On the equations of motion for particles with arbitrary spin in nonrelativistic mechanics

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It is well known that the electron motion in the external electromagnetic field is described by the relativistic Dirac equation. In this case, in the Foldy–Wouthuysen representation, the Hamiltonian includes the terms corresponding to the interaction of the particle magnetic moment with a magnetic field ($\sim (1/m)(\sigma\mathcal{H})$) and the terms which are interpreted as a spin-orbit coupling ($\sim (\sigma/m^2)\{(\boldsymbol{p}-e\boldsymbol{\mathcal{A}})\times\boldsymbol{E}\}$). Apart from these constituents the Hamiltonian includes the Darwin term ($\sim (1/m^2)\,\mathrm{div}\,\boldsymbol{E}$) [1].

Such a description is in good accordance with the experimental data.

It was shown by Bargman [2] that it is possible to introduce the particle spin in the nonrelativistic quantum mechanics by performing the central extension of the Galilei group. In connection with this Bargman result the problem of finding the motion equations, which are invariant with respect to the extended Galilei group G, arises naturally.

Such a problem has been considered in [3–5]. The equations obtained in [5] have redundant components and, besides, these equations do not describe the spin-orbit and the Darwin couplings if one makes the replacement $p_\mu \to \pi_\mu = p_\mu - e\mathcal{A}_\mu$ in them.

The aim of this note is to find such motion equations for a particle with spin which are invariant relative to the group G, have no redundant components and describe the spin-orbit and the Darwin couplings of the particle with the field. It is reached by the supposition that the free nonrelativistic particle Hamiltonian has two energy signs just as the Dirac Hamiltonian. This is equivalent to the requirement that the theory (equations) be invariant under such a transformation:

$$t \to -t, \qquad T\Psi(t,\boldsymbol{x}) = \tau T\Psi(-t,\boldsymbol{x}), \tag{1}$$

where τ is some unitary matrix. In terms of group-theoretical language this means some extension of the group G to the group $\overline{G} \supset G$, which includes the transformation (1).

In order to find the motion equations in the Schrödinger form

$$i\frac{\partial}{\partial t}\Psi = \mathcal{H}_s\Psi(t,\boldsymbol{x}), \tag{2}$$

which are invariant under the group G and the transformation (1), we have used the method of the work [6], where the same problem has been solved for the full Poincaré group. Equations (2) with an unknown operator function $\mathcal{H}_s$ will be invariant under the group G if the following relations are satisfied:

$$\begin{aligned}
&[\mathcal{H}_s, P_a] = [\mathcal{H}_s, J_{ab}] = 0, \qquad [\mathcal{H}_s, G_a] = iP_a, \qquad [G_a, G_b] = 0,\\
&[P_a, G_b] = i\varepsilon\delta_{ab}m, \qquad [J_{ab}, G_a] = i\delta_{ac}G_b - i\delta_{bc}G_a,\\
&[J_{ab}, J_{cd}] = i(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad}),\\
&[\varepsilon, P_a] = [\varepsilon, J_{ab}] = [\varepsilon, G_a] = 0,
\end{aligned} \tag{3}$$

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$$[T, J_{ab}] = \{T, \varepsilon\} = \{T, G_a\} = [T, P_a] = 0, \qquad \varepsilon = \frac{P_0}{|P_0|},$$

where P_a, $P_0 = \mathcal{H}_s$, J_{ab} and G_a are the generators of the Galilei group, T is the operator of time reflection.

To determine all possible (up to unitary transformations independent of the particle momentum) operators $\mathcal{H}_s$, which satisfy relations (3), we use the following realization of the generators P_a and J_{ab}:

$$P_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_a + S_{ab},$$
$$S_{ab} = \begin{pmatrix} s_{ab} & 0 \\ 0 & s_{ab} \end{pmatrix}, \quad a, b = 1, 2, 3, \tag{4}$$

where s_{ab} are the generators of the irreducible representation $D(s)$ of the O_3 group.

We require the Hamiltonian $\mathcal{H}_s$ to be the differential and Hermitian operator with respect to the usual scalar product

$$(\Psi_1, \Psi_2) = \int d^3x\ \Psi_1^\dagger(t, \boldsymbol{x})\Psi_2(t, \boldsymbol{x}), \tag{5}$$

where Ψ is the $2(2s+1)$-component function, which satisfies eq. (1).

Expanding the operator $\mathcal{H}_s$ in a complete system of the ortoprojectors

$$\Lambda_r = \prod_{\substack{r'=-s \\ r'\neq r}}^{s} \frac{S_p - r}{r' - r}, \qquad S_p = \frac{S_{12}p_3 + S_{31}p_2 + S_{23}p_1}{|\boldsymbol{p}|}, \tag{6}$$

and using the results of the work [6], we have obtained that the Hamiltonians $\mathcal{H}_s$ which satisfy conditions (3) are represented by the formulae

$$\mathcal{H}_{\frac12} = \rho_1\left(m + \frac{\boldsymbol{p}^2}{2m} - \frac{\boldsymbol{p}^2}{m}\sin^2\theta_{\frac12}\right) + \rho_2\frac{\boldsymbol{p}^2}{2m}\sin 2\theta_{\frac12} + \rho_3\sqrt{2}\frac{\boldsymbol{Sp}}{s}\sin\theta_{\frac12}, \tag{7}$$

$$\mathcal{H}_1 = \rho_1\left[m + \frac{\boldsymbol{p}^2}{2m} - \frac{(\boldsymbol{Sp})^2}{s^2 m}\sin^2\theta_1\right] + \rho_2\frac{(\boldsymbol{Sp})^2}{2ms^2}\sin 2\theta_1 + \rho_3\sqrt{2}\frac{\boldsymbol{Sp}}{s}\sin\theta_1, \tag{8}$$

$$\begin{aligned}\mathcal{H}_{\frac32} = {}& \rho_1\left[m + \frac{\boldsymbol{p}^2}{2m} - \frac{(\boldsymbol{Sp})^2}{s^2 m}\sin^2\theta_{\frac32}\right] + \\ & +\rho_2\left\{\frac{\boldsymbol{p}^2}{2m}\left[-\frac18\sin 2\theta_{\frac32} - \frac14\sin\theta_{\frac32}\sqrt{9 - \sin^2\theta_{\frac32}}\right] + \right. \\ & \left. + \frac{(\boldsymbol{Sp})^2}{2ms^2}\left[\frac98\sin 2\theta_{\frac32} - \frac14\sin\theta_{\frac32}\sqrt{9 - \sin^2\theta_{\frac32}}\right]\right\} + \rho_3\sqrt{2}\frac{\boldsymbol{Sp}}{s}\sin\theta_{\frac32},\end{aligned} \tag{9}$$

$$\mathcal{H}_s = \rho_1\left(m + \frac{\boldsymbol{p}^2}{2m}\right), \qquad s > \frac32, \tag{10}$$

where θ_s are arbitrary parameters, ρ_a are the $2(2s+1)$-dimensional Pauli matrices. The Hamiltonians (7)–(10) are the square roots of the operators

$$\mathcal{H}^2 = m^2 + \boldsymbol{p}^2 + \frac{\boldsymbol{p}^4}{4m^2}. \tag{11}$$

The operator $\mathcal{H}_{\frac{1}{2}}$ is the nonrelativistic analogue of the Dirac Hamiltonian for an electron. As will be shown in what follows, the parameters $\cos 2\theta_s$ must be interpreted as the anomalous magnetic moment of the particle.

If $\theta = \pi/4$, eq. (2) with the Hamiltonian (7) may be written in the compact form

$$[m + \gamma_\mu p^\mu]\Psi = i\gamma_4 \frac{\boldsymbol{p}^4}{2m}\Psi, \tag{12}$$

where γ_μ are the 4×4 dimensional Dirac matrices

$$\gamma_0 = \rho_1, \qquad \gamma_a = i\rho_2 \frac{S_a}{s}, \qquad \gamma_4 = \rho_3.$$

The operators $\mathcal{H}_1$ and $\mathcal{H}_{\frac{3}{2}}$ are the nonrelativistic Hamiltonians for the particles with spins $s = 1$ and $s = \frac{3}{2}$ respectively.

The Hamiltonians (7)–(9) are not diagonal. They may be diagonalized by the unitary transformation

$$\mathcal{H}_s \to \mathcal{H}'_s = \mathcal{U}\mathcal{H}_s\mathcal{U}^\dagger, \tag{13}$$

where

$$\mathcal{U} = \sum_r \frac{E + \rho_1 \mathcal{H}_s}{\sqrt{2E\left(E + m + \frac{\boldsymbol{p}^2}{2m}\left(1 - \frac{2r^2}{s^2}\sin^2\theta_s\right)\right)}}\Lambda_r, \qquad E = \sqrt{\boldsymbol{p}^2 + m^2 + \frac{\boldsymbol{p}^2}{4m^2}}.$$

For the case $s > \frac{3}{2}$ the only trivial Hamiltonians (10) which result in no spin effects are possible in our statement of the problem.

The description of the behaviour of the nonrelativistic particle with spin in the external electromagnetic field is made by the replacement $p_\mu \to \pi_\mu$ in eq. (1). In order to preserve the explicit Hermiticity of the Hamiltonians it is necessary to symmetrize previously the formulae (7)–(9) using the identity

$$p_a p_b = \frac{1}{2}(p_a p_b + p_b p_a). \tag{14}$$

After such a symmetrization and the replacement $p_\mu \to \pi_\mu$ in (7)–(9) we obtain

$$\begin{aligned} \mathcal{H}_s = \rho_1 &\left[m + \frac{\boldsymbol{\pi}^2}{2m} - \frac{(\boldsymbol{S\pi})^2}{ms^2}\sin^2\theta_s + \frac{e(\boldsymbol{S\mathcal{H}})}{2ms^2}\sin^2\theta_s\right] + e\varphi + \\ &+ \rho_2\left[a_s\frac{\boldsymbol{\pi}^2}{2m} + b_s\frac{(\boldsymbol{S\pi})^2}{2ms^2} - \frac{e(\boldsymbol{S\mathcal{H}})}{4ms^2}b_s\right] + \rho_3\sqrt{2}\frac{\boldsymbol{S\pi}}{s}\sin\theta_s, \end{aligned} \tag{15}$$

where $\mathcal{H}_a = i\varepsilon_{abc}[\pi_a, \pi_c]$ is the magnitude of the magnetic field,

$$a_{\frac{1}{2}} = \sin 2\theta_{\frac{1}{2}}, \qquad b_{\frac{1}{2}} = 0, \qquad a_1 = 0, \qquad b_a = \sin 2\theta_1,$$

$$a_{\frac{3}{2}} = -\frac{1}{8}\sin 2\theta_{\frac{3}{2}} - \frac{3}{4}\sin\theta_{\frac{3}{2}}\sqrt{1 - \frac{1}{9}\sin^2\theta_{\frac{3}{2}}},$$

$$b_{\frac{3}{2}} = \frac{9}{8}\sin 2\theta_{\frac{3}{2}} - \frac{3}{4}\sin\theta_{\frac{3}{2}}\sqrt{1 - \frac{1}{9}\sin^2\theta_{\frac{3}{2}}}.$$

For $s \leq \frac{3}{2}$ it is impossible to diagonalize the Hamiltonians $\mathcal{H}_s$ for the interacting particles exactly. The diagonalization of the Hamiltonians (15) to terms of the power $1/m^2$ is made by the unitary operator

$$\mathcal{U} = \exp\left[i(B_3^s)\right]\exp\left[i(B_2^s)\right]\exp\left[i(B_1^s)\right],$$

where

$$\begin{aligned}
B_1^s &= -\frac{\sqrt{2}}{2ms}\sin\theta_s\rho_2(\boldsymbol{S}\boldsymbol{\pi}),\\
B_2^s &= \rho_3\frac{1}{4m^2}\left[a_s\boldsymbol{\pi}^2 + b_s\frac{(\boldsymbol{S}\boldsymbol{\pi})^2}{s^2} - \frac{eb_s(\boldsymbol{S}\boldsymbol{\mathcal{H}})}{2s^2} + \frac{e\sqrt{2}\sin\theta_s}{s}\boldsymbol{S}\boldsymbol{E}\right],\\
B_3^s &= \rho_2\frac{1}{8m^3}\left[\frac{\sqrt{2}\sin\theta_s}{s}\{\boldsymbol{S}\boldsymbol{\pi},\boldsymbol{\pi}^2\} + \frac{\sqrt{2}\sin^3\theta_s}{s^3}\{\boldsymbol{S}\boldsymbol{\pi},\boldsymbol{S}\boldsymbol{\mathcal{H}}\} - \right.\\
&\quad\left. - \frac{4\sqrt{2}\sin^3\theta_s}{s^3}(\boldsymbol{S}\boldsymbol{\pi})^3 - iea_s[\boldsymbol{\pi}^2,\varphi] - \frac{ieb_s}{s^2}[(\boldsymbol{S}\boldsymbol{\pi})^2,\varphi]\right].
\end{aligned} \tag{16}$$

After the transformation (16) the Hamiltonian (15) takes the form

$$\begin{aligned}
\mathcal{H}_s &= \rho_1\left[m + \frac{\boldsymbol{\pi}^2}{2m} + \frac{e(\boldsymbol{S}\boldsymbol{\mathcal{H}})}{2ms^2}\sin^2\theta_s\right] + e\varphi + \frac{e\sqrt{2}\sin\theta_s}{4m^2s}\left[-a_s + \frac{b_s}{4s^2}\right]\times\\
&\times\boldsymbol{S}(\boldsymbol{\pi}\times\boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}}\times\boldsymbol{\pi}) + \frac{e\sqrt{2}b_s\sin\theta_s}{24s^3}Q_{ab}\nabla_a\mathcal{H}_b - \frac{\sqrt{2}eb_s\sin\theta_s}{24m^2s^3}\boldsymbol{S}^2\cdot\operatorname{div}\boldsymbol{\mathcal{H}} +\\
&+ \frac{e\sin^2\theta_s}{8m^2s^2}\boldsymbol{S}(\boldsymbol{\pi}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{\pi}) - \frac{e\sin^2\theta_s}{12s^2}Q_{ab}\nabla_aE_b - \frac{e\sin^2\theta_s}{12m^2s^2}\boldsymbol{S}^2\cdot\operatorname{div}\boldsymbol{E},
\end{aligned} \tag{17}$$

where Q_{ab} is the tensor of the quadrupole coupling [7]

$$Q_{ab} = \frac{e}{2m^2}[3\{S_a,S_b\} - \delta_{ab}s(s+1)].$$

To elucidate the physical meaning of the constituents which are included in (17) we consider in detail the case $s = \frac{1}{2}$. Substituting into (17) the spin matrices from the representation $D\left(\frac{1}{2}\right)$ and using (15), we obtain

$$\mathcal{H}^{\frac{1}{2}} = \mathcal{H}^{\text{F-W}} - \mathcal{H}', \tag{18}$$

where

$$\mathcal{H}^{\text{F-W}} = \rho_1\left(m + \frac{\boldsymbol{\pi}^2}{2m} + \frac{e(\boldsymbol{S}\boldsymbol{\mathcal{H}})}{m}\right) + e\varphi + \frac{e\boldsymbol{S}(\boldsymbol{\pi}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{\pi})}{4m^2} - \frac{e}{8m^2}\operatorname{div}\boldsymbol{E} \tag{19}$$

is the Hamiltonian which has been previously obtained by Foldy and Wouthuysen [1] by the diagonalization of the Dirac Hamiltonian, and

$$\begin{aligned}
\mathcal{H}' &= \rho_1\frac{e(\boldsymbol{S}\boldsymbol{\mathcal{H}})}{m}\cos 2\theta_s + \frac{e\boldsymbol{S}(\boldsymbol{\pi}\times\boldsymbol{E} - \boldsymbol{E}\times\boldsymbol{\pi})}{4m^2}\cos 2\theta_s +\\
&\quad + \frac{e}{8m^2}\cos 2\theta_s\operatorname{div}\boldsymbol{E} + \frac{e\sqrt{2}\sin\theta_s\sin 2\theta_s}{2m^2}\boldsymbol{S}(\boldsymbol{\pi}\times\boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}}\times\boldsymbol{\pi}).
\end{aligned} \tag{20}$$

It follows from (18), (20) that the parameter $\cos 2\theta_s$ plays the role of the anomalous magnetic moment of a particle. Choosing in (18), (20) $\cos 2\theta_s = 0$ one obtains the operator which differs from the Foldy–Wouthuysen Hamiltonian by the existence of the additional constituent $e\boldsymbol{S}(\boldsymbol{\pi} \times \boldsymbol{\mathcal{H}} - \boldsymbol{\mathcal{H}} \times \boldsymbol{\pi})/2m^2$, which describes the magnetic spin-orbit coupling of the particle with the external field. The analogous situation takes place for the spins.

The Hamiltonian (17) for $s = 1, \frac{3}{2}$ includes the constituents which correspond to the electrical quadrupole $(\sqrt{2}b_s \sin\theta_s/24s^3) \times Q_{ab}\nabla_a\mathcal{H}_b$ and the magnetic quadrupole $(-(1-\cos^2\theta_s)/24s^2) \times Q_{ab}\nabla_a E_b$ couplings.

Equation (1) with the Hamiltonians (15) can be solved exactly for many important classes of external fields. Thus the energy spectrum of the particle with spin $s = \frac{1}{2}$ in the homogeneous external magnetic field, has the form

$$E = \pm\left[m^2 + \xi^2 + p_3^2 + \left(\frac{\xi^2 + p_3^2}{2m}\right)^2 + \left(\frac{e\mathcal{H}}{2m}\right)^2 - \right.$$
$$\left. -\sigma_3\frac{e\mathcal{H}}{m}\left[m^2\cos^2 2\theta - p_3^2\cos 2\theta + \xi^2 + \left(\frac{\xi^2 + p_3^2}{2m}\right)^2\right]^{1/2}\right]^{1/2}, \tag{21}$$
$$\xi^2 = (2n+1)e\mathcal{H}.$$

Thus, within the framework of the nonrelativistic quantum mechanics, when one uses the equation in the form (1), the successive description of the particle with the spin $s \le \frac{3}{2}$ movement in the external electromagnetic fields is obtained which includes the dipole, quadrupole and the spin-orbit couplings of the particle with the field.

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