On the Poincaré-invariant equations for particles with variable spin and mass

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The Poincaré-invariant equations without redundant components, describing the motion of a particle which can be in different spin and mass states are obtained. The quasi-relativistic equation for a particle with arbitrary spin in external electromagnetic field is found. The group-theoretical analysis of these equations in carried out.

1. Introduction

Many papers are devoted to the problem of construction of relativistic equations for a particle which can be in different spin and mass states. There are various approaches to this problem. The fundamentals of the equation theory describing a particle with an infinite number of spin states have been developed by Majorana [12], and later by Gelfand and Yaglom [8] who used infinite-dimensional unitary representations of the homogeneous Lorentz group $O(1,3)$. (For the actual situation of this theory see e.g. [10].) In other works [1, 9, 17] the wave function of such a particle is supposed to possess some additional variables (inner degrees of freedom) besides three space variables. On the basis of this assumption some relativistic equations for particles with variable spin and mass (for instance, for particles of rotator type) are constructed. By extending the four-dimensional Minkowsky space to the five-dimensional one and by using the representations of the inhomogeneous de Sitter group $P(1,4)$ which includes the Poincaré group $P(1,3)$ as a sub-group, equations were derived [4, 6, 7] which can be interpreted as the motion equations for a particle (or for a system of two particles) with variable discrete spin and variable continuous mass. In contrast to the above-mentioned papers [1, 8, 17], where a particle has always an infinite number of spin states, in the framework of the group $P(1,4)$ the particle has only a finite number of spin states. This is connected with the fact that any irreducible representation of the group $O(4)$ which is a small group of the group $P(1,4)$ is decomposed into a finite direct sum of irreducible representations of the rotation group $O(3)$.

In the present paper, without going beyond the scope of the Poincaré group and using the irreducible representations of the group $O(4)$, we find the relativistic equations of motion in the Schrödinger form, describing a particle which can be in finite spin states. The spin s of such a particle can take the values

$$|j-\tau| \leq s \leq j+\tau, \tag{1.1}$$

where j and τ are the integers or half-integers labelling the irreducible representations of the group $O(4)$. The particle mass m can be either fixed or given by the formulas

$$m = a_1 + b_1 \cdot s(s+1) \qquad \text{or} \qquad m^2 = a_2^2 + b_2^2 \cdot s(s+1), \tag{1.2}$$

where a_1, a_2, b_1, b_2 are constants. The wave functions in the motion equation obtained have $2(2j+1)(2\tau+1)$ components which corresponds to the number of the degrees

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of freedom of the system described. This means that the equations proposed do not contain redundant components and hence do not lead to the well-known difficulties [15, 18, 19].

For the construction of the equations the algebraic (nonspinor) approach developed in [5, 11, 13, 16, 20] is used. The group-theoretical analysis of the equations is performed not in terms of the Lorentz group $O(1,3)$ representations used traditionally but in terms of the Poincaré group $P(1,3) \supset O(1,3)$ representations. This is stipulated by the fact that only the invariants of the group $P(1,3)$ have distinct physical meaning.

2. Statement of the problem

We shall investigate the equations for a particle with variable spin and mass in the form

$$i\frac{\partial\Psi(t,\vec{x})}{\partial t} = H_{j\tau}\Psi(t,\vec{x}), \tag{2.1}$$

where $H_{j\tau}$ is the unknown operator function (the Hamiltonian of a particle) which depends on the momenta and spin matrices, Ψ is the wave function which transforms under the four-dimensional rotations and translations according to the reducible representation of the group $P(1,3)$ and contains $2(2j+1)(2\tau+1)$ components. In the previous paper [5] describing (up to unitary equivalence) all the Poincaré-invariant equations of the form (2.1) for a particle with fixed mass m and fixed spin s we imposed the conditions

$$P_\mu P^\mu \Psi(t,\vec{x}) = m^2\Psi(t,\vec{x}); \tag{2.2}$$

$$W_\mu W^\mu \Psi(t,\vec{x}) = m^2 s(s+1)\Psi(t,\vec{x}) \tag{2.3}$$

on the solutions of equation (2,2), where P_μ is the energy-momentum operator on the mass shell and W_μ is the Pauli–Lubansky vector. If the spin and mass of a particle are not fixed, conditions (2.2), (2.3) should be omitted.

We resolve the problem of finding the Poincaré-invariant equations of the form (2.1) using two different approaches. This is connected with the fact that the equations obtained using the first approach may prove to be convenient in terms of quantum mechanics and the equations derived in the second approach are useful in terms of field theory.

In the first approach (I) the problem is reduced to the following: one has to find all Hamiltonians $H^{\mathrm{I}}_{j\tau}$ such that the operators

$$\begin{aligned} &P^{\mathrm{I}}_0 = H^{\mathrm{I}}_{j\tau}, \qquad P^{\mathrm{I}}_a = p_a = -i\frac{\partial}{\partial x_a}; \\ &J^{\mathrm{I}}_{ab} = x_a p_b - x_a p_b + S_{ab}, \qquad J^{\mathrm{I}}_{0a} = tp_a - \frac{1}{2}[x_a, P^{\mathrm{I}}_0]_+ \end{aligned} \tag{2.4}$$

should satisfy the Poincaré algebra. The matrices S_{ab}, have the following structure

$$S_{ab} = j_c + \tau_c, \qquad j_c = \begin{pmatrix} \hat{j}_c & 0 \\ 0 & \hat{j}_c \end{pmatrix}, \qquad \tau_c = \begin{pmatrix} \hat{\tau}_c & 0 \\ 0 & \hat{\tau}_c \end{pmatrix}, \tag{2.5}$$

where $\hat{j}_c$ and $\hat{\tau}_c$ are the $(2j+1)(2\tau+1)$-dimensional matrices satisfying the commutation relations of the algebra $O(4)$

$$[\hat{j}_a, \hat{j}_b]_- = i\hat{j}_c, \qquad [\hat{\tau}_a, \hat{\tau}_b]_- = i\hat{\tau}_c, \qquad [\hat{j}_a, \hat{\tau}_b]_- = 0;$$

(a, b, c) is the cycle $(1, 2, 3)$,

$$\hat{j}_a^2 = j(j+1), \qquad \hat{\tau}_a^2 = \tau(\tau+1). \tag{2.6}$$

In the second approach (II) the problem[1] is formulated as follows: one has to find all the Hamiltonians $H^{\mathrm{II}}_{j\tau}$ such that the operators

$$\begin{gathered} P_0^{\mathrm{II}} = H^{\mathrm{II}}_{j\tau}, \qquad P_a^{\mathrm{II}} = p_a = -i\frac{\partial}{\partial x_a}; \\ J^{\mathrm{II}}_{ab} = x_a p_b - x_a p_b + S_{ab}, \qquad J^{\mathrm{II}}_{0a} = t p_a - x_a P_0^{\mathrm{II}} + i\sigma_3 S_{4a} \end{gathered} \tag{2.7}$$

should satisfy the Poincaré algebra.

Here

$$\sigma_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad S_{4a} = j_a + \lambda_a, \qquad \lambda_a = \pm\tau_a, \tag{2.8}$$

I is the $(2j+1)(2\tau+1)$-dimensional unit matrix. In particular, when $j = 0$, $\tau = \frac{1}{2}$, the operators (2.4) and (2.7) coincide, since

$$H^{\mathrm{I}}_{0\frac{1}{2}} = H^{\mathrm{II}}_{0\frac{1}{2}} = \sigma_1 m + 2\sigma_3 \vec{\tau}\cdot\vec{p}. \tag{2.9}$$

The operator (2.9) is the Dirac Hamiltonian. For other numbers j and τ, as will be shown later, the representations (2.4) and (2.7) do not coincide. The choice of the representation structures for the algebra $P(1,3)$ in the form (2.4) and (2.7) is stipulated by the fact that on the set of solutions of the Dirac equation the Poincaré algebra may be represented either in the form (2.4) or (2.7). One of the principal differences between the operators (2.4) and (2.7) consists in the fact that all the operators (2.4) are Hermitian with respect to the usual scalar product

$$(\Psi_1, \Psi_2) = \int d^3x\, \Psi_1^\dagger(t, \vec{x})\Psi_2(t, \vec{x}), \tag{2.10}$$

while the operators (2.7) are non-Hermitian with respect to (2.10). But the operators (2.7) are Hermitian with respect to the scalar product

$$(\Psi_1, \Psi_2) = \int d^3x\, \Psi_1^\dagger(t, \vec{x})\hat{M}\Psi_2(t, \vec{x}), \tag{2.11}$$

where $\hat{M}(j, \tau, p)$ is some metric operator whose form will be found later.

Equation (2.1) will obviously be Poincaré-invariant if the operators (2.4) or (2.7) satisfy the Poincaré algebra, as in this case the condition

$$\left[i\frac{\partial}{\partial t} - H_{j\tau}, \exp(i\omega Q)\right]_- \Psi(t, \vec{x}) = 0 \tag{2.12}$$

is satisfied, where Q is an arbitrary generator and ω is the parameter of the group $P(1,3)$. For the infinitisimal transformations this condition takes the form

$$\left[i\frac{\partial}{\partial t} - H_{j\tau}, Q\right]_- \Psi(t, \vec{x}) = 0. \tag{2.13}$$

[1]The operators with indices I and II refer to the approaches I and II, respectively. When these indices are omitted, the corresponding relations are true both for the approach I and the approach II.

On the set $\{\Psi\}$ of the solutions of (2.1) we define the operators of discrete transformations

$$\begin{aligned} &P\Psi(t,\vec{x}) = r_1\Psi(t,-\vec{x}); \\ &T\Psi(t,\vec{x}) = r_2\Psi^*(-t,\vec{x}); \\ &C\Psi(t,\vec{x}) = r_3\Psi^*(t,\vec{x}), \end{aligned} \tag{2.14}$$

where r_1, r_2, r_3 are the matrices which can be chosen (without loss of generality) in the form

$$\begin{aligned} &r_1^{\mathrm{I}} = \sigma_1 \quad \text{or} \quad r_1^{\mathrm{I}} = \hat{I} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad r_1^{\mathrm{II}} = \sigma_1; \\ &r_2^{\mathrm{I}} = r_2^{\mathrm{II}} = \Delta, \qquad r_3^{\mathrm{I}} = r_3^{\mathrm{II}} = \sigma_2\Delta, \qquad \Delta = \begin{pmatrix} \Delta' & 0 \\ 0 & \Delta' \end{pmatrix}, \end{aligned} \tag{2.15}$$

where Δ' is the matrix satisfying the relations

$$\Delta'\hat{j}_a = -\hat{j}_a\Delta', \qquad \Delta'\hat{\tau}_a = -\hat{\tau}_a^*\Delta'. \tag{2.16}$$

The proof of existence of such a matrix is not given here.

The operators P, T, C and the generators P_μ, $J_{\mu\nu}$ must satisfy the relations

$$\begin{aligned} &[P,P_0]_- = [P,P_a]_+ = [P,J_{ab}]_- = [P,J_{0a}]_+ = [C,P_a]_+ = [C,P_0]_+ = 0; \\ &[C,J_{ab}]_+ = [C,J_{0a}]_+ = [T,P_0]_- = [T,P_a]_+ = [T,J_{ab}]_+ = [T,J_{0a}]_- = 0. \end{aligned} \tag{2.17}$$

Let equation (2.1) be invariant under P, T, C-transformations. In this case the relations (2.17) must be added to the Poincaré algebra.

Thus *the problem of finding all the Poincaré-invariant equations for a particle with variable spin and mass is reduced to that of finding the operators $H_{j\tau}$ which satisfy the relations*

$$\begin{aligned} &[H_{j\tau},P_a]_- = [H_{j\tau},J_{0a}]_- = 0, \qquad [P_a,J_{0b}]_- = i\delta_{ab}H_{j\tau}; \\ &[J_{ab},J_{0c}]_- = i(\delta_{ac}J_{0b} - \delta_{bc}J_{0a}); \end{aligned} \tag{2.18}$$

$$[H_{j\tau},J_{0a}]_- = ip_a; \tag{2.19}$$

$$[J_{0a},J_{0b}]_- = -iJ_{ab}; \tag{2.20}$$

$$[P,H_{j\tau}]_- = [C,H_{j\tau}]_+ = [T,H_{j\tau}]_- = 0. \tag{2.21}$$

3. Explicit form of the operators $H_{j\tau}^{\mathrm{I}}$

In this section we solve problem I, i.e. we find those operators $H_{j\tau}^{\mathrm{I}}$ which satisfy the set of relations (2.18)–(2.21) in the case where the representation of the Poincaré algebra has the structure (2.4).

The squared-mass operator for the representation (2.4) (which, generally speaking, is not a multiple of the unit one) is of the form

$$M^2 = P_\mu P^\mu = (H_{j\tau}^{\mathrm{I}})^2 - p^2. \tag{3.1}$$

The commutativity of M^2 with the operators (2.4) and (2.14) yields the following relations

$$[P_a, M^2]_- = [x_a, M^2]_- = [(j_a + \tau_a), M^2]_- = 0; \tag{3.2}$$

$$[P, M^2]_- = [C, M^2]_- = [T, M^2]_- = 0; \tag{3.3}$$

$$[H^{\rm I}_{j\tau}, M^2]_- = 0. \tag{3.4}$$

From (2.14), (2.15), (2.16) it follows that the general form of the operator M^2 satisfying the conditions (3.2), (3.3) is given by the formula

$$M^2 = a_0 + a_1(\vec{j}\cdot\vec{\tau}) + a_2(\vec{j}\cdot\vec{\tau})^2 + \cdots, \tag{3.5}$$

where $a_0, a_1, \ldots$ are real numbers. The series (3.5) contains only a finite number of terms, as follows from the relation

$$\prod_{|j-\tau|\le s\le j+\tau} [\vec{j}\cdot\vec{\tau} - \varkappa_s] = 0, \tag{3.6}$$

where $\varkappa_s$ is the eigenvalue of the operator $\vec{j}\cdot\vec{\tau}$. Between the numbers $\varkappa_s$ and j, τ, s there is the relation

$$2\varkappa_s = -j(j+1) - \tau(\tau+1) + s(s+1). \tag{3.7}$$

From (3.6) it is seen that the particle system capable of being in different mass states may be compared with the representation (2.4) and hence with the equation (2.1). As will be shown later, with a proper choice of the coefficients a_n in (3.5) we can obtain the mass formula (1.2).

In order to find the explicit structure of the operator $H^{\rm I}_{j\tau}$ satisfying relations (2.18)–(2.21) and condition (3.1) we expand it in a complete system of ortoprojectors

$$H^{\rm I}_{j\tau} = \sum_{j_3\tau_3} \left(d^{\rm I}_{j_3\tau_3}(p) + \sigma_1 g^{\rm I}_{j_3\tau_3}(p) + \sigma_2 h^{\rm I}_{j_3\tau_3}(p) + \sigma_3 f^{\rm I}_{j_3\tau_3}(p)\right)\Lambda_{j_3}\Lambda_{\tau_3}, \tag{3.8}$$

where

$$\begin{gathered}
\Lambda_{j_3} = \prod_{j_3\ne j_3'} \frac{j_p - j_3'}{j_3 - j_3'}; \qquad \Lambda_{\tau_3} = \prod_{\tau_3\ne\tau_3'} \frac{\tau_p - \tau_3'}{\tau_3 - \tau_3'}; \\
j_p = \frac{\vec{j}\cdot\vec{p}}{p}, \qquad \tau_p = \frac{\vec{\tau}\cdot\vec{p}}{p}, \qquad j_3 = -j, -j+1, \ldots, j, \quad \tau_3 = -\tau, -\tau+1, \ldots, \tau,
\end{gathered} \tag{3.9}$$

$d^{\rm I}_{j_3\tau_3}$, $h^{\rm I}_{j_3\tau_3}$, $f^{\rm I}_{j_3\tau_3}$, $g^{\rm I}_{j_3\tau_3}$ are the unknown functions which depend on p and M.

It is easy to see that the operators (3.9) are the orthoprojectors on the proper subspaces of the operators j_p and τ_p, i.e. they satisfy the relations

$$\begin{gathered}
\Lambda_{j_3}\Lambda_{j_3'} = \delta_{j_3 j_3'}\Lambda_{j_3'}; \qquad \sum_{j_3=-j}^{j} \Lambda_{j_3} = 1; \qquad j_p = \sum_{j_3=-j}^{j} j_3\Lambda_{j_3}; \\
\Lambda_{\tau_3}\Lambda_{\tau_3'} = \delta_{\tau_3\tau_3'}\Lambda_{\tau_3'}; \qquad \sum_{\tau_3=-\tau}^{\tau} \Lambda_{\tau_3} = 1; \qquad \tau_p = \sum_{\tau_3=-\tau}^{\tau} \tau_3\Lambda_{\tau_3}.
\end{gathered} \tag{3.10}$$

The condition (2.21) is satisfied if

$$\begin{aligned} &d^{\mathrm{I}}_{j_3\tau_3} = h^{\mathrm{I}}_{j_3\tau_3} = 0; \quad f^{\mathrm{I}}_{j_3\tau_3} = f^{\mathrm{I}}_{-j_3-\tau_3}; \quad g^{\mathrm{I}}_{j_3\tau_3} = g^{\mathrm{I}}_{-j_3-\tau_3} \quad \text{for} \quad r^{\mathrm{I}}_1 = \hat{I}, \\ &d^{\mathrm{I}}_{j_3\tau_3} = h^{\mathrm{I}}_{j_3\tau_3} = 0; \quad f^{\mathrm{I}}_{j_3\tau_3} = -f^{\mathrm{I}}_{-j_3-\tau_3}; \quad g^{\mathrm{I}}_{j_3\tau_3} = g^{\mathrm{I}}_{-j_3-\tau_3} \quad \text{for} \quad r^{\mathrm{I}}_1 = \sigma_1. \end{aligned} \tag{3.11}$$

In order that (3.4) be fulfilled with an arbitrary choice of the coefficients a_n in (3.6), it is necessary to set

$$f^{\mathrm{I}}_{j_3\tau_3} = \varphi_1(j_3+\tau_3), \qquad g^{\mathrm{I}}_{j_3\tau_3} = \varphi_2(j_3+\tau_3), \tag{3.12}$$

i.e. $f^{\mathrm{I}}_{j_3\tau_3}$, $g^{\mathrm{I}}_{j_3\tau_3}$ may depend only on the sum of the indices. If (3.12) is not fulfilled, then the operator M^2 commutes with the Hamiltonian (the condition (3.4)) only in the case where $a_1 = a_2 = a_3 = \cdots = 0$.

The condition (3.1) imposes the additional restriction

$$\left(f^{\mathrm{I}}_{j_3\tau_3}\right)^2 + \left(g^{\mathrm{I}}_{j_3\tau_3}\right)^2 = p^2 + M^2 \tag{3.13}$$

on the functions $f^{\mathrm{I}}_{j_3\tau_3}$ and $g^{\mathrm{I}}_{j_3\tau_3}$. Direct verification shows that if the conditions (3.4), (3.13) are fulfilled, the relations (2.18), (2.19), (2.21) are satisfied. Thus, it remains to satisfy the relation (2.18) which together with (3.11)–(3.13) will determine the ultimate structure of the operator $H^{\mathrm{I}}_{j\tau}$, i.e. the explicit form of the functions $f^{\mathrm{I}}_{j_3\tau_3}$ and $g^{\mathrm{I}}_{j_3\tau_3}$.

The relations (2.20) for the operator (2.4) may be reduced to the form [5]

$$\left[[H^{\mathrm{I}}_{j\tau}, x_a]_-, [H^{\mathrm{I}}_{j\tau}, x_b]_-\right]_- = -4iS_{ab}. \tag{3.14}$$

Substituting (3.8) into (3.14), using the commutation relations (A.1) and taking into account the linear independence of the vectors (A.2), we obtain the following equations for $f^{\mathrm{I}}_{j_3\tau_3}$, $g^{\mathrm{I}}_{j_3\tau_3}$

$$g^{\mathrm{I}}_{j_3\tau_3}g^{\mathrm{I}}_{j_3+1\tau_3} + f^{\mathrm{I}}_{j_3\tau_3}f^{\mathrm{I}}_{j_3+1\tau_3} = g^{\mathrm{I}}_{j_3\tau_3}g^{\mathrm{I}}_{j_3\tau_3+1} + f^{\mathrm{I}}_{j_3\tau_3}f^{\mathrm{I}}_{j_3\tau_3+1} = M^2 - p^2. \tag{3.15}$$

From (3.13) it is seen that the functions $g^{\mathrm{I}}_{j_3\tau_3}$, $f^{\mathrm{I}}_{j_3\tau_3}$ can be represented in the form

$$f^{\mathrm{I}}_{j_3\tau_3} = E\sin\varphi_{j_3\tau_3}, \qquad g^{\mathrm{I}}_{j_3\tau_3} = E\cos\varphi_{j_3\tau_3}, \qquad E = \sqrt{p^2+M^2}. \tag{3.16}$$

Inserting (3.16) into (3.15), we obtain for $\varphi_{j_3\tau_3}$ the following recurrence formulas

$$\varphi_{j_3+1\tau_3} = \varphi_{j_3\tau_3} \pm 2\theta^{\mathrm{I}}, \qquad \varphi_{j_3\tau_3+1} = \varphi_{j_3\tau_3} \pm 2\theta^{\mathrm{I}}, \qquad \theta^{\mathrm{I}} = \operatorname{arctg}\frac{P}{M}. \tag{3.17}$$

By means of (3.16), (3.17) we can define all the coefficients $g^{\mathrm{I}}_{j_3\tau_3}$, $f^{\mathrm{I}}_{j_3\tau_3}$ of the operator (3.8) if at least one of the functions of the set $f^{\mathrm{I}}_{j_3\tau_3}$ (or $g^{\mathrm{I}}_{j_3\tau_3}$) is known. This initial function can be found from the relations (3.17), (3.11) which, taking into account (3.16), can be written in the form:

$$\varphi_{j_3\tau_3} = \begin{cases} \varphi_{-j_3-\tau_3} & \text{for} \quad r^{\mathrm{I}}_1 = I; \\ -\varphi_{-j_3-\tau_3} & \text{for} \quad r^{\mathrm{I}}_1 = \sigma_1. \end{cases} \tag{3.18}$$

Finally, we are led to the following result: the operator (3.8) with coefficient functions (3.16), (3.17) satisfies the relations (2.18)–(2.21) and this means that the problem I is completely solved. The equation (2.1) with such $H^{\rm I}_{j\tau}$ will be invariant with respect to the full Poincaré group $\widetilde{P}(1,3)$.

Let as present the simplest solutions for the system of the recurrence relations (3.17), (3.18) (for the details of the solutions for equations such as (3.15) see [5])

$$\varphi_{j_3\tau_3}=\begin{cases}(-1)^{j_3+\tau_3+\frac12}\theta^{\rm I}, & j+\tau \text{ — half-integers},\\ (-1)^{j_3+\tau_3}\theta^{\rm I}, & j+\tau \text{ — integers},\\ 2(j_3+\lambda_3)\theta^{\rm I},\ \lambda_3=\pm\tau_3, & j+\tau \text{ — arbitrary numbers.}\end{cases} \tag{3.19}$$

Substituting (3.19), (3.11), (3.16) into (3.8), we obtain

$$H^{\rm I}_{j\tau}=\begin{cases}\sigma_1M+\sigma_3\sum\limits_{j_3\tau_3}p(-1)^{j_3+\tau_3+\frac12}\Lambda_{j_3}\Lambda_{\tau_3}, & j+\tau \text{ — half-integers}, \qquad (3.20a)\\ \sigma_1M+\sigma_3\sum\limits_{j_3\tau_3}p(-1)^{j_3+\tau_3}\Lambda_{j_3}\Lambda_{\tau_3}, & j+\tau \text{ — integers}, \qquad (3.20b)\\ E\sum\limits_{j_3\tau_3}\left\{\sigma_1\cos\left[2(j_3+\lambda_3)\theta^{\rm I}\right]+\right. & \\ \qquad \left.+\sigma_3\sin\left[2(j_3+\lambda_3)\theta^{\rm I}\right]\right\}\Lambda_{j_3}\Lambda_{\tau_3}, & j+\tau \text{ — arbitrary numbers.}\end{cases}$$

Choosing other solutions of the system (3.17), (3.18) we arrive at Hamiltonians which are unitarily equivalent to (3.20) but differ from them in form.

Let us write the explicit expressions for the operators (3.20) in terms of $\vec{j}\cdot\vec{p}$, $\vec{\tau}\cdot\vec{p}$ for $j,\tau\le 1$. Using (3.9), (3.20a), we find

$$\begin{aligned}&H^{\rm I}_{\frac12\frac12}=\sigma_1M+2\sigma_3(\vec{\tau}\cdot\vec{p})(\vec{j}\cdot\vec{p})p^{-1};\\ &H^{\rm I}_{\frac12 1}=H^{\rm I}_{\frac12 0}-4\sigma_3(\vec{j}\cdot\vec{p})(\vec{\tau}\cdot\vec{p})^2p^{-2};\\ &H^{\rm I}_{11}=-H^{\rm I}_{00}+H^{\rm I}_{10}+H^{\rm I}_{01}+2\sigma_3(\vec{j}\cdot\vec{p})^2(\vec{\tau}\cdot\vec{p})^2p^{-3},\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}&H^{\rm I}_{00}=\sigma_1M+\sigma_3p; \qquad H^{\rm I}_{\frac12 0}=\sigma_1M+2\sigma_3(\vec{j}\cdot\vec{p});\\ &H^{\rm I}_{01}=H^{\rm I}_{00}-2\sigma_3(\vec{\tau}\cdot\vec{p})^2p^{-1}; \qquad H^{\rm I}_{10}=H^{\rm I}_{00}-2\sigma_3(\vec{j}\cdot\vec{p})^2p^{-1}\end{aligned} \tag{3.22}$$

are the Hamiltonians of particles with the fixed spin found earlier in [5].

Substituting (3.10) into (3.20b), we obtain

$$\begin{aligned}&H^{\rm I}_{\frac12\frac12}=H^{\rm I}_{\frac12 0}\sigma_3H^{\rm I}_{0\frac12}E^{-1};\\ &H^{\rm I}_{\frac12 1}=H^{\rm I}_{\frac12 0}\sigma_3H^{\rm I}_{0\frac12}E^{-1};\\ &H^{\rm I}_{11}=H^{\rm I}_{10}\sigma_3H^{\rm I}_{01}E^{-1},\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}&H^{\rm I}_{0\frac12}=\sigma_1M+2\sigma_3(\vec{\lambda}\cdot\vec{p}); \qquad H^{\rm I}_{\frac12 0}=\sigma_1M+2\sigma_3(\vec{j}\cdot\vec{p});\\ &H^{\rm I}_{01}=\sigma_1E+2(\vec{\lambda}\cdot\vec{p})[\sigma_3M-\sigma_1(\vec{\lambda}\cdot\vec{p})]E^{-1};\\ &H^{\rm I}_{10}=\sigma_1E+2(\vec{j}\cdot\vec{p})[\sigma_3M-\sigma_1(\vec{j}\cdot\vec{p})]E^{-1}\end{aligned} \tag{3.24}$$

also coincide with the Hamiltonians obtained in [5]. The operator $H^{\rm I}_{\frac{1}{2}0}$ is the Dirac Hamiltonian.

The equation (2.1) together with the Hamiltonians $H^{\rm I}_{\frac{1}{2}\frac{1}{2}}$, $H^{\rm I}_{\frac{1}{2}1}$, $H^{\rm I}_{11}$ describes the particles with spins 0 and 1, $\frac{1}{2}$ and $\frac{3}{2}$, 0, 1 and 2, respectively. As it is seen from (3.21)–(3.24), the operators $H^{\rm I}_{j\tau}$ can be expressed by $H^{\rm I}_{j-1\tau}$, $H^{\rm I}_{j\tau-1}$. *Thus, the form of the Hamiltonian for arbitrary j and τ is completely defined by the Hamiltonians for $j,\tau = 0, \frac{1}{2}$.*

4. Explicit form of the operators $H^{\rm II}_{j\tau}$

In this section we solve the problem II, i.e. find all the operators $H^{\rm II}_{j\tau}$ satisfying the system (2.18)–(2.21), when the representation of the algebra $P(1,3)$ has the structure (2.7).

Using the representation (2.7) for the special case when $j = 0$, τ is an arbitrary number, Weaver, Hammer, and Good, and then, for a more general statement of the problem, Mathews [13, 16], found equations of the type (2.1) for a particle with fixed spin and mass. The results given below are a generalization of [13, 16, 20] to the case of particles with variable spin and mass.

By analogy with the previous section, we seek $H^{\rm II}_{j\tau}$ in the form

$$H^{\rm II}_{j\tau} = \sum_{j_3\tau_3} (\sigma_1 g^{\rm II}_{j_3\tau_3} + \sigma_3 f^{\rm II}_{j_3\tau_3})\Lambda_{j_3}\Lambda_{\tau_3}, \tag{4.1}$$

where the unknown functions $g^{\rm II}_{j_3\tau_3}$, $f^{\rm II}_{j_3\tau_3}$ depending only on p, M have the following properties

$$f^{\rm II}_{j_3\tau_3} = -f^{\rm II}_{-j_3-\tau_3}; \qquad g^{\rm II}_{j_3\tau_3} = g^{\rm II}_{-j_3-\tau_3}; \tag{4.2}$$

$$\left(f^{\rm II}_{j_3\tau_3}\right)^2 + \left(g^{\rm II}_{j_3\tau_3}\right)^2 = p^2 + M^2. \tag{4.3}$$

We can verify directly that the relations (2.18), (2.20), (2.21) are fulfilled, provided (4.1), (4.3), (2.19) are satisfied. Using (2.7), we reduce (2.19) to the form

$$-[H^{\rm II}_{j\tau}, x_a]_- H^{\rm II}_{j\tau} + i(j_a + \lambda_a)[H^{\rm II}_{j\tau}, \sigma_3]_- + i[H^{\rm II}_{j\tau}, (j_a + \lambda_a)]_- \sigma_3 = ip_a. \tag{4.4}$$

Substituting (4.1) into (4.4), using the values of the commutators Λ_{j_3}, Λ_{τ_3} with x_a, j_a, τ_a, (A.1) and equating linearly independent terms, we obtain the following set of equations

$$\begin{aligned}
&g^{\rm II}_{j_3\tau_3} g^{\rm II}_{j_3+1\tau_3} + f^{\rm II}_{j_3\tau_3} f^{\rm II}_{j_3+1\tau_3} = E^2 + p\left(f^{\rm II}_{j_3+1\tau_3} - f^{\rm II}_{j_3\tau_3}\right);\\
&g^{\rm II}_{j_3\tau_3} g^{\rm II}_{j_3\tau_3+1} + f^{\rm II}_{j_3\tau_3} f^{\rm II}_{j_3\tau_3+1} = E^2 + \varepsilon p\left(f^{\rm II}_{j_3\tau_3+1} - f^{\rm II}_{j_3\tau_3}\right);\\
&g^{\rm II}_{j_3\tau_3}\left(f^{\rm II}_{j_3+1\tau_3} + p\right) = g^{\rm II}_{j_3+1\tau_3}\left(f^{\rm II}_{j_3\tau_3} - p\right);\\
&g^{\rm II}_{j_3\tau_3}\left(f^{\rm II}_{j_3\tau_3+1} + \varepsilon p\right) = g^{\rm II}_{j_3\tau_3+1}\left(f^{\rm II}_{j_3\tau_3} - \varepsilon p\right);\\
&g^{\rm II}_{j_3\tau_3}\frac{\partial g^{\rm II}_{j_3\tau_3}}{\partial p} - f^{\rm II}_{j_3\tau_3}\frac{\partial f^{\rm II}_{j_3\tau_3}}{\partial p} = 2p(j_3 + \lambda_3), \qquad \varepsilon = \frac{\lambda_3}{\tau_3} = \pm 1.
\end{aligned} \tag{4.5}$$

In the case $j = 0$, the set (4.5) coincides with the set of equations for the coefficient functions obtained in [13, 16].

Omitting rather cumbersome calculations, we give the solution of this system

$$f^{\rm II}_{j_3\tau_3} = E\,{\rm th}\,\left[2(j_3+\lambda_3)\theta^{\rm II}\right], \qquad \theta^{\rm II} = {\rm arcth}\,\frac{p}{E},$$
$$g^{\rm II}_{j_3\tau_3} = E\,{\rm sech}\,\left[2(j_3+\lambda_3)\theta^{\rm II}\right]. \tag{4.6}$$

By means of (4.6) and (4.1) we obtain the following explicit form of the Hamiltonians $H^{\rm II}_{j\tau}$ for the representation (2.7)

$$H^{\rm II}_{j\tau} = E\sum_{j_3\tau_3}\left\{\sigma_1\,{\rm sech}\,\left[2(j_3+\lambda_3)\theta^{\rm II}\right] + \sigma_3\,{\rm th}\,\left[2(j_3+\lambda_3)\theta^{\rm II}\right]\right\}\Lambda_{j_3}\Lambda_{\tau_3}. \tag{4.7}$$

Formula (4.7) gives the solution to the problem II. Let us write out the Hamiltonians $H^{\rm II}_{j\tau}$ for $j,\tau\le 1$. According to (4.7), (3.12), we have

$$H^{\rm II}_{\frac12\frac12} = H^{\rm II}_{\frac12 0}\sigma_1 H^{\rm II}_{0\frac12}E^{-1};$$
$$H^{\rm II}_{\frac12 1} = H^{\rm II}_{\frac12 0}\sigma_1 H^{\rm II}_{01}E^{-1}; \tag{4.8}$$
$$H^{\rm II}_{11} = H^{\rm II}_{10}\sigma_1 H^{\rm II}_{01}E^{-1},$$

where

$$H^{\rm II}_{\frac12 0} = \sigma_1 M + 2\sigma_3(\vec{j}\cdot\vec{p}); \qquad H^{\rm II}_{0\frac12} = \sigma_1 M + 2\sigma_3(\vec{\lambda}\cdot\vec{p});$$
$$H^{\rm II}_{01} = H^{\rm II}_{00} - 2E(\vec{\lambda}\cdot\vec{p})[\sigma_1(\vec{\lambda}\cdot\vec{p}) - \sigma_3 E](E^2+p^2)^{-1}; \qquad H^{\rm II}_{00} = \sigma_1 E; \tag{4.9}$$
$$H^{\rm II}_{10} = H^{\rm II}_{00} - 2E(\vec{j}\cdot\vec{p})[\sigma_1(\vec{j}\cdot\vec{p}) - \sigma_3 E](E^2+p^2)^{-1}$$

are the Hamiltonians of the particles with fixed spin and mass obtained in [13, 16, 20].

5. Transition to the canonical representation

To give an unambiguous answer to the question what particles are described by the equation (2.1) with the Hamiltonians obtained in Sections 3, 4 it is necessary to find the explicit form of the Cazimir operators $W_\mu W^\mu$ and $P_\mu P^\mu$ of the group $P(1,3)$. These operators prove to be of the simplest form in the canonical representation of the Foldy–Shirokov type. Let us pass from the representations (2.4) and (2.7) to the canonical one. Such a transition for the representation (2.4) is performed by means of the operator

$$U^{\rm I}_{j\tau} = \exp\left(i\sigma_2\sum_{j_3\tau_3}\frac12\varphi_{j_3\tau_3}\Lambda_{j_3}\Lambda_{\tau_3}\right). \tag{5.1}$$

For the Hamiltonian (3.20a) the operator (5.1) has the simple form

$$U^{\rm I}_{j\tau} = \frac{E+\sigma_1 H^{\rm I}_{j\tau}}{\sqrt{2E(E+M)}}, \tag{5.2}$$

for the Hamiltonians (3.20b) the operator (5.1) is of the form

$$U^{\rm I}_{j\tau} = \exp\left[i\sigma_2\frac{(\vec{j}+\vec{\lambda})}{p}\vec{p}\theta^{\rm I}\right]. \tag{5.3}$$

The transition from the representation (2.7) to the canonical one is performed by the isometric operator

$$U^{\mathrm{II}}_{j\tau} = \sqrt{\frac{E}{M}}\sum_{j_3\tau_3}\left\{\mathrm{ch}\,\left[(j_3+\lambda_3)\theta^{\mathrm{II}}\right] - i\sigma_2\,\mathrm{sh}\,\left[(j_3+\lambda_3)\theta^{\mathrm{II}}\right]\right\}\times$$
$$\times\mathrm{sech}\,\left[2(j_3+\lambda_3)\theta^{\mathrm{II}}\right]\Lambda_{j_3}\Lambda_{\tau_3}; \tag{5.4}$$
$$\left(U^{\mathrm{II}}_{j\tau}\right)^{-1} = \sqrt{\frac{E}{M}}\sum_{j_3\tau_3}\left\{\mathrm{ch}\,\left[(j_3+\lambda_3)\theta^{\mathrm{II}}\right] + i\sigma_2\,\mathrm{sh}\,\left[(j_3+\lambda_3)\theta^{\mathrm{II}}\right]\right\}\Lambda_{j_3}\Lambda_{\tau_3}.$$

The operators (2.4) and (2.7) under the transformations (5.1) and (5.4) take the form

$$\begin{aligned} &P_a = p_a = -i\frac{\partial}{\partial x_a}, \qquad J_{ab} = x_a p_b - x_b p_b + S_{ab};\\ &P_0 = \sigma_1 E, \qquad J_{0a} = tp_a - \frac{1}{2}\sigma_1[x_a,E]_+ - \sigma_1\frac{S_{ab}P_b}{E+M}. \end{aligned} \tag{5.5}$$

The operators (5.5) are Hermitian operators with respect to the usual scalar product

$$(\Phi_1,\Phi_2) = \int d^3x\ \Phi_1^\dagger(,\vec{x})\Phi_2(t,\vec{x}), \tag{5.6}$$

where the functions Φ are related to the solutions of the equation (2.1) by

$$\Phi(t,\vec{x}) = U^{\mathrm{I}}_{j\tau}\Psi^{\mathrm{I}}(t,\vec{x}) = U^{\mathrm{II}}_{j\tau}\Psi^{\mathrm{II}}(t,\vec{x}). \tag{5.7}$$

In the representation (5.5) the Casimir operators are expressed by the matrices of spin (2.5)

$$W_\mu W^\mu = M^2(\vec{j}+\vec{\tau})^2 = M^2\left[j(j+1)+\tau(\tau+1)+2(\vec{j},\vec{\tau})\right], \tag{5.8}$$

$$P_\mu P^\mu = M^2, \tag{5.9}$$

where the squared-mass operator in the general case is given by formula (3.6). It follows from (5.8) that equation (2.1) in the case of arbitrary j and τ describes a particle whose spin can have the values $|j-\tau|\le s\le j+\tau$. Equation (2.1) describes a particle with the fixed spin S_0, if one imposes additional relativistically-invariant conditions on the solutions $\Psi(t,\vec{x})$. These conditions have the form

$$\left(U^{\mathrm{I}}_{j\tau}\right)^\dagger(\vec{j}\cdot\vec{\tau})U^{\mathrm{I}}_{j\tau}\Psi^{\mathrm{I}}(t,\vec{x}) = \varkappa_s\Psi^{\mathrm{I}}(t,\vec{x}); \tag{5.10}$$

$$\left(U^{\mathrm{II}}_{j\tau}\right)^{-1}(\vec{j}\cdot\vec{\tau})U^{\mathrm{II}}_{j\tau}\Psi^{\mathrm{II}}(t,\vec{x}) = \varkappa_s\Psi^{\mathrm{II}}(t,\vec{x}). \tag{5.11}$$

The solutions of equations (2.1) and (5.10), (2,1) and (5.11) are eigenfunctions of the operators $W_\mu W^\mu$ and $P_\mu P^\mu$

$$W_\mu W^\mu\Psi(t,\vec{x}) = m_s^2 s(s+1)\Psi(t,\vec{x}), \qquad P_\mu P^\mu\Psi(t,\vec{x}) = m_s^2\Psi(t,\vec{x}), \tag{5.12}$$

where

$$m_s = a + bs(s+1), \tag{5.13}$$

if we set in (3.6)

$$a_0 = (a + b\eta)^2, \qquad a_1 = 4b(a + b\eta), \qquad a_2 = 4b^2,$$
$$a_3 = a_4 = \cdots = 0, \qquad \eta = j(j+1) + \tau(\tau+1), \tag{5.14}$$

or

$$m_s^2 = a^2 + b^2 s(s+1) \tag{5.15}$$

if in (3.6)

$$a_0 = a + b\eta, \qquad a_1 = 2b, \qquad a_2 = a_3 = \cdots = 0. \tag{5.16}$$

Choosing the other values for the coefficients a_n in (3.6), one obtains another dependence of the mass on spin.

At the conclusion of this section we present the explicit form of the metric operator for the representation (2.7)

$$\hat{M}_{j\tau} = \left(U_{j\tau}^{\rm II}\right)^\dagger U_{j\tau}^{\rm II} = \frac{E}{M}\sum_{j_3\tau_3} \text{sech}\,\left[2(j_2 + \lambda_3)\theta^{\rm II}\right]\Lambda_{j_3}\Lambda_{\tau_3}. \tag{5.17}$$

For the problem of external motion of a charged particle in an external electromagnetic field to be solved, it is more convenient to use the equation (2.1) with the Hamiltonian $H_{j\tau}^{\rm I}$, and for the second quantization the equation (2.1) with the Hamiltonian $H_{j\tau}^{\rm II}$ is more preferable. The first problem is considered in the next section, the second one is solved in [14] for the case $j = 0$.

Notation. When formulating the problems I and II, we restricted ourselves to the case where the matrices S_{kl} in (2.4), (2.7) form the irreducible representation $D(j,\tau)$ or the direct sum $\sum\limits_{s=|j-\tau|}^{j+\tau} \oplus D(s,0)$ of group $O(4)$. Using the reducible representations with more complex structure, we obtain qualitatively new equations of the type (1).

Let, for instance, the matrices S_{kl} in (2.7) realize the representation $D(0, s-\frac{1}{2})$ $\otimes[D(\frac{1}{2},0) \oplus D(0,\frac{1}{2})]$ of the group $O(4)$. This means that these matrices have been represented in the form

$$S_{kl} = \hat{S}_{kl} + \frac{i}{2}\gamma_k\gamma_l, \qquad [S_{kl}, \gamma_\lambda]_- = 0, \qquad \gamma_k\gamma_l + \gamma_l\gamma_k = -2\delta_{kl}, \tag{5.18}$$

where $\hat{S}_{kl}$ are the generators of the representation $D(0, s-\frac{1}{2})$. In this case the Hamiltonian H, which satisfies the conditions (2.17)–(2.21), has the form

$$H = \gamma_0\gamma_a p_a + \gamma_0 m, \qquad \gamma_0 = \sigma_3\gamma_4, \tag{5.19}$$

and the Poincaré-invariant supplementary condition (5.11), which selects the spin S, may be written down as

$$[(m + \gamma_\mu p^\mu)[S_{\mu\nu}S^{\mu\nu} - 2s^2](1 - i\gamma_4) - 4ms]\psi = 0. \tag{5.20}$$

The system of equations (1) with the Hamiltonian (5.19) and (5.20) describes the free motion of a particle with the fixed spin s. When the interaction is included into these equations by the standard substitution $p_\mu \to \pi_\mu = p_\mu - eA_\mu$, no difficulties typical for the equations of Bargman–Wigner, Rarita–Schwinger and other equations [18, 19] arise. This question will be discussed in the next paper.

6. The equation for a charged particle in an external electromagnetic field

The generalization of equation (2.1) to the case of a charged particle in an external electromagnetic field proves to be a difficult problem owing to the complicated dependence of the Hamiltonians $H_{j\tau}$ on momenta. In this section the problem is solved, the assumption being that the particle momenta are small compared with the particles masses which are considered to be fixed. With the help of successive unitary transformations we found the equation for the positive energy states of the particle with arbitrary spin and fixed mass just as it had been done by Foldy and Wouthuysen [3] for $s = \frac{1}{2}$.

For $p \ll m$ the Hamiltonians $H_{j\tau}$ have been represented as series in powers of $1/m$ (Compton wavelength). Restricting ourselves to the constituents of power $1/m^2$ and using the relation

$$\sum_{j_3\tau_3}(j_2-\tau_3)^l\Lambda_{j_3}\Lambda_{\tau_3} = \sum_a\left(\frac{S_{4a}p_a}{p}\right)^l, \qquad S_{4a} = j_a - \tau_a, \quad l = 0, 1, \ldots, \tag{6.1}$$

we write the operators (3.20b), (4.7) in the form

$$\begin{aligned} H^{\alpha}_{j\tau} = \sigma_1\left[m + \frac{1}{m}\sum_{a,b} d_{ab}(p_ap_b - p_bp_a)\right] + \\ +\sigma_3\left[\sum_a b_ap_a + \frac{1}{m^2}h^{\alpha}(\vec{p})\right] + o\left(\frac{1}{m^3}\right), \end{aligned} \tag{6.2}$$

where

$$\alpha = \mathrm{I},\ \mathrm{II}; \qquad d_{ab} = \frac{1}{4}\delta_{ab} - S_{4a}S_{4b}, \qquad b_a = 2S_{4a},$$
$$h^{\mathrm{I}}(\vec{p}) = -2h^{\mathrm{II}}(\vec{p}) = \frac{2}{3}S_{4a}d_{bc}p_ap_bp_c; \qquad a, b, c = 1, 2, 3.$$

It is seen from (6.2) that the Hamiltonians (3.20b), (4,7) *coincide in the approximation to terms of power* $1/m$ *and are polinomials in* p_a. Equation (2.1) with the Hamiltonian (6.2) describes the free motion of a particle without any spin. In order to describe the motion of a charged particle in an external electromagnetic field we make in (2.1), (6.2) the usual replacement $p_\mu \to \pi_\mu = p_\mu - cA_\mu$. The result is

$$H^{\alpha}_{j\tau}(\vec{\pi})\Psi(t,\vec{x}) = i\frac{\partial}{\partial t}\Psi(t,\vec{x}); \tag{6.3}$$

$$\begin{aligned} H^{\alpha}_{j\tau}(\vec{\pi}) = \sigma_1\left[m + \frac{\pi^2}{2m} - 2\sum_a\frac{(S_{4a}\pi_a)^2}{m} - e\frac{\vec{S}\cdot\vec{H}}{m}\right] + \\ +eA_0 + \sigma_3\left[2\sum_a S_{4a}\pi_a + \frac{1}{m^2}h^{\alpha}(\vec{\pi})\right] + o\left(\frac{1}{m^3}\right); \end{aligned} \tag{6.4}$$

$$\vec{H} = \mathrm{curl}\,\vec{A}, \qquad \vec{S} = \vec{j} + \vec{\tau}.$$

One can verify directly that the Hamiltonian (6.4) has positive energy eigenvalues as well as negative ones. We obtain from (6,3) the equation for the positive energy states. It is achieved by the unitary transformation

$$\Psi \to \Psi' = U\Psi, \qquad H_{j\tau}(\vec{\pi}) \to H^{\mathrm{I}}_{j\tau}(\vec{\pi}) = UH^{\alpha}_{j\tau}(\vec{\pi})U^{\dagger} - i\frac{\partial U}{\partial t}U^{\dagger}, \tag{6.5}$$

where

$$\begin{gathered}U^{\alpha} = \exp(iS_3^{\alpha})\exp(iS_2)\exp(iS_1);\\ S_1 = -\sigma_2\sum_a \frac{S_{4a}\pi_a}{m}, \qquad S_2 = \sigma_3 e\sum_a \frac{S_{4a}E_a}{2m^2};\\ S_3^{\alpha} = -\frac{\sigma_2}{2m^3}\left\{h^{\alpha}(\vec{\pi}) + \frac{4}{3}\sum_a (S_{4a}\pi_a)^3 - \sum_a [\pi^2, S_{4a}\pi_a]_+ + \right.\\ \left. +\frac{e}{2}\frac{\partial}{\partial t}\sum_a S_{4a}E_a + e\sum_a [\vec{S}\cdot\vec{H}, S_{4a}\pi_a]_+\right\}, \qquad E_a = -\frac{\partial A_a}{\partial t} - \frac{\partial A_0}{\partial x_a}.\end{gathered} \tag{6.6}$$

From (6.5), (6.6) one obtains

$$H'_{j\tau}(\vec{\pi}) = \sigma_1\left(m + \frac{\pi^2}{2m} - e\frac{\vec{S}\cdot\vec{H}}{m}\right) + eA_0 + \frac{e}{2m^2}\sum_{a,b}[S_{4a}E_a, S_{4b}\pi_b]_-. \tag{6.7}$$

The operator (6.7) commutes with σ_1. On the set of functions Φ^+ which satisfy to the condition

$$\sigma_1\Phi^+ = \Phi^+ \tag{6.8}$$

the Hamiltonian (6.7) is positive definite and equals

$$\begin{gathered}H'_{j\tau}(\vec{\pi})\Phi^+ = \left(m + \frac{\pi^2}{2m} - e\frac{\vec{S}\cdot\vec{H}}{m}\right) + eA_0 + \\ +\frac{e}{2m^2}\sum_{a,b}[S_{4a}E_a, S_{4b}\pi_b]_-\Phi^+ = i\frac{\partial}{\partial t}\Phi^+.\end{gathered} \tag{6.9}$$

Formula (6.9) should be considered as a generalization of the Pauli equation for a particle of spin $\frac{1}{2}$ to the case of a particle with an arbitrary (in general, variable) spin.

To inquire into the physical sense of the constituents which are included in $H'_{j\tau}(\vec{\pi})$, we consider in detail the special case of the equation (6.9) when $j = 0$, $S_{4a} = S_a = \tau_a$. According to (1.1) it corresponds to the particle with a fixed spin. Using the identity

$$\begin{gathered}\frac{e}{2m^2}\sum_{a,b}[S_{4a}E_a, S_{4b}\pi_b]_- \equiv -\frac{i}{12m^2}Q^{ab}_{j\tau}\frac{\partial E_a}{\partial x_b} - \frac{e\sum\limits_a (S_{4a})^2 \operatorname{div}\vec{E}}{6m^2} - \\ -\frac{e}{4m^2}\vec{S}\cdot(\vec{E}\times\vec{p} - \vec{p}\times\vec{E}); \qquad Q^{ab}_{j\tau} = 3[S_{4a}, S_{4b}]_+ - 2\sum_c \delta_{ab}(S_{4c})^2,\end{gathered} \tag{6.10}$$

we write the equation (6.9) for $j=0$ as

$$H'_{0\tau}(\vec{\pi})\Phi^+ = \left\{m + \frac{\pi^2}{2m} + eA_0 - e\frac{\vec{S}\cdot\vec{H}}{m} - e\frac{s(s+1)}{6m^2}\,\mathrm{div}\,\vec{E} - \right.$$
$$\left. -\frac{i}{12m^2}Q^{ab}_{0\tau}\frac{\partial E_a}{\partial x_b} + \frac{e}{4m^2}\vec{S}\cdot(\vec{E}\times\vec{p}-\vec{p}\times\vec{E})\right\}\Phi^+ = i\frac{\partial}{\partial t}\Phi^+, \tag{6.11}$$
$$Q^{ab}_{0\tau} = 3[S_a,S_b]_+ - 2\delta_{ab}S(S+1).$$

Equation (6.11) describes in the quasi-relativistic approximation the movement of a charged particle with an arbitrary fixed spin in an external electromagnetic field. The Hamiltonian $H'_{0\tau}(\vec{\pi})$ includes the constituents which corresponds to the dipole $\left(-\frac{e}{m}\vec{S}\cdot\vec{H}\right)$, quadrupole $\left(-\frac{1}{12m^2}\sum\limits_{a,b}Q^{ab}_{0\tau}\frac{\partial E_a}{\partial x_b}\right)$, spin-orbital $\left(-\frac{e}{4m^2}\vec{S}\cdot(\vec{p}\times\vec{E}-\vec{E}\times\vec{p})\right)$ and Darwin $\left(-\frac{1}{6m^2}s(s+1)\,\mathrm{div}\,\vec{E}\right)$ interactions. By substituting (6.10) into (6.9) one verifies that similar constituents are included in the Hamiltonian $H'_{j\tau}(\vec{\pi})$ of a particle with variable spin.

Thus using the equations for a free particle with an arbitrary spin obtained in Sections 1–4 we found the quasi-relativistic equations (6.9), (6.11) for a charged particle in an external electromagnetic field. We established that in the approximation of $1/m^2$ both Hamiltonians (3.20b) and (4.7) are formally equivalent to (6.7). However, the operator $H^{\mathrm{II}}_{j\tau}$ is determined in the Hilbert space where the scalar product has the complicated structure (2.11), (5.17), and that's why the Hamiltonian $H^{\mathrm{I}}_{j\tau}$ is more convenient for the description of the motion of a charged particle in an external electromagnetic field.

In the case $s=\tau=\frac{1}{2}$ (6.11) coincides with the equation obtained by Foldy and Wouthuysen [3]. For $s=\tau=1$ (6.11) has the structure analogous to the equation obtained in [2], but in addition it takes into account the quadrupole interaction of the particle with a field.

Appendix

Here we present, without proof, some relations used in the paper. In [5] it is shown that for the projectors of the type (3.9) the following formulas hold:

$$[\vec{j},\Lambda_{j_3}] = \frac{i}{2p}\vec{p}\times\vec{j}(\Lambda_{j_3-1}-\Lambda_{j_3+1}) - \frac{1}{2}\left(\vec{j}-\frac{\vec{p}}{p}j_p\right)(\Lambda_{j_3-1}+\Lambda_{j_3+1}-2\Lambda_{j_3});$$
$$[\vec{x},\Lambda_{j_3}] = \frac{1}{2p^2}\vec{p}\times\vec{j}(\Lambda_{j_3-1}+\Lambda_{j_3+1}-2\Lambda_{j_3}) - \frac{i}{2p}\left(\vec{j}-\frac{\vec{p}}{p}j_p\right)(\Lambda_{j_3-1}-\Lambda_{j_3+1}). \tag{A.1}$$

The corresponding relations for Λ_{τ_3} can be obtained from (A.1) by making replacement $\vec{j}\to\vec{\tau}$, $j_3\to\vec{\tau}_3$, $j_p\to\tau_p$. These relations are used in solving equations (3.14) and (4.4). Besides, the following fact is taken into account: the linear combination

$$L = \left(b_1\frac{\vec{p}\times\vec{j}}{p} + b_2\frac{\vec{p}\times\vec{\tau}}{p} + b_3\vec{j} + b_4\vec{\tau} + b_s\frac{\vec{p}}{p}\right)\Lambda_{j_3}\Lambda_{\tau_3} \tag{A.2}$$

equals zero if and only if either $j_3 = \pm j_3$, or $\tau_3 = \pm\tau$, and then

$$\begin{aligned} &\left(\mp i\frac{\vec{p}\times\vec{j}}{p} + \vec{j} - \frac{\vec{p}}{p}j_3\right)\Lambda_{j_3}\Lambda_{\tau_3} = 0, \qquad j_3 = \pm j, \\ &\left(\mp i\frac{\vec{p}\times\vec{\tau}}{p} + \vec{\tau} - \frac{\vec{p}}{p}\tau_3\right)\Lambda_{j_3}\Lambda_{\tau_3} = 0, \qquad \tau_3 = \pm\tau \end{aligned} \tag{A.3}$$

is either fulfilled, or all numbers b_n equal zero.

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