On the new invariance groups of the Dirac and Kemmer–Duffin–Petiau equations

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In works [1–6] the canonical-transformation method has been proposed for the investigations of the group properties of the differential equations of the quantum mechanics. This method essence in that the system of differential equation is first transformed to the diagonal or Jordan form and then the invariance algebra of the transformed equation is established. The explicit form of this algebra basis elements for the starting equations is found by the inverse transformation.

The main distinguishing feature of this method from the intensively developed during last years classical Lie method [7, 8] is that the basis elements of the invariance algebra of the corresponding equations do not belong to the class of the differential operators, but are as a rule integrodifferential operators. The new invariance algebras of the Dirac [1, 2][1], Maxwell [2], Klein–Gordon [3], Kemmer–Duffin–Petiau (KDP) and Rarita–Schwinger [4] equations have been found just in the class of integrodifferential operators.

The aim of this note is to establish the Dirac and the KDP equation invariance algebras in the class of differential operators. The theorems given below (which establish new group properties of the Dirac and of the KDP equations) are proved with the help of the canonical-transformation method.

To establish an invariance of the equation

$$L(p_0,p_1,p_2,p_3)\Psi(x_0,\boldsymbol{x}) \equiv L\Psi = 0, \qquad p_\mu = i\frac{\partial}{\partial x^\mu} \tag{1}$$

under the set of transformations $\Psi \to \Psi'_A = Q_A\Psi$ is to found a set of operators $Q \equiv \{Q_A\}$ such that

$$[L, Q_A]_-\Psi = 0, \qquad \forall\, Q_A \in Q, \tag{2}$$

where Ψ is a function which satisfies eq. (1). Condition (2) may be written in the operator form

$$[L, Q_A]_- = F \cdot L, \tag{3}$$

where F is some set of operators, which are defined in the space of equation (1) solutions.

Theorem 1. *The Dirac equation*

$$L_{\frac{1}{2}}\Psi \equiv (\gamma_\mu p^\mu + m)\Psi = 0 \tag{4}$$

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[1]The results of the work [2] have been generalized by Jayaraman (*J. Phys. A*, 1976, **9**, 1181) to the case of the equation without redundant components for any spin. See also [1].

is invariant under the 16-dimensional Lie algebra A_{16}, whose basis elements are the differential operators

$$P_\mu = p_\mu = i\frac{p}{\partial x^\mu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{2}\gamma_\mu\gamma_\nu, \tag{5}$$

$$Q_{\mu\nu} = i\gamma_\mu\gamma_\nu + \frac{i}{m}(1+i\gamma_4)(\gamma_\mu p_\nu - \gamma_\nu p_\mu), \qquad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3. \tag{6}$$

Proof. If one does not ask himself about the way to found the operators (6) (the operators (5), whith form the $P_{1,3}$ algebra, are well-known), the theorem validity may be established by direct verification. Indeed, one obtains by direct calculation that $Q_{\mu\nu}$ satisfies the invariance condition (3)

$$[Q_{\mu\nu}, L_{\frac{1}{2}}]_- = F^{\frac{1}{2}}_{\mu\nu} L_{\frac{1}{2}}, \qquad F^{\frac{1}{2}}_{\mu\nu} = \frac{i}{m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) \tag{7}$$

and form together with P_μ, $J_{\mu\nu}$ the Lie algebra

$$\begin{gathered}
[P_\mu, P_\nu]_- = 0, \qquad [P_\lambda, J_{\mu\nu}]_- = i(g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu), \qquad [P_\lambda, Q_{\mu\nu}]_- = 0,\\
[J_{\mu\nu}, J_{\lambda\sigma}]_- = i(g_{\mu\lambda}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\lambda} - g_{\mu\sigma}J_{\nu\lambda} - g_{\nu\lambda}J_{\mu\sigma}),\\
[Q_{\mu\nu}, J_{\lambda\sigma}]_- = \frac{1}{2}[Q_{\mu\nu}, Q_{\lambda\sigma}]_- = i(g_{\mu\lambda}Q_{\nu\sigma} + g_{\nu\sigma}Q_{\mu\lambda} - g_{\mu\sigma}Q_{\nu\lambda} - g_{\nu\lambda}Q_{\mu\sigma}).
\end{gathered} \tag{8}$$

But such calculations are very cumbersome. A more elegant and constructive way, which shown the method to obtain the operators (6) is to transform eq. (4) to the diagonal form. After such a transformation the theorem statements become obvious ones.

Such a transformation may be carried out in two steps. First eq. (4) is multiplied by the inversible differential operator

$$\begin{gathered}
W = 1 - \frac{1}{m}\gamma_\mu p^\mu - \frac{1}{2m^2}(1+i\gamma_4)p_\mu p^\mu,\\
W = 1 + \frac{1}{m}\gamma_\mu p^\mu - \frac{1}{2m^2}(1-i\gamma_4)p_\mu p^\mu.
\end{gathered} \tag{9}$$

As a result one obtains the equation

$$WL_{\frac{1}{2}}\Psi = 0, \tag{10}$$

which is equivalent to the starting eq. (4). Then with the help of the isomeric operator

$$V = \exp\left[\frac{1}{2m}(1+i\gamma_4)\gamma_\mu p^\mu\right] \equiv 1 + \frac{1}{2m}(1+i\gamma_4)\gamma_\mu p^\mu \tag{11}$$

one reduces eq. (10) to the diagonal form

$$L'\Phi \equiv V(WL_{\frac{1}{2}})V^{-1}\Phi = \left[\lambda^+ m + \frac{\lambda^-}{m}(p_\mu p^\mu - m^2)\right]\Phi = 0, \tag{12}$$

where $\Phi = V\Psi$, $\lambda^\pm = \frac{1}{2}(1 \pm i\gamma_4)$.

Equation (12) is equivalent to the starting eq. (4) and contains the only matrix γ_4, which may be taken in the diagonal form without loss of generality. So it is evident that the matrices $Q'_{\mu\nu} = i\gamma_\mu\gamma_\nu$ commute with the operator $L_{\frac{1}{2}}^{1}$. These matrices satisfy the commutation relations of the Lie algebra of the $SU_2 \otimes SU_2$ group and satisfy the relations (8) together with the generators $P'_\mu = VP_\mu V^{-1} = P_\mu$ and $J'_{\mu\nu} = VJ_{\mu\nu}V^{-1} = J_{\mu\nu}$.

To complete the proof it is sufficient to find the explicit form of the matrices $Q'_{\mu\nu}$ in the starting Ψ-representation. Calculating $Q_{\mu\nu} = V^{-1}Q'_{\mu\nu}V$, one obtains the operators (6). The theorem is proved.

Corollary 1. If one makes in (4), (9)–(12) the substitution

$$\gamma_\mu p^\mu \to \gamma_\mu = (p_\mu - eA_\mu)\gamma_\mu, \qquad p_\mu p^\mu \to \pi_\mu \pi^\mu - \frac{ie}{2}\gamma_\mu\gamma_\nu F_{\mu\nu},$$

where A_μ is the vector potential, and $F_{\mu\nu}$ is the tensor of the electromagnetic field, the transformations (9)–(12) establish the one-to-one correspondence between the solutions of the Dirac and of the Zaitsev–Gell–Mann equations [9].

Corollary 2. The above founded operators $Q_{\mu\nu}$ may be used to find the constants of motion for the particle interacting with external field. For instance the operator the $Q = \varepsilon_{abc}Q_{bc}(\boldsymbol{\pi})(H_a - iE_a)$ is the constant of motion for a particle moving in the homogeneous constant magnetic field $\boldsymbol{H}$ and the electric field $\boldsymbol{E}(Q_{bc}(\boldsymbol{\pi}))$ is obtained from (6) by the substitution $p_\mu \to \pi_\mu$.

Corollary 3. In theorem 1 the invariance condition of eq. (4) is formulated by the language of Lie algebras, i.e. on the infinitesimal level. The natural question arises: what sort of finite transformations are generated by $Q_{\mu\nu}$? Using the explicit form of the generators (6), one obtains these transformations in the form

$$\begin{gathered}
\Psi(x) \to \Psi'(x) = \exp[iQ_{ab}\theta_{ab}]\Psi(x) = (\cos\theta_{ab} - \gamma_a\gamma_b\sin\theta_{ab})\Psi(x) + \\
+ \frac{1}{m}(1 + i\gamma_4)\sin\theta_{ab}\left(\gamma_a\frac{\partial\Psi(x)}{\partial x_b} - \gamma_b\frac{\partial\Psi(x)}{\partial x_a}\right), \\
\Psi(x) \to \Psi'(x) = \exp[iQ_{0a}\theta_{ab}]\Psi(x) = (\cosh\theta_{0a} - i\sinh\theta_{0a}\gamma_0\gamma_a)\Psi(x) + \\
+ \frac{i}{m}(1 + i\gamma_4)\sinh\theta_{0a}\left(\gamma_0\frac{\partial\Psi(x)}{\partial x_a} - \gamma_a\frac{\partial\Psi(x)}{\partial x_0}\right), \\
x_\mu \to x'_\mu = \exp[iQ_{ab}\theta_{ab}]x_\mu\exp[-iQ_{ab}\theta_{ab}] = x_\mu + \frac{1}{m}(1 + i\gamma_4)\sin\theta_{ab}\times \\
\times(\gamma_a g_{\mu b} - \gamma_b g_{\mu a})(\cos\theta_{ab} - \gamma_a\gamma_b\sin\theta_{ab}), \\
x_\mu \to x'_\mu = \exp[iQ_{0a}\theta_{0a}]x_\mu\exp[-iQ_{0a}\theta_{0a}] = x_\mu + \frac{i}{m}(1 + i\gamma_4)\sinh\theta_{0a}\times \\
\times(\gamma_0 g_{\mu a} - \gamma_a g_{\mu 0})(\cosh\theta_{0a} - i\gamma_0\gamma_a\sinh\theta_{0a}),
\end{gathered} \tag{13}$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu}$ are the six transformation parameters (there is no sum by a, b). Transformations (13) together with the Lorentz transformations form the 16-parameter invariance group of the Dirac equation.

In the quantum field theory not only the Dirac equation (4) but the system of two four-component equations for the two independent functions Ψ and $\bar{\Psi}$ is considered usually. Such a system is equivalent to one eight-component Dirac equation

$$(\Gamma_\mu p^\mu + m)\Psi(x_0, \boldsymbol{x}) = 0, \tag{14}$$

where Γ_μ are (8×8)-dimensional matrices, which satisfy together with Γ_4, Γ_5, Γ_6 the Clifford algebra (one can see details e.g. in [5]).

The system of eq. (14) has the higher symmetry in comparison with the four-component Dirac equation. It is shown in [5] that the additional invariance algebra of eq. (4) (apart from $P_{1,3}$) is the Lie algebra of the O_6-group. This result admits the following strengthening:

Theorem 2. *Equation (14) is invariant under the 40-dimensional Lie algebra A_{40}. The basis elements of this algebra have the form*

$$P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{i}{2}\Gamma_\mu\Gamma_\nu,$$
$$\tilde{Q}_{mn} = i\Gamma_m\Gamma_n + \frac{i}{m}(1+i\Gamma_6)(\Gamma_m\Gamma_n - \Gamma_n\Gamma_m), \qquad m,n = 1,2,\ldots 5, \tag{15}$$
$$\tilde{\tilde{Q}}_{mn} = \left[\Gamma_6 + \frac{i}{m}(1+i\Gamma_6)\Gamma_\mu p^\mu\right]\tilde{Q}_{mn},$$

where, by the definition,

$$p_{a+3}\Psi(x_0,\boldsymbol{x}) = -i\frac{\partial\Psi(x_0,\boldsymbol{x})}{\partial x_{a+3}} \equiv 0.$$

Proof may be carried out in full analogy with the proof of theorem 1. We only draw attention to the fact, that Q_{mn} satisfies the Lie algebra of the group SU_4.

Let us now consider the group properties of the KDP equation, which describes the particles with spin $s=1$. This equation has the form

$$L_1\Psi(x_0,\boldsymbol{x}) = 0, \qquad L_1 = \beta_\mu p^\mu + m, \tag{16}$$

where β_μ are the ten-row KDP matrices.

It follows from the above that the KDP equation has to possess the more high symmetry then eq. (4) do. This conclusion is supported by the following

Theorem 3. *The KDP equation in invariant under the 26-dimensional Lie algebra A_{26}, basis elements of which belong to the class of differential operators and have the form*

$$P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + i[\beta_\mu,\beta_\nu]_-,$$
$$\lambda_a = [c_{ab}, c_{ac}]_+, \qquad \lambda_{a+3} = c_{bc}, \qquad \lambda_7 = -i[c_{12}c_{23}c_{31} - c_{23}c_{31}c_{12}],$$
$$\lambda_8 = -\frac{i}{\sqrt{3}}(c_{12}c_{23}c_{31} + c_{23}c_{31}c_{12} - 2c_{31}c_{12}c_{23}), \qquad \lambda_{8+a} = c_{ab}c_{0b}, \tag{17}$$
$$\lambda_{11+a} = ic_{0a}, \qquad \lambda_{15} = (c_{12}c_{23}c_{02} - c_{23}c_{31}c_{03}),$$
$$\lambda_{16} = \frac{1}{\sqrt{3}}(c_{12}c_{23}c_{02} + c_{23}c_{31}c_{03} - 2c_{31}c_{12}c_{01}),$$

where

$$c_{\mu\nu} = i[\beta_\mu,\beta_\nu]_- + \frac{1}{m}(a_\mu p_\nu - a_\nu p_\mu), \qquad (a,b,c) - \text{cycl}\,(1,2,3),$$
$$a_\mu = i[\beta_5,\beta_\mu]_- + i\beta_\mu, \qquad \beta_5 = \frac{1}{4!}\varepsilon_{\mu\nu\rho\sigma}\beta_\mu\beta_\nu\beta_\rho\beta_\sigma. \tag{18}$$

Proof. First we shall show, that the operators λ_f satisfy the invariance condition (3). By direct verification one obtains

$$[c_{\mu\nu}, L_1]_- = F^1_{\mu\nu} L - 1, \qquad F^1_{\mu\nu} = (L_1 - 2m)\frac{i}{m^2}(\beta_\mu p_\nu - \beta_\nu p_\mu). \tag{19}$$

It follows from eq. (19) that the operators $c_{\mu\nu}$ (and hence all λ_f) satisfy eq. (3).

The operators (17b) satisfy the commutation relations of the Lie algebra of the $SU_3 \otimes SU_3$ group. This fact may be verified immediately, but the more simple way is to make previously the transformation $\lambda \to V\lambda_f V^{-1} = \lambda'_f$, where

$$V = \exp\left[\frac{i}{m} a_\mu p^\mu\right], \qquad c_{\mu\nu} \to c'_{\mu\nu} = V c_{\mu\nu} V^{-1} = i[\beta_\mu, \beta_\nu]_-. \tag{20}$$

By means of eq. (20) it is not difficult to make sure that the operators λ'_f and $p'_\mu = V p_\mu V^{-1} = p_\mu$, $J'_{\mu\nu} = V J_{\mu\nu} V^{-1} = J_{\mu\nu}$ form the Lie algebra. The theorem in proved.

In conclusion let us note that the main part of the theorems 1, 2, 3 (i.e. the invariance of eqs. (4), (14), (16) under the corresponding algebras) may be proved also by the transformation $L_s \to \tilde{V} L_s \tilde{V}^{-1}$, where $\tilde{V}$ is the integrodifferential operator

$$\tilde{V} = \exp\left[i\frac{S_{4a}p_a}{p}\,\mathrm{arctg}\,\frac{p}{m}\right]\exp\left[\frac{S_{ab}p_c}{p}\,\mathrm{arctgh}\,\frac{p}{E}\right]. \tag{21}$$

The preference of this transformation is that it may be easily generalized for the case of an arbitrary spin, but the basis elements $Q_{\mu\nu}$ of the new invariance algebra have to be integrodifferential operators (as like as (21)). Thus for the Dirac equation one obtains

$$Q_{ab} = i\gamma_a\gamma_b + \frac{i}{m}(\gamma_a p_b - \gamma_b p_a)(1 + i\gamma_4\hat{\varepsilon}), \qquad Q_{0a} = i\hat{\varepsilon}Q_{bc},$$

where $\hat{\varepsilon}$ is the integrodifferential operator of energy sign

$$\hat{\varepsilon} = \frac{H^D}{|H^D|} = (\gamma_0\gamma_a p_a + \gamma_0 m)(m^2 + p^2)^{-1/2}.$$

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