Glauber dynamics in the continuum via generating functionals evolution

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Abstract

We construct the time evolution for states of Glauber dynamics for a spatial infinite particle system in terms of generating functionals. This is carried out by an Ovsjannikov-type result in a scale of Banach spaces, leading to a local (in time) solution which, under certain initial conditions, might be extended to a global one. An application of this approach to Vlasov-type scaling in terms of generating functionals is considered as well.

Keywords: Generating functional, Glauber dynamics, Interacting particle system, Continuous system, Ovsjannikov's method, Vlasov scaling

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1 Introduction

Originally, Bogoliubov generating functionals (shortly GF) were introduced by N. N. Bogoliubov in [Bog46] to define correlation functions for statistical mechanics systems. Apart from this specific application, and many others, GF are, by themselves, a subject of interest in infinite dimensional analysis. This is partially due to the fact that to a probability measure μ defined on the space Γ of locally finite configurations $\gamma \subset \mathbb{R}^d$ one may associate a GF

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x\in\gamma}(1 + \theta(x)),$$

yielding an alternative method to study the stochastic dynamics of an infinite particle system in the continuum by exploiting the close relation between measures and GF [FKO09, KKO06].

Within the semigroups theory, a non-equilibrium Glauber dynamics has been constructed through evolution equations for correlation functions in [FKK09, FKKZ09, KKZ06]. However, within the GF context, semigroup techniques seem do not work. Alternatively, existence and uniqueness results for the Glauber dynamics through GF arise naturally from Picard-type approximations and a method suggested in [GS58, Appendix 2, A2.1] in a scale of Banach spaces (Theorem 2.5). This method, originally presented for equations with coefficients time independent, has been extended to an abstract and general framework by T. Yamanaka in [Yam60] and L. V. Ovsjannikov in [Ovs65] in the linear case, and many applications were exposed by F. Treves in [Tre68]. As an aside, within an analytical framework outside of our setting, all these statements are very closely related to variants of the abstract Cauchy-Kovalevskaya theorem. However, all these abstract forms, namely, Theorem 2.5, only yield a local solution, that is, a solution which is defined on a finite time interval. Moreover, starting with an initial condition from a certain Banach space, in general the solution evolves on larger Banach spaces. It is only for a certain class of initial conditions that the solution does not leave the initial Banach space. In this case, the solution might be extended to a global solution (Corollary 3.7).

As a particular application, this work concludes with the study of the Vlasov-type scaling proposed in [FKK10a] for generic continuous particle systems and accomplished in [FKK10b] for the Glauber dynamics. The general scheme proposed in [FKK10a] for correlation functions yields a limiting hierarchy which possesses a chaos preservation property, namely, starting with a Poissonian (non-homogeneous) initial state this structural property is preserved during the time evolution. In Section 4 the same problem is formulated in terms of GF and its analysis is carried out by Ovsjannikov-type approximations in a scale of Banach spaces (Theorem 4.3).

For further applications, let us pointing out that the alternative technical standpoint presented in this work shows to be efficient as well on the treatment of other types of stochastic dynamics of infinite particle systems, namely, the Kawasaki type dynamics in the continuum. This and other cases are now being studied and will be reported in forthcoming publications.

2 General Framework

In this section we briefly recall the concepts and results of combinatorial harmonic analysis on configuration spaces and Bogoliubov generating functionals needed throughout this work (for a detailed explanation see [KK02, KKO06]).

2.1 Harmonic analysis on configuration spaces

Let $\Gamma := \Gamma_{\mathbb{R}^d}$ be the configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$,

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \text{ for every compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ denotes the cardinality of a set. We identify each $\gamma \in \Gamma$ with the non-negative Radon measure $\sum_{x\in\gamma} \delta_x$ on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$, where δ_x is the Dirac measure with mass at x, which allows to endow Γ with the vague topology and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$.

For any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ let

$$\Gamma^{(n)} := \{\gamma \in \Gamma : |\gamma| = n\}, \ n \in \mathbb{N}, \quad \Gamma^{(0)} := \{\emptyset\}.$$

Clearly, each $\Gamma^{(n)}$, $n \in \mathbb{N}$, can be identify with the symmetrization of the set $\{(x_1, ..., x_n) \in (\mathbb{R}^d)^n : x_i \neq x_j \text{ if } i \neq j\}$ under the permutation group over $\{1, ..., n\}$, which induces a natural (metrizable) topology on $\Gamma^{(n)}$ and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma^{(n)})$ as well. This leads to the space of finite configurations

$$\Gamma_0 := \bigsqcup_{n=0}^{\infty} \Gamma^{(n)}$$

endowed with the topology of disjoint union of topological spaces and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0)$.

Let now $\mathcal{B}_c(\mathbb{R}^d)$ be the set of all bounded Borel sets in $\mathbb{R}^d$, and for each $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ let $\Gamma_\Lambda := \{\eta \in \Gamma : \eta \subset \Lambda\}$. Evidently $\Gamma_\Lambda = \bigsqcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}$, where $\Gamma_\Lambda^{(n)} := \Gamma_\Lambda \cap \Gamma^{(n)}$, $n \in \mathbb{N}_0$, leading to a situation similar to the one for Γ_0, described above. Given a complex-valued $\mathcal{B}(\Gamma_0)$-measurable function G such that $G\!\upharpoonright_{\Gamma\setminus\Gamma_\Lambda}\equiv 0$ for some $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$, the K-transform of G is a mapping $KG:\Gamma\to\mathbb{C}$ defined at each $\gamma\in\Gamma$ by

$$(KG)(\gamma) := \sum_{\substack{\eta\subset\gamma\\ |\eta|<\infty}} G(\eta). \tag{2.1}$$

It has been shown in [KK02] that the K-transform is a linear and invertible mapping.

Among the functions in the domain of the K-transform we distinguish the so-called coherent states $e_\lambda(f)$, defined for complex-valued $\mathcal{B}(\mathbb{R}^d)$-measurable functions f by

$$e_\lambda(f,\eta) := \prod_{x\in\eta} f(x)\,,\ \eta\in\Gamma_0\setminus\{\emptyset\},\quad e_\lambda(f,\emptyset):=1.$$

The special role of these functions is partially due to the fact that their image under the K-transform coincides with the integrand functions of generating functionals (Subsection 2.2 below). More precisely, for any f described as before, having in addition compact support, for all $\gamma\in\Gamma$

$$(Ke_\lambda(f))(\gamma) = \prod_{x\in\gamma}(1+f(x)). \tag{2.2}$$

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ on $(\Gamma,\mathcal{B}(\Gamma))$ with finite local moments of all orders, i.e.,

$$\int_\Gamma d\mu(\gamma)\,|\gamma\cap\Lambda|^n<\infty\quad \text{for all } n\in\mathbb{N} \text{ and all } \Lambda\in\mathcal{B}_c(\mathbb{R}^d),$$

and let $B_{\mathrm{bs}}(\Gamma_0)$ be the set of all complex-valued bounded $\mathcal{B}(\Gamma_0)$-measurable functions with bounded support, i.e., $G\!\upharpoonright_{\Gamma_0\setminus\left(\bigsqcup_{n=0}^N\Gamma_\Lambda^{(n)}\right)}\equiv 0$ for some $N\in\mathbb{N}_0,\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$. Given a $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, the so-called correlation measure ρ_μ corresponding to μ is a measure on $(\Gamma_0,\mathcal{B}(\Gamma_0))$ defined for all $G\in B_{\mathrm{bs}}(\Gamma_0)$ by

$$\int_{\Gamma_0} d\rho_\mu(\eta)\,G(\eta) = \int_\Gamma d\mu(\gamma)\ (KG)(\gamma). \tag{2.3}$$

Observe that under the above conditions $K|G|$ is μ-integrable. In terms of correlation measures this means that $B_{\mathrm{bs}}(\Gamma_0)\subset L^1(\Gamma_0,\rho_\mu)$.[1]

[1]Throughout this work all L^p-spaces, $p\geq 1$, consist of complex-valued functions.

Actually, $B_{\rm bs}(\Gamma_0)$ is dense in $L^1(\Gamma_0, \rho_\mu)$. Moreover, still by (2.3), on $B_{\rm bs}(\Gamma_0)$ the inequality $\|KG\|_{L^1(\Gamma,\mu)} \leq \|G\|_{L^1(\Gamma_0,\rho_\mu)}$ holds, allowing an extension of the K-transform to a bounded operator $K : L^1(\Gamma_0, \rho_\mu) \to L^1(\Gamma, \mu)$ in such a way that equality (2.3) still holds for any $G \in L^1(\Gamma_0, \rho_\mu)$. For the extended operator the explicit form (2.1) still holds, now μ-a.e. This means, in particular, that for all $\mathcal{B}(\mathbb{R}^d)$-measurable functions f such that $e_\lambda(f) \in L^1(\Gamma_0, \rho_\mu)$ equality (2.2) holds for μ-a.a. $\gamma \in \Gamma$.

Example 2.1. *The Poisson measure* $\pi := \pi_{dx}$ *with intensity the Lebesgue measure* dx *on* $\mathbb{R}^d$ *is the probability measure defined on* $(\Gamma, \mathcal{B}(\Gamma))$ *by*

$$\int_\Gamma d\pi(\gamma) \exp\left(\sum_{x\in\gamma} \varphi(x)\right) = \exp\left(\int_{\mathbb{R}^d} dx \, \left(e^{\varphi(x)} - 1\right)\right)$$

for all real-valued smooth functions φ *on* $\mathbb{R}^d$ *with compact support. The correlation measure corresponding to* π *is the so-called Lebesgue–Poisson measure,*

$$\lambda := \lambda_{dx} := \sum_{n=0}^\infty \frac{1}{n!} m^{(n)},$$

where $m^{(n)}$, $n \in \mathbb{N}$, *is the measure on* $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$ *obtained by symmetrization of the Lebesgue product measure* $(dx)^{\otimes n}$ *through the symmetrization procedure described above. For* $n = 0$ *we set* $m^{(0)}(\{\emptyset\}) := 1$. *This special case emphasizes the technical role of coherent states in our setting, namely, due to the fact* $e_\lambda(f) \in L^p(\Gamma_0, \lambda)$ *whenever* $f \in L^p := L^p(\mathbb{R}^d, dx)$ *for some* $p \geq 1$, *and, moreover,* $\|e_\lambda(f)\|^p_{L^p} = \exp(\|f\|^p_{L^p})$. *In particular, for* $p = 1$, *one additionally has*

$$\int_{\Gamma_0} d\lambda(\eta) \, e_\lambda(f, \eta) = \exp\left(\int_{\mathbb{R}^d} dx \, f(x)\right), \tag{2.4}$$

for all $f \in L^1$. *For more details see [KKO04].*

2.2 Bogoliubov generating functionals

Given a probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ the so-called Bogoliubov generating functional (shortly GF) B_μ corresponding to μ is the functional defined at each $\mathcal{B}(\mathbb{R}^d)$-measurable function θ by

$$B_\mu(\theta) := \int_\Gamma d\mu(\gamma) \prod_{x\in\gamma} (1 + \theta(x)), \tag{2.5}$$

provided the right-hand side exists. It is clear from (2.5) that one cannot define the GF for all probability measures on Γ but, if it exists for some measure μ, then the domain of B_μ depends on μ and, conversely, the domain of B_μ reflects special properties over the underlying measure μ [KKO06]. For instance, if μ has finite local exponential moments, i.e.,

$$\int_\Gamma d\mu(\gamma)\, e^{\alpha|\gamma\cap\Lambda|} < \infty \quad \text{for all } \alpha > 0 \text{ and all } \Lambda \in \mathcal{B}_c(\mathbb{R}^d),$$

then B_μ is well-defined, for instance, on all bounded functions θ with compact support. According to the previous subsection, this implies that to such a measure μ one may associate the correlation measure ρ_μ, leading to a description of the functional B_μ in terms of either the measure ρ_μ:

$$B_\mu(\theta) = \int_\Gamma d\mu(\gamma)\ (Ke_\lambda(\theta))\,(\gamma) = \int_{\Gamma_0} d\rho_\mu(\eta)\, e_\lambda(\theta,\eta),$$

or the so-called correlation function $k_\mu := \frac{d\rho_\mu}{d\lambda}$ corresponding to the measure μ, if ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure λ:

$$B_\mu(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta)k_\mu(\eta). \tag{2.6}$$

Throughout this work we will consider GF defined on the whole complex L^1 space. Furthermore, we will assume that the GF are entire. We recall that a functional $A : L^1 \to \mathbb{C}$ is entire on L^1 whenever A is locally bounded and for all $\theta_0, \theta \in L^1$ the mapping $\mathbb{C} \ni z \mapsto A(\theta_0 + z\theta) \in \mathbb{C}$ is entire. Thus, at each $\theta_0 \in L^1$, every entire functional A on L^1 has a representation in terms of its Taylor expansion,

$$A(\theta_0 + z\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} d^n A(\theta_0; \theta, ..., \theta), \quad z \in \mathbb{C}, \theta \in L^1.$$

The next theorem states properties specific for entire functionals A on L^1 and their higher order derivatives $d^n A(\theta_0; \cdot)$ (for a detailed explanation see [KKO06] and the references therein).

Theorem 2.2. *Let A be an entire functional on L^1. Then each differential $d^n A(\theta_0; \cdot), n \in \mathbb{N}, \theta_0 \in L^1$ is defined by a symmetric kernel*

$$\delta^n A(\theta_0; \cdot) \in L^\infty(\mathbb{R}^{dn}) := L^\infty\big((\mathbb{R}^d)^n, (dx)^{\otimes n}\big)$$

called the variational derivative of n-th order of A at the point θ_0. More precisely,

$$\begin{aligned} d^n A(\theta_0; \theta_1, ..., \theta_n) \ &:= \ \frac{\partial^n}{\partial z_1 ... \partial z_n} A\left(\theta_0 + \sum_{i=1}^n z_i\theta_i\right)\Bigg|_{z_1=...=z_n=0} \\ &=: \ \int_{(\mathbb{R}^d)^n} dx_1 \dots dx_n\, \delta^n A(\theta_0; x_1, \dots, x_n) \prod_{i=1}^n \theta_i(x_i) \end{aligned}$$

for all $\theta_1, ..., \theta_n \in L^1$. Moreover, the operator norm of the bounded n-linear functional $d^n A(\theta_0; \cdot)$ is equal to $\|\delta^n A(\theta_0;\cdot)\|_{L^\infty(\mathbb{R}^{dn})}$ and for all $r > 0$ one has

$$\|\delta A(\theta_0;\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{1}{r} \sup_{\|\theta'\|_{L^1}\le r} |A(\theta_0+\theta')| \tag{2.7}$$

and, for $n \ge 2$,

$$\|\delta^n A(\theta_0;\cdot)\|_{L^\infty(\mathbb{R}^{dn})} \le n! \left(\frac{e}{r}\right)^n \sup_{\|\theta'\|_{L^1}\le r} |A(\theta_0+\theta')|. \tag{2.8}$$

The first part of Theorem 2.2 stated for GF and their variational derivatives at $\theta_0 = 0$ yields the next result.

Proposition 2.3. *Let B_μ be an entire GF on L^1. Then the measure ρ_μ is absolutely continuous with respect to the Lebesgue–Poisson measure λ and the Radon–Nykodim derivative $k_\mu = \dfrac{d\rho_\mu}{d\lambda}$ is given by*

$$k_\mu(\eta) = \delta^{|\eta|} B_\mu(0;\eta) \quad \textit{for } \lambda\textit{-a.a. } \eta \in \Gamma_0.$$

Concerning the second part of Theorem 2.2, namely, estimates (2.7) and (2.8), we note that A being entire does not ensure that for every $r > 0$ the supremum appearing on the right-hand side of (2.7), (2.8) is always finite. This will hold if, in addition, the entire functional A is of bounded type, that is,

$$\forall\, r > 0, \ \sup_{\|\theta\|_{L^1}\le r} |A(\theta_0+\theta)| < \infty, \ \forall\, \theta_0 \in L^1.$$

Hence, as a consequence of Proposition 2.3, it follows from (2.7) and (2.8) that the correlation function k_μ of an entire GF of bounded type on L^1 fulfills the so-called generalized Ruelle bound, that is, for any $0 \le \varepsilon \le 1$ and any $r > 0$ there is some constant $C \ge 0$ depending on r such that

$$k_\mu(\eta) \le C\,(|\eta|!)^{1-\varepsilon} \left(\frac{e}{r}\right)^{|\eta|}, \quad \lambda\text{–a.a. } \eta \in \Gamma_0. \tag{2.9}$$

In our case, $\varepsilon = 0$. We observe that if (2.9) holds for $\varepsilon = 1$ and for at least one $r > 0$, then condition (2.9) is the classical Ruelle bound. In terms of GF, the latter means that

$$|B_\mu(\theta)| \le C \exp\left(\frac{e}{r}\|\theta\|_{L^1}\right),$$

as can be easily checked using the representation (2.6) and (2.4). This special case motivates the definition of the following family of Banach spaces, see [KKO06, Proposition 23].

Definition 2.4. *For each $\alpha > 0$, let $\mathcal{E}_\alpha$ be the Banach space of all entire functionals B on L^1 such that*

$$\|B\|_\alpha := \sup_{\theta \in L^1} \left(|B(\theta)| \, e^{-\frac{1}{\alpha}\|\theta\|_{L^1}} \right) < \infty.$$

2.3 Time evolution equations

Informally, the stochastic evolution of an interacting particle system on $\mathbb{R}^d$ may be described through a Markov generator L defined on a proper space of functions on Γ. The problem of construction of the corresponding Markov process on Γ is related to the existence (on a proper space of functions) of the semigroup corresponding to L, which will be the solution to a Cauchy problem

$$\frac{dF_t}{dt} = LF_t, \qquad F_t \big|_{t=0} = F_0.$$

However, from the technical point of view, to show that L is the generator of a semigroup on some reasonable space of functions defined on Γ seems to be often a difficult question.

In applications, the properties of the evolution of the system through its states, that is, probability measures on Γ, is a subject of interest. Informally, such a time evolution is given by the dual Kolmogorov equation, the so-called Fokker-Planck equation,

$$\frac{d\mu_t}{dt} = L^*\mu_t, \qquad \mu_t \big|_{t=0} = \mu_0, \tag{2.10}$$

where L^* is the dual operator of L. Technically, the use of definition (2.3) allows an alternative approach to the study of (2.10) through the corresponding correlation functions $k_t := k_{\mu_t}$, $t \geq 0$, provided they exist. This leads to the Cauchy problem

$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t, \quad k_{t|t=0} = k_0,$$

where k_0 is the correlation function corresponding to the initial distribution μ_0 and $\hat{L}^*$ is the dual operator of $\hat{L} := K^{-1}LK$ in the sense

$$\int_{\Gamma_0} d\lambda(\eta) \, (\hat{L}G)(\eta)k(\eta) = \int_{\Gamma_0} d\lambda(\eta) \, G(\eta)(\hat{L}^*k)(\eta).$$

Through the representation (2.6), this gives us a way to express the dynamics also in terms of the GF B_t corresponding to μ_t, i.e., informally,

$$\frac{\partial}{\partial t} B_t(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) \left(\frac{\partial}{\partial t} k_t(\eta)\right) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta)(\hat{L}^* k_t)(\eta)$$
$$= \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L} e_\lambda(\theta))(\eta) k_t(\eta) =: (\tilde{L} B_t)(\theta). \tag{2.11}$$

Concerning the evolution equation

$$\frac{\partial B_t}{\partial t} = \tilde{L} B_t, \tag{2.12}$$

we observe that from the previous construction follows that if a solution B_t, $t \geq 0$, exists for some GF as an initial condition, then one may expect that each B_t is the GF corresponding to the state of the system at the time t. However, besides the existence problem, at this point it is opportune to underline that if a solution to (2.12) exists, a priori it does not have to be a GF (corresponding to some measure). This verification requests an additional analysis, see e.g. [KKO06], [Kun99].

In most concrete applications, to find a solution to (2.12) on a Banach space seems to be often a difficult question. However, the problem may be simplified within the framework of scales of Banach spaces. We recall that a scale of Banach spaces is a one-parameter family of Banach spaces $\{\mathbb{B}_s : 0 < s \leq s_0\}$ such that

$$\mathbb{B}_{s''} \subseteq \mathbb{B}_{s'}, \quad \|\cdot\|_{s'} \leq \|\cdot\|_{s''}$$

for any pair s', s'' such that $0 < s' < s'' \leq s_0$, where $\|\cdot\|_s$ denotes the norm in $\mathbb{B}_s$. As an example, it is clear from Definition 2.4 that for any $\alpha_0 > 0$ the family $\{\mathcal{E}_\alpha : 0 < \alpha \leq \alpha_0\}$ is a scale of Banach spaces.

Within this framework, one has the following existence and uniqueness result (see e.g. [Tre68]).

Theorem 2.5. *On a scale of Banach spaces $\{\mathbb{B}_s : 0 < s \leq s_0\}$ consider the initial value problem*

$$\frac{du(t)}{dt} = Au(t), \quad u(0) = u_0 \in \mathbb{B}_{s_0} \tag{2.13}$$

where, for each $s \in (0, s_0)$ fixed and for each pair s', s'' such that $s \leq s' < s'' \leq s_0$, $A : \mathbb{B}_{s''} \to \mathbb{B}_{s'}$ is a linear mapping so that there is an $M > 0$ such that for all $u \in \mathbb{B}_{s''}$

$$\|Au\|_{s'} \leq \frac{M}{s'' - s'} \|u\|_{s''}.$$

Here M is independent of s', s'' and u, however it might depend continuously on s, s_0.

Then, for each $s \in (0, s_0)$, there is a constant $\delta > 0$ (which depends on M) such that there is a unique function $u : \big[0, \delta(s_0 - s)\big) \to \mathbb{B}_s$ which is continuously differentiable on $\big(0, \delta(s_0 - s)\big)$ in $\mathbb{B}_s$, $Au \in \mathbb{B}_s$, and solves (2.13) *in the time-interval $0 \le t < \delta(s_0 - s)$.*

In Appendix we present a sketch of the proof of Theorem 2.5, which will be used to prove Theorem 4.3 below.

3 The Glauber dynamics

The Glauber dynamics is an example of a birth-and-death model where, in this special case, particles appear and disappear according to a death rate identically equal to 1 and to a birth rate depending on the interaction between particles. More precisely, let $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ be a pair potential, that is, a $\mathcal{B}(\mathbb{R}^d)$-measurable function such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$, which we will assume to be non-negative and integrable. Given a configuration $\gamma \in \Gamma$, the birth rate of a new particle at a site $x \in \mathbb{R}^d \setminus \gamma$ is given by $\exp(-E(x, \gamma))$, where $E(x, \gamma)$ is a relative energy of interaction between a particle located at x and the configuration γ defined by

$$E(x, \gamma) := \sum_{y \in \gamma} \phi(x - y) \in [0, +\infty].$$

Informally, in terms of Markov generators, this means that the behavior of such an infinite particle system is described by

$$\begin{aligned}(LF)(\gamma) := & \sum_{x \in \gamma} \big(F(\gamma \setminus \{x\}) - F(\gamma)\big) \\ & + z \int_{\mathbb{R}^d} dx\, e^{-E(x,\gamma)} \big(F(\gamma \cup \{x\}) - F(\gamma)\big), \end{aligned} \tag{3.1}$$

where $z > 0$ is an activity parameter (for more details see e.g. [FKO09, KKZ06]). As a consequence of Subsection 2.3, this implies that the operator $\tilde{L}$ defined in (2.11) is given cf. [FKO09] by

$$(\tilde{L}B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \Big(\delta B(\theta; x) - zB\left(\theta e^{-\phi(x-\cdot)} + e^{-\phi(x-\cdot)} - 1\right)\Big). \tag{3.2}$$

Theorem 3.1. *Given an $\alpha_0 > 0$, let $B_0 \in \mathcal{E}_{\alpha_0}$. For each $\alpha \in (0, \alpha_0)$ there is a $T > 0$ (which depends on α, α_0) such that there is a unique solution B_t, $t \in [0, T)$, to the initial value problem $\frac{\partial B_t}{\partial t} = \tilde{L}B_t$, $B_{t|t=0} = B_0$ in the space $\mathcal{E}_\alpha$.*

This theorem follows as a concrete application of Theorem 2.5 and the following estimate of norms.

Proposition 3.2. *Let $\alpha_0 > \alpha > 0$ be given. If $B \in \mathcal{E}_{\alpha''}$ for some $\alpha'' \in (\alpha, \alpha_0]$, then $\tilde{L}B \in \mathcal{E}_{\alpha'}$ for all $\alpha \leq \alpha' < \alpha''$, and we have*

$$\|\tilde{L}B\|_{\alpha'} \leq \frac{\alpha_0}{\alpha'' - \alpha'} \left(1 + z\alpha_0 e^{\frac{\|\phi\|_{L^1}}{\alpha} - 1}\right) \|B\|_{\alpha''}.$$

To prove this result, the next two lemmata show to be useful.

Lemma 3.3. *Given an $\alpha > 0$, for all $B \in \mathcal{E}_\alpha$ let*

$$(L_0 B)(\theta) := \int_{\mathbb{R}^d} dx\, \theta(x) \delta B(\theta; x), \quad \theta \in L^1.$$

Then, for all $\alpha' < \alpha$, we have $L_0 B \in \mathcal{E}_{\alpha'}$ and, moreover, the following estimate of norms holds:

$$\|L_0 B\|_{\alpha'} \leq \frac{\alpha'}{\alpha - \alpha'} \|B\|_\alpha.$$

Proof. First we observe that an application of Theorem 2.2 shows that $L_0 B$ is an entire functional on L^1 and, in addition, that for all $r > 0$ and all $\theta \in L^1$,

$$|(L_0 B)(\theta)| \leq \|\theta\|_{L^1} \|\delta B(\theta; \cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{\|\theta\|_{L^1}}{r} \sup_{\|\theta_0\|_{L^1} \leq r} |B(\theta + \theta_0)|,$$

where, for all $\theta_0 \in L^1$ such that $\|\theta_0\|_{L^1} \leq r$,

$$|B(\theta + \theta_0)| \leq \|B\|_\alpha e^{\frac{\|\theta\|_{L^1}}{\alpha} + \frac{r}{\alpha}}.$$

Thus,

$$\|L_0 B\|_{\alpha'} = \sup_{\theta \in L^1} \left(e^{-\frac{1}{\alpha'}\|\theta\|_{L^1}} |(L_0 B)(\theta)|\right) \leq \frac{e^{\frac{r}{\alpha}}}{r} \|B\|_\alpha \sup_{\theta \in L^1} \left(e^{-\left(\frac{1}{\alpha'} - \frac{1}{\alpha}\right)\|\theta\|_{L^1}} \|\theta\|_{L^1}\right),$$

where the latter supremum is finite provided $\frac{1}{\alpha'} - \frac{1}{\alpha} > 0$. In such a situation, the use of the inequality $xe^{-mx} \leq \frac{1}{em}$, $x \geq 0$, $m > 0$ leads for each $r > 0$ to

$$\|L_0 B\|_{\alpha'} \leq \frac{e^{\frac{r}{\alpha}}}{r} \frac{\alpha\alpha'}{e(\alpha - \alpha')} \|B\|_\alpha.$$

The required estimate of norms follows by minimizing the expression $\frac{e^{\frac{r}{\alpha}}}{r} \frac{\alpha\alpha'}{e(\alpha - \alpha')}$ in the parameter r, that is, $r = \alpha$. □

Lemma 3.4. *Let $\varphi, \psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be such that, for a.a. $x \in \mathbb{R}^d$, $\varphi(x, \cdot) \in L^\infty := L^\infty(\mathbb{R}^d)$, $\psi(x, \cdot) \in L^1$ and $\|\varphi(x, \cdot)\|_{L^\infty} \leq c_0$, $\|\psi(x, \cdot)\|_{L^1} \leq c_1$ for some constants $c_0, c_1 > 0$ independent of x. For each $\alpha > 0$ and all $B \in \mathcal{E}_\alpha$ let*

$$(L_1 B)(\theta) := \int_{\mathbb{R}^d} dx\, \theta(x) B(\varphi(x, \cdot)\theta + \psi(x, \cdot)), \quad \theta \in L^1.$$

Then, for all $\alpha' > 0$ such that $c_0\alpha' < \alpha$, we have $L_1 B \in \mathcal{E}_{\alpha'}$ and

$$\|L_1 B\|_{\alpha'} \leq \frac{\alpha\alpha'}{\alpha - c_0\alpha'} e^{\frac{c_1}{\alpha} - 1} \|B\|_\alpha.$$

Proof. As before, it follows from Theorem 2.2 that $L_1 B$ is an entire functional on L^1. Hence, given a $B \in \mathcal{E}_\alpha$, for all $\theta \in L^1$ one has

$$|B(\varphi(x, \cdot)\theta + \psi(x, \cdot))| \leq \|B\|_\alpha\, e^{\frac{1}{\alpha}\left(\|\varphi(x,\cdot)\theta\|_{L^1} + \|\psi(x,\cdot)\|_{L^1}\right)},$$

and thus

$$\begin{aligned} \|L_1 B\|_{\alpha'} &\leq \sup_{\theta \in L^1} \left(e^{-\frac{1}{\alpha'}\|\theta\|_{L^1}} \int_{\mathbb{R}^d} dx\, |\theta(x) B(\varphi(x, \cdot)\theta + \psi(x, \cdot))| \right) \\ &\leq e^{\frac{c_1}{\alpha}} \|B\|_\alpha \sup_{\theta \in L^1} \left(e^{-\left(\frac{1}{\alpha'} - \frac{c_0}{\alpha}\right)\|\theta\|_{L^1}} \|\theta\|_{L^1} \right). \end{aligned}$$

The proof follows as in the proof of Lemma 3.3. □

Proof of Proposition 3.2. In Lemma 3.4 replace φ by $e^{-\phi}$ and ψ by $e^{-\phi} - 1$. Due to the positiveness and integrability properties of ϕ one has $e^{-\phi} \leq 1$ and $|e^{-\phi} - 1| = 1 - e^{-\phi} \leq \phi \in L^1$, ensuring the conditions to apply Lemma 3.4. This together with Lemma 3.3 leads to the required result. □

Remark 3.5. *It follows from the proof of Theorem 2.5 that for each $t \in (0, T)$ there is an $\alpha_t \in (\alpha, \alpha_0)$ such that $B_t \in \mathcal{E}_\beta$ for all $\beta \in [\alpha, \alpha_t)$, cf. [FKK11].*

Remark 3.6. *Concerning the initial conditions considered in Theorem 3.1, observe that, in particular, B_0 can be an entire GF B_{μ_0} on L^1 such that, for some constants $\alpha_0, C > 0$, $|B_{\mu_0}(\theta)| \leq C \exp(\frac{\|\theta\|_{L^1}}{\alpha_0})$ for all $\theta \in L^1$. As we have mentioned before, in such a situation an additional analysis is required in order to guarantee that for each time t the local solution B_t given by Theorem 3.1 is a GF. Such an analysis is outside of our goal, but it may be done using e.g. [GK06, Theorem 2.13], which yields the existence of a proper $\mathcal{S} \subset \Gamma$ and a $\mathcal{S}$-valued process with sample paths in the Skorokhod space $D_{\mathcal{S}}([0, +\infty))$ associated with L defined in (3.1). This shows the existence of the time evolution $\mu_0 \mapsto \mu_t$, where μ_t is the law of the $\mathcal{S}$-valued process,* *leading apart from existence problems to the time evolution $B_{\mu_0} \mapsto B_{\mu_t}$ of the corresponding GF. The latter will be a solution to the initial value problem (2.12), (3.2) with $B_{t|t=0} = B_{\mu_0}$. By the uniqueness stated in Theorem 3.1, this implies that for each $t \in [0,T)$ we will have $B_t = B_{\mu_t}$, and thus B_t is a GF.*

Theorem 3.1 only ensures the existence of a local solution. However, under certain initial conditions, such a solution might be extend to a global one, that is, to a solution defined on the whole time interval $[0,+\infty)$, as follows. Assume that the initial condition B_0 is an entire GF on L^1. Then, by Proposition 2.3, B_0 can be written in terms of the corresponding correlation function k_0,

$$B_0(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta,\eta) k_0(\eta), \quad \theta \in L^1.$$

Assuming, in addition, that k_0 fulfills the Ruelle bound $k_0(\eta) \leq z^{|\eta|}$, $\eta \in \Gamma_0$, being z the activity parameter appearing in definition (3.1), then, in terms of B_0, this leads to $|B_0(\theta)| \leq e^{z\|\theta\|_{L^1}}$, $\theta \in L^1$, showing that $B_0 \in \mathcal{E}_{1/z}$ and, moreover, $\|B_0\|_{\frac{1}{z}} \leq 1$. Thus, fixing an $\alpha \in (0,1/z)$, an application of Theorem 3.1 yields a solution B_t, $t \in [0,\delta(1/z-\alpha))$, to the initial value problem (2.12), (3.2) with $B_{t|t=t_0} = B_0$. Assume that each B_t is an entire GF on L^1. As shown in [FKK11, Lemma 3.10], in this case the corresponding correlation function k_t still fulfills the Ruelle bound with the same constant z, meaning that the local solution B_t, $t \in [0,\delta(1/z-\alpha))$, does not leave the initial Banach space $\mathcal{E}_{1/z}$. This allows us to consider any $t_0 \in [0,\delta(1/z-\alpha))$ sufficiently close to $\delta(1/z-\alpha)$ as an initial time and, as before, to study the initial value problem (2.12), (3.2) with $B_{t|t=t_0} = B_{t_0}$ in the same scale of Banach spaces. This will give a solution B_t on the time-interval $[t_0, t_0+\delta(1/z-\alpha))$. Assuming again that each B_t, $t \in [t_0, t_0+\delta(1/z-\alpha))$, is an entire GF on L^1, naturally that $B_t \in \mathcal{E}_{1/z}$ with $\|B_t\|_{1/z} \leq 1$, for all $t \in [t_0, t_0+\delta(1/z-\alpha))$. Therefore, one may repeat the above arguments.

This argument iterated yields at the end a solution to the initial value problem $\frac{\partial B_t}{\partial t} = \tilde{L}B_t$, $B_{t|t=0} = B_0$ defined on $[0,+\infty)$. Of course, by the uniqueness stated in Theorem 3.1, the global solution constructed in this way is necessarily unique. In this context, one may state the following result.

Corollary 3.7. *Given an entire GF B_0 on L^1 such that the corresponding correlation function k_0 fulfills the Ruelle bound $k_0(\eta) \leq z^{|\eta|}$, $\eta \in \Gamma_0$, for the activity parameter z appearing in definition (3.1), the local solution to the initial value problem $\dfrac{\partial B_t}{\partial t} = \tilde{L}B_t$, $B_{t|t=0} = B_0$ (given by Theorem 3.1) might be extended to a global solution which, for each time $t \geq 0$, is an entire GF on L^1.*

4 Vlasov scaling

We proceed to investigate the Vlasov-type scaling proposed in [FKK10a] for generic continuous particle systems and accomplished in [FKK10b] for the Glauber dynamics, now in terms of GF. As explained in both references, the aim is to construct a scaling of the operator L defined in (3.1), L_ε, $\varepsilon > 0$, in such a way that the rescale of a starting correlation function k_0, denote by $k_0^{(\varepsilon)}$, provides a singularity with respect to ε of the type $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\eta \in \Gamma_0$, being r_0 a function independent of ε, and, moreover, the following two conditions are fulfilled. The first one is that under the scaling $L \mapsto L_\varepsilon$ the solution $k_t^{(\varepsilon)}$, $t \geq 0$, to

$$\frac{\partial}{\partial t} k_t^{(\varepsilon)} = \hat{L}_\varepsilon^* k_t^{(\varepsilon)}, \quad k_t^{(\varepsilon)}{}_{|t=0} = k_0^{(\varepsilon)}$$

preserves the order of the singularity with respect to ε, that is, $k_t^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta)$, $\eta \in \Gamma_0$. The second condition is that the dynamics $r_0 \mapsto r_t$ preserves the Lebesgue-Poisson exponents, that is, if r_0 is of the form $r_0 = e_\lambda(\rho_0)$, then each r_t, $t > 0$, is of the same type, i.e., $r_t = e_\lambda(\rho_t)$, where ρ_t is a solution to a non-linear equation (called a Vlasov-type equation). As shown in [FKK10a, Example 8], [FKK10b], this equation is given by

$$\frac{\partial}{\partial t} \rho_t(x) = -\rho_t(x) + z e^{-(\rho_t * \phi)(x)}, \quad x \in \mathbb{R}^d, \tag{4.1}$$

where $*$ denotes the usual convolution of functions. Existence of classical solutions $0 \leq \rho_t \in L^\infty$ to (4.1) has been discussed in [FKK10b], [FKK11]. Therefore, it is natural to consider the same scaling, but in GF.

The previous scheme was accomplished in [FKK10b] through the scale transformations $z \mapsto \varepsilon^{-1} z$ and $\phi \mapsto \varepsilon \phi$ of the operator L, that is,

$$(L_\varepsilon F)(\gamma) := \sum_{x \in \gamma} \big(F(\gamma \setminus \{x\}) - F(\gamma)\big) + \frac{z}{\varepsilon} \int_{\mathbb{R}^d} dx\, e^{-\varepsilon E(x,\gamma)} \big(F(\gamma \cup \{x\}) - F(\gamma)\big).$$

To proceed towards GF, we consider $k_t^{(\varepsilon)}$ defined as before and $k_{t,\mathrm{ren}}^{(\varepsilon)}(\eta) := \varepsilon^{|\eta|} k_t^{(\varepsilon)}(\eta)$. In terms of GF, these yield

$$B_t^{(\varepsilon)}(\theta) := \int_{\Gamma_0} d\lambda(\eta) e_\lambda(\theta, \eta) k_t^{(\varepsilon)}(\eta),$$

and

$$B_{t,\mathrm{ren}}^{(\varepsilon)}(\theta) := \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\theta, \eta) k_{t,\mathrm{ren}}^{(\varepsilon)}(\eta) = \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda(\varepsilon\theta, \eta) k_t^{(\varepsilon)}(\eta) = B_t^{(\varepsilon)}(\varepsilon\theta),$$

leading, as in (2.11), to the initial value problem

$$\frac{\partial}{\partial t} B^{(\varepsilon)}_{t,\mathrm{ren}} = \tilde{L}_{\varepsilon,\mathrm{ren}} B^{(\varepsilon)}_{t,\mathrm{ren}}, \quad B^{(\varepsilon)}_{t,\mathrm{ren}|t=0} = B^{(\varepsilon)}_{0,\mathrm{ren}}. \tag{4.2}$$

Proposition 4.1. *For all $\varepsilon > 0$ and all $\theta \in L^1$, we have*

$$(\tilde{L}_{\varepsilon,\mathrm{ren}} B)(\theta) = -\int_{\mathbb{R}^d} dx\, \theta(x) \left(\delta B(\theta, x) - zB\Big(\theta e^{-\varepsilon\phi(x-\cdot)} + \frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}\Big)\right).$$

Proof. As shown in [FKK10b, Proposition 3.1],

$$\begin{aligned}
&(\hat{L}_{\varepsilon,\mathrm{ren}} e_\lambda(\theta))(\eta) \\
=\ & -|\eta| e_\lambda(\theta,\eta) + z \sum_{\xi\subseteq\eta} e_\lambda(\theta,\xi) \int_{\mathbb{R}^d} dx\, \theta(x) e^{-\varepsilon E(x,\xi)} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}, \eta\setminus\xi\right).
\end{aligned}$$

Therefore,

$$(\tilde{L}_{\varepsilon,\mathrm{ren}} B)(\theta) = \int_{\Gamma_0} d\lambda(\eta)\, (\hat{L}_{\varepsilon,\mathrm{ren}} e_\lambda(\theta))(\eta) k(\eta),$$

with

$$\int_{\Gamma_0} d\lambda(\eta)\, |\eta| e_\lambda(\theta,\eta) k(\eta) = \int_{\mathbb{R}^d} dx\, \theta(x) \delta B(\theta; x)$$

and

$$\begin{aligned}
&\int_{\Gamma_0} d\lambda(\eta)\, k(\eta) \sum_{\xi\subseteq\eta} e_\lambda(\theta,\xi) \int_{\mathbb{R}^d} dx\, \theta(x) e^{-\varepsilon E(x,\xi)} e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}, \eta\setminus\xi\right) \\
=\ & \int_{\mathbb{R}^d} dx\, \theta(x) \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}, \eta\right) \int_{\Gamma_0} d\lambda(\xi)\, k(\eta\cup\xi) e_\lambda(\theta e^{-\varepsilon\phi(x-\cdot)}, \xi) \\
=\ & \int_{\mathbb{R}^d} dx\, \theta(x) \int_{\Gamma_0} d\lambda(\eta)\, e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}, \eta\right) \delta^{|\eta|} B(\theta e^{-\varepsilon\phi(x-\cdot)}; \eta) \\
=\ & \int_{\mathbb{R}^d} dx\, \theta(x) B\left(\theta e^{-\varepsilon\phi(x-\cdot)} + \frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}\right),
\end{aligned}$$

where we have used the relation between variational derivatives derived in [KKO06, Proposition 11]. □

Proposition 4.2. *(i) If $B \in \mathcal{E}_\alpha$ for some $\alpha > 0$, then, for all $\theta \in L^1$, $(\tilde{L}_{\varepsilon,\mathrm{ren}} B)(\theta)$ converges as ε tends zero to*

$$(\tilde{L}_V B)(\theta) := -\int_{\mathbb{R}^d} dx\, \theta(x) \left(\delta B(\theta; x) - zB\left(\theta - \phi(x-\cdot)\right)\right).$$

(ii) Let $\alpha_0 > \alpha > 0$ be given. If $B \in \mathcal{E}_{\alpha''}$ for some $\alpha'' \in (\alpha, \alpha_0]$, then $\left\{\tilde{L}_{\varepsilon,\mathrm{ren}}B, \tilde{L}_V B\right\} \subset \mathcal{E}_{\alpha'}$ for all $\alpha \leq \alpha' < \alpha''$, and we have

$$\|\tilde{L}_{\#}B\|_{\alpha'} \leq \frac{\alpha_0}{\alpha'' - \alpha'}\left(1 + z\alpha_0 e^{\frac{\|\phi\|_{L^1}}{\alpha} - 1}\right)\|B\|_{\alpha''},$$

where $\tilde{L}_{\#} = \tilde{L}_{\varepsilon,\mathrm{ren}}$ or $\tilde{L}_{\#} = \tilde{L}_V$.

Proof. (i) First we observe that for a.a. $x \in \mathbb{R}^d$ one clearly has

$$\lim_{\varepsilon \searrow 0}\left(\theta e^{-\varepsilon\phi(x-\cdot)} + \frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}\right) = \theta - \phi(x - \cdot) \text{ in } L^1,$$

and thus, due to the continuity of B in L^1 (B is even entire on L^1), the following limit holds

$$\lim_{\varepsilon \searrow 0} B\left(\theta e^{-\varepsilon\phi(x-\cdot)} + \frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}\right) = B\left(\theta - \phi(x - \cdot)\right), \quad a.a.\ x \in \mathbb{R}^d.$$

This shows the pointwise convergence of the integrand functions which appear in the definition of $(\tilde{L}_{\varepsilon,\mathrm{ren}}B)(\theta)$ and $(\tilde{L}_V B)(\theta)$. In addition, for all $\varepsilon > 0$ we have

$$\left|B\left(\theta e^{-\varepsilon\phi(x-\cdot)} + \frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}\right)\right| \leq \|B\|_\alpha \exp\left(\frac{1}{\alpha}\|\theta\|_{L^1} + \frac{1}{\alpha}\|\phi\|_{L^1}\right),$$

leading through an application of the Lebesgue dominated convergence theorem to the required limit.

(ii) In Lemma 3.4 replace φ by $e^{-\varepsilon\phi}$ and ψ by $\frac{e^{-\varepsilon\phi}-1}{\varepsilon}$. Arguments similar to prove Proposition 3.2 together with Lemma 3.3 complete the proof for $\tilde{L}_{\varepsilon,\mathrm{ren}}$. For $\tilde{L}_V$, the proof follows similarly. □

Proposition 4.2 (ii) provides similar estimate of norms for $\tilde{L}_{\varepsilon,\mathrm{ren}}$, $\varepsilon > 0$, and the limiting mapping $\tilde{L}_V$, namely, $\|\tilde{L}_{\varepsilon,\mathrm{ren}}B\|_{\alpha'}, \|\tilde{L}_V B\|_{\alpha'} \leq \frac{M}{\alpha''-\alpha'}\|B\|_{\alpha''}$, $0 < \alpha \leq \alpha' < \alpha'' \leq \alpha_0$, with

$$M := \alpha_0\left(1 + z\alpha_0 e^{\frac{\|\phi\|_{L^1}}{\alpha} - 1}\right).$$

Therefore, given any $B_{0,V}, B_{0,\mathrm{ren}}^{(\varepsilon)} \in \mathcal{E}_{\alpha_0}$, $\varepsilon > 0$, it follows from Theorem 3.1 and its proof that for each $\alpha \in (0, \alpha_0)$ and $\delta = \frac{1}{eM}$ there is a unique solution $B_{t,\mathrm{ren}}^{(\varepsilon)} : [0, \delta(\alpha_0 - \alpha)) \to \mathcal{E}_\alpha$, $\varepsilon > 0$, to each initial value problem (4.2) and a unique solution $B_{t,V} : [0, \delta(\alpha_0 - \alpha)) \to \mathcal{E}_\alpha$ to the initial value problem

$$\frac{\partial}{\partial t}B_{t,V} = \tilde{L}_V B_{t,V}, \quad B_{t,V}{}_{|t=0} = B_{0,V}. \tag{4.3}$$

That is, independent of the initial value problem under consideration, the solutions obtained are defined on the same time-interval and with values in the same Banach space. Therefore, it is natural to analyze under which conditions the solutions to (4.2) converge to the solution to (4.3). This follows straightforwardly from a general result which proof (see Appendix) follows closely the lines of the proof of Theorem 2.5.

Theorem 4.3. *On a scale of Banach spaces* $\{\mathbb{B}_s : 0 < s \leq s_0\}$ *consider a family of initial value problems*

$$\frac{du_\varepsilon(t)}{dt} = A_\varepsilon u_\varepsilon(t), \quad u_\varepsilon(0) = u_\varepsilon \in \mathbb{B}_{s_0}, \quad \varepsilon \geq 0, \tag{4.4}$$

where, for each $s \in (0, s_0)$ *fixed and for each pair* s', s'' *such that* $s \leq s' < s'' \leq s_0$, $A_\varepsilon : \mathbb{B}_{s''} \to \mathbb{B}_{s'}$ *is a linear mapping so that there is an* $M > 0$ *such that for all* $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u\|_{s'} \leq \frac{M}{s'' - s'} \|u\|_{s''}.$$

Here M *is independent of* ε, s', s'' *and* u, *however it might depend continuously on* s, s_0. *Assume that there is a* $p \in \mathbb{N}$ *and for each* $\varepsilon > 0$ *there is an* $N_\varepsilon > 0$ *such that for each pair* s', s'', $s \leq s' < s'' \leq s_0$, *and all* $u \in \mathbb{B}_{s''}$

$$\|A_\varepsilon u - A_0 u\|_{s'} \leq \sum_{k=1}^{p} \frac{N_\varepsilon}{(s'' - s')^k} \|u\|_{s''}.$$

In addition, assume that $\lim_{\varepsilon \to 0} N_\varepsilon = 0$ *and* $\lim_{\varepsilon \to 0} \|u_\varepsilon(0) - u_0(0)\|_{s_0} = 0$.

Then, for each $s \in (0, s_0)$, *there is a constant* $\delta > 0$ *(which depends on* M*) such that there is a unique solution* $u_\varepsilon : [0, \delta(s_0 - s)) \to \mathbb{B}_s$, $\varepsilon \geq 0$, *to each initial value problem (4.4) and for all* $t \in [0, \delta(s_0 - s))$ *we have*

$$\lim_{\varepsilon \to 0} \|u_\varepsilon(t) - u_0(t)\|_s = 0.$$

To proceed to an application of this general result one needs the following estimate of norms.

Proposition 4.4. *Assume that* $0 \leq \phi \in L^1 \cap L^\infty$ *and let* $\alpha_0 > \alpha > 0$ *be given. Then, for all* $B \in \mathcal{E}_{\alpha''}$, $\alpha'' \in (\alpha, \alpha_0]$, *the following estimate holds*

$$\|\tilde{L}_{\varepsilon,\mathrm{ren}} B - \tilde{L}_V B\|_{\alpha'} \leq \varepsilon z \|\phi\|_{L^\infty} \|B\|_{\alpha''} e^{\frac{\|\phi\|_{L^1}}{\alpha}} \left(\frac{\|\phi\|_{L^1} \alpha_0}{\alpha'' - \alpha'} + \frac{4\alpha_0^3}{(\alpha'' - \alpha')^2 e} \right)$$

for all α' *such that* $\alpha \leq \alpha' < \alpha''$ *and all* $\varepsilon > 0$.

Proof. First we observe that

$$\Big|(\tilde{L}_{\varepsilon,\mathrm{ren}}B)(\theta)-(\tilde{L}_V B)(\theta)\Big| \leq z\int_{\mathbb{R}^d} dx\ |\theta(x)| \\ \times\left|B\left(\theta e^{-\varepsilon\phi(x-\cdot)}+\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}\right)-B\left(\theta-\phi(x-\cdot)\right)\right|. \quad (4.5)$$

In order to estimate (4.5), given any $\theta_1,\theta_2\in L^1$, let us consider the function $C_{\theta_1,\theta_2}(t)=B\left(t\theta_1+(1-t)\theta_2\right)$, $t\in[0,1]$. One has

$$\frac{\partial}{\partial t}C_{\theta_1,\theta_2}(t)=\frac{\partial}{\partial s}C_{\theta_1,\theta_2}(t+s)\Big|_{s=0}=\frac{\partial}{\partial s}B\big(\theta_2+t(\theta_1-\theta_2)+s(\theta_1-\theta_2)\big)\Big|_{s=0} \\ =\int_{\mathbb{R}^d} dx\,(\theta_1(x)-\theta_2(x))\,\delta B(\theta_2+t(\theta_1-\theta_2);x),$$

leading to

$$\big|B(\theta_1)-B(\theta_2)\big|=\big|C_{\theta_1,\theta_2}(1)-C_{\theta_1,\theta_2}(0)\big| \\ \leq \max_{t\in[0,1]}\int_{\mathbb{R}^d} dx\,\big|\theta_1(x)-\theta_2(x)\big|\big|\delta B(\theta_2+t(\theta_1-\theta_2);x)\big| \\ \leq \|\theta_1-\theta_2\|_{L^1}\max_{t\in[0,1]}\|\delta B(\theta_2+t(\theta_1-\theta_2);\cdot)\|_{L^\infty},$$

where, through similar arguments to prove Lemma 3.3,

$$\big\|\delta B(\theta_2+t(\theta_1-\theta_2);\cdot)\big\|_{L^\infty}\leq\frac{e}{\alpha''}\|B\|_{\alpha''}\exp\left(\frac{\|\theta_2+t(\theta_1-\theta_2)\|_{L^1}}{\alpha''}\right).$$

As a result

$$\big|B(\theta_1)-B(\theta_2)\big|\leq\frac{e}{\alpha''}\|\theta_1-\theta_2\|_{L^1}\|B\|_{\alpha''}\max_{t\in[0,1]}\exp\left(\frac{t\|\theta_1\|_{L^1}+(1-t)\|\theta_2\|_{L^1}}{\alpha''}\right),$$

for all $\theta_1,\theta_2\in L^1$. In particular, this shows that

$$\left|B\Big(\theta e^{-\varepsilon\phi(x-\cdot)}+\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}\Big)-B\big(\theta-\phi(x-\cdot)\big)\right| \\ \leq\varepsilon\frac{e}{\alpha''}\|\phi\|_{L^\infty}\|B\|_{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right) \\ \times\max_{t\in[0,1]}\exp\left(\frac{1}{\alpha''}\left(t\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)+(1-t)\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\right)\right) \\ =\varepsilon\frac{e}{\alpha''}\|\phi\|_{L^\infty}\|B\|_{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\exp\left(\frac{1}{\alpha''}\left(\|\theta\|_{L^1}+\|\phi\|_{L^1}\right)\right),$$

where we have used the inequalities

$$\begin{aligned}
\|\theta e^{-\varepsilon\phi(x-\cdot)}-\theta\|_{L^1} &\leq \varepsilon\|\phi\|_{L^\infty}\|\theta\|_{L^1},\\
\Big\|\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}+\phi(x-\cdot)\Big\|_{L^1} &\leq \varepsilon\|\phi\|_{L^\infty}\|\phi\|_{L^1},\\
\Big\|\theta e^{-\varepsilon\phi(x-\cdot)}+\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}\Big\|_{L^1} &\leq \|\theta\|_{L^1}+\|\phi\|_{L^1}.
\end{aligned}$$

In this way we obtain

$$\begin{aligned}
&\|\tilde{L}_{\varepsilon,\mathrm{ren}}B-\tilde{L}_V B\|_{\alpha'}\\
\leq\ & \varepsilon\frac{ze}{\alpha''}\|\phi\|_{L^\infty}\|B\|_{\alpha''}e^{\frac{\|\phi\|_{L^1}}{\alpha''}}\left\{\sup_{\theta\in L^1}\left(\|\theta\|_{L^1}^2\exp\left(\|\theta\|_{L^1}\left(\frac{1}{\alpha''}-\frac{1}{\alpha'}\right)\right)\right)\right.\\
&\left.+\|\phi\|_{L^1}\sup_{\theta\in L^1}\left(\|\theta\|_{L^1}\exp\left(\|\theta\|_{L^1}\left(\frac{1}{\alpha''}-\frac{1}{\alpha'}\right)\right)\right)\right\},
\end{aligned}$$

and the proof follows using the inequalities $xe^{-mx}\leq\frac{1}{me}$ and $x^2e^{-mx}\leq\frac{4}{m^2e^2}$ for $x\geq 0$, $m>0$. □

We are now in conditions to state the following result.

Theorem 4.5. *Given an $0<\alpha<\alpha_0$, let $B_{t,\mathrm{ren}}^{(\varepsilon)}, B_{t,V}$, $t\in[0,\delta(\alpha_0-\alpha))$, be the local solutions in $\mathcal{E}_\alpha$ to the initial value problems* (4.2), (4.3) *with $B_{0,\mathrm{ren}}^{(\varepsilon)}, B_{0,V}\in\mathcal{E}_{\alpha_0}$. If $0\leq\phi\in L^1\cap L^\infty$ and $\lim_{\varepsilon\to0}\|B_{0,\mathrm{ren}}^{(\varepsilon)}-B_{0,V}\|_{\alpha_0}=0$, then, for each $t\in[0,\delta(\alpha_0-\alpha))$,*

$$\lim_{\varepsilon\to0}\|B_{t,\mathrm{ren}}^{(\varepsilon)}-B_{t,V}\|_\alpha=0.$$

Moreover, if $B_{0,V}(\theta)=\exp\left(\int_{\mathbb{R}^d}dx\,\rho_0(x)\theta(x)\right)$, $\theta\in L^1$, for some function $0\leq\rho_0\in L^\infty$ such that $\|\rho_0\|_{L^\infty}\leq\frac{1}{\alpha_0}$, and if $\max\{\frac{1}{\alpha_0},z\}<\frac{1}{\alpha}$ then, for each $t\in[0,\delta(\alpha_0-\alpha))$,

$$B_{t,V}(\theta)=\exp\left(\int_{\mathbb{R}^d}dx\,\rho_t(x)\theta(x)\right),\quad \theta\in L^1, \tag{4.6}$$

where $0\leq\rho_t\in L^\infty$ is a classical solution to the equation (4.1) *such that, for each $t\in[0,\delta(\alpha_0-\alpha))$, $\|\rho_t\|_{L^\infty}\leq\frac{1}{\alpha}$.*

Proof. The first part follows directly from Proposition 4.4 and Theorem 4.3 for $p=2$ and $N_\varepsilon=\varepsilon z\|\phi\|_{L^\infty}\alpha_0 e^{\frac{\|\phi\|_{L^1}}{\alpha}}\max\{\|\phi\|_{L^1},\frac{4\alpha_0^2}{e}\}$.

Concerning the last part, we begin by observing that it has been shown in [FKK10b, Proof of Theorem 3.3] that given a $0\leq\rho_0\in L^\infty$ such that

$\|\rho_0\|_{L^\infty} \leq \frac{1}{\alpha_0}$, the solution ρ_t to (4.1) (which existence has been proved in [FKK11]) fulfills $0 \leq \rho_t \in L^\infty$, $\|\rho_t\|_{L^\infty} \leq \max\{\frac{1}{\alpha_0}, z\}$. In this way, the assumption $\max\{\frac{1}{\alpha_0}, z\} < \frac{1}{\alpha}$ implies that $B_{t,V}$, given by (4.6), fulfills $B_{t,V} \in \mathcal{E}_\alpha$. Then, by an argument of uniqueness, to prove the last assertion amounts to show that $B_{t,V}$ solves equation (4.3). For this purpose we note that for any $\theta, \theta_1 \in L^1$ we have

$$\frac{\partial}{\partial z_1} B_{t,V}(\theta + z_1\theta_1)\bigg|_{z_1=0} = B_{t,V}(\theta) \int_{\mathbb{R}^d} dx \rho_t(x)\theta_1(x),$$

and thus $\delta B_{t,V}(\theta; x) = B_{t,V}(\theta)\rho_t(x)$. Hence, for all $\theta \in L^1$,

$$(\tilde{L}_V B_{t,V})(\theta) = -B_{t,V}(\theta) \left(\int_{\mathbb{R}^d} dx\, \theta(x)\rho_t(x) - z \int_{\mathbb{R}^d} dx\, \theta(x) \exp\left(-(\rho_t * \phi)(x)\right) \right).$$

Since ρ_t is a classical solution to (4.1), ρ_t solves a weak form of equation (4.1), that is, the right-hand side of the latter equality is equal to

$$B_{t,V}(\theta) \frac{d}{dt} \int_{\mathbb{R}^d} dx\, \rho_t(x)\theta(x) = \frac{\partial}{\partial t} B_{t,V}(\theta). \qquad \square$$

Appendix: Proofs of Theorems 2.5 and 4.3

Sketch of the proof of Theorem 2.5. For some $t > 0$ which later on will be properly chosen, let us consider the sequence of functions $(u_n)_{n\in\mathbb{N}_0}$ with $u_0(t) \equiv u_0 \in \mathbb{B}_{s_0}$ and

$$u_n(t) := u_0 + \int_0^t (Au_{n-1})(s)ds, \quad n \in \mathbb{N}.$$

By an induction argument, it is easy to check that $u_n(t) \in \mathbb{B}_s$ for any $s < s_0$ and, in an equivalent way, the sequence may be rewritten as

$$u_n(t) = u_0 + \sum_{m=1}^{n} \frac{t^m}{m!} A^m u_0. \tag{4.7}$$

Fixed an $0 < s < s_0$, let us now consider a partition of the interval $[s, s_0]$ into m equals parts, $m \in \mathbb{N}$. That is, we define $s_l := s_0 - \frac{l(s_0-s)}{m}$ for $l = 0, \ldots, m$. By assumption, observe that for each $l = 0, \ldots, m$ the linear mapping $A : \mathbb{B}_{s_l} \to \mathbb{B}_{s_{l+1}}$ verifies

$$\|A\|_{s_l s_{l+1}} := \|A\|_{\mathbb{B}_{s_l} \mapsto \mathbb{B}_{s_{l+1}}} \leq \frac{mM}{s_0 - s},$$

and thus

$$\|A^m\|_{s_0 s} \leq \|A\|_{s_0 s_1} \cdots \|A\|_{s_{m-1} s} \leq \left(\frac{mM}{s_0 - s}\right)^m. \tag{4.8}$$

From this and the Stirling formula follow the convergence of the series

$$\sum_{m=1}^{n} \frac{t^m}{m!} \|A^m u_0\|_s \leq \|u_0\|_{s_0} \sum_{m=1}^{n} \frac{m^m}{m!} \left(\frac{Mt}{s_0 - s}\right)^m$$

whenever $\frac{tM}{s_0 - s} < \frac{1}{e}$. This means that for all $t < \frac{s_0 - s}{eM}$ the sequence (4.7) converges in $\mathbb{B}_s$ to the function

$$u(t) := u_0 + \sum_{m=1}^{\infty} \frac{1}{m!} t^m A^m u_0.$$

Moreover, setting $\delta := \frac{1}{eM}$, $M = M(s, s_0)$, this convergence is uniform on any interval $[0, T] \subset [0, \delta(s_0 - s))$. Similar arguments show that an analogous situation occurs for the series

$$\sum_{m=1}^{\infty} \frac{1}{m!} \frac{d}{dt} t^m A^m u_0 = \sum_{m=0}^{\infty} \frac{1}{m!} t^m A^{m+1} u_0. \tag{4.9}$$

This shows that on the time-interval $(0, \delta(s_0 - s))$ the function u is continuously differentiable in $\mathbb{B}_s$.

Of course, these considerations hold for any $s_1 \in (s, s_0)$, showing that the sequence (4.7) also converges in the space $\mathbb{B}_{s_1}$ uniformly to a function $\tilde{u}$ on any time interval $[0, T] \subset [0, \delta_1(s_0 - s_1))$, $\delta_1 := \frac{1}{eM_1}$, $M_1 = M_1(s_0, s_1)$. On the other hand, due to the continuity of $M(s_0, \cdot)$ on $(0, s_0)$, for each $t \in [0, \delta(s_0 - s))$ fixed there is an $s_1 \in (s, s_0)$ such that $t \in [0, \delta_1(s_0 - s_1))$. As a result, $u_n(t)$ converges to a $\tilde{u}(t)$ in the space $\mathbb{B}_{s_1} \subset \mathbb{B}_s$. Since

$$\|\tilde{u}(t) - u(t)\|_{\mathbb{B}_s} \leq \|\tilde{u}(t) - u_n(t)\|_{\mathbb{B}_{s_1}} + \|u(t) - u_n(t)\|_{\mathbb{B}_s},$$

it follows that $\tilde{u}(t) = u(t)$ in $\mathbb{B}_s$. In other words, $u(t) \in \mathbb{B}_{s_1}$. Therefore, $u(t)$ is in the domain of $A : \mathbb{B}_{s_1} \to \mathbb{B}_s$, and thus $Au(t) \in \mathbb{B}_s$. Since this holds for every $t \in [0, \delta(s_0 - s))$, the convergence of the series (4.9) then implies that u is a solution to the initial value problem (2.13). To check the uniqueness see e.g. [Tre68, pp. 16–17]. □

Proof of Theorem 4.3. To prove this result amounts to check the convergence. Following the scheme used to prove Theorem 2.5, we begin by recalling that in that proof each solution u_ε, $\varepsilon \geq 0$, to (4.4) was obtained as a limit in $\mathbb{B}_s$ of

$$u_{\varepsilon,n}(t) = u_\varepsilon + \sum_{m=1}^{n} \frac{1}{m!} t^m A_\varepsilon^m u_\varepsilon,$$

where $t \in [0, \delta(s_0 - s))$ with $\delta = \frac{1}{eM}$. Thus, for each $\varepsilon' > 0$, there is an $n \in \mathbb{N}$ such that

$$\begin{aligned}
\|u_\varepsilon(t) - u_0(t)\|_s \leq & \|u_\varepsilon(t) - u_{\varepsilon,n}(t)\|_s + \|u_{\varepsilon,n}(t) - u_{0,n}(t)\|_s + \|u_{0,n}(t) - u_0(t)\|_s \\
< & \frac{\varepsilon'}{2} + \|u_\varepsilon - u_0\|_s + \sum_{m=1}^{n} \frac{t^m}{m!} \|A_\varepsilon^m u_\varepsilon - A_0^m u_0\|_s \\
\leq & \frac{\varepsilon'}{2} + \|u_\varepsilon - u_0\|_s + \sum_{m=1}^{n} \frac{t^m}{m!} \|A_\varepsilon^m (u_\varepsilon - u_0)\|_s \\
& + \sum_{m=1}^{n} \frac{t^m}{m!} \|(A_\varepsilon^m - A_0^m) u_0)\|_s. \qquad (4.10)
\end{aligned}$$

Observe that by (4.8)

$$\|A_\varepsilon^m (u_\varepsilon - u_0)\|_s \leq \left(\frac{mM}{s_0 - s} \right)^m \|u_\varepsilon - u_0\|_{s_0}.$$

To estimate (4.10) we proceed as in the proof of Theorem 2.5. For this purpose, we will use the decomposition

$$\begin{aligned}
A_\varepsilon^m - A_0^m = & (A_\varepsilon - A_0) A_\varepsilon^{m-1} + A_0 (A_\varepsilon - A_0) A_\varepsilon^{m-2} + \\
& + \cdots + A_0^{m-2} (A_\varepsilon - A_0) A_\varepsilon + A_0^{m-1} (A_\varepsilon - A_0).
\end{aligned}$$

Then, considering again a partition of the interval $[s, s_0]$ into m parts and the points $s_l = s_0 - \frac{l(s_0 - s)}{m}$, $l = 0, \ldots, m$, one finds the estimate

$$\begin{aligned}
\|(A_\varepsilon^m - A_0^m) u_0)\|_s \leq & \sum_{l=0}^{m-1} \|A_\varepsilon - A_0\|_{s_l s_{l+1}} \left(\frac{mM}{s_0 - s} \right)^{m-1} \|u_0\|_{s_0} \\
\leq & \sum_{k=1}^{p} \frac{N_\varepsilon}{(s_0 - s)^{k-1}} \left(\frac{mM}{s_0 - s} \right)^m \frac{m^k}{M} \|u_0\|_{s_0}.
\end{aligned}$$

As a result, defining for each $t \in \big[0, \delta(s_0 - s)\big)$, $\delta = \frac{1}{eM}$,

$$f_q(t) := \sum_{m=1}^{\infty} \frac{m^q}{m!} \left(\frac{tmM}{s_0 - s} \right)^m < \infty, \quad q \geq 0,$$

we obtain from the previous considerations the estimate

$$\|u_\varepsilon(t) - u_0(t)\|_s < \frac{\varepsilon'}{2} + \|u_\varepsilon - u_0\|_s + \|u_\varepsilon - u_0\|_{s_0} f_0(t) + \sum_{k=1}^{p} \frac{N_\varepsilon}{M(s_0 - s)^{k-1}} \|u_0\|_{s_0} f_k(t).$$

Here we observe that, by assumption, u_ε converges in $\mathbb{B}_{s_0}$ to u_0. Thus, by the definition of a scale of Banach spaces, this convergence also holds in $\mathbb{B}_s$. Therefore, for small enough ε, one has $\|u_\varepsilon(t) - u_0(t)\|_s < \varepsilon'$, which completes the proof. □

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