

# KAWASAKI DYNAMICS IN THE CONTINUUM VIA GENERATING FUNCTIONALS EVOLUTION

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ABSTRACT. We construct the time evolution of Kawasaki dynamics for a spatial infinite particle system in terms of generating functionals. This is carried out by an Ovsjannikov-type result in a scale of Banach spaces, which leads to a local (in time) solution. An application of this approach to Vlasov-type scaling in terms of generating functionals is considered as well.

## 1. INTRODUCTION

Originally, Bogoliubov generating functionals (GF for short) were introduced by N. N. Bogoliubov in [2] to define correlation functions for statistical mechanics systems. Apart from this specific application, and many others, GF are, by themselves, a subject of interest in infinite dimensional analysis. This is partially due to the fact that to a probability measure  $\mu$  defined on the space  $\Gamma$  of locally finite configurations  $\gamma \subset \mathbb{R}^d$  one may associate a GF

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)),$$

yielding an alternative method to study the stochastic dynamics of an infinite particle system in the continuum by exploiting the close relation between measures and GF [4, 9].

Existence and uniqueness results for the Kawasaki dynamics through GF arise naturally from Picard-type approximations and a method suggested in [6, Appendix 2, A2.1] in a scale of Banach spaces (see e.g. [5, Theorem 2.5]). This method, originally presented for equations with coefficients time independent, has been extended to an abstract and general framework by T. Yamanaka in [12] and L. V. Ovsjannikov in [10] in the linear case, and many applications were exposed by F. Treves in [11]. As an aside, within an analytical framework outside of our setting, all these statements are very closely related to variants of the abstract Cauchy-Kovalevskaya theorem. However, all these abstract forms only yield a local solution, that is, a solution which is defined on a finite time interval. Moreover, starting with an initial condition from a certain Banach space, in general the solution evolves on larger Banach spaces.

As a particular application, this work concludes with the study of the Vlasov-type scaling proposed in [3] for general continuous particle systems and accomplished in [1] for the Kawasaki dynamics. The general scheme proposed in [3] for correlation functions yields a limiting hierarchy which possesses a chaos preservation property, namely, starting with a Poissonian (non-homogeneous) initial state this structural property is preserved during the time evolution. In Section 4 the same problem is formulated in terms of GF

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Received by the editors DD/MM/2011.

2000 *Mathematics Subject Classification*. Primary 82C22; Secondary 46G20, 46E50.

*Key words and phrases*. Generating functional, Kawasaki dynamics, Interacting particle system, Continuous system, Ovsjannikov's method, Vlasov scaling.

Financial support of DFG through SFB 701 (Bielefeld University), German-Ukrainian Project 436 UKR 113/97 and FCT through PTDC/MAT/100983/2008 and ISFL-1-209 are gratefully acknowledged.

and its analysis is carried out by the general Ovsjannikov-type result in a scale of Banach spaces presented in [5, Theorem 4.3].

## 2. GENERAL FRAMEWORK

In this section we briefly recall the concepts and results of combinatorial harmonic analysis on configuration spaces and Bogoliubov generating functionals needed throughout this work (for a detailed explanation see [7, 9]).

**2.1. Harmonic analysis on configuration spaces.** Let  $\Gamma := \Gamma_{\mathbb{R}^d}$  be the configuration space over  $\mathbb{R}^d$ ,  $d \in \mathbb{N}$ ,

$$\Gamma := \{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \},$$

where  $|\cdot|$  denotes the cardinality of a set. We identify each  $\gamma \in \Gamma$  with the non-negative Radon measure  $\sum_{x \in \gamma} \delta_x$  on the Borel  $\sigma$ -algebra  $\mathcal{B}(\mathbb{R}^d)$ , where  $\delta_x$  is the Dirac measure with mass at  $x$ , which allows to endow  $\Gamma$  with the vague topology and the corresponding Borel  $\sigma$ -algebra  $\mathcal{B}(\Gamma)$ .

For any  $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$  let

$$\Gamma^{(n)} := \{ \gamma \in \Gamma : |\gamma| = n \}, \quad n \in \mathbb{N}, \quad \Gamma^{(0)} := \{\emptyset\}.$$

Clearly, each  $\Gamma^{(n)}$ ,  $n \in \mathbb{N}$ , can be identify with the symmetrization of the set  $\{(x_1, \dots, x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$ , which induces a natural (metrizable) topology on  $\Gamma^{(n)}$  and the corresponding Borel  $\sigma$ -algebra  $\mathcal{B}(\Gamma^{(n)})$ . In particular, for the Lebesgue product measure  $(dx)^{\otimes n}$  fixed on  $(\mathbb{R}^d)^n$ , this identification yields a measure  $m^{(n)}$  on  $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$ . For  $n = 0$  we set  $m^{(0)}(\{\emptyset\}) := 1$ . This leads to the definition of the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}$$

endowed with the topology of disjoint union of topological spaces and the corresponding Borel  $\sigma$ -algebra  $\mathcal{B}(\Gamma_0)$ , and to the so-called Lebesgue-Poisson measure on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$ ,

$$(2.1) \quad \lambda := \lambda_{dx} := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}.$$

Let  $\mathcal{B}_c(\mathbb{R}^d)$  be the set of all bounded Borel sets in  $\mathbb{R}^d$  and, for each  $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ , let  $\Gamma_\Lambda := \{ \eta \in \Gamma : \eta \subset \Lambda \}$ . Evidently  $\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$ , where  $\Gamma_\Lambda^{(n)} := \Gamma_\Lambda \cap \Gamma^{(n)}$ ,  $n \in \mathbb{N}_0$ . Given a complex-valued  $\mathcal{B}(\Gamma_0)$ -measurable function  $G$  such that  $G \upharpoonright_{\Gamma \setminus \Gamma_\Lambda} \equiv 0$  for some  $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ , the  $K$ -transform of  $G$  is a mapping  $KG : \Gamma \rightarrow \mathbb{C}$  defined at each  $\gamma \in \Gamma$  by

$$(2.2) \quad (KG)(\gamma) := \sum_{\substack{\eta \subset \gamma \\ |\eta| < \infty}} G(\eta).$$

It has been shown in [7] that the  $K$ -transform is a linear and invertible mapping.

Let  $\mathcal{M}_{\text{fm}}^1(\Gamma)$  be the set of all probability measures  $\mu$  on  $(\Gamma, \mathcal{B}(\Gamma))$  with finite local moments of all orders, i.e.,

$$\int_{\Gamma} d\mu(\gamma) |\gamma \cap \Lambda|^n < \infty \quad \text{for all } n \in \mathbb{N} \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

and let  $B_{\text{bs}}(\Gamma_0)$  be the set of all complex-valued bounded  $\mathcal{B}(\Gamma_0)$ -measurable functions with bounded support, i.e.,  $G \upharpoonright_{\Gamma_0 \setminus (\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)})} \equiv 0$  for some  $N \in \mathbb{N}_0, \Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ . Given

a  $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ , the so-called correlation measure  $\rho_\mu$  corresponding to  $\mu$  is a measure on  $(\Gamma_0, \mathcal{B}(\Gamma_0))$  defined for all  $G \in B_{\text{bs}}(\Gamma_0)$  by

$$(2.3) \quad \int_{\Gamma_0} d\rho_\mu(\eta) G(\eta) = \int_{\Gamma} d\mu(\gamma) (KG)(\gamma).$$

This definition implies, in particular, that  $B_{\text{bs}}(\Gamma_0) \subset L^1(\Gamma_0, \rho_\mu)$ .<sup>1</sup> Moreover, still by (2.3), on  $B_{\text{bs}}(\Gamma_0)$  the inequality  $\|KG\|_{L^1(\Gamma, \mu)} \leq \|G\|_{L^1(\Gamma_0, \rho_\mu)}$  holds, allowing an extension of the  $K$ -transform to a bounded operator  $K : L^1(\Gamma_0, \rho_\mu) \rightarrow L^1(\Gamma, \mu)$  in such a way that equality (2.3) still holds for any  $G \in L^1(\Gamma_0, \rho_\mu)$ . For the extended operator the explicit form (2.2) still holds, now  $\mu$ -a.e. In particular, for coherent states  $e_\lambda(f)$  of complex-valued  $\mathcal{B}(\mathbb{R}^d)$ -measurable functions  $f$ ,

$$(2.4) \quad e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1.$$

Additionally, if  $f$  has compact support we have

$$(2.5) \quad (Ke_\lambda(f))(\gamma) = \prod_{x \in \gamma} (1 + f(x))$$

for all  $\gamma \in \Gamma$ , while for functions  $f$  such that  $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$  equality (2.5) holds, but only for  $\mu$ -a.a.  $\gamma \in \Gamma$ . Concerning the Lebesgue-Poisson measure (2.1), we observe that  $e_\lambda(f) \in L^p(\Gamma_0, \lambda)$  whenever  $f \in L^p := L^p(\mathbb{R}^d, dx)$  for some  $p \geq 1$ . In this case,  $\|e_\lambda(f)\|_{L^p}^p = \exp(\|f\|_{L^p}^p)$ . In particular, for  $p = 1$ , in addition we have

$$\int_{\Gamma_0} d\lambda(\eta) e_\lambda(f, \eta) = \exp\left(\int_{\mathbb{R}^d} dx f(x)\right),$$

for all  $f \in L^1$ . For more details see [8].

**2.2. Bogoliubov generating functionals.** Given a probability measure  $\mu$  on  $(\Gamma, \mathcal{B}(\Gamma))$  the so-called Bogoliubov generating functional (GF for short)  $B_\mu$  corresponding to  $\mu$  is the functional defined at each  $\mathcal{B}(\mathbb{R}^d)$ -measurable function  $\theta$  by

$$(2.6) \quad B_\mu(\theta) := \int_{\Gamma} d\mu(\gamma) \prod_{x \in \gamma} (1 + \theta(x)),$$

provided the right-hand side exists. It is clear from (2.6) that the domain of a GF  $B_\mu$  depends on the underlying measure  $\mu$  and, conversely, the domain of  $B_\mu$  reflects special properties over the measure  $\mu$ . Throughout this work we will consider GF defined on the whole complex  $L^1$  space. This implies, in particular, that the underlying measure  $\mu$  has finite local exponential moments, i.e.,

$$\int_{\Gamma} d\mu(\gamma) e^{\alpha|\gamma \cap \Lambda|} < \infty \quad \text{for all } \alpha > 0 \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

and thus  $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ . According to the previous subsection, this implies that to such a measure  $\mu$  one may associate the correlation measure  $\rho_\mu$ , which leads to a description of the functional  $B_\mu$  in terms of either the measure  $\rho_\mu$ :

$$B_\mu(\theta) = \int_{\Gamma} d\mu(\gamma) (Ke_\lambda(\theta))(\gamma) = \int_{\Gamma_0} d\rho_\mu(\eta) e_\lambda(\theta, \eta),$$

or the so-called correlation function  $k_\mu := \frac{d\rho_\mu}{d\lambda}$  corresponding to the measure  $\mu$ , if  $\rho_\mu$  is absolutely continuous with respect to the Lebesgue-Poisson measure  $\lambda$ :

$$(2.7) \quad B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_\mu(\eta).$$

<sup>1</sup>Throughout this work all  $L^p$ -spaces,  $p \geq 1$ , consist of complex-valued functions.

Throughout this work we will assume, in addition, that GF are entire on the  $L^1$  space [9], which is a natural environment, namely, to recover the notion of correlation function. For a generic entire functional  $B$  on  $L^1$ , this assumption implies that  $B$  has a representation in terms of its Taylor expansion,

$$B(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n B(\theta_0; \theta, \dots, \theta), \quad z \in \mathbb{C}, \theta \in L^1,$$

being each differential  $d^n B(\theta_0; \cdot)$ ,  $n \in \mathbb{N}$ ,  $\theta_0 \in L^1$  defined by a symmetric kernel

$$\delta^n B(\theta_0; \cdot) \in L^\infty(\mathbb{R}^{dn}) := L^\infty((\mathbb{R}^d)^n, (dx)^{\otimes n}),$$

called the variational derivative of  $n$ -th order of  $B$  at the point  $\theta_0$ . That is,

$$(2.8) \quad \begin{aligned} d^n B(\theta_0; \theta_1, \dots, \theta_n) &:= \frac{\partial^n}{\partial z_1 \dots \partial z_n} B \left( \theta_0 + \sum_{i=1}^n z_i \theta_i \right) \Big|_{z_1 = \dots = z_n = 0} \\ &=: \int_{(\mathbb{R}^d)^n} dx_1 \dots dx_n \delta^n B(\theta_0; x_1, \dots, x_n) \prod_{i=1}^n \theta_i(x_i) \end{aligned}$$

for all  $\theta_1, \dots, \theta_n \in L^1$ . Moreover, the operator norm of the bounded  $n$ -linear functional  $d^n B(\theta_0; \cdot)$  is equal to  $\|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})}$  and for all  $r > 0$  one has

$$(2.9) \quad \|\delta B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{r} \sup_{\|\theta'\|_{L^1} \leq r} |B(\theta_0 + \theta')|$$

and, for  $n \geq 2$ ,

$$(2.10) \quad \|\delta^n B(\theta_0; \cdot)\|_{L^\infty(\mathbb{R}^{dn})} \leq n! \left(\frac{e}{r}\right)^n \sup_{\|\theta'\|_{L^1} \leq r} |B(\theta_0 + \theta')|.$$

In particular, if  $B$  is an entire GF  $B_\mu$  on  $L^1$  then, in terms of the underlying measure  $\mu$ , the entireness property of  $B_\mu$  implies that the correlation measure  $\rho_\mu$  is absolutely continuous with respect to the Lebesgue-Poisson measure  $\lambda$  and the Radon-Nykodim derivative  $k_\mu = \frac{d\rho_\mu}{d\lambda}$  is given by

$$k_\mu(\eta) = \delta^{|\eta|} B_\mu(0; \eta) \quad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0.$$

In what follows, for each  $\alpha > 0$ , we consider the Banach space  $\mathcal{E}_\alpha$  of all entire functionals  $B$  on  $L^1$  such that

$$\|B\|_\alpha := \sup_{\theta \in L^1} \left( |B(\theta)| e^{-\frac{1}{\alpha} \|\theta\|_{L^1}} \right) < \infty,$$

see [9]. This class of Banach spaces has the particularity that, for each  $\alpha_0 > 0$ , the family  $\{\mathcal{E}_\alpha : 0 < \alpha \leq \alpha_0\}$  is a scale of Banach spaces, that is,

$$\mathcal{E}_{\alpha''} \subseteq \mathcal{E}_{\alpha'}, \quad \|\cdot\|_{\alpha'} \leq \|\cdot\|_{\alpha''}$$

for any pair  $\alpha', \alpha''$  such that  $0 < \alpha' < \alpha'' \leq \alpha_0$ .

### 3. THE KAWASAKI DYNAMICS

The Kawasaki dynamics is an example of a hopping particle model where, in this case, particles randomly hop over the space  $\mathbb{R}^d$  according to a rate depending on the interaction between particles. More precisely, let  $a : \mathbb{R}^d \rightarrow [0, +\infty)$  be an even and integrable function and let  $\phi : \mathbb{R}^d \rightarrow [0, +\infty]$  be a pair potential, that is, a  $\mathcal{B}(\mathbb{R}^d)$ -measurable function such that  $\phi(-x) = \phi(x) \in \mathbb{R}$  for all  $x \in \mathbb{R}^d \setminus \{0\}$ , which we will assume to be integrable. A particle located at a site  $x$  in a given configuration  $\gamma \in \Gamma$

hops to a site  $y$  according to a rate given by  $a(x-y)\exp(-E(y,\gamma))$ , where  $E(y,\gamma)$  is a relative energy of interaction between the site  $y$  and the configuration  $\gamma$  defined by

$$E(y,\gamma) := \sum_{x \in \gamma} \phi(x-y) \in [0, +\infty].$$

Informally, the behavior of such an infinite particle system is described by

$$(3.1) \quad (LF)(\gamma) = \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy a(x-y) e^{-E(y,\gamma)} (F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma)).$$

Given an infinite particle system, as the Kawasaki dynamics, its time evolution in terms of states is informally given by the so-called Fokker-Planck equation,

$$(3.2) \quad \frac{d\mu_t}{dt} = L^* \mu_t, \quad \mu_t|_{t=0} = \mu_0,$$

where  $L^*$  is the dual operator of  $L$ . Technically, the use of definition (2.3) allows an alternative approach to the study of (3.2) through the corresponding correlation functions  $k_t := k_{\mu_t}$ ,  $t \geq 0$ , provided they exist. This leads to the Cauchy problem

$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t, \quad k_t|_{t=0} = k_0,$$

where  $k_0$  is the correlation function corresponding to the initial distribution  $\mu_0$  and  $\hat{L}^*$  is the dual operator of  $\hat{L} := K^{-1}LK$  in the sense

$$\int_{\Gamma_0} d\lambda(\eta) (\hat{L}G)(\eta) k(\eta) = \int_{\Gamma_0} d\lambda(\eta) G(\eta) (\hat{L}^* k)(\eta).$$

Through the representation (2.7), this gives us a way to express the dynamics also in terms of the GF  $B_t$  corresponding to  $\mu_t$ , i.e., informally,

$$(3.3) \quad \begin{aligned} \frac{\partial}{\partial t} B_t(\theta) &= \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) \left( \frac{\partial}{\partial t} k_t(\eta) \right) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) (\hat{L}^* k_t)(\eta) \\ &= \int_{\Gamma_0} d\lambda(\eta) (\hat{L}e_\lambda(\theta))(\eta) k_t(\eta) =: (\tilde{L}B_t)(\theta). \end{aligned}$$

This leads to the time evolution equation

$$(3.4) \quad \frac{\partial B_t}{\partial t} = \tilde{L}B_t,$$

where, in the case of the Kawasaki dynamics,  $\tilde{L}$  is given cf. [4] by

$$(3.5) \quad \begin{aligned} &(\tilde{L}B)(\theta) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) e^{-\phi(x-y)} (\theta(y) - \theta(x)) \delta B(\theta e^{-\phi(y-\cdot)} + e^{-\phi(y-\cdot)} - 1; x). \end{aligned}$$

**Theorem 3.1.** *Given an  $\alpha_0 > 0$ , let  $B_0 \in \mathcal{E}_{\alpha_0}$ . For each  $\alpha \in (0, \alpha_0)$  there is a  $T > 0$  (which depends on  $\alpha, \alpha_0$ ) such that there is a unique solution  $B_t$ ,  $t \in [0, T)$ , to the initial value problem (3.4), (3.5),  $B_t|_{t=0} = B_0$  in the space  $\mathcal{E}_\alpha$ .*

This theorem follows as a particular application of an abstract Ovsjannikov-type result in a scale of Banach spaces which can be found e.g. in [5, Theorem 2.5], and the following estimate of norms.

**Proposition 3.2.** *Let  $0 < \alpha < \alpha_0$  be given. If  $B \in \mathcal{E}_{\alpha'}$  for some  $\alpha' \in (\alpha, \alpha_0]$ , then  $\tilde{L}B \in \mathcal{E}_{\alpha'}$  for all  $\alpha \leq \alpha' < \alpha''$ , and we have*

$$\|\tilde{L}B\|_{\alpha'} \leq 2e^{\frac{\|\phi\|_{L^1}}{\alpha}} \|a\|_{L^1} \frac{\alpha_0}{\alpha'' - \alpha'} \|B\|_{\alpha''}.$$

To prove this result as well as other forthcoming ones the next lemma shows to be useful.

**Lemma 3.3.** *Let  $\varphi, \psi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be such that, for a.a.  $y \in \mathbb{R}^d$ ,  $\varphi(y, \cdot) \in L^\infty := L^\infty(\mathbb{R}^d)$ ,  $\psi(y, \cdot) \in L^1$  and  $\|\varphi(y, \cdot)\|_{L^\infty} \leq c_0$ ,  $\|\psi(y, \cdot)\|_{L^1} \leq c_1$  for some constants  $c_0, c_1 > 0$  independent of  $y$ . For each  $\alpha > 0$  and all  $B \in \mathcal{E}_\alpha$  let*

$$(L_0 B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) e^{-k\phi(x-y)} (\theta(y) - \theta(x)) \delta B(\varphi(y, \cdot)\theta + \psi(y, \cdot); x),$$

$\theta \in L^1$ . Here  $a$  and  $\phi$  are defined as before and  $k \geq 0$  is a constant. Then, for all  $\alpha' > 0$  such that  $c_0\alpha' < \alpha$ , we have  $L_0 B \in \mathcal{E}_{\alpha'}$  and

$$\|L_0 B\|_{\alpha'} \leq 2e^{\frac{c_1}{\alpha}} \|a\|_{L^1} \frac{\alpha'}{\alpha - c_0\alpha'} \|B\|_\alpha.$$

*Proof.* First we observe that from the considerations done in Subsection 2.2 it follows that  $L_0 B$  is an entire functional on  $L^1$  and, in addition, that for all  $r > 0$ ,  $\theta \in L^1$ , and a.a.  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} |\delta B(\varphi(y, \cdot)\theta + \psi(y, \cdot); x)| &\leq \|\delta B(\varphi(y, \cdot)\theta + \psi(y, \cdot); \cdot)\|_{L^\infty} \\ &\leq \frac{1}{r} \sup_{\|\theta_0\|_{L^1} \leq r} |B(\varphi(y, \cdot)\theta + \psi(y, \cdot) + \theta_0)|, \end{aligned}$$

where, for all  $\theta_0 \in L^1$  such that  $\|\theta_0\|_{L^1} \leq r$ ,

$$|B(\varphi(y, \cdot)\theta + \psi(y, \cdot) + \theta_0)| \leq \|B\|_\alpha e^{\frac{\|\varphi(y, \cdot)\theta + \psi(y, \cdot)\|_{L^1} + r}{\alpha}} \leq \|B\|_\alpha e^{\frac{c_0\|\theta\|_{L^1} + c_1 + r}{\alpha}}.$$

As a result, due to the positiveness of  $\phi$  and to the fact that  $a$  is an even function, for all  $\theta \in L^1$  one has

$$\begin{aligned} |(L_0 B)(\theta)| &\leq \frac{1}{r} e^{\frac{c_0\|\theta\|_{L^1} + c_1 + r}{\alpha}} \|B\|_\alpha \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) e^{-k\phi(x-y)} |\theta(y) - \theta(x)| \\ &\leq \frac{2}{r} e^{\frac{c_1 + r}{\alpha}} \|a\|_{L^1} \|\theta\|_{L^1} e^{\frac{c_0\|\theta\|_{L^1}}{\alpha}} \|B\|_\alpha. \end{aligned}$$

Thus,

$$\begin{aligned} \|L_0 B\|_{\alpha'} &= \sup_{\theta \in L^1} \left( e^{-\frac{1}{\alpha'} \|\theta\|_{L^1}} |(L_0 B)(\theta)| \right) \\ &\leq \frac{2}{r} e^{\frac{c_1 + r}{\alpha}} \|a\|_{L^1} \|B\|_\alpha \sup_{\theta \in L^1} \left( e^{-\left(\frac{1}{\alpha'} - \frac{c_0}{\alpha}\right) \|\theta\|_{L^1}} \|\theta\|_{L^1} \right), \end{aligned}$$

where the supremum is finite provided  $\frac{1}{\alpha'} - \frac{c_0}{\alpha} > 0$ . In such a situation, the use of the inequality  $x e^{-mx} \leq \frac{1}{em}$ ,  $x \geq 0$ ,  $m > 0$  leads for each  $r > 0$  to

$$\|L_0 B\|_{\alpha'} \leq \frac{2}{r} \|a\|_{L^1} e^{\frac{c_1 + r}{\alpha}} \frac{\alpha\alpha'}{e(\alpha - c_0\alpha')} \|B\|_\alpha.$$

The required estimate of norms follows by minimizing the expression  $\frac{1}{r} e^{\frac{c_1 + r}{\alpha}}$  in the parameter  $r$ , that is,  $r = \alpha$ .  $\square$

*Proof of Proposition 3.2.* In Lemma 3.3 replace  $\varphi$  by  $e^{-\phi}$  and  $\psi$  by  $e^{-\phi} - 1$ , and consider  $k = 1$ . Due to the positiveness and integrability properties of  $\phi$  one has  $e^{-\phi} \leq 1$  and  $|e^{-\phi} - 1| = 1 - e^{-\phi} \leq \phi \in L^1$ , ensuring the conditions to apply Lemma 3.3.  $\square$

**Remark 3.4.** *Concerning the initial conditions considered in Theorem 3.1, observe that, in particular,  $B_0$  can be an entire GF  $B_{\mu_0}$  on  $L^1$  such that, for some constants  $\alpha_0, C > 0$ ,  $|B_{\mu_0}(\theta)| \leq C \exp\left(\frac{\|\theta\|_{L^1}}{\alpha_0}\right)$  for all  $\theta \in L^1$ . In such a situation an additional analysis is need in order to guarantee that for each  $t$  the local solution  $B_t$  given by Theorem 3.1 is a GF (corresponding to some measure). For more details see e.g. [5, 9] and references therein.*

## 4. VLASOV SCALING

We proceed to investigate the Vlasov-type scaling proposed in [3] for generic continuous particle systems and accomplished in [1] for the Kawasaki dynamics. As explained in both references, we start with a rescaling of an initial correlation function  $k_0$ , denoted by  $k_0^{(\varepsilon)}$ ,  $\varepsilon > 0$ , which has a singularity with respect to  $\varepsilon$  of the type  $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$ ,  $\eta \in \Gamma_0$ , being  $r_0$  a function independent of  $\varepsilon$ . The aim is to construct a scaling of the operator  $L$  defined in (3.1),  $L_\varepsilon$ ,  $\varepsilon > 0$ , in such a way that the following two conditions are fulfilled. The first one is that under the scaling  $L \mapsto L_\varepsilon$  the solution  $k_t^{(\varepsilon)}$ ,  $t \geq 0$ , to

$$\frac{\partial}{\partial t} k_t^{(\varepsilon)} = \hat{L}_\varepsilon^* k_t^{(\varepsilon)}, \quad k_t^{(\varepsilon)}|_{t=0} = k_0^{(\varepsilon)}$$

preserves the order of the singularity with respect to  $\varepsilon$ , that is,  $k_t^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta)$ ,  $\eta \in \Gamma_0$ . The second condition is that the dynamics  $r_0 \mapsto r_t$  preserves the Lebesgue-Poisson exponents, that is, if  $r_0$  is of the form  $r_0 = e_\lambda(\rho_0)$ , then each  $r_t$ ,  $t > 0$ , is of the same type, i.e.,  $r_t = e_\lambda(\rho_t)$ , where  $\rho_t$  is a solution to a non-linear equation (called a Vlasov-type equation).

The previous scheme was accomplished in [1] through the scale transformation  $\phi \mapsto \varepsilon\phi$  of the operator  $L$ , that is,

$$(L_\varepsilon F)(\gamma) := \sum_{x \in \gamma} \int_{\mathbb{R}^d} dy a(x-y) e^{-\varepsilon E(y, \gamma)} (F(\gamma \setminus \{x\} \cup \{y\}) - F(\gamma)).$$

As shown in [3, Example 12], [1], the corresponding Vlasov-type equation is given by

$$(4.1) \quad \frac{\partial}{\partial t} \rho_t(x) = (\rho_t * a)(x) e^{-(\rho_t * \phi)(x)} - \rho_t(x) (a * e^{-(\rho_t * \phi)})(x), \quad x \in \mathbb{R}^d,$$

where  $*$  denotes the usual convolution of functions. Existence of classical solutions  $0 \leq \rho_t \in L^\infty$  to (4.1) has been discussed in [1]. Therefore, it is natural to consider the same scaling, but in GF.

To proceed towards GF, we consider  $k_t^{(\varepsilon)}$  defined as before and  $k_{t, \text{ren}}^{(\varepsilon)}(\eta) := \varepsilon^{|\eta|} k_t^{(\varepsilon)}(\eta)$ . In terms of GF, these yield

$$B_t^{(\varepsilon)}(\theta) := \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_t^{(\varepsilon)}(\eta),$$

and

$$B_{t, \text{ren}}^{(\varepsilon)}(\theta) := \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_{t, \text{ren}}^{(\varepsilon)}(\eta) = \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\varepsilon\theta, \eta) k_t^{(\varepsilon)}(\eta) = B_t^{(\varepsilon)}(\varepsilon\theta),$$

leading, as in (3.3), to the initial value problem

$$(4.2) \quad \frac{\partial}{\partial t} B_{t, \text{ren}}^{(\varepsilon)} = \tilde{L}_{\varepsilon, \text{ren}} B_{t, \text{ren}}^{(\varepsilon)}, \quad B_{t, \text{ren}}^{(\varepsilon)}|_{t=0} = B_{0, \text{ren}}^{(\varepsilon)}.$$

**Proposition 4.1.** *For all  $\varepsilon > 0$  and all  $\theta \in L^1$ , we have*

$$(4.3) \quad \begin{aligned} (\tilde{L}_{\varepsilon, \text{ren}} B)(\theta) &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x)) \\ &\times \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right). \end{aligned}$$

*Proof.* Since

$$(\tilde{L}_{\varepsilon, \text{ren}} B)(\theta) = \int_{\Gamma_0} d\lambda(\eta) (\hat{L}_{\varepsilon, \text{ren}} e_\lambda(\theta))(\eta) k(\eta),$$

first we have to calculate  $(\hat{L}_{\varepsilon, \text{ren}} e_\lambda(\theta))(\eta) := \varepsilon^{-|\eta|} \hat{L}_\varepsilon(e_\lambda(\varepsilon\theta, \eta))$ ,  $\hat{L}_\varepsilon = K^{-1} L_\varepsilon K$  cf. [3]. Similar calculations done in [4, Subsection 4.2.1] show

$$\begin{aligned} (\hat{L}_{\varepsilon, \text{ren}} e_\lambda(\theta))(\eta) &= \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x)) \\ &\quad \times e_\lambda \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}, \eta \setminus \{x\} \right), \end{aligned}$$

and thus, using the relation between variational derivatives derived in [9, Proposition 11], one finds

$$\begin{aligned} (\tilde{L}_{\varepsilon, \text{ren}} B)(\theta) &= \int_{\Gamma_0} d\lambda(\eta) k(\eta) \sum_{x \in \eta} \int_{\mathbb{R}^d} dy a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x)) \\ &\quad \times e_\lambda \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}, \eta \setminus \{x\} \right) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x)) \\ &\quad \int_{\Gamma_0} d\lambda(\eta) k(\eta \cup \{x\}) e_\lambda \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}, \eta \right) \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x)) \\ &\quad \times \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right). \quad \square \end{aligned}$$

**Proposition 4.2.** (i) If  $B \in \mathcal{E}_\alpha$  for some  $\alpha > 0$ , then, for all  $\theta \in L^1$ ,  $(\tilde{L}_{\varepsilon, \text{ren}} B)(\theta)$  converges as  $\varepsilon$  tends to zero to

$$(\tilde{L}_V B)(\theta) := \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) (\theta(y) - \theta(x)) \delta B(\theta - \phi(y-\cdot); x).$$

(ii) Let  $\alpha_0 > \alpha > 0$  be given. If  $B \in \mathcal{E}_{\alpha''}$  for some  $\alpha'' \in (\alpha, \alpha_0]$ , then  $\{\tilde{L}_{\varepsilon, \text{ren}} B, \tilde{L}_V B\} \subset \mathcal{E}_{\alpha'}$  for all  $\alpha \leq \alpha' < \alpha''$ , and we have

$$\|\tilde{L}_\# B\|_{\alpha'} \leq 2 \|a\|_{L^1} \frac{\alpha_0}{(\alpha'' - \alpha')} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \|B\|_{\alpha''}$$

where  $\tilde{L}_\# = \tilde{L}_{\varepsilon, \text{ren}}$  or  $\tilde{L}_\# = \tilde{L}_V$ .

*Proof.* (i) To prove this result we first analyze the pointwise convergence of the variational derivative (4.3) appearing in  $\tilde{L}_{\varepsilon, \text{ren}}$ . For this purpose we will use the relation between variational derivatives derived in [9, Proposition 11], i.e.,

$$\delta B(\theta_1 + \theta_2; x) = \int_{\Gamma_0} d\lambda(\eta) \delta^{|\eta|+1} B(\theta_1; \eta \cup \{x\}) e_\lambda(\theta_2, \eta), \quad a.a. x \in \mathbb{R}^d, \theta_1, \theta_2 \in L^1,$$

which allows to rewrite (4.3) as

$$\begin{aligned} &\delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right) \\ (4.4) \quad &= \int_{\Gamma_0} d\lambda(\eta) \delta^{|\eta|+1} B(\theta - \phi(y-\cdot); \eta \cup \{x\}) \\ &\quad \times e_\lambda \left( \theta \left( e^{-\varepsilon\phi(y-\cdot)} - 1 \right) + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon} + \phi(y-\cdot), \eta \right), \end{aligned}$$



for a.a.  $x, y \in \mathbb{R}^d$ . Concerning the function

$$f_\varepsilon := f_\varepsilon(\theta, \phi, y) := \theta \left( e^{-\varepsilon\phi(y-\cdot)} - 1 \right) + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon} + \phi(y-\cdot)$$

which appears in (4.4), for a.a.  $y \in \mathbb{R}^d$ , one clearly has  $\lim_{\varepsilon \rightarrow 0} f_\varepsilon = 0$  a.e. in  $\mathbb{R}^d$ . By definition (2.4), the latter implies that  $e_\lambda(f_\varepsilon)$  converges  $\lambda$ -a.e. to  $e_\lambda(0)$ . Moreover, for the whole integrand function in (4.4), estimates (2.9), (2.10) yield for any  $r > 0$  and  $\lambda$ -a.a.  $\eta \in \Gamma_0$ ,

$$\begin{aligned} & \left| \delta^{|\eta|+1} B(\theta - \phi(y-\cdot); \eta \cup \{x\}) e_\lambda(f_\varepsilon, \eta) \right| \\ & \leq \left\| \delta^{|\eta|+1} B(\theta - \phi(y-\cdot); \cdot) \right\|_{L^\infty(\mathbb{R}^{d(|\eta|+1)})} e_\lambda(|f_\varepsilon|, \eta) \\ & \leq (|\eta|+1)! \left( \frac{e}{r} \right)^{|\eta|+1} e_\lambda(|f_\varepsilon|, \eta) \sup_{\|\theta_0\|_{L^1} \leq r} |B(\theta - \phi(y-\cdot) + \theta_0)| \\ & \leq (|\eta|+1)! \left( \frac{e}{r} \right)^{|\eta|+1} e_\lambda(|\theta| + 2|\phi(y-\cdot)|, \eta) e^{\frac{\|\theta - \phi(y-\cdot)\|_{L^1} + r}{\alpha}} \|B\|_\alpha \end{aligned}$$

with

$$\int_{\Gamma_0} d\lambda(\eta) (|\eta|+1)! \left( \frac{e}{r} \right)^{|\eta|+1} e_\lambda(|\theta| + 2|\phi(y-\cdot)|, \eta) = \sum_{n=0}^{\infty} (n+1) \left( \frac{e}{r} \right)^{n+1} (\|\theta\|_{L^1} + 2\|\phi\|_{L^1})^n$$

being finite for any  $r > e(\|\theta\|_{L^1} + 2\|\phi\|_{L^1})$ .

As a result, by an application of the Lebesgue dominated convergence theorem we have proved that, for a.a.  $x, y \in \mathbb{R}^d$ , (4.4) converges as  $\varepsilon$  tends to zero to

$$\int_{\Gamma_0} d\lambda(\eta) \delta^{|\eta|+1} B(\theta - \phi(y-\cdot); \eta \cup \{x\}) e_\lambda(0, \eta) = \delta B(\theta - \phi(y-\cdot); x).$$

In addition, for the integrand function which appears in  $(\tilde{L}_{\varepsilon, \text{ren}} B)(\theta)$  we have

$$\begin{aligned} & \left| a(x-y) e^{-\varepsilon\phi(x-y)} (\theta(y) - \theta(x)) \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right) \right| \\ & \leq \frac{e}{\alpha} a(x-y) |\theta(y) - \theta(x)| \|B\|_\alpha \exp \left( \frac{1}{\alpha} \|\theta\|_{L^1} + \frac{1}{\alpha} \|\phi\|_{L^1} \right), \end{aligned}$$

for all  $\varepsilon > 0$  and a.a.  $x, y \in \mathbb{R}^d$ , leading through a second application of the Lebesgue dominated convergence theorem to the required limit.

(ii) In Lemma 3.3 replace  $\varphi$  by  $e^{-\varepsilon\phi}$ ,  $\psi$  by  $\frac{e^{-\varepsilon\phi}-1}{\varepsilon}$ , and  $k$  by  $\varepsilon$ . Arguments similar to prove Proposition 3.2 complete the proof for  $\tilde{L}_{\varepsilon, \text{ren}}$ . A similar proof holds for  $\tilde{L}_V$ .  $\square$

Proposition 4.2 (ii) provides similar estimate of norms for  $\tilde{L}_{\varepsilon, \text{ren}}$ ,  $\varepsilon > 0$ , and the limiting mapping  $\tilde{L}_V$ . According to the Ovsjannikov-type result used to prove Theorem 3.1, this means that given any  $B_{0,V}, B_{0, \text{ren}}^{(\varepsilon)} \in \mathcal{E}_{\alpha_0}$ ,  $\varepsilon > 0$ , for each  $\alpha \in (0, \alpha_0)$  there is a  $T > 0$  such that there is a unique solution  $B_{t, \text{ren}}^{(\varepsilon)} : [0, T) \rightarrow \mathcal{E}_\alpha$ ,  $\varepsilon > 0$ , to each initial value problem (4.2) and a unique solution  $B_{t,V} : [0, T) \rightarrow \mathcal{E}_\alpha$  to the initial value problem

$$(4.5) \quad \frac{\partial}{\partial t} B_{t,V} = \tilde{L}_V B_{t,V}, \quad B_{t,V}|_{t=0} = B_{0,V}.$$

In other words, independent of the initial value problem under consideration, the solutions obtained are defined on the same time-interval and with values in the same Banach space. For more details see e.g. Theorem 2.5 and its proof in [5]. Therefore, it is natural to analyze under which conditions the solutions to (4.2) converge to the solution to (4.5). This follows from a general result presented in [5] (Theorem 4.3). However, to proceed to an application of this general result one needs the following estimate of norms.

**Proposition 4.3.** *Assume that  $0 \leq \phi \in L^1 \cap L^\infty$  and let  $\alpha_0 > \alpha > 0$  be given. Then, for all  $B \in \mathcal{E}_{\alpha''}$ ,  $\alpha'' \in (\alpha, \alpha_0]$ , the following estimate holds*

$$\begin{aligned} & \|\tilde{L}_{\varepsilon, \text{ren}} B - \tilde{L}_V B\|_{\alpha'} \\ & \leq 2\varepsilon \|a\|_{L^1} \|\phi\|_{L^\infty} \frac{e\alpha_0}{\alpha} \|B\|_{\alpha''} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \left( \left( 2e\|\phi\|_{L^1} + \frac{\alpha_0}{e} \right) \frac{1}{\alpha'' - \alpha'} + \frac{8\alpha_0^2}{(\alpha'' - \alpha')^2} \right) \end{aligned}$$

for all  $\alpha'$  such that  $\alpha \leq \alpha' < \alpha''$  and all  $\varepsilon > 0$ .

*Proof.* First we observe that

$$\begin{aligned} & \left| (\tilde{L}_{\varepsilon, \text{ren}} B)(\theta) - (\tilde{L}_V B)(\theta) \right| \leq \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) |\theta(y) - \theta(x)| \\ & \times \left| e^{-\varepsilon\phi(x-y)} \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right) - \delta B(\theta - \phi(y-\cdot); x) \right| \end{aligned}$$

with

$$\begin{aligned} & \left| e^{-\varepsilon\phi(x-y)} \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right) - \delta B(\theta - \phi(y-\cdot); x) \right| \\ (4.6) \quad & \leq \left| \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; x \right) - \delta B(\theta - \phi(y-\cdot); x) \right| \\ & + \left( 1 - e^{-\varepsilon\phi(x-y)} \right) |\delta B(\theta - \phi(y-\cdot); x)|. \end{aligned}$$

In order to estimate (4.6), given any  $\theta_0, \theta_1, \theta_2 \in L^1$ , let us consider the function  $C_{\theta_0, \theta_1, \theta_2}(t) = dB(t\theta_1 + (1-t)\theta_2; \theta_0)$ ,  $t \in [0, 1]$ , where  $dB$  is the first order differential of  $B$ , defined in (2.8). One has

$$\begin{aligned} \frac{\partial}{\partial t} C_{\theta_0, \theta_1, \theta_2}(t) &= \frac{\partial}{\partial s} C_{\theta_0, \theta_1, \theta_2}(t+s) \Big|_{s=0} \\ &= \frac{\partial}{\partial s} dB(\theta_2 + t(\theta_1 - \theta_2) + s(\theta_1 - \theta_2); \theta_0) \Big|_{s=0} \\ &= \frac{\partial^2}{\partial s_1 \partial s_2} B(\theta_2 + t(\theta_1 - \theta_2) + s_1(\theta_1 - \theta_2) + s_2\theta_0) \Big|_{s_1=s_2=0} \\ &= \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy (\theta_1(x) - \theta_2(x)) \theta_0(y) \delta^2 B(\theta_2 + t(\theta_1 - \theta_2); x, y), \end{aligned}$$

leading to

$$\begin{aligned} & |dB(\theta_1; \theta_0) - dB(\theta_2; \theta_0)| \\ &= |C_{\theta_0, \theta_1, \theta_2}(1) - C_{\theta_0, \theta_1, \theta_2}(0)| \\ &\leq \max_{t \in [0, 1]} \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy |\theta_1(x) - \theta_2(x)| |\theta_0(y)| |\delta^2 B(\theta_2 + t(\theta_1 - \theta_2); x, y)| \\ &\leq \|\theta_1 - \theta_2\|_{L^1} \|\theta_0\|_{L^1} \max_{t \in [0, 1]} \|\delta^2 B(\theta_2 + t(\theta_1 - \theta_2); \cdot)\|_{L^\infty(\mathbb{R}^{2d})}, \end{aligned}$$

where, through estimate (2.10) with  $r = \alpha''$ ,

$$\|\delta^2 B(\theta_2 + t(\theta_1 - \theta_2); \cdot)\|_{L^\infty(\mathbb{R}^{2d})} \leq 2 \frac{e^3}{\alpha''^2} \|B\|_{\alpha''} \exp\left(\frac{\|\theta_2 + t(\theta_1 - \theta_2)\|_{L^1}}{\alpha''}\right).$$

As a result,

$$\begin{aligned} & |dB(\theta_1; \theta_0) - dB(\theta_2; \theta_0)| \\ & \leq 2 \frac{e^3}{\alpha''^2} \|\theta_1 - \theta_2\|_{L^1} \|\theta_0\|_{L^1} \|B\|_{\alpha''} \max_{t \in [0, 1]} \exp\left(\frac{t\|\theta_1\|_{L^1} + (1-t)\|\theta_2\|_{L^1}}{\alpha''}\right), \end{aligned}$$

for all  $\theta_0, \theta_1, \theta_2 \in L^1$ . In particular, this shows that for all  $\theta_0 \in L^1$ ,

$$\begin{aligned} & \left| dB \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; \theta_0 \right) - dB(\theta - \phi(y-\cdot); \theta_0) \right| \\ & \leq 2\varepsilon \frac{e^3}{\alpha''^2} \|\phi\|_{L^\infty} \|B\|_{\alpha''} (\|\theta\|_{L^1} + \|\phi\|_{L^1}) \|\theta_0\|_{L^1} \\ & \quad \times \max_{t \in [0,1]} \exp \left( \frac{1}{\alpha''} (t(\|\theta\|_{L^1} + \|\phi\|_{L^1}) + (1-t)(\|\theta\|_{L^1} + \|\phi\|_{L^1})) \right) \\ & = 2\varepsilon \frac{e^3}{\alpha''^2} \|\phi\|_{L^\infty} \|B\|_{\alpha''} (\|\theta\|_{L^1} + \|\phi\|_{L^1}) \exp \left( \frac{1}{\alpha''} (\|\theta\|_{L^1} + \|\phi\|_{L^1}) \right) \|\theta_0\|_{L^1}, \end{aligned}$$

where we have used the inequalities

$$\begin{aligned} \|\theta e^{-\varepsilon\phi(y-\cdot)} - \theta\|_{L^1} & \leq \varepsilon \|\phi\|_{L^\infty} \|\theta\|_{L^1}, \\ \left\| \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon} + \phi(y-\cdot) \right\|_{L^1} & \leq \varepsilon \|\phi\|_{L^\infty} \|\phi\|_{L^1}, \\ \left\| \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon} \right\|_{L^1} & \leq \|\theta\|_{L^1} + \|\phi\|_{L^1}. \end{aligned}$$

In other words, we have shown that the norm of the bounded linear functional on  $L^1$

$$L^1 \ni \theta_0 \mapsto dB \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; \theta_0 \right) - dB(\theta - \phi(y-\cdot); \theta_0)$$

is bounded by

$$Q := 2\varepsilon \frac{e^3}{\alpha''^2} \|\phi\|_{L^\infty} \|B\|_{\alpha''} (\|\theta\|_{L^1} + \|\phi\|_{L^1}) \exp \left( \frac{1}{\alpha''} (\|\theta\|_{L^1} + \|\phi\|_{L^1}) \right).$$

Since this operator norm is given by

$$\left\| \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; \cdot \right) - \delta B(\theta - \phi(y-\cdot); \cdot) \right\|_{L^\infty}$$

cf. Subsection 2.2, this means that

$$\left\| \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; \cdot \right) - \delta B(\theta - \phi(y-\cdot); \cdot) \right\|_{L^\infty} \leq Q.$$

In this way we obtain

$$\begin{aligned} & \left| (\tilde{L}_{\varepsilon, \text{ren}} B)(\theta) - (\tilde{L}_V B)(\theta) \right| \\ & \leq \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) |\theta(y) - \theta(x)| \\ & \quad \times \left\{ \left\| \delta B \left( \theta e^{-\varepsilon\phi(y-\cdot)} + \frac{e^{-\varepsilon\phi(y-\cdot)} - 1}{\varepsilon}; \cdot \right) - \delta B(\theta - \phi(y-\cdot); \cdot) \right\|_{L^\infty} \right. \\ & \quad \left. + \varepsilon \|\phi\|_{L^\infty} \|\delta B(\theta - \phi(y-\cdot); \cdot)\|_{L^\infty} \right\} \\ & \leq 2\varepsilon \|\phi\|_{L^\infty} \|a\|_{L^1} \frac{e}{\alpha''} \exp \left( \frac{1}{\alpha''} (\|\theta\|_{L^1} + \|\phi\|_{L^1}) \right) \|\theta\|_{L^1} \\ & \quad \times \left\{ 2 \frac{e^2}{\alpha''} (\|\theta\|_{L^1} + \|\phi\|_{L^1}) + 1 \right\} \|B\|_{\alpha''}, \end{aligned}$$

and thus

$$\begin{aligned} & \|\tilde{L}_{\varepsilon, \text{ren}} B - \tilde{L}_V B\|_{\alpha'} \\ & \leq 2\varepsilon \|\phi\|_{L^\infty} \|a\|_{L^1} \frac{e}{\alpha''} e^{\frac{\|\phi\|_{L^1}}{\alpha''}} \left\{ 2 \frac{e^2}{\alpha''} \sup_{\theta \in L^1} \left( \|\theta\|_{L^1}^2 \exp \left( \|\theta\|_{L^1} \left( \frac{1}{\alpha''} - \frac{1}{\alpha'} \right) \right) \right) \right. \\ & \quad \left. + \left( 2 \frac{e^2}{\alpha''} \|\phi\|_{L^1} + 1 \right) \sup_{\theta \in L^1} \left( \|\theta\|_{L^1} \exp \left( \|\theta\|_{L^1} \left( \frac{1}{\alpha''} - \frac{1}{\alpha'} \right) \right) \right) \right\} \|B\|_{\alpha''}, \end{aligned}$$

and the proof follows using the inequalities  $xe^{-mx} \leq \frac{1}{me}$  and  $x^2e^{-mx} \leq \frac{4}{m^2e^2}$  for  $x \geq 0$ ,  $m > 0$ .  $\square$

We are now in conditions to state the following result.

**Theorem 4.4.** *Given an  $0 < \alpha < \alpha_0$ , let  $B_{t, \text{ren}}^{(\varepsilon)}, B_{t, V}$ ,  $t \in [0, T)$ , be the local solutions in  $\mathcal{E}_\alpha$  to the initial value problems (4.2), (4.5) with  $B_{0, \text{ren}}^{(\varepsilon)}, B_{0, V} \in \mathcal{E}_{\alpha_0}$ . If  $0 \leq \phi \in L^1 \cap L^\infty$  and  $\lim_{\varepsilon \rightarrow 0} \|B_{0, \text{ren}}^{(\varepsilon)} - B_{0, V}\|_{\alpha_0} = 0$ , then, for each  $t \in [0, T)$ ,*

$$\lim_{\varepsilon \rightarrow 0} \|B_{t, \text{ren}}^{(\varepsilon)} - B_{t, V}\|_\alpha = 0.$$

Moreover, if  $B_{0, V}(\theta) = \exp \left( \int_{\mathbb{R}^d} dx \rho_0(x) \theta(x) \right)$ ,  $\theta \in L^1$ , for some function  $0 \leq \rho_0 \in L^\infty$  such that  $\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$ , then for each  $t \in [0, T)$ ,

$$(4.7) \quad B_{t, V}(\theta) = \exp \left( \int_{\mathbb{R}^d} dx \rho_t(x) \theta(x) \right), \quad \theta \in L^1,$$

where  $0 \leq \rho_t \in L^\infty$  is a classical solution to the equation (4.1).

*Proof.* The first part follows directly from Proposition 4.3 and [5, Theorem 4.3], taking in [5, Theorem 4.3]  $p = 2$  and

$$N_\varepsilon = 2\varepsilon \|a\|_{L^1} \|\phi\|_{L^\infty} \frac{e\alpha_0}{\alpha} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \max \left\{ 2e \|\phi\|_{L^1} + \frac{\alpha_0}{e}, 8\alpha_0^2 \right\}.$$

Concerning the last part, we begin by observing that it has been shown in [1, Subsection 4.2] that given a  $0 \leq \rho_0 \in L^\infty$  such that  $\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$ , there is a solution  $0 \leq \rho_t \in L^\infty$  to (4.1) such that  $\|\rho_t\|_{L^\infty} \leq \frac{1}{\alpha_0}$ . This implies that  $B_{t, V}$ , given by (4.7), does not leave the initial Banach space  $\mathcal{E}_{\alpha_0} \subset \mathcal{E}_\alpha$ . Then, by an argument of uniqueness, to prove the last assertion amounts to show that  $B_{t, V}$  solves equation (4.5). For this purpose we note that for any  $\theta, \theta_1 \in L^1$  we have

$$\frac{\partial}{\partial z_1} B_{t, V}(\theta + z_1 \theta_1) \Big|_{z_1=0} = B_{t, V}(\theta) \int_{\mathbb{R}^d} dx \rho_t(x) \theta_1(x),$$

and thus  $\delta B_{t, V}(\theta; x) = B_{t, V}(\theta) \rho_t(x)$ . Hence, for all  $\theta \in L^1$ ,

$$\begin{aligned} (\tilde{L}_V B_{t, V})(\theta) &= B_{t, V}(\theta) \left( \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy a(x-y) (\theta(y) - \theta(x)) \rho_t(x) e^{-(\rho_t * \phi)(y)} \right) \\ &= B_{t, V}(\theta) \left( \int_{\mathbb{R}^d} dy \theta(y) (a * \rho_t)(y) e^{-(\rho_t * \phi)(y)} \right. \\ & \quad \left. - \int_{\mathbb{R}^d} dx \theta(x) \left( a * e^{-(\rho_t * \phi)(y)} \right)(x) \rho_t(x) \right). \end{aligned}$$

Since  $\rho_t$  is a classical solution to (4.1),  $\rho_t$  solves a weak form of equation (4.1), that is, the right-hand side of the latter equality is equal to

$$B_{t, V}(\theta) \frac{d}{dt} \int_{\mathbb{R}^d} dx \rho_t(x) \theta(x) = \frac{\partial}{\partial t} B_{t, V}(\theta). \quad \square$$

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