beta_5 Q_a', \qquad a = 1, 2, 3. \end{gathered} \tag{16}$$

The operators (16) obey the relations

$$\begin{gathered} [Q_A', Q_B']_- = 0, \qquad (Q_A')^2\Phi = \Phi, \qquad Q_a'Q_b' = Q_c', \\ Q_{3+a}'Q_{3+b}' = Q_c', \qquad Q_{3+a}'Q_b' = Q_{3+c}', \qquad a \neq b \neq c \neq a, \end{gathered}$$

i.e. form the six-dimensional Klein group.

If $k = 1$, there exist ten linearly independent matrices, which commute with H^c and β_5. These matrices may be chosen in the form

$$\begin{gathered} N_{12}' = (1 - 2\beta_5^2)S_{12}^2\beta_5^2, \qquad N_{31}' = i\beta_5 S_{12}'^2, \qquad N_{32}' = iN_{31}'N_{12}', \\ N_{4a}' = i\beta_5 S_{12} N_{bc}', \qquad B_1' = i\beta_5\left(1 - S_{12}^2\right), \qquad B_{1+a}' = Q_{3+a}'. \end{gathered}$$

Operators B_k' commute with $B_{k'}'$ and with $N_{k'l}'$, and the operators N_{kl}' form the representation $D\left(\frac{1}{2}, 0\right) \oplus D\left(0, \frac{1}{2}\right) \oplus 6D(0,0)$ of the Lie algebra of the group $SU_2 \otimes SU_2$. The exact form of the operators Q_A, N_{kl}, N_k in the original Ψ-representation may be obtained by the formulae

$$Q_A = \tilde{W}^{-1}Q_A\tilde{W}, \qquad N_{kl} = \tilde{W}^{-1}N_{kl}'\tilde{W}, \qquad B_k = \tilde{W}^{-1}B_k'\tilde{W}, \tag{17}$$

where $\tilde{W} = \tilde{V}_1\tilde{V}_2\tilde{V}_3$ and $\tilde{V}_1$, $\tilde{V}_2$, $\tilde{V}_3$ given in (11), (14). The theorem is proved.

Remark 2. The analogous theorem may be proved for the KDP equation, which describes the motion of a charged particle with anomalous moment in a constant homogeneous elwctric field E. Such an equation has the form (9), where

$$\pi_0 = p_0 - Ex_3, \qquad \pi_1 = p_1, \qquad \pi_2 = p_2, \qquad \pi_3 = p_3, \qquad S_{\mu\nu}F^{\mu\nu} = -2ES_{03}.$$

Let us consider the equation for a particle with an arbitrary spin [5]

$$H_s\Psi(t, \boldsymbol{x}) = i\frac{\partial}{\partial t}\Psi(t, \boldsymbol{x}), \tag{18}$$

where $\Psi(t,\boldsymbol{x})$ is a $2(2s+1)$-component wave function,

$$H_s = \sigma_1 m + \sigma_3 p \sum_{\nu=-s}^{s} (-1)^{[\nu]}\Lambda_\nu + (1+\sigma_1)\varphi(t,\boldsymbol{x}),$$
$$\Lambda_\nu = \prod_{\mu\neq\nu}\left(\frac{\boldsymbol{S}\cdot\boldsymbol{p}}{p} - \nu\right)(\mu-\nu)^{-1}, \qquad p = \left(p_1^2+p_2^2+p_3^2\right)^{1/2}, \tag{19}$$

S_a are generators of the direct sum $D(s)\oplus D(s)$ of the irreducible representation of the SU_2 group, σ_1 and σ_3 are $2(2s+1)\times 2(2s+1)$-dimensional Pauli matrices, commuting with S_a, $\varphi(t,\boldsymbol{x})$ is an arbitrary potential. If $\varphi(t,\boldsymbol{x})=0$ eq.(18) coincides with the one obtained in [5] and describes a free motion of a relativistic spin-s particle.

Theorem 4. *Equation (18) is invariant under SU_2 algebra. The basis elements of this algebra have the form*

$$\Sigma_a = O_{bc} = \sigma_1 S_a + (1-\sigma_1)p_a \boldsymbol{S}\cdot\boldsymbol{p}p^{-2}. \tag{20}$$

Proof. Using the transformation

$$H_s \to V H_s V^{-1} = \sigma_1 m + \sigma_3 p + (1+\sigma_1)\varphi(t,\boldsymbol{x}),$$
$$\Sigma_a \to V\Sigma_a V^{-1} = S_a, \qquad V = V^{-1} = \frac{1}{2}\left[1+\sigma_1 + (1-\sigma_1)\sum_{\nu=-s}^{s}(-1)^{[\nu]}\Lambda_\nu\right],$$

one reduces the Hamiltonian (19) and the operators (20) to such a form, that the theorem statements become obvious.

For $s=\frac{1}{2}$ eq.(18) coinsides with the Dirac equation with a semirelativistic potential $(1+\sigma_1)\varphi \equiv (1+\gamma_0)\varphi$. The SU_2-invariance of such equation has been established in [6].

Theorem 5. *The Tamm–Sakata–Taketani equation with a semirelativistic potential*

$$i\frac{\partial}{\partial t}\Psi = \left[\sigma_1\left(m+\frac{p^2}{2m}\right) + i\sigma_3\left(\frac{p^2}{2m} - \frac{(\boldsymbol{S}\cdot\boldsymbol{p})^2}{m}\right) + (1+\sigma_1)\varphi(t,\boldsymbol{x})\right]\Psi, \tag{21}$$

is invariant under the Lie algebra of the SU_2 group. The basis elements λ_A of this algebra have the form

$$\lambda_a = [O_{ab}, O_{ac}]_+, \qquad \lambda_{3+a} = O_{bc}, \qquad \lambda_7 = i(O_{23}O_{31}O_{12} - O_{12}O_{23}O_{31}),$$
$$\lambda_8 = -\frac{i}{\sqrt{3}}(O_{12}O_{23}O_{31} + O_{23}O_{31}O_{12} - 2O_{31}O_{12}O_{23}),$$

where O_{ab} are given in (20).

We do not give the proof here. The analogous theorem may be formulated for the KDP equation with the potential $\beta_0(1+\beta_0)\varphi(t,\boldsymbol{x})$.

In conclusion we note that the obtained invariance algebrae may be used for deriving of new solutions of the equations considered above, if certain partial solution of these equations is known.

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