Vlasov scaling for the Glauber dynamics in continuum

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Abstract

We consider Vlasov-type scaling for the Glauber dynamics in continuum with a positive integrable potential, and construct rescaled and limiting evolutions of correlation functions. Convergence to the limiting evolution for the positive density system in infinite volume is shown. Chaos preservation property of this evolution gives a possibility to derive a non-linear Vlasov-type equation for the particle density of the limiting system.

1 Introduction

Kinetic equations are a useful approximation for the description of dynamical processes in multi-body systems, see, e.g., the reviews by H.Spohn [32], [33]. Among them, the Vlasov equation has important role in physics (in particular, physics of plasma). It describes the Hamiltonian motion of an infinite particle system in the mean field scaling limit when the influence of weak long-range forces is taken into account. The convergence of the Vlasov scaling limit was shown rigorously by W.Braun and K.Hepp [1] (for the Hamiltonian dynamics) and by R.L.Dobrushin [3] (for more general deterministic dynamical systems). However, the resulting Vlasov-type equations for particle densities are considered in classes of integrable functions (or, in the weak form, of finite measures). This, in fact, restricts us to the case of finite volume systems or systems with zero mean density in an infinite volume. Detailed analysis of Vlasov-type equations for integrable functions is presented in the recent paper by V.V.Kozlov [25].

In [9], we proposed a general approach to study the Vlasov-type scaling for some classes of stochastic evolutions in the continuum, in particular, for spatial birth-and-death Markov processes. The approaches mentioned above are not

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applicable to these dynamics (even in a finite volume) due to essential reasons (see [9] for details). One of them is a possible variation of the particle number during the evolution. More essentially is that for these processes the possibility of their descriptions in terms of proper stochastic evolutional equations for particle motion is, generally speaking, absent. There are only few works concerning general spatial birth-and-death evolutions, see [30], [16], [13], [14], [29], [31]. However, the conditions for the existence (in different senses) of the evolutions considered therein are quite far from the general form.

Therefore, we looked for an alternative approach to the derivation of kinetic Vlasov-type equations from stochastic dynamics. The correct Vlasov limit can be easily guessed from the BBGKY hierarchy for the Hamiltonian system, see, e.g., [32]. Such a heuristic derivation does not assume the integrability condition for the density, but until now, it could not be made rigorously due to the lack of detailed information about the properties of solutions to the BBGKY hierarchy. Our approach is based on this observation applied in a new dynamical framework. Note that we already know that many stochastic evolutions in continuum admit effective descriptions in terms of hierarchical equations for correlation functions which generalize the BBGKY hierarchy from Hamiltonian to Markov setting, see, e.g., [12] and the references therein. Even more, these hierarchical equations are often the only available technical tools for a construction of considered dynamics [20], [21], [8].

Developing this point of view, our scheme for the Vlasov scaling of stochastic dynamics is based on the proper scaling of the hierarchical equations. This scheme has also a clear interpretation in the terms of scaled Markov generators. An application of the considered scaling leads to the limiting hierarchy which posses a chaos preservation property. Namely, if we start from a Poissonian (non-homogeneous) initial state of the system, then during the time evolution this property will be preserved. Moreover, a special structure of the interaction in the resulting virtual Vlasov system gives a non-linear evolutional equation for the density of the evolving Poisson state.

The control of the convergence of Vlasov scalings for the considered hierarchies is a quite difficult technical problem which should be analyzed for any particular model separately. In the present paper, we solve this problem for the Glauber dynamics in continuum. These dynamics have given reversible states which are grand canonical Gibbs measures. The corresponding equilibrium dynamics which preserve the initial Gibbs state in the time evolution were considered in, e.g., [22], [23], [24], [11]. Note that, in applications, the time evolution of initial state is the subject of the primary interest. Therefore, we understand the considered stochastic (non-equilibrium) dynamics as the evolution of initial distributions for the system. Actually, the corresponding Markov process (provided it exists) itself gives a general technical equipment to study this problem. Moreover, using the techniques developed in [13], it is possible to construct this Markov process as a solution of a stochastic differential equation. Unfortunately, this approach does not give any information about the properties of the corresponding correlation functions which we need for the study of Vlasov scaling as was mentioned above.

However, we note that the transition from the micro-state evolution corresponding to the given initial configuration to the macro-state dynamics is the well developed concept in the theory of infinite particle systems. This point of view appeared initially in the framework of the Hamiltonian dynamics of classical gases, see, e.g., [4]. Again, the lack of the general Markov processes techniques for the considered systems makes it necessary to develop alternative approaches to study the state evolutions in the Glauber dynamics. Such approaches we realized in [20], [21], [10], [7]. The description of the time evolutions for measures on configuration spaces in terms of an infinite system of evolutional equations for the corresponding correlation functions was used there. The latter system is a Glauber evolution's analog of the famous BBGKY-hierarchy for the Hamiltonian dynamics.

Here we extend the approximation approach proposed in [10], [7] to the Vlasov scaling for the Glauber dynamics in continuum. We construct and study semigroups corresponding to properly rescaled Markov generator of the Glauber dynamics (Propositions 3.8 and 3.11). We prove for the integrable and bounded potential the convergence of these semigroups to the limiting semigroup which describe Vlasov evolution (Theorem 3.12). We derive the corresponding Vlasov-type equation from this evolution (Theorem 3.14). Note that the stationary solution of this equation will satisfied the well-known Kirkwood–Monroe equation in the freezing theory (Remark 3.15).

2 Glauber dynamics in continuum

2.1 Basic facts and notation

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$; $\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ denotes the system of all bounded sets in $\mathcal{B}(\mathbb{R}^d)$.

The configuration space over space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$, namely,

$$\Gamma = \Gamma_{\mathbb{R}^d} := \Big\{ \gamma \subset \mathbb{R}^d \;\Big|\; |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d) \Big\}. \tag{2.1}$$

Here $\gamma_\Lambda := \gamma \cap \Lambda$, and $|\cdot|$ means the cardinality of a finite set. The space Γ is equipped with the vague topology, i.e., the minimal topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support; note that the summation in $\sum_{x\in\gamma} f(x)$ is taken over finitely many points of γ which belong to the support of f. In [19], it was shown that Γ with the vague topology may be metrizable and it becomes a Polish space (i.e., complete separable metric space). Corresponding to this topology, the Borel σ-algebra $\mathcal{B}(\Gamma)$ is the smallest σ-algebra for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are measurable for any $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$.

The space of n-point configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_Y^{(n)} := \Big\{ \eta \subset Y \;\Big|\; |\eta| = n \Big\}, \quad n \in \mathbb{N}.$$

We set also $\Gamma_Y^{(0)} := \{\emptyset\}$. As a set, $\Gamma_Y^{(n)}$ may be identified with the symmetrization of

$$\widetilde{Y^n} = \Big\{(x_1, \ldots, x_n) \in Y^n \;\Big|\; x_k \neq x_l \text{ if } k \neq l\Big\}.$$

Hence one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}(\Gamma_Y^{(n)})$. The space of finite configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_{0,Y} := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_Y^{(n)}.$$

This space is equipped with the topology of disjoint unions. Therefore, one can introduce the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_{0,Y})$. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the notation, namely, $\Gamma_0 := \Gamma_{0,\mathbb{R}^d}$, $\Gamma^{(n)} := \Gamma_{\mathbb{R}^d}^{(n)}$.

The restriction of the Lebesgue product measure $(dx)^n$ to $\big(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\big)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2.2}$$

For any $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ the restriction of λ to $\Gamma_\Lambda := \Gamma_{0,\Lambda}$ will be also denoted by λ. The space $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is the projective limit of the family of spaces $\big\{(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))\big\}_{\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)}$. The Poisson measure π on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is given as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is the probability measure on $\big(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)\big)$. Here $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$.

For any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ we define a *Lebesgue–Poisson exponent*

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0; \qquad e_\lambda(f, \emptyset) := 1. \tag{2.3}$$

Then, by (2.2), for $f \in L^1(\mathbb{R}^d, dx)$ we obtain $e_\lambda(f) \in L^1(\Gamma_0, d\lambda)$ and

$$\int_{\Gamma_0} e_\lambda(f, \eta) d\lambda(\eta) = \exp\left\{ \int_{\mathbb{R}^d} f(x) dx \right\}. \tag{2.4}$$

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. The set of bounded measurable functions with bounded support we denote by $B_\mathrm{bs}(\Gamma_0)$, i.e., $G \in B_\mathrm{bs}(\Gamma_0)$ if $G \restriction_{\Gamma_0 \setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0, in fact, is a sequence of functions $\big\{G^{(n)}\big\}_{n \in \mathbb{N}_0}$ where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$. We consider also the set $\mathcal{F}_\mathrm{cyl}(\Gamma)$ of *cylinder functions* on Γ. Each $F \in \mathcal{F}_\mathrm{cyl}(\Gamma)$ is characterized by the following relation: $F(\gamma) = F \restriction_{\Gamma_\Lambda} (\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$.

There is the following mapping from $B_\mathrm{bs}(\Gamma_0)$ into $\mathcal{F}_\mathrm{cyl}(\Gamma)$, which plays the key role in our further considerations:

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.5}$$

where $G \in B_{\mathrm{bs}}(\Gamma_0)$, see, e.g., [18, 26, 27]. The summation in (2.5) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol, $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.6}$$

We denote the restriction of K onto functions on Γ_0 by K_0.

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to (w.r.t. for short) the Poisson measure π if for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the projection of μ onto Γ_Λ is absolutely continuous w.r.t. the projection of π onto Γ_Λ. By [18], in this case, there exists a *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ such that for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the following equality holds

$$\int_\Gamma (KG)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta). \tag{2.7}$$

The restrictions $k_\mu^{(n)}$ of this functional on $\Gamma_0^{(n)}$, $n \in \mathbb{N}_0$ are called *correlation functions* of the measure μ. Note that $k_\mu^{(0)} = 1$.

We recall now without a proof the partial case of the well-known technical lemma (cf., [24]) which plays very important role in our calculations.

Lemma 2.1. *For any measurable function $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \to \mathbb{R}$*

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta)\, d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi)\, d\lambda(\xi)\, d\lambda(\eta) \tag{2.8}$$

if only both sides of the equality make sense.

2.2 Non-equilibrium Glauber dynamics in continuum

Let $\phi : \mathbb{R}^d \to \mathbb{R}_+ := [0; +\infty)$ be an even non-negative function which satisfies the following integrability condition

$$C_\phi := \int_{\mathbb{R}^d} \bigl(1 - e^{-\phi(x)}\bigr) dx < +\infty. \tag{2.9}$$

For any $\gamma \in \Gamma$, $x \in \mathbb{R}^d \setminus \gamma$ we set

$$E^\phi(x, \gamma) := \sum_{y \in \gamma} \phi(x - y) \in [0; \infty]. \tag{2.10}$$

Let us define the (pre-)generator of the Glauber dynamics: for any $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ we set

$$\begin{aligned}(LF)(\gamma) :=& \sum_{x \in \gamma} \bigl[F(\gamma \setminus x) - F(\gamma)\bigr] \\ &+ z \int_{\mathbb{R}^d} \bigl[F(\gamma \cup x) - F(\gamma)\bigr] \exp\bigl\{-E^\phi(x, \gamma)\bigr\} dx, \qquad \gamma \in \Gamma.\end{aligned} \tag{2.11}$$

Here $z > 0$ is the *activity* parameter. Note that for any $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$ there exists a $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ such that $F(\gamma \setminus x) = F(\gamma)$ for all $x \in \gamma_{\Lambda^c}$ and $F(\gamma \cup x) = F(\gamma)$ for all $x \in \Lambda^c$; note also that $\exp\{-E^\phi(x,\gamma)\} \le 1$, therefore, sum and integral in (2.11) are finite.

For any fixed $C > 1$ we consider the following Banach space of $\mathcal{B}(\Gamma_0)$-measurable functions

$$\mathcal{L}_C := \left\{ G : \Gamma_0 \to \mathbb{R} \;\middle|\; \|G\|_C := \int_{\Gamma_0} |G(\eta)| C^{|\eta|} d\lambda(\eta) < \infty \right\}.$$

In [10, Proposition 3.1], it was shown that the mapping $\hat{L} := K^{-1}LK$ given on $B_{\mathrm{bs}}(\Gamma_0)$ by

$$\begin{aligned}(\hat{L}G)(\eta) = &- |\eta| G(\eta) \\ &+ z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e^{-E^\phi(x,\xi)} G(\xi \cup x) e_\lambda(e^{-\phi(x-\cdot)} - 1, \eta \setminus \xi) dx\end{aligned} \tag{2.12}$$

is a linear operator on $\mathcal{L}_C$ with the dense domain $\mathcal{L}_{2C} \subset \mathcal{L}_C$. If additionally,

$$z \le \min\{Ce^{-CC_\phi}; 2Ce^{-2CC_\phi}\}, \tag{2.13}$$

then $(\hat{L}, \mathcal{L}_{2C})$ is closable linear operator in $\mathcal{L}_C$ and its closure $(\hat{L}, D(\hat{L}))$ generates a strongly continuous contraction semigroup $\hat{T}(t)$ on $\mathcal{L}_C$ (see [10, Theorem 3.8] for details).

Let us set $d\lambda_C := C^{|\cdot|} d\lambda$; then the dual space $(\mathcal{L}_C)' = \big(L^1(\Gamma_0, d\lambda_C)\big)' = L^\infty(\Gamma_0, d\lambda_C)$. The space $(\mathcal{L}_C)'$ is isometrically isomorphic to the Banach space

$$\mathcal{K}_C := \left\{ k : \Gamma_0 \to \mathbb{R} \;\middle|\; k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda) \right\}$$

with the norm $\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|} k(\cdot)\|_{L^\infty(\Gamma_0,\lambda)}$ where the isomorphism is provided by the isometry R_C

$$(\mathcal{L}_C)' \ni k \longmapsto R_C k := k \cdot C^{|\cdot|} \in \mathcal{K}_C. \tag{2.14}$$

In fact, one may consider the duality between the Banach spaces $\mathcal{L}_C$ and $\mathcal{K}_C$ given by the following expression

$$\langle\!\langle G, k \rangle\!\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda, \quad G \in \mathcal{L}_C, \ k \in \mathcal{K}_C \tag{2.15}$$

with $|\langle\!\langle G, k \rangle\!\rangle| \le \|G\|_C \cdot \|k\|_{\mathcal{K}_C}$. It is clear that $k \in \mathcal{K}_C$ implies $|k(\eta)| \le \|k\|_{\mathcal{K}_C} C^{|\eta|}$ for λ-a.a. $\eta \in \Gamma_0$.

Let $(\hat{L}', D(\hat{L}'))$ be an operator in $(\mathcal{L}_C)'$ which is dual to the closed operator $(\hat{L}, D(\hat{L}))$. We consider also its image on $\mathcal{K}_C$ under the isometry R_C, namely, let $\hat{L}^* = R_C \hat{L}' R_{C^{-1}}$ with the domain $D(\hat{L}^*) = R_C D(\hat{L}')$. It was noted in [7] that $\hat{L}^*$ is the dual operator to $\hat{L}$ w.r.t. the duality (2.15) and that for any $k \in D(\hat{L}^*)$

$$(\hat{L}^* k)(\eta) = -|\eta| k(\eta) + z \sum_{x\in\eta} e^{-E^\phi(x,\eta\setminus x)} \int_{\Gamma_0} e_\lambda(e^{-\phi(x-\cdot)} - 1, \xi) k((\eta \setminus x) \cup \xi)\, d\lambda(\xi). \tag{2.16}$$

Under condition (2.13), we consider the adjoint semigroup $\hat{T}'(t)$ in $(\mathcal{L}_C)'$ and its image $\hat{T}^*(t)$ in $\mathcal{K}_C$. By the general results from [28, Sections 1.2, 1.3], the restriction $\hat{T}^\odot(t)$ of the semigroup $\hat{T}^*(t)$ onto its invariant Banach subspace $\overline{D(\hat{L}^*)}$ is a contraction strongly continuous semigroup. By [7, Proposition 3.1], for any $\alpha \in (0;1)$ we have $\mathcal{K}_{\alpha C} \subset D(\hat{L}^*)$ and, moreover, by [7, Proposition 3.3], there exists $\alpha_0 = \alpha_0(z, \phi, C) \in (0;1)$ such that for any $\alpha \in (\alpha_0; 1)$ the set $\mathcal{K}_{\alpha C}$ will be also a $\hat{T}^*(t)$-invariant linear subspace. As a result, for any $\overline{D(\hat{L}^*)}$ the Cauchy problem in $\mathcal{K}_C$

$$\begin{cases} \dfrac{\partial}{\partial t} k_t = \hat{L}^* k_t \\ k_t\big|_{t=0} = k_0 \end{cases} \tag{2.17}$$

is well-defined and solvable: $k_t = \hat{T}^*(t) k_0 = \hat{T}^\odot(t) k_0 \in \overline{D(\hat{L}^*)}$; moreover, $k_0 \in \mathcal{K}_{\alpha C}$ implies $k_t \in \mathcal{K}_{\alpha C}$.

3 Vlasov-type scaling

3.1 Description of scaling

We start from the explanation of the idea of the Vlasov-type scaling. We want to construct some scaling of the generator L, say, L_ε, $\varepsilon > 0$, such that the following scheme holds. Suppose that we have a semigroup $\hat{T}_\varepsilon(t)$ with generator $\hat{L}_\varepsilon$ in some $\mathcal{L}_{C_\varepsilon}$. Consider the dual semigroup $\hat{T}^*_\varepsilon(t)$. Let us choose an initial function of the corresponding Cauchy problem with a big singularity by ε, namely, $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\varepsilon \to 0$, $\eta \in \Gamma_0$ with some function r_0, independent of ε. Our first demand to the scaling $L \mapsto L_\varepsilon$ is that the semigroup $\hat{T}^*_\varepsilon(t)$ preserves the order of the singularity:

$$(\hat{T}^*_\varepsilon(t) k_0^{(\varepsilon)})(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0. \tag{3.1}$$

And the second one is that the dynamics $r_0 \mapsto r_t$ should preserve Lebesgue–Poisson exponents, namely, if $r_0(\eta) = e_\lambda(\rho_0, \eta)$ then $r_t(\eta) = e_\lambda(\rho_t, \eta)$ and there exists explicit (nonlinear, in general) differential equation for ρ_t:

$$\frac{\partial}{\partial t} \rho_t(x) = \upsilon(\rho_t)(x) \tag{3.2}$$

which we will call the Vlasov-type equation.

Now let us explain an informal way for the realization of this scheme. Let us consider for any $\varepsilon > 0$ the following mapping (cf. (2.14)) on functions on Γ_0

$$(R_\varepsilon r)(\eta) := \varepsilon^{|\eta|} r(\eta). \tag{3.3}$$

This mapping is "self-dual" w.r.t. the duality (2.15), moreover, $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$. Then we have $k_0^{(\varepsilon)} \sim R_{\varepsilon^{-1}} r_0$, and we need $r_t \sim R_\varepsilon \hat{T}_\varepsilon^*(t) k_0^{(\varepsilon)} \sim R_\varepsilon \hat{T}_\varepsilon^*(t) R_{\varepsilon^{-1}} r_0$. Therefore, we have to show that for any $t \geq 0$ the operator family $R_\varepsilon \hat{T}_\varepsilon^*(t) R_{\varepsilon^{-1}}$, $\varepsilon > 0$ has limiting (in a proper sense) operator $U(t)$ and

$$U(t) e_\lambda(\rho_0) = e_\lambda(\rho_t). \tag{3.4}$$

But, informally, $\hat{T}_\varepsilon^*(t) = \exp\{t\hat{L}_\varepsilon^*\}$ and $R_\varepsilon \hat{T}_\varepsilon^*(t) R_{\varepsilon^{-1}} = \exp\{tR_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}\}$. Let us consider the "renormalized" operator

$$\hat{L}_{\varepsilon,\,\mathrm{ren}}^* := R_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}. \tag{3.5}$$

In fact, we need that there exists an operator $\hat{L}_V^*$ such that $\exp\{tR_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}\} \to \exp\{t\hat{L}_V^*\} =: U(t)$ for which (3.4) holds. Therefore, a heuristic way to produce such a scaling $L \mapsto L_\varepsilon$ is to demand that

$$\lim_{\varepsilon \to 0} \left(\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) - \hat{L}_{\varepsilon,\,\mathrm{ren}}^* e_\lambda(\rho_t, \eta) \right) = 0, \qquad \eta \in \Gamma_0$$

if only ρ_t is satisfied (3.2). The point-wise limit of $\hat{L}_{\varepsilon,\,\mathrm{ren}}^*$ will be natural candidate for $\hat{L}_V^*$.

Note that (3.5) implies $\hat{L}_{\varepsilon,\,\mathrm{ren}} = R_{\varepsilon^{-1}} \hat{L}_\varepsilon R_\varepsilon$. Hence, we will use the following scheme to give rigorous meaning to all considerations above. We consider, for a proper scaling L_ε, the "renormalized" operator $\hat{L}_{\varepsilon,\,\mathrm{ren}}$ and prove that it is a generator of a strongly continuous contraction semigroup $\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)$ in $\mathcal{L}_C$. Next, we show that the formal limit $\hat{L}_V$ of $\hat{L}_{\varepsilon,\,\mathrm{ren}}$ is also a generator of a strongly continuous contraction semigroup $\hat{T}_V(t)$ in $\mathcal{L}_C$ also. Then, we consider the dual semigroups $\hat{T}_{\varepsilon,\,\mathrm{ren}}^*(t)$ and $\hat{T}_V^*(t)$ in the proper Banach subspace of the space $\mathcal{K}_C$. Finally, we prove that $\hat{T}_{\varepsilon,\,\mathrm{ren}}^*(t) \to \hat{T}_V^*(t)$ strongly on this subspace and explain in which sense $\hat{T}_V^*(t)$ satisfies the properties above. Below we try to realize this scheme.

3.2 Construction and convergence of the evolutions in $\mathcal{L}_C$

Let us consider for any $F \in \mathcal{F}_{\mathrm{cyl}}(\Gamma)$, $\varepsilon > 0$

$$\begin{aligned}(L_\varepsilon F)(\gamma) :=& \sum_{x \in \gamma} \big[F(\gamma \setminus x) - F(\gamma)\big] \\ &+ \varepsilon^{-1} z \int_{\mathbb{R}^d} \big[F(\gamma \cup x) - F(\gamma)\big] \exp\big\{-\varepsilon E^\phi(x, \gamma)\big\} dx, \quad \gamma \in \Gamma.\end{aligned} \tag{3.6}$$

We define also for any $G \in B_{\mathrm{bs}}(\Gamma_0)$, $\varepsilon > 0$

$$\hat{L}_\varepsilon G := K^{-1} L_\varepsilon K G; \qquad \hat{L}_{\varepsilon,\,\mathrm{ren}} G := R_{\varepsilon^{-1}} \hat{L}_\varepsilon R_\varepsilon G.$$

Let ϕ be integrable function on the whole $\mathbb{R}^d$, namely,

$$\beta := \int_{\mathbb{R}^d} \phi(x) dx < +\infty. \tag{3.7}$$

We fix this notation for our considerations below.

Then, by the elementary inequality

$$1 - e^{-t} \leq t, \quad t \geq 0 \tag{3.8}$$

(which we will use often), ϕ will satisfy (2.9) and $C_\phi \leq \beta$.

Proposition 3.1. *For any* $G \in B_{bs}(\Gamma_0)$

$$\left(\hat{L}_{\varepsilon,\mathrm{ren}} G\right)(\eta) = (L_1 G)(\eta) + (L_{2,\varepsilon} G)(\eta), \tag{3.9}$$

where

$$\begin{aligned}
(L_1 G)(\eta) &= -|\eta| G(\eta), \\
(L_{2,\varepsilon} G)(\eta) &= z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)}, \xi\right) \\
&\qquad \times e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)} - 1}{\varepsilon}, \eta \setminus \xi\right) G(\xi \cup x)\, dx.
\end{aligned}$$

Moreover, the expression (3.9) *defines a linear operator in* $\mathcal{L}_C$ *with dense domain* $\mathcal{L}_{2C}$.

Proof. By (2.12), for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ we have

$$\begin{aligned}
(\hat{L}_\varepsilon G)(\eta) =& -|\eta| G(\eta) \\
&+ \varepsilon^{-1} z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e^{-\varepsilon E^\phi(x,\xi)} G(\xi \cup x) e_\lambda(e^{-\varepsilon\phi(x-\cdot)} - 1, \eta \setminus \xi) dx.
\end{aligned} \tag{3.10}$$

Then

$$\begin{aligned}
(\hat{L}_{\varepsilon,\,\mathrm{ren}} G)(\eta) &= (R_{\varepsilon^{-1}} \hat{L}_\varepsilon R_\varepsilon G)(\eta) \\
&= -\varepsilon^{-|\eta|} |\eta| \varepsilon^{|\eta|} G(\eta) \\
&\quad + \varepsilon^{-|\eta|} \varepsilon^{-1} z \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e^{-\varepsilon E^\phi(x,\xi)} \varepsilon^{|\xi \cup x|} G(\xi \cup x) e_\lambda(e^{-\varepsilon\phi(x-\cdot)} - 1, \eta \setminus \xi) dx \\
&= (L_1 G)(\eta) + (L_{2,\varepsilon} G)(\eta).
\end{aligned}$$

Next, for any $G\in\mathcal{L}_{2C}$ we obtain

$$\begin{aligned}\|L_1G\|_C=&\int_{\Gamma_0}|\eta|\,|G(\eta)|\,C^{|\eta|}d\lambda(\eta)\\ \leq&\int_{\Gamma_0}2^{|\eta|}\,|G(\eta)|\,C^{|\eta|}d\lambda(\eta)=\|G\|_{2C}\,.\end{aligned}\tag{3.11}$$

From (3.8) and the estimate $e^{-\phi}\leq 1$ we get

$$\begin{aligned}&\|L_{2,\varepsilon}G\|_C\\ \leq& z\int_{\Gamma_0}\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}|G(\xi\cup x)|\,e_\lambda\left(\frac{1-e^{-\varepsilon\phi(x-\cdot)}}{\varepsilon},\eta\setminus\xi\right)dxC^{|\eta|}d\lambda(\eta)\\ \leq& z\int_{\Gamma_0}\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}|G(\xi\cup x)|\,e_\lambda(\phi(x-\cdot),\eta\setminus\xi)\,dxC^{|\eta|}d\lambda(\eta)\,,\end{aligned}$$

then, by Lemma 2.1, one may continue,

$$\leq z\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d}|G(\xi\cup x)|\,e_\lambda(\phi(x-\cdot),\eta)\,dxC^{|\eta|}d\lambda(\eta)\,C^{|\xi|}d\lambda(\xi)$$

and (2.4) yields

$$=z\exp\{C\beta\}\int_{\Gamma_0}\int_{\mathbb{R}^d}|G(\xi\cup x)|\,dxC^{|\xi|}d\lambda(\xi)\,,$$

then, using Lemma 2.1 again,

$$\begin{aligned}=&z\exp\{C\beta\}\,C^{-1}\int_{\Gamma_0}|G(\xi)|\cdot|\xi|\,C^{|\xi|}d\lambda(\xi)\\ \leq& z\exp\{C\beta\}\,C^{-1}\,\|G\|_{2C}\,.\end{aligned}\tag{3.12}$$

The estimates (3.11) and (3.12) provide the statement. □

Proposition 3.2. *Let for any* $G\in B_{\mathrm{bs}}(\Gamma_0)$

$$(\hat{L}_VG)(\eta):=\lim_{\varepsilon\to 0}(\hat{L}_{\varepsilon,\mathrm{ren}}G)(\eta)=(L_1G)(\eta)+(L_2^VG)(\eta)\,,\quad \eta\in\Gamma_0,\tag{3.13}$$

where

$$(L_2^VG)(\eta)=z\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}G(\xi\cup x)\,e_\lambda(-\phi(x-\cdot),\eta\setminus\xi)\,dx.$$

Then, the expression (3.13) *defines a linear operator in* $\mathcal{L}_C$ *with dense domain* $\mathcal{L}_{2C}$.

Proof. Since, by the definition,

$$\left\|L_2^VG\right\|_C\leq z\int_{\Gamma_0}\sum_{\xi\subset\eta}\int_{\mathbb{R}^d}|G(\xi\cup x)|\,e_\lambda(\phi(x-\cdot),\eta\setminus\xi)\,dxC^{|\eta|}d\lambda(\eta)$$

the statement follows from (3.11) and (3.12). □

Let us set (cf. [10, (3.12)]) for any $\delta \in (0;1)$, $\varepsilon > 0$, $G \in B_{bs}(\Gamma_0)$, $\eta \in \Gamma_0$

$$\begin{aligned}\left(\hat{P}_{\delta,\varepsilon} G\right)(\eta) := &\sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|} G(\xi \cup \omega) \\ &\times e_\lambda \left(e^{-\varepsilon E^\phi(\cdot,\omega)}, \xi\right) e_\lambda \left(\frac{e^{-\varepsilon E^\phi(\cdot,\omega)} - 1}{\varepsilon}, \eta \setminus \xi\right) d\lambda(\omega).\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\left(\hat{Q}_\delta G\right)(\eta) := &\sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|} G(\xi \cup \omega) \\ &\times e_\lambda \left(-E^\phi(\cdot,\omega), \eta \setminus \xi\right) d\lambda(\omega).\end{aligned} \tag{3.15}$$

Proposition 3.3. *Let*

$$ze^{\beta C} \le C. \tag{3.16}$$

Then $\hat{P}_{\delta,\varepsilon}$ and $\hat{Q}_\delta$ given by (3.14) *and* (3.15) *are well defined linear contractions on $\mathcal{L}_C$.*

Proof. By (3.8), Lemma 2.1, and (2.4), we get for any $G \in \mathcal{L}_C$

$$\begin{aligned}&\max\left\{\left\|\hat{P}_{\delta,\varepsilon} G\right\|_C;\ \left\|\hat{Q}_\delta G\right\|_C\right\} \\
\le &\int_{\Gamma_0} \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|} |G(\xi \cup \omega)| e_\lambda \left(E^\phi(\cdot,\omega), \eta \setminus \xi\right) d\lambda(\omega) C^{|\eta|} d\lambda(\eta) \\
= &\int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|} |G(\xi \cup \omega)| e_\lambda \left(E^\phi(\cdot,\omega), \eta\right) d\lambda(\omega) C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi) \\
= &\int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} (z\delta)^{|\omega|} |G(\xi \cup \omega)| \exp\{C\beta |\omega|\} d\lambda(\omega) C^{|\xi|} d\lambda(\xi) \\
= &\int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} |G(\xi \cup \omega)| \left(z\delta \exp\{C\beta\} C^{-1}\right)^{|\omega|} C^{|\omega|} C^{|\xi|} d\lambda(\xi) d\lambda(\omega) \\
= &\int_{\Gamma_0} |G(\xi)| \left(1 - \delta + z\delta \exp\{C\beta\} C^{-1}\right)^{|\xi|} C^{|\xi|} d\lambda(\xi) \le \|G\|_C,\end{aligned}$$

that proves the contraction property; then, in particular,

$$\left(\hat{P}_{\delta,\varepsilon} G\right)(\eta) < +\infty, \qquad \left(\hat{Q}_\delta G\right)(\eta) < +\infty$$

for λ-a.a. $\eta \in \Gamma_0$. □

Now let us construct the approximations for the operators $\hat{L}_V$ and $\hat{L}_{\varepsilon,\mathrm{ren}}$.

Proposition 3.4. *Let for $\delta \in (0;1)$*

$$\hat{L}_{\delta,V} := \frac{1}{\delta}\left(\hat{Q}_\delta - 1\right); \quad \hat{L}_{\delta,\varepsilon} := \frac{1}{\delta}\left(\hat{P}_{\delta,\varepsilon} - 1\right),\ \varepsilon > 0.$$

Let (3.16) *holds, then*

$$\left\|(\hat{L}_{\delta,V}-\hat{L}_V)G\right\|_C<3\delta\left\|G\right\|_{2C}$$

and for any $\varepsilon>0$

$$\left\|(\hat{L}_{\delta,\varepsilon}-\hat{L}_{\varepsilon,\mathrm{ren}})G\right\|_C\leq 3\delta\left\|G\right\|_{2C}.$$

Proof. Let us denote

$$\begin{aligned}\left(\hat{Q}_\delta^{(0)}G\right)(\eta):=&\sum_{\xi\subset\eta}(1-\delta)^{|\xi|}G(\xi)\,0^{|\eta\setminus\xi|}=(1-\delta)^{|\eta|}G(\eta),\\ \left(\hat{Q}_\delta^{(1)}G\right)(\eta):=&z\delta\sum_{\xi\subset\eta}(1-\delta)^{|\xi|}\int_{\mathbb{R}^d}G(\xi\cup x)\,e_\lambda\left(-\phi(x-\cdot),\eta\setminus\xi\right)dx,\end{aligned}$$

and

$$\hat{Q}_\delta^{(\geq 2)}:=\hat{Q}_\delta-\left(\hat{Q}_\delta^{(0)}+\hat{Q}_\delta^{(1)}\right).$$

Clearly, we have

$$\begin{aligned}\left\|(\hat{L}_{\delta,V}-\hat{L}_V)G\right\|_C\leq&\left\|\frac{1}{\delta}(\hat{Q}_\delta^{(0)}-1)G-L_1G\right\|_C\\ &+\left\|\frac{1}{\delta}\hat{Q}_\delta^{(1)}G-L_2^VG\right\|_C+\left\|\frac{1}{\delta}\hat{Q}_\delta^{(\geq 2)}G\right\|_C.\end{aligned}$$

It follows from the simple inequality

$$0\leq n-\frac{1-(1-\delta)^n}{\delta}<\delta\cdot 2^n,\ n\in\mathbb{N},\ \delta>0, \tag{3.17}$$

that

$$\left\|\frac{1}{\delta}(\hat{Q}_\delta^{(0)}-1)G-L_1G\right\|_C=\left\|\frac{1}{\delta}\left((1-\delta)^{|\cdot|}-1\right)G+|\cdot|\,G\right\|_C<\delta\left\|G\right\|_{2C}$$

and

$$\begin{aligned}&\left\|\frac{1}{\delta}\hat{Q}_\delta^{(1)}G-L_2^VG\right\|_C\\ \leq& z\int_{\Gamma_0}\left|\sum_{\xi\subset\eta}\left[(1-\delta)^{|\xi|}-1\right]\int_{\mathbb{R}^d}G(\xi\cup x)\,e_\lambda\left(-\phi(x-\cdot),\eta\setminus\xi\right)dx\right|C^{|\eta|}d\lambda(\eta)\\ \leq& z\int_{\Gamma_0}\int_{\Gamma_0}\left[1-(1-\delta)^{|\xi|}\right]\int_{\mathbb{R}^d}|G(\xi\cup x)|\,e_\lambda\left(\phi(x-\cdot),\eta\right)dxC^{|\eta|}C^{|\xi|}d\lambda(\eta)\,d\lambda(\xi)\\ =& z\exp\{C\beta\}\int_{\Gamma_0}\left[1-(1-\delta)^{|\xi|}\right]\int_{\mathbb{R}^d}|G(\xi\cup x)|\,dxC^{|\xi|}d\lambda(\xi)\\ \leq& \delta z\exp\{C\beta\}\int_{\Gamma_0}|\xi|\int_{\mathbb{R}^d}|G(\xi\cup x)|\,dxC^{|\xi|}d\lambda(\xi)\\ =& \delta z\exp\{C\beta\}\,C^{-1}\int_{\Gamma_0}|\xi|\,(|\xi|-1)\,|G(\xi)|\,C^{|\xi|}d\lambda(\xi)<\delta\left\|G\right\|_{2C},\end{aligned}$$

since, $n(n-1)\le 2^n$, $n\in\mathbb{N}$. And, if we denote

$$\Gamma_0^{(\ge 2)} := \bigsqcup_{n\ge 2}\Gamma_0^{(n)},$$

we obtain

$$\begin{aligned}
&\left\|\frac{1}{\delta}\hat{Q}_\delta^{(\ge 2)}G\right\|_C\\
\le&\frac{1}{\delta}\int_{\Gamma_0}\sum_{\xi\subset\eta}(1-\delta)^{|\xi|}\int_{\Gamma_0^{(\ge 2)}}(z\delta)^{|\omega|}\left|G(\xi\cup\omega)\right|\\
&\qquad\times e_\lambda\left(E^\phi(\cdot,\omega),\eta\setminus\xi\right)d\lambda(\omega)\,C^{|\eta|}d\lambda(\eta)\\
=&\frac{1}{\delta}\int_{\Gamma_0}\int_{\Gamma_0}(1-\delta)^{|\xi|}\int_{\Gamma_0^{(\ge 2)}}(z\delta)^{|\omega|}\left|G(\xi\cup\omega)\right|\\
&\qquad\times e_\lambda\left(E^\phi(\cdot,\omega),\eta\right)d\lambda(\omega)\,C^{|\eta|}d\lambda(\eta)\,C^{|\xi|}d\lambda(\xi)\\
\le&\delta\int_{\Gamma_0}(1-\delta)^{|\xi|}\int_{\Gamma_0}(z\exp\{C\beta\})^{|\omega|}\left|G(\xi\cup\omega)\right|d\lambda(\omega)\,C^{|\xi|}d\lambda(\xi)\\
=&\delta\int_{\Gamma_0}(C-\delta C+z\exp\{C\beta\})^{|\xi|}\left|G(\xi)\right|d\lambda(\xi)\\
\le&\delta\int_{\Gamma_0}(2C-\delta C)^{|\xi|}\left|G(\xi)\right|d\lambda(\xi)<\delta\left\|G\right\|_{2C}.
\end{aligned}$$

The same considerations may be done for $\hat{P}_{\delta,\varepsilon}$. Namely, let

$$\begin{aligned}
\left(\hat{P}_{\delta,\varepsilon}^{(0)}G\right)(\eta):=&\sum_{\xi\subset\eta}(1-\delta)^{|\xi|}G(\xi)\,1^{|\xi|}0^{|\eta\setminus\xi|}=(1-\delta)^{|\eta|}G(\eta),\\
\left(\hat{P}_{\delta,\varepsilon}^{(1)}G\right)(\eta):=&z\delta\sum_{\xi\subset\eta}(1-\delta)^{|\xi|}\int_{\mathbb{R}^d}G(\xi\cup x)\\
&\qquad\times e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)},\xi\right)e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon},\eta\setminus\xi\right)dx,
\end{aligned}$$

and

$$\hat{P}_{\delta,\varepsilon}^{(\ge 2)} := \hat{P}_{\delta,\varepsilon}-\left(\hat{P}_{\delta,\varepsilon}^{(0)}+\hat{P}_{\delta,\varepsilon}^{(1)}\right).$$

Then

$$\left\|\frac{1}{\delta}\left(\hat{P}_{\delta,\varepsilon}^{(0)}-1\right)G-L_1G\right\|_C=\left\|\frac{1}{\delta}\left(\hat{Q}_\delta^{(0)}-1\right)G-L_1G\right\|_C<\delta\left\|G\right\|_{2C},$$

next, by (3.8), (3.16) and Lemma 2.1,

$$\begin{aligned}
&\left\|\frac{1}{\delta}\hat{P}_{\delta,\varepsilon}^{(1)}G-L_{2,\varepsilon}G\right\|_C\\
\le&z\int_{\Gamma_0}\int_{\Gamma_0}\left[1-(1-\delta)^{|\xi|}\right]\int_{\mathbb{R}^d}\left|G(\xi\cup x)\right|e_\lambda\left(\phi(x-\cdot),\eta\right)dx\,C^{|\eta|}C^{|\xi|}d\lambda(\eta)\,d\lambda(\xi)\\
<&\left\|G\right\|_{2C}\cdot\delta e^{C\beta}C^{-1}z\le\delta\left\|G\right\|_{2C},
\end{aligned}\tag{3.18}$$

and, finally,

$$\left\| \frac{1}{\delta} \hat{P}^{(\geq 2)}_{\delta,\varepsilon} G \right\|_C \leq \frac{1}{\delta} \int_{\Gamma_0} \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0^{(\geq 2)}} (z\delta)^{|\omega|} \left| G(\xi \cup \omega) \right| \times \left| e_\lambda \left(E^\phi(\cdot,\omega), \eta \setminus \xi \right) \right| d\lambda(\omega) C^{|\eta|} d\lambda(\eta) < \delta \|G\|_{2C}.$$

Combining all these inequalities we obtain the assertion. □

We will need the following results in the sequel.

Lemma 3.5 ([6, Corollary 3.8]). *Let A be a linear operator on a Banach space L with $D(A)$ dense in L, and let $|||\cdot|||$ be a norm on $D(A)$ with respect to which $D(A)$ is a Banach space. For $n \in \mathbb{N}$ let T_n be a linear $|||\cdot|||$-contraction on L such that $T_n : D(A) \to D(A)$, and define $A_n = n(T_n - 1)$. Suppose there exist $\omega \geq 0$ and a sequence $\{\varepsilon_n\} \subset (0; +\infty)$ tending to zero such that for $n \in \mathbb{N}$*

$$\|(A_n - A) f\| \leq \varepsilon_n |||f|||, \ \ f \in D(A) \tag{3.19}$$

and

$$|||T_n \upharpoonright_{D(A)} ||| \leq 1 + \frac{\omega}{n}. \tag{3.20}$$

Then A is closable and the closure of A generates a strongly continuous contraction semigroup on L.

Lemma 3.6 (cf. [6, Theorem 6.5]). *Let L, L_n, $n \in \mathbb{N}$ be Banach spaces, and $p_n : L \to L_n$ be bounded linear transformation, such that $\sup_n \|p_n\| < \infty$. For any $n \in \mathbb{N}$, let T_n be a linear contraction on L_n, let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty} \varepsilon_n = 0$, and put $A_n = \varepsilon_n^{-1}(T_n - 1\!\!1)$. Let T_t be a strongly continuous contraction semigroup on L with generator A and let D be a core for A. Then the following are equivalent:*

1. *For each $f \in L$, $T_n^{[t/\varepsilon_n]} p_n f \to p_n T_t f$ in L_n for all $t \geq 0$ uniformly on bounded intervals. Here and below $[\cdot]$ mean the entire part of a real number.*
2. *For each $f \in D$, there exists $f_n \in L_n$ for each $n \in \mathbb{N}$ such that $f_n \to p_n f$ and $A_n f_n \to p_n A f$ in L_n.*

Lemma 3.7. *Let X be a Banach space with a norm $\|\cdot\|_X$; A and B be linear contraction mappings on X. Let Y with a norm $\|\cdot\|_Y$ be a Banach subspace of X such that Y is invariant w.r.t. B. Suppose that the restriction of B on Y is also a contraction w.r.t. $\|\cdot\|_Y$. Suppose also that there exists $c > 0$ such that for any $f \in Y$*

$$\|Af - Bf\|_X \leq c \|f\|_Y. \tag{3.21}$$

Then for any $m \in \mathbb{N}$ and for any $f \in Y$

$$\|A^m f - B^m f\|_X \leq cm \|f\|_Y. \tag{3.22}$$

Proof. For any $f \in Y$, $m \geq 2$ we have

$$\begin{aligned}
&\|A^m f - B^m f\|_X \\
\leq &\|A^m f - AB^{m-1} f\|_X + \|AB^{m-1} f - B^m f\|_X \\
\leq &\|A\| \cdot \|A^{m-1} f - B^{m-1} f\|_X + \|(A - B)B^{m-1} f\|_X
\end{aligned}$$

(where $\|A\|$ means the norm of the operator A on X); since $\|A\| \leq 1$ and $B^{m-1} f \in Y$, condition (3.21) yields

$$\leq \|A^{m-1} f - B^{m-1} f\|_X + c\|B^{m-1} f\|_Y,$$

but, B is a contraction on Y, therefore, one get

$$\leq \|A^{m-1} f - B^{m-1} f\|_X + c\|f\|_Y,$$

that gives (3.22) by induction principle. □

And now one can construct the corresponding semigroups rigorously.

Proposition 3.8. *Let*

$$z \leq \min\left\{Ce^{-C\beta}, 2Ce^{-2C\beta}\right\}. \tag{3.23}$$

Then, $\left(\hat{L}_V, \mathcal{L}_{2C}\right)$ *and* $\left(\hat{L}_{\varepsilon,\mathrm{ren}}, \mathcal{L}_{2C}\right)$ *are closable linear operators in* $\mathcal{L}_C$ *and their closures* $\left(\hat{L}_V, D(\hat{L}_V)\right)$ *and* $\left(\hat{L}_{\varepsilon,\mathrm{ren}}, D(\hat{L}_{\varepsilon,\mathrm{ren}})\right)$ *generate strongly continuous contraction semigroups* $\hat{T}_V(t)$ *and* $\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)$ *on* $\mathcal{L}_C$*, respectively. Moreover, for any* $G \in \mathcal{L}_C$*,* $\varepsilon > 0$

$$\hat{Q}_{\frac{1}{n}}^{[nt]} G \to \hat{T}_V(t)G, \qquad \hat{P}_{\frac{1}{n},\varepsilon}^{[nt]} G \to \hat{T}_{\varepsilon,\,\mathrm{ren}}(t)G, \qquad n \to \infty \tag{3.24}$$

for any $t \geq 0$ *uniformly on bounded intervals.*

Proof. Note that (3.23) provides that $\hat{Q}_\delta$ and $\hat{P}_{\delta,\varepsilon}$ are also contractions on $\mathcal{L}_{2C}$. Then the first part of the statement follows from Lemma 3.5. Therefore, $\mathcal{L}_{2C}$ will be a core for the generators and, by Lemma 3.6, we obtain the convergence (3.24). □

The definition (3.13) of $\hat{L}_V$ together with Proposition 3.8 allow us to expect that the semigroup $\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)$ converges to $\hat{T}_V(t)$ in a proper sense. The next theorem improve this statement. However, this result is not crucial in the context of the our paper. Moreover, its proof is quite technical and, on the other hand, is very similar to the proof of the main Theorem 3.12 concerning the dual semigroups. Hence, we give the sketch of the proof only.

Theorem 3.9. *Let* (3.23) *holds and suppose that* $\bar{\phi} := \sup_{\mathbb{R}^d} \phi(x) < +\infty$*. Then for any* $G \in \mathcal{L}_{2C}$

$$\left\|\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)G - \hat{T}_V(t)G\right\|_C \leq \varepsilon t\, \bar{\phi}\,(1+\beta)\,\|G\|_{2C}$$

for any $t \geq 0$*,* $\varepsilon > 0$*. In particular, it means that* $\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)G \to \hat{T}_V(t)G$ *in* $\mathcal{L}_C$ *as* $\varepsilon \to 0$ *for any* $t \geq 0$ *uniformly on bounded intervals.*

Proof. By the triangle inequality,

$$\left\|\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)G - \hat{T}_V(t)G\right\|_C \le \left\|\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)G - \hat{P}^{[nt]}_{\frac{1}{n},\varepsilon}G\right\|_C + \left\|\hat{P}^{[nt]}_{\frac{1}{n},\varepsilon}G - \hat{Q}^{[nt]}_{\frac{1}{n}}G\right\|_C + \left\|\hat{Q}^{[nt]}_{\frac{1}{n}}G - \hat{T}_V(t)G\right\|_C . \tag{3.25}$$

By (3.24), the first and third norms in the r.h.s. of (3.25) are tend to 0 as $n \to \infty$. Next, in a similar way as for the proof of (3.45) one can show that for any $G \in \mathcal{L}_{2C}$

$$\left\|\hat{P}_{\frac{1}{n},\varepsilon}G - \hat{Q}_{\frac{1}{n}}G\right\|_C \le \frac{1}{n}\,\varepsilon\,\bar{\phi}\,(1+\beta)\,\|G\|_{2C} . \tag{3.26}$$

By Proposition 3.3 and condition (3.23), the subspace $\mathcal{L}_{2C}$ is $\hat{Q}_{\frac{1}{n}}$-invariant, hence, by Lemma 3.7, we obtain

$$\begin{aligned}\left\|\hat{P}^{[nt]}_{\frac{1}{n},\varepsilon}G - \hat{Q}^{[nt]}_{\frac{1}{n}}G\right\|_C &\le [nt]\frac{\bar{\phi}\,(1+\beta)}{n}\varepsilon\,\|G\|_{2C}\\ &< \bar{\phi}\,(1+\beta)\left(t+\frac{1}{n}\right)\varepsilon\,\|G\|_{2C} ,\end{aligned}$$

that fulfilled the first assertion. And, clearly, $\mathcal{L}_{2C}$ is a dense subspace of $\mathcal{L}_C$. □

3.3 Convergence of the evolutions in $\mathcal{K}_C$

Let $\varepsilon > 0$ be given. Let $\big(\hat{L}'_{\varepsilon,\,\mathrm{ren}}, D(\hat{L}'_{\varepsilon,\,\mathrm{ren}})\big)$ and $\big(\hat{L}'_V, D(\hat{L}'_V)\big)$ be dual operators to the closed operators $\big(\hat{L}_{\varepsilon,\,\mathrm{ren}}, D(\hat{L}_{\varepsilon,\,\mathrm{ren}})\big)$ and $\big(\hat{L}_V, D(\hat{L}_V)\big)$ in the Banach space $(\mathcal{L}_C)'$. Let the operators $\big(\hat{L}^*_{\varepsilon,\,\mathrm{ren}}, D(\hat{L}^*_{\varepsilon,\,\mathrm{ren}})\big)$ and $\big(\hat{L}^*_V, D(\hat{L}^*_V)\big)$ be their images in the space $\mathcal{K}_C$ under the isometry (2.14). Our aim is to transfer the previous results onto $*$-objects. However, similarly to the case of the operator $\hat{L}^*$ (see Subsection 2.2), the space $\mathcal{K}_C$ is too big. The reason is that the dual semigroup in a non-reflexive case (namely, L^1 case) will not be a strongly continuous semigroup on the whole dual space. Hence, we consider some Banach subspace of $\mathcal{K}_C$ which will be useful for the strong continuity property.

Proposition 3.10. *For any $\alpha \in (0;1)$, $\varepsilon > 0$, and $k \in \mathcal{K}_{\alpha C}$ we have that*

$$\left\{\hat{L}^*_{\varepsilon,\,\mathrm{ren}}k,\ \hat{L}^*_V k\right\} \subset \mathcal{K}_C. \tag{3.27}$$

Moreover, for any $k \in \mathcal{K}_{\alpha C}$

$$\begin{aligned}(\hat{L}^*_{\varepsilon,\,\mathrm{ren}}k)\,(\eta) &= -\,|\eta|\,k\,(\eta)\\ &\quad + z\sum_{x\in\eta}\int_{\Gamma_0} e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)}, \eta\setminus x\right)\\ &\qquad \times e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}, \xi\right) k\,(\xi\cup\eta\setminus x)\,d\lambda(\xi)\end{aligned} \tag{3.28}$$

and

$$(\hat{L}_V^* k)(\eta) = -|\eta|\, k(\eta) \qquad (3.29)$$
$$+ z \sum_{x\in\eta} \int_{\Gamma_0} e_\lambda\left(-\phi(x-\cdot), \xi\right) k(\xi\cup\eta\setminus x)\, d\lambda(\xi).$$

Proof. By Lemma 2.1, for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_{\Gamma_0} \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)}, \xi\right) e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}, \eta\setminus\xi\right)$$
$$\times G(\xi\cup x)\, dx k(\eta) d\lambda(\eta)$$
$$= \int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)}, \xi\right) e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}, \eta\right)$$
$$\times G(\xi\cup x)\, dx k(\eta\cup\xi) d\lambda(\xi) d\lambda(\eta)$$
$$= \int_{\Gamma_0}\int_{\Gamma_0}\sum_{x\in\xi} e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)}, \xi\setminus x\right) e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}, \eta\right)$$
$$\times G(\xi)\, dx k(\eta\cup\xi\setminus x) d\lambda(\xi) d\lambda(\eta),$$

that implies (3.28). The equality (3.29) may be obtained in the same way or just as a point-wise limit of (3.28) as $\varepsilon \to 0$.

The inclusion (3.27) follows from the estimate ($k \in \mathcal{K}_{\alpha C}$)

$$\frac{1}{C^{|\eta|}} \sum_{x\in\eta} \int_{\Gamma_0} e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)}, \eta\setminus x\right) e_\lambda\left(\left|\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}\right|, \xi\right) \left|k(\xi\cup\eta\setminus x)\right| d\lambda(\xi)$$
$$\le \frac{\|k\|_{\mathcal{K}_{\alpha C}}}{C^{|\eta|}} \sum_{x\in\eta}\int_{\Gamma_0} e_\lambda\left(\phi(x-\cdot), \xi\right) (\alpha C)^{|\xi\cup\eta\setminus x|} d\lambda(\xi)$$
$$= \frac{\|k\|_{\mathcal{K}_{\alpha C}} \cdot \exp\{\alpha C\beta\}}{\alpha C} |\eta| \alpha^{|\eta|} \le \frac{\|k\|_{\mathcal{K}_{\alpha C}} \cdot \exp\{\alpha C\beta\}}{\alpha C} \cdot \frac{-1}{e\ln\alpha},$$

where we used that $x\alpha^x \le -\dfrac{1}{e\ln\alpha}$ for any $\alpha \in (0;1)$ and $x \ge 0$; and the similar estimates for

$$\frac{1}{C^{|\eta|}} |\eta|\, |k(\eta)|\,, \quad \frac{1}{C^{|\eta|}} \sum_{x\in\eta}\int_{\Gamma_0} e_\lambda\left(\phi(x-\cdot), \xi\right) \left|k(\xi\cup\eta\setminus x)\right| d\lambda(\xi). \qquad (3.30)$$

□

Let now (3.23) holds. By Proposition 3.8, there exist strongly continuous contraction semigroups $\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)$ and $\hat{T}_V(t)$ on $\mathcal{L}_C$. Then the corresponding dual semigroups $\hat{T}'_{\varepsilon,\,\mathrm{ren}}(t)$ and $\hat{T}'_V(t)$ act in the space $(\mathcal{L}_C)'$. Let us denote by $\hat{T}^*_{\varepsilon,\,\mathrm{ren}}(t)$ and $\hat{T}^*_V(t)$ their corresponding images in $\mathcal{K}_C$ under the isometry (2.14).

Proposition 3.10 yields that for any $\alpha \in (0;1)$ the following inclusion holds

$$\overline{\mathcal{K}_{\alpha C}} \subset \Big(\bigcap_{\varepsilon>0} \overline{D(\hat{L}^*_{\varepsilon,\,\mathrm{ren}})}\Big) \bigcap \overline{D(\hat{L}^*_V)} \qquad (3.31)$$

(all closures are in $\mathcal{K}_C$; in particular, $\overline{\mathcal{K}_{\alpha C}}$ is a Banach space with norm $\|\cdot\|_C$).

Moreover, by, e.g., [28, Sections 1.2, 1.3] or [5, Subsection II.2.5], for any $\varepsilon > 0$ the restrictions $\hat{T}^{\odot}_{\varepsilon,\,\mathrm{ren}}(t)$ and $\hat{T}^{\odot}_{V}(t)$ of $\hat{T}^{*}_{\varepsilon,\,\mathrm{ren}}(t)$ and $\hat{T}^{*}_{V}(t)$ onto $\overline{D(\hat{L}^{*}_{\varepsilon,\,\mathrm{ren}})}$ and $\overline{D(\hat{L}^{*}_{V})}$, correspondingly, are strongly continuous semigroups; their generators $\hat{L}^{\odot}_{\varepsilon,\,\mathrm{ren}}$ and $\hat{L}^{\odot}_{V}$ are the parts of $\hat{L}^{*}_{\varepsilon,\,\mathrm{ren}}$ and $\hat{L}^{*}_{V}$, correspondingly. Namely,

$$D(\hat{L}^{\odot}_{\varepsilon,\,\mathrm{ren}}) = \bigl\{k \in D(\hat{L}^{*}_{\varepsilon,\,\mathrm{ren}}) \bigm| \hat{L}^{*}_{\varepsilon,\,\mathrm{ren}} k \in \overline{D(\hat{L}^{*}_{\varepsilon,\,\mathrm{ren}})}\bigr\},$$
$$D(\hat{L}^{\odot}_{V}) = \bigl\{k \in D(\hat{L}^{*}_{V}) \bigm| \hat{L}^{*}_{V} k \in \overline{D(\hat{L}^{*}_{V})}\bigr\},$$

and

$$\hat{L}^{*}_{\varepsilon,\,\mathrm{ren}} k = \hat{L}^{\odot}_{\varepsilon,\,\mathrm{ren}} k, \qquad k \in D(\hat{L}^{\odot}_{\varepsilon,\,\mathrm{ren}}),$$
$$\hat{L}^{*}_{V} k = \hat{L}^{\odot}_{V} k, \qquad k \in D(\hat{L}^{\odot}_{V}).$$

Proposition 3.11. *Assume that, as before,*

$$z \le \min\left\{Ce^{-C\beta}, 2Ce^{-2C\beta}\right\}. \tag{3.32}$$

If $C\beta = \ln 2$ we suppose additionally that $z < \frac{C}{2}$. Then, there exists $\alpha_1 = \alpha_1(z,\beta,C) \in (0;1)$ such that for any $\alpha \in (\alpha_1;1)$ the space $\overline{\mathcal{K}_{\alpha C}}$ will be $\hat{T}^{\odot}_{V}(t)$- and $\hat{T}^{\odot}_{\varepsilon,\,\mathrm{ren}}(t)$-invariant, $\varepsilon > 0$.

Proof. The proof is fully analogous to that of [7, Proposition 3.3]. For readers convince we explain it in details.

By (3.32), $z\beta \le \min\{C\beta e^{-C\beta}, 2C\beta e^{-2C\beta}\}$. Note that the function $f(x) = xe^{-x}$, $x \ge 0$ is increasing on $(0;1)$ from 0 to e^{-1} and it is asymptotically decreasing on $(1;+\infty)$ from e^{-1} to 0. Therefore, if $C\beta e^{-C\beta} \ne 2C\beta e^{-2C\beta}$ then (3.32) with necessity implies $z\beta < e^{-1}$. Otherwise, if $C\beta = \ln 2$ then the condition $2z < C$ implies $z\beta < \frac{C\beta}{2} = C\beta e^{-C\beta} = 2C\beta e^{-2C\beta}$, and, again, $z\beta < e^{-1}$. As a result, the equation $f(x) = z\beta$ has exactly two roots, say, $0 < x_1 < 1 < x_2 < +\infty$. Therefore, $x_1 < C\beta < 2C\beta < x_2$.

If $C\beta > 1$ then we set $\alpha_1 := \max\left\{\frac{1}{2}; \frac{1}{C\beta}; \frac{1}{C}\right\} < 1$. This yields $2\alpha C\beta > C\beta$ and $\alpha C\beta > 1 > x_1$. If $x_1 < C\beta \le 1$ then we set $\alpha_1 := \max\left\{\frac{1}{2}; \frac{x_1}{C\beta}; \frac{1}{C}\right\} < 1$ that gives $2\alpha C\beta > C\beta$ and $\alpha C\beta > x_1$. As a result,

$$x_1 < \alpha C\beta < C\beta < 2\alpha C\beta < 2C\beta < x_2 \tag{3.33}$$

and $1 < \alpha C < C < 2\alpha C < 2C$. The last inequality shows that $\mathcal{L}_{2C} \subset \mathcal{L}_{2\alpha C} \subset \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$.

By (3.33), $z\beta < \min\{f(\alpha C\beta), f(2\alpha C\beta)\}$, hence, $z < \min\{\alpha Ce^{-\alpha C\beta}, 2\alpha Ce^{-2\alpha C\beta}\}$. Then, analogously to Proposition 3.8, we obtain that the operators $\bigl(\hat{L}_V, \mathcal{L}_{2\alpha C}\bigr)$ and $\bigl(\hat{L}_{\varepsilon,\mathrm{ren}}, \mathcal{L}_{2\alpha C}\bigr)$ are closable in $\mathcal{L}_{\alpha C}$ and their closures are generators of contraction semigroups, say, $\hat{T}_{\alpha,V}(t)$ and $\hat{T}_{\alpha,\varepsilon,\,\mathrm{ren}}(t)$ on $\mathcal{L}_{\alpha C}$, correspondingly.

It is easy to see, that $\hat{T}_{\alpha,V}(t)G = \hat{T}_V(t)G$ and $\hat{T}_{\alpha,\varepsilon,\,\mathrm{ren}}(t)G = \hat{T}_{\varepsilon,\,\mathrm{ren}}(t)G$ for any $G \in \mathcal{L}_C$. Indeed, since the contraction mappings $\hat{Q}_\delta$ and $\hat{P}_{\delta,\varepsilon}$, $\delta,\varepsilon > 0$ do not depend on α, we obtain, by Proposition 3.8, that for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ we have that $\hat{T}_V(t)G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ and $\hat{T}_{\alpha,V}(t)G \in \mathcal{L}_{\alpha C}$ and

$$\begin{aligned}
&\|\hat{T}_V(t)G - \hat{T}_{\alpha,V}(t)G\|_{\alpha C} \\
\leq &\left\|\hat{T}_V(t)G - \hat{Q}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{\alpha C} + \left\|\hat{T}_{\alpha,V}(t)G - \hat{Q}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{\alpha C} \\
\leq &\left\|\hat{T}_V(t)G - \hat{Q}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{C} + \left\|\hat{T}_{\alpha,V}(t)G - \hat{Q}_\delta^{\left[\frac{t}{\delta}\right]}G\right\|_{\alpha C} \to 0,
\end{aligned}$$

as $\delta \to 0$. Therefore, $\hat{T}_V(t)G = \hat{T}_{\alpha,V}(t)G$ in $\mathcal{L}_{\alpha C}$ (recall that $G \in \mathcal{L}_C$) that yields $\hat{T}_V(t)G(\eta) = \hat{T}_{\alpha,V}(t)G(\eta)$ for λ-a.a. $\eta \in \Gamma_0$ and, therefore, $\hat{T}_V(t)G = \hat{T}_{\alpha,V}(t)G$ in $\mathcal{L}_C$.

Note that for any $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ and for any $k \in \mathcal{K}_{\alpha C} \subset \mathcal{K}_C$ we have $\hat{T}_{\alpha,V}(t)G \in \mathcal{L}_{\alpha C}$ and

$$\left\langle\!\!\left\langle \hat{T}_{\alpha,V}(t)G, k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle G, \hat{T}^*_{\alpha,V}(t)k \right\rangle\!\!\right\rangle,$$

where, by construction, $\hat{T}^*_{\alpha,V}(t)k \in \mathcal{K}_{\alpha C}$. But $G \in \mathcal{L}_C$, $k \in \mathcal{K}_C$ implies

$$\left\langle\!\!\left\langle \hat{T}_{\alpha,V}(t)G, k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \hat{T}_V(t)G, k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle G, \hat{T}^*(t)k \right\rangle\!\!\right\rangle.$$

Hence, $\hat{T}^*_V(t)k = \hat{T}^*_{\alpha,V}(t)k \in \mathcal{K}_{\alpha C}$ that is what we need.

Since $\hat{T}^\odot_V(t)$ and $\hat{T}^\odot_{\varepsilon,\,\mathrm{ren}}(t)$ are restrictions of $\hat{T}^*_V(t)$ and $\hat{T}^*_{\varepsilon,\,\mathrm{ren}}(t)$ onto $\overline{D(\hat{L}^*_V)}$ and $\overline{D(\hat{L}^*_{\varepsilon,\,\mathrm{ren}})}$, correspondingly, one has, by (3.31), that the corresponding semigroups coincide on $\mathcal{K}_{\alpha C}$. Therefore, $\mathcal{K}_{\alpha C}$ is $\hat{T}^\odot_V(t)$- and $\hat{T}^\odot_{\varepsilon,\,\mathrm{ren}}(t)$-invariant, $\varepsilon > 0$; and the result follows from the continuity of operators which formed semigroups. □

Let now $\hat{T}^{\odot\alpha}_V(t)$ and $\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}(t)$ be restrictions of the strongly continuous semigroups $\hat{T}^\odot_V(t)$ and $\hat{T}^\odot_{\varepsilon,\,\mathrm{ren}}(t)$ (which acting on the Banach spaces $\overline{D(\hat{L}^*_V)}$ and $\overline{D(\hat{L}^*_{\varepsilon,\,\mathrm{ren}})}$, correspondingly) onto the closed linear subspace $\overline{\mathcal{K}_{\alpha C}}$ of all these Banach spaces which are invariant w.r.t. all these $\odot$-semigroups. By the general result (see, e.g., [5, Subsection II.2.3]), $\hat{T}^{\odot\alpha}_V(t)$ and $\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}(t)$ are strongly continuous semigroups on $\overline{\mathcal{K}_{\alpha C}}$ with generators $\hat{L}^{\odot\alpha}_V$ and $\hat{L}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}$ which are restrictions of the corresponding operators $\hat{L}^\odot_V$ and $\hat{L}^\odot_{\varepsilon,\,\mathrm{ren}}$. Namely,

$$D(\hat{L}^{\odot\alpha}_V) = \left\{k \in \overline{\mathcal{K}_{\alpha C}} \;\middle|\; \hat{L}^*_V k \in \overline{\mathcal{K}_{\alpha C}}\right\}, \tag{3.34}$$

$$D(\hat{L}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}) = \left\{k \in \overline{\mathcal{K}_{\alpha C}} \;\middle|\; \hat{L}^*_{\varepsilon,\,\mathrm{ren}} k \in \overline{\mathcal{K}_{\alpha C}}\right\}, \quad \varepsilon > 0, \tag{3.35}$$

and

$$\hat{L}^{\odot\alpha}_V k = \hat{L}^*_V k, \qquad k \in D(\hat{L}^{\odot\alpha}_V), \tag{3.36}$$

$$\hat{L}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}} k = \hat{L}^*_{\varepsilon,\,\mathrm{ren}} k, \qquad k \in D(\hat{L}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}). \tag{3.37}$$

By Proposition 3.8, $\hat{T}_V(t)$ and $\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)$ are contraction semigroups on $\mathcal{L}_C$, then, $\hat{T}'_V(t)$ and $\hat{T}'_{\varepsilon,\,\mathrm{ren}}(t)$ are also contraction semigroups on $(\mathcal{L}_C)'$; but isomorphism (2.14) is isometrical, therefore, $\hat{T}^*_V(t)$ and $\hat{T}^*_{\varepsilon,\,\mathrm{ren}}(t)$ are contraction semigroups on $\mathcal{K}_C$. As a result, their restrictions $\hat{T}^{\odot\alpha}_V(t)$ and $\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}(t)$ are contraction semigroups on $\overline{\mathcal{K}_{\alpha C}}$.

To summarize, we have the Banach space $\overline{\mathcal{K}_{\alpha C}}$ and the family of the strongly continuous contraction semigroups $\hat{T}^{\odot\alpha}_V(t)$ and $\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}(t)$, $\varepsilon>0$ on this space. The generators of these semigroups are satisfied (3.34)–(3.37). Moreover, by construction, $\hat{T}^{\odot\alpha}_V(t)k=\hat{T}^*_V(t)k$ and $\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}(t)k=\hat{T}^*_{\varepsilon,\,\mathrm{ren}}(t)k$ for any $k\in\overline{\mathcal{K}_{\alpha C}}$.

Theorem 3.12. *Let C,z,β,α_1 be as in Proposition 3.11. Suppose additionally that $\bar{\phi}:=\sup_{\mathbb{R}^d}\phi(x)<+\infty$. Then, for any $\alpha\in(\alpha_1;1)$ and for any $k\in\mathcal{K}_{\alpha C}$*

$$\left\|\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}(t)k-\hat{T}^{\odot\alpha}_V(t)k\right\|_{\mathcal{K}_C}\leq\varepsilon tA\|k\|_{\mathcal{K}_{\alpha C}},\quad\varepsilon>0,\tag{3.38}$$

where A is depend on α, C, $\bar{\phi}$ only.

Proof. Let $\hat{Q}^*_\delta$, $\hat{P}^*_{\delta,\varepsilon}$, $\delta\in(0;1)$, $\varepsilon>0$ be the images of the dual operators $\hat{Q}'_\delta$, $\hat{P}'_{\delta,\varepsilon}$ under the isometrical isomorphism (2.14). Since the norms of dual operators are equal we have that $\hat{Q}^*_\delta$ and $\hat{P}^*_{\delta,\varepsilon}$ are linear contractions on $\mathcal{K}_C$. Moreover, for any $k\in\mathcal{K}_{\alpha C}$ we have

$$\begin{aligned}&\int_{\Gamma_0}(\hat{Q}_\delta G)(\eta)\,k(\eta)\,d\lambda(\eta)\\
&=\int_{\Gamma_0}\sum_{\xi\subset\eta}(1-\delta)^{|\xi|}\int_{\Gamma_0}(z\delta)^{|\omega|}\,G(\xi\cup\omega)\\
&\qquad\times e_\lambda\left(-E^\phi(\cdot,\omega),\eta\setminus\xi\right)d\lambda(\omega)\,k(\eta)\,d\lambda(\eta)\\
&=\int_{\Gamma_0}\int_{\Gamma_0}\int_{\Gamma_0}(1-\delta)^{|\xi|}\,(z\delta)^{|\omega|}\,G(\xi\cup\omega)\\
&\qquad\times e_\lambda\left(-E^\phi(\cdot,\omega),\eta\right)d\lambda(\omega)\,k(\eta\cup\xi)\,d\lambda(\eta)\,d\lambda(\xi)\\
&=\int_{\Gamma_0}\int_{\Gamma_0}\sum_{\omega\subset\xi}(1-\delta)^{|\xi\setminus\omega|}\,(z\delta)^{|\omega|}\,G(\xi)\\
&\qquad\times e_\lambda\left(-E^\phi(\cdot,\omega),\eta\right)k(\eta\cup\xi\setminus\omega)\,d\lambda(\eta)\,d\lambda(\xi)\end{aligned}$$

and, therefore,

$$\begin{aligned}(\hat{Q}^*_\delta k)(\eta)=&\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\,(z\delta)^{|\omega|}\\
&\times\int_{\Gamma_0}e_\lambda\left(-E^\phi(\cdot,\omega),\xi\right)k(\xi\cup\eta\setminus\omega)\,d\lambda(\xi).\end{aligned}\tag{3.39}$$

Then, by (3.32),

$$
\begin{aligned}
&(\alpha C)^{-|\eta|}\left|(\hat{Q}^*_\delta k)(\eta)\right| \\
&\leq \|k\|_{\mathcal{K}_{\alpha C}}(\alpha C)^{-|\eta|}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|} \\
&\qquad\times\int_{\Gamma_0} e_\lambda\left(E^\phi(\cdot,\omega),\xi\right)(\alpha C)^{|\xi\cup\eta\setminus\omega|}\,d\lambda(\xi) \\
&=\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\left(\frac{z\delta}{\alpha C}\right)^{|\omega|}\exp\left\{\alpha C\int_{\mathbb{R}^d}E^\phi(x,\omega)\,dx\right\} \\
&=\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\left(\frac{z\delta}{\alpha C}\right)^{|\omega|}\exp\{\alpha C\,|\omega|\,\beta\} \\
&\leq \|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\delta^{|\omega|}=\|k\|_{\mathcal{K}_{\alpha C}}.
\end{aligned}
$$

Therefore, $\mathcal{K}_{\alpha C}$ is $\hat{Q}^*_V$-invariant, hence, $\overline{\mathcal{K}_{\alpha C}}$ is also $\hat{Q}^*_V$-invariant due to continuity of $\hat{Q}^*_V$; moreover, $\hat{Q}^*_V$ is a contraction in $\mathcal{L}_{\alpha C}$. Absolutely in the same way we may obtain that for any $k\in\mathcal{K}_{\alpha C}$

$$
\begin{aligned}
(\hat{P}^*_{\delta,\varepsilon}k)(\eta) = &\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|}e_\lambda\left(e^{-\varepsilon E^\phi(\cdot,\omega)},\eta\setminus\omega\right) \\
&\times\int_{\Gamma_0}e_\lambda\left(\frac{e^{-\varepsilon E^\phi(\cdot,\omega)}-1}{\varepsilon},\xi\right)k(\xi\cup\eta\setminus\omega)\,d\lambda(\xi)
\end{aligned}
\tag{3.40}
$$

and that the set $\mathcal{K}_{\alpha C}$, and, therefore, the set $\overline{\mathcal{K}_{\alpha C}}$ are $\hat{P}^*_{\delta,\varepsilon}$-invariant; moreover, $\hat{P}^*_{\delta,\varepsilon}$ is a contraction in $\mathcal{L}_{\alpha C}$. We preserve the same notations for the restrictions of this contractions onto $\overline{\mathcal{K}_{\alpha C}}$.

Now, for any fixed $\varepsilon>0$ we consider a set $D_\varepsilon:=\left\{k\in\mathcal{K}_{\alpha C}\;\middle|\;\hat{L}^*_{\varepsilon,\,\mathrm{ren}}k\in\overline{\mathcal{K}_{\alpha C}}\right\}$. By (3.35), D_ε is a core for the operator $\hat{L}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}$. Next, let us show that for any $k\in D_\varepsilon$

$$
\lim_{\delta\to0}\left\|\frac{1}{\delta}(\hat{P}^*_{\delta,\varepsilon}-1\!\!1)k-\hat{L}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}k\right\|_{\mathcal{K}_C}=0. \tag{3.41}
$$

Indeed, let

$$
\begin{aligned}
(\hat{P}^{*,(0)}_{\delta,\varepsilon}k)(\eta) =&(1-\delta)^{|\eta|}k(\eta); \\
(\hat{P}^{*,(1)}_{\delta,\varepsilon}k)(\eta) =&\sum_{x\in\eta}(1-\delta)^{|\eta|-1}z\delta e_\lambda\left(e^{-\varepsilon E^\phi(\cdot,x)},\eta\setminus x\right) \\
&\times\int_{\Gamma_0}e_\lambda\left(\frac{e^{-\varepsilon E^\phi(\cdot,x)}-1}{\varepsilon},\xi\right)k(\xi\cup\eta\setminus x)\,d\lambda(\xi);
\end{aligned}
$$

and $\hat{P}^{*,(\geq 2)}_{\delta,\varepsilon} = \hat{P}^{*}_{\delta,\varepsilon} - \hat{P}^{*,(0)}_{\delta,\varepsilon} - \hat{P}^{*,(1)}_{\delta,\varepsilon}$. One may improve inequality (3.17), namely, for any $n \in \mathbb{N} \cup \{0\}$, $\delta \in (0;1)$

$$0 \leq n - \frac{1-(1-\delta)^n}{\delta} \leq \delta \frac{n(n-1)}{2}.$$

Then, for any $k \in \mathcal{K}_{\alpha C}$, $\eta \neq \emptyset$

$$\begin{aligned} &C^{-|\eta|}\left|\frac{1}{\delta}(\hat{P}^{*,(0)}_{\delta,\varepsilon} - 1\!\!1)k(\eta) + |\eta|k(\eta)\right| \\ \leq &\|k\|_{\mathcal{K}_{\alpha C}}\alpha^{|\eta|}\left||\eta| - \frac{1-(1-\delta)^{|\eta|}}{\delta}\right| \leq \frac{\delta}{2}\|k\|_{\mathcal{K}_{\alpha C}}\alpha^{|\eta|}|\eta|(|\eta|-1) \end{aligned} \tag{3.42}$$

and the function $\alpha^x x(x-1)$ is bounded for $x \geq 1$, $\alpha \in (0;1)$. Next, for any $k \in \mathcal{K}_{\alpha C}$, $\eta \neq \emptyset$

$$\begin{aligned} &C^{-|\eta|}\left|\frac{1}{\delta}\hat{P}^{*,(1)}_{\delta,\varepsilon}k(\eta) - z\sum_{x\in\eta}\int_{\Gamma_0} e_\lambda\left(e^{-\varepsilon\phi(x-\cdot)}, \eta\setminus x\right)\right. \\ &\qquad \left.\times\, e_\lambda\left(\frac{e^{-\varepsilon\phi(x-\cdot)}-1}{\varepsilon}, \xi\right) k\left(\xi\cup\eta\setminus x\right) d\lambda(\xi)\right| \\ \leq &\frac{z}{\alpha C}\alpha^{|\eta|}\sum_{x\in\eta}\left((1-\delta)^{|\eta|-1}-1\right)e_\lambda\left(e^{-\varepsilon E^\phi(\cdot,x)}, \eta\setminus x\right) \\ &\qquad \times\int_{\Gamma_0} e_\lambda\left(\alpha C\frac{|e^{-\varepsilon E^\phi(\cdot,x)}-1|}{\varepsilon}, \xi\right) d\lambda\left(\xi\right) \\ \leq &\frac{z}{\alpha C}\alpha^{|\eta|}\sum_{x\in\eta}\left|(1-\delta)^{|\eta|-1}-1\right|\exp\left\{\alpha C\beta\right\} \\ \leq &\frac{z}{\alpha C}\alpha^{|\eta|}\delta|\eta|(|\eta|-1)\exp\left\{\alpha C\beta\right\} \end{aligned} \tag{3.43}$$

that is smaller then δ uniformly in $|\eta|$. And, finally,

$$\begin{aligned} &\frac{1}{\delta C^{|\eta|}}\sum_{\substack{\omega\subset\eta\\|\omega|\geq 2}}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|}e_\lambda\left(e^{-\varepsilon E^\phi(\cdot,\omega)}, \eta\setminus\omega\right) \\ &\qquad\times\int_{\Gamma_0} e_\lambda\left(\left|\frac{e^{-\varepsilon E^\phi(\cdot,\omega)}-1}{\varepsilon}\right|, \xi\right)|k(\xi\cup\eta\setminus\omega)|d\lambda\left(\xi\right) \\ \leq &\frac{1}{\delta C^{|\eta|}}\sum_{\substack{\omega\subset\eta\\|\omega|\geq 2}}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|} \\ &\qquad\times\int_{\Gamma_0} e_\lambda\left(E^\phi\left(\cdot,\omega\right), \xi\right)(\alpha C)^{|\xi|}(\alpha C)^{|\eta|-|\omega|}d\lambda\left(\xi\right) \\ = &\alpha^{|\eta|}\frac{1}{\delta}\sum_{\substack{\omega\subset\eta\\|\omega|\geq 2}}(1-\delta)^{|\eta\setminus\omega|}\left(\frac{z\delta}{\alpha C}\exp\left\{\alpha C\beta\right\}\right)^{|\omega|} \end{aligned}$$

but recall that $\alpha > \alpha_1$, therefore, $z \exp\{\alpha C \beta\} \le \alpha C$, and one may continue

$$\begin{aligned}
\le&\alpha^{|\eta|} \frac{1}{\delta} \sum_{\substack{\omega \subset \eta \\ |\omega| \ge 2}} (1-\delta)^{|\eta \setminus \omega|} \delta^{|\omega|} = \delta \alpha^{|\eta|} \sum_{k=2}^{|\eta|} \frac{|\eta|!}{k! \, (|\eta|-k)!} (1-\delta)^{|\eta|-k} \delta^{k-2} \\
=&\delta \alpha^{|\eta|} \sum_{k=0}^{|\eta|-2} \frac{|\eta|!}{(k+2)! \, (|\eta|-k-2)!} (1-\delta)^{|\eta|-k-2} \delta^{k} \\
=&\delta \alpha^{|\eta|} \, |\eta| \, (|\eta|-1) \sum_{k=0}^{|\eta|-2} \frac{(|\eta|-2)!}{(k+2)! \, (|\eta|-k-2)!} (1-\delta)^{|\eta|-2-k} \delta^{k} \\
\le&\delta \alpha^{|\eta|} \, |\eta| \, (|\eta|-1) \sum_{k=0}^{|\eta|-2} \frac{(|\eta|-2)!}{k! \, (|\eta|-k-2)!} (1-\delta)^{|\eta|-2-k} \delta^{k} \\
=&\delta \alpha^{|\eta|} \, |\eta| \, (|\eta|-1) \, .
\end{aligned}$$

Combining these inequalities, we obtain (3.41).

Analogously, one may obtain that for any $k \in D_V := \bigl\{ k \in \mathcal{K}_{\alpha C} \bigm| \hat{L}_V^* k \in \overline{\mathcal{K}_{\alpha C}} \bigr\}$ (that is core for $\hat{L}_V^{\odot \alpha}$)

$$\lim_{\delta \to 0} \Bigl\| \frac{1}{\delta} (\hat{Q}_\delta^* - 1\!\!1) k - \hat{L}_V^{\odot \alpha} k \Bigr\|_{\mathcal{K}_C} = 0. \tag{3.44}$$

By Lemma 3.6, we obtain that for any $k \in \overline{\mathcal{K}_{\alpha C}}$

$$(\hat{P}_{\delta,\varepsilon}^*)^{\left[\frac{t}{\delta}\right]} k \to \hat{T}_{\varepsilon,\,\mathrm{ren}}^{\odot \alpha}(t) k; \qquad (\hat{Q}_\delta^*)^{\left[\frac{t}{\delta}\right]} k \to \hat{T}_V^{\odot \alpha}(t) k$$

(convergence in $\overline{\mathcal{K}_{\alpha C}}$, recall that norm in this space is $\| \cdot \|_{\mathcal{K}_C}$).

Therefore, to use the same arguments as in the proof of Theorem 3.9 and to apply Lemma 3.7, we need only to show that for any $k \in \mathcal{K}_{\alpha C}$

$$\bigl\| \hat{P}_{\delta,\varepsilon}^* k - \hat{Q}_\delta^* k \bigr\|_{\mathcal{K}_C} \le \varepsilon \delta A \|k\|_{\mathcal{K}_{\alpha C}}. \tag{3.45}$$

We have the following elementary inequalities. For any $\{a_k\}_{k=1}^n \subset [0;1]$, $n \in \mathbb{N}$

$$1 - \prod_{k=1}^n a_k \le \sum_{k=1}^n (1 - a_k) \, , \tag{3.46}$$

which can be easily checked by the induction principle. Next, since

$$x + e^{-x} - 1 \le x^2, \quad x \ge 0,$$

we obtain

$$E^\phi(x,\omega) \left(1 - \frac{1 - e^{-\varepsilon E^\phi(x,\omega)}}{\varepsilon E^\phi(x,\omega)} \right) \le \varepsilon \left(E^\phi(x,\omega) \right)^2 . \tag{3.47}$$

Hence,

$$
\begin{aligned}
&\frac{1}{C^{|\eta|}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|}\\
&\qquad\times\int_{\Gamma_0}\left|\left(e^{-\varepsilon E^{\phi}(\cdot,\omega)},\eta\setminus\omega\right)e_\lambda\left(\frac{e^{-\varepsilon E^{\phi}(\cdot,\omega)}-1}{\varepsilon},\xi\right)-e_\lambda\left(-E^{\phi}(\cdot,\omega),\xi\right)\right|\\
&\qquad\qquad\times k(\xi\cup\eta\setminus\omega)\,d\lambda(\xi)\\
\leq&\frac{\|k\|_{\mathcal{K}_{\alpha C}}}{C^{|\eta|}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|}\\
&\qquad\times\int_{\Gamma_0}e_\lambda\left(E^{\phi}(\cdot,\omega),\xi\right)\left|\left(e^{-\varepsilon E^{\phi}(\cdot,\omega)},\eta\setminus\omega\right)e_\lambda\left(\frac{1-e^{-\varepsilon E^{\phi}(\cdot,\omega)}}{\varepsilon E^{\phi}(\cdot,\omega)},\xi\right)-1\right|\\
&\qquad\times(\alpha C)^{|\xi\cup\eta\setminus\omega|}\,d\lambda(\xi)
\end{aligned}
$$

and, by (3.46), one may continue

$$
\begin{aligned}
\leq&\alpha^{|\eta|}\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|}\int_{\Gamma_0}e_\lambda\left(E^{\phi}(\cdot,\omega),\xi\right)\\
&\qquad\times\sum_{x\in\eta\setminus\omega}\left(1-e^{-\varepsilon E^{\phi}(x,\omega)}\right)(\alpha C)^{|\xi\setminus\omega|}\,d\lambda(\xi)\\
&+\alpha^{|\eta|}\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}(z\delta)^{|\omega|}\int_{\Gamma_0}e_\lambda\left(E^{\phi}(\cdot,\omega),\xi\right)\\
&\qquad\times\sum_{x\in\xi}\left(1-\frac{1-e^{-\varepsilon E^{\phi}(x,\omega)}}{\varepsilon E^{\phi}(x,\omega)}\right)(\alpha C)^{|\xi\setminus\omega|}\,d\lambda(\xi)
\end{aligned}
$$

and, by (3.47),

$$
\begin{aligned}
\leq&\alpha^{|\eta|}\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\left(\frac{z\delta}{\alpha C}\exp\{\alpha C\beta\}\right)^{|\omega|}\sum_{x\in\eta\setminus\omega}\varepsilon E^{\phi}(x,\omega)\\
&+\alpha^{|\eta|}\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\left(\frac{z\delta}{\alpha C}\right)^{|\omega|}\\
&\qquad\times\int_{\Gamma_0}\int_{\mathbb{R}^d}\varepsilon\left(E^{\phi}(x,\omega)\right)^2e_\lambda\left(E^{\phi}(\cdot,\omega),\xi\right)(\alpha C)^{|\xi|}\alpha C\,dx\,d\lambda(\xi)
\end{aligned}
$$

again, $z\exp\{\alpha C\beta\}\leq\alpha C$ and we continue

$$
\begin{aligned}
\leq&\varepsilon\,\bar{\phi}\,\alpha^{|\eta|}\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\delta^{|\omega|}|\eta\setminus\omega|\cdot|\omega|\\
&+\varepsilon\alpha C\,\bar{\phi}\alpha^{|\eta|}\|k\|_{\mathcal{K}_{\alpha C}}\sum_{\omega\subset\eta}(1-\delta)^{|\eta\setminus\omega|}\delta^{|\omega|}|\omega|^2=:J.
\end{aligned}
$$

To complete the proof we need to use the following simple estimates: for any $|\xi| = n \geq 2$ one has

$$\begin{aligned}
&\sum_{\omega\subset\xi} |\omega|\,|\xi\setminus\omega|\,(1-\delta)^{|\xi\setminus\omega|}\,\delta^{|\omega|} \qquad (3.48)\\
=&\sum_{k=1}^{n-1} \frac{n!}{k!\,(n-k)!} k\,(n-k)\,(1-\delta)^{n-k}\,\delta^k\\
=&\sum_{k=1}^{n-1} \frac{n!}{(k-1)!\,(n-k-1)!}\,(1-\delta)^{n-k}\,\delta^k\\
=&\sum_{k=0}^{n-2} \frac{n!}{k!\,(n-(k+1)-1)!}\,(1-\delta)^{n-(k+1)}\,\delta^{k+1}\\
=&\delta\,(1-\delta)\,n\,(n-2)\sum_{k=0}^{n-2} \frac{(n-2)!}{k!\,(n-2-k)!}\,(1-\delta)^{n-2-k}\,\delta^k\\
=&\delta\,(1-\delta)\,n\,(n-2)\,(1-\delta+\delta)^{n-2} \leq \delta\cdot 2^n = \delta\cdot 2^{|\xi|}
\end{aligned}$$

(and this estimate is trivial for $|\xi| \leq 1$); and, for any $n = |\xi| \geq 1$

$$\begin{aligned}
&\sum_{\omega\subset\xi} (1-\delta)^{|\xi\setminus\omega|}\,\delta^{|\omega|}\,|\omega|^2 \qquad (3.49)\\
=&\sum_{k=1}^{n} \frac{n!}{k!\,(n-k)!} k^2\,(1-\delta)^{n-k}\,\delta^k\\
=&\delta\sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-1-(k-1))!} k\,(1-\delta)^{(n-1)-(k-1)}\,\delta^{k-1}\\
=&\delta\sum_{k=0}^{n-1} \frac{n!}{k!\,(n-1-k)!} k\,(1-\delta)^{(n-1)-k}\,\delta^k\\
\leq&\delta n\,(n-1)\,(1-\delta+\delta)^{n-1} < \delta\cdot 2^n
\end{aligned}$$

(and, again, it is trivial for $\xi = \emptyset$).

Then, by (3.48), (3.49), we obtain for any $|\eta| \geq 2$

$$J \leq \varepsilon\,\bar{\phi}\,\alpha^{|\eta|}\,\|k\|_{\mathcal{K}_{\alpha C}}\,\delta|\eta|(|\eta|-1) + \varepsilon\alpha C\,\bar{\phi}\alpha^{|\eta|}\,\|k\|_{\mathcal{K}_{\alpha C}}\,\delta|\eta|(|\eta|-1) \leq \varepsilon\delta A,$$

where A is independent on η. □

Corollary 3.13. *Let the conditions of Theorem 3.12 hold. Then for any* $\{k^{(\varepsilon)}, k\} \subset \mathcal{K}_{\alpha C}$, $\varepsilon > 0$

$$\left\|\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}(t)k^{(\varepsilon)} - \hat{T}^{\odot\alpha}_{V}(t)k\right\|_{\mathcal{K}_C} \leq \left\|k^{(\varepsilon)} - k\right\|_{\mathcal{K}_C} + \varepsilon t A\|k\|_{\mathcal{K}_{\alpha C}}. \qquad (3.50)$$

Proof. The proof follows directly from the triangle inequality and the contractive property of the semigroup $\hat{T}^{\odot\alpha}_{\varepsilon,\,\mathrm{ren}}$. □

And now we will show that our Vlasov limiting dynamics has the properties described in the Subsection 3.1.

Theorem 3.14. *Let C, z, β, α_1 be as in Proposition 3.11, and $\alpha_2 := \max\bigl\{\alpha_1, \frac{z}{C}\bigr\} \in (0;1)$. Let ρ_0 be a measurable function on $\mathbb{R}^d$ such that there exists $\alpha \in (\alpha_2;1)$ such that $0 \le \rho_0(x) \le \alpha C$ for a.a. $x \in \mathbb{R}^d$. Then the Cauchy problem*

$$\begin{cases} \dfrac{\partial}{\partial t} k_t = \hat{L}_V^* k_t \\ k_0 = e_\lambda(\rho_0) \end{cases} \tag{3.51}$$

is well-defined on $\overline{\mathcal{K}_{\alpha C}}$ and has a solution $k_t = e_\lambda(\rho_t) \in \mathcal{K}_{\alpha C}$, where ρ_t is a solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} \rho_t(x) = -\rho_t(x) + z \exp\Bigl\{ - \displaystyle\int_{\mathbb{R}^d} \rho_t(y)\phi(x-y)dy \Bigr\}, \\ \rho_t\big|_{t=0}(x) = \rho_0(x), \end{cases} \tag{3.52}$$

for a.a. $x \in \mathbb{R}^d$ such that $0 \le \rho_t(x) \le \alpha C$ for a.a. $x \in \mathbb{R}^d$.

Proof. First of all, we note that (3.32) implies $z < C$, therefore, the condition $\frac{z}{C} < 1$ holds. Next, if (3.52) has a solution $\rho_t(x) \ge 0$ then $\frac{\partial}{\partial t}\rho_t(x) \le -\rho_t(x) + z$ and, therefore, $\rho_t(x) \le r_t(x)$ where $r_t(x)$ is a solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} r_t(x) = -r_t(x) + z, \\ r_t\big|_{t=0}(x) = \rho_0(x), \end{cases}$$

for a.a. $x \in \mathbb{R}^d$, hence,

$$r_t(x) = e^{-t}\rho_0(x) + z(1-e^{-t}) = z + e^{-t}(\rho_0(x) - z) \le \max\{z, \rho_0(x)\} \le \alpha C,$$

that yields $0 \le \rho_t(x) \le \alpha C$.

To prove the existence of the solution of (3.52) let us fix some $T > 0$ and define the Banach space $X_T = C([0;T], L^\infty(\mathbb{R}^d))$ of all continuous functions on $[0;T]$ with values in $L^\infty(\mathbb{R}^d)$; the norm on X_T is given by $\|u\|_T := \max\limits_{t\in[0;T]} \|u_t\|_{L^\infty(\mathbb{R}^d)}$. We denote by X_T^+ the cone of the all nonnegative functions from X_T.

Let Φ be a mapping which assign to any $v \in X_T$ the solution u_t of the linear Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t} u_t(x) = -u_t(x) + z \exp\{-(v_t * \phi)(x)\}, \\ u_t\big|_{t=0}(x) = \rho_0(x), \end{cases} \tag{3.53}$$

for a.a. $x \in \mathbb{R}^d$, where we use the usual notation for convolution on $\mathbb{R}^d$: $(f*g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)dy$. Therefore,

$$(\Phi v)_t(x) = e^{-t}\rho_0(x) + z\int_0^t e^{-(t-s)} \exp\{-(v_t * \phi)(x)\}ds \ge 0. \tag{3.54}$$

Similarly as before we obtain that $v \in X_T^+$ implies the estimate $|(\Phi v)_t(x)| \leq \max\{z, \rho_0(x)\}$; in particular, $\Phi v \in X_T^+$. Next, using elementary inequality $|e^{-a} - e^{-b}| \leq |a - b|$ for any $a, b \geq 0$, we obtain that for any $v, w \in X_T^+$

$$\begin{aligned}
\big|(\Phi v)_t(x) - (\Phi w)_t(x)\big| &\leq z \int_0^t e^{-(t-s)} \Big|\exp\{-(v_t * \phi)(x) - \exp\{-(w_t * \phi)(x)\}\Big| ds \\
&\leq z \int_0^t e^{-(t-s)} \big|(v_t * \phi)(x) - (w_t * \phi)(x)\big| ds \\
&\leq z \int_0^t e^{-(t-s)} (|v_t - w_t| * \phi)(x) ds \\
&\leq z\beta \|v - w\|_T (1 - e^{-t}),
\end{aligned}$$

where we used the inequality $|(f * g)(x)| \leq \|f\|_{L^\infty(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$ and condition (3.7). Therefore, $\|\Phi v - \Phi w\|_T \leq z\beta \|v - w\|_T$. Since (3.32) implies $z\beta \leq e^{-1}$ (see the proof of Proposition 3.11), hence, Φ is a contraction mapping on the cone X_T^+. Taking, as usual, $v^{(n)} = \Phi^n v^{(0)}$, $n \geq 1$ for $v^{(0)} \in X_T^+$ we obtain that $\{v^{(n)}\} \subset X_T^+$ is a fundamental sequence in X_T which has, therefore, a unique limit point $v \in X_T$. Since X_T^+ is a closed cone we have that $v \in X_T^+$. Then, identically to the classical Banach fixed point theorem, v will be a fixed point of Φ on X_T and a unique fixed point on X_T^+. Then, this v is the nonnegative solution of (3.52) on the interval $[0; T]$. By the note above, $v_t(x) \leq \alpha C$. Changing initial value in (3.52) onto $\rho_t\big|_{t=T}(x) = v_T(x)$ we may extend all our considerations on the time-interval $[T; 2T]$ with the same estimate $v_t(x) \leq \alpha C$; and so on. As a a result, (3.52) has a global bounded solution $\rho_t(x)$ on $\mathbb{R}_+$.

Clearly, $k_0 = e_\lambda(\rho_0) \in \mathcal{K}_{\alpha C} \subset \overline{\mathcal{K}_{\alpha C}}$. Then $k_t = \hat{T}_V^{\odot\alpha}(t) k_0$ will be a strongly differentiable function (in the sense of norm $\|\cdot\|_{\mathcal{K}_C}$ in $\overline{\mathcal{K}_{\alpha C}}$); moreover, $k_t \in \mathcal{K}_{\alpha C}$. Next, if we substitute $k_t = e_\lambda(\rho_t)$ into (3.51), then, by (3.28), we obtain

$$\begin{aligned}
&\sum_{x \in \eta} \frac{\partial}{\partial t} \rho_t(x) e_\lambda(\rho_t, \eta \setminus x) \\
&= - |\eta| \, e_\lambda(\rho_t, \eta) \\
&\quad + z \sum_{x \in \eta} e_\lambda(\rho_t, \eta \setminus x) \int_{\Gamma_0} e_\lambda\left(-\phi\left(x - \cdot\right), \xi\right) e_\lambda(\rho_t, \xi) d\lambda(\xi) \\
&= - \sum_{x \in \eta} \rho_t(x) e_\lambda(\rho_t, \eta \setminus x) \\
&\quad + z \sum_{x \in \eta} e_\lambda(\rho_t, \eta \setminus x) \exp\bigg\{- \int_{\mathbb{R}^d} \phi(x - y) \rho_t(y) dy\bigg\},
\end{aligned}$$

that holds since ρ_t is satisfied (3.52). □

Remark 3.15. Note that the stationary equation for (3.52) has the following form

$$\rho(x) = z \exp\bigg\{- \int_{\mathbb{R}^d} \rho(y) \phi(x - y) dy\bigg\} \tag{3.55}$$

and coincides with the famous Kirkwood–Monroe equation ([17], see also, e.g., [15] and references therein, and the recent work [2]).

3.4 Further considerations

We have realized the scheme proposed at the end of Subsection 3.1. But let us explain also the rigorous meaning of the equivalence (3.1) which was background to all our consideration.

Let C, z, β, α_2 be as in Theorem 3.14. Then, for any fixed $\varepsilon > 0$ we have $1 - \exp\{-\varepsilon\phi\} \in L^1(\mathbb{R}^d)$ and, by [10, Proposition 3.2], $\hat{L}_\varepsilon$, given by (3.10), is a linear operator in $\mathcal{L}_{\varepsilon^{-1}C}$ with dense domain $\mathcal{L}_{2\varepsilon^{-1}C}$. Consider the image $\big(\hat{L}_\varepsilon^*, D(\hat{L}_\varepsilon^*)\big)$ in $\mathcal{K}_{\varepsilon^{-1}C} = R_{\varepsilon^{-1}}\mathcal{K}_C$ under the isometrical isomorphism $R_{\varepsilon^{-1}C}$ of the dual operator $\big(\hat{L}_\varepsilon', D(\hat{L}_\varepsilon')\big)$ in $(\mathcal{L}_{\varepsilon^{-1}C})'$.

We are not able to show that $\hat{L}_\varepsilon$ is a generator of a strongly continuous semigroup in $\mathcal{L}_{\varepsilon^{-1}C}$ since a condition like (2.13) (with $\varepsilon^{-1}C$ instead of C) cannot be fulfilled uniformly in $\varepsilon > 0$. But one can do in the following manner.

Let $\alpha \in (\alpha_2; 1)$ and let us consider the space $\mathbb{K}_\varepsilon^\alpha = \overline{\mathcal{K}_{\varepsilon^{-1}\alpha C}}^{\mathcal{K}_{\varepsilon^{-1}C}}$. Note that for any $r^{(\varepsilon)} \in \mathbb{K}_\varepsilon^\alpha$ there exist $\{r_n^{(\varepsilon)}\} \subset \mathcal{K}_{\varepsilon^{-1}\alpha C}$ such that

$$0 = \lim_{n\to\infty} \|r_n^{(\varepsilon)} - r^{(\varepsilon)}\|_{\mathcal{K}_{\varepsilon^{-1}C}} = \lim_{n\to\infty} \|R_\varepsilon r_n^{(\varepsilon)} - R_\varepsilon r^{(\varepsilon)}\|_{\mathcal{K}_C}$$

and the inclusion $R_\varepsilon r_n^{(\varepsilon)} \in \mathcal{K}_{\alpha C}$, $n \in \mathbb{N}$ yields $R_\varepsilon r^{(\varepsilon)} \in \overline{\mathcal{K}_{\alpha C}}$. Vise versa, for any $k^{(\varepsilon)} \in \overline{\mathcal{K}_{\alpha C}}$ we see that $R_{\varepsilon^{-1}} k^{(\varepsilon)} \in \mathbb{K}_\varepsilon^\alpha$. As a result, R_ε provides an isometrical isomorphism between the Banach spaces $\mathbb{K}_\varepsilon^\alpha$ and $\overline{\mathcal{K}_{\alpha C}}$. Then, $U_\varepsilon^\alpha(t) := R_{\varepsilon^{-1}} \hat{T}_{\varepsilon,\,\mathrm{ren}}^{\odot\alpha}(t) R_\varepsilon$ will be a strongly continuous contraction semigroup on $\mathbb{K}_\varepsilon^\alpha$ with the generator $A_\varepsilon^\alpha = R_{\varepsilon^{-1}} \hat{L}_{\varepsilon,\,\mathrm{ren}}^{\odot\alpha} R_\varepsilon$ and the domain $D(A_\varepsilon^\alpha) = R_{\varepsilon^{-1}} D(\hat{L}_{\varepsilon,\,\mathrm{ren}}^{\odot\alpha})$. Moreover, since $\mathcal{K}_{\alpha C} \cap D(\hat{L}_{\varepsilon,\,\mathrm{ren}}^{\odot\alpha})$ is a core for $\hat{L}_{\varepsilon,\,\mathrm{ren}}^{\odot\alpha}$, the set $K_{\varepsilon^{-1}\alpha C} \cap D(A_\varepsilon^\alpha)$ is a core for A_ε^α and on this core the operator A_ε^α coincides with $\hat{L}_\varepsilon^*$. Note that, the semigroup $U_\varepsilon^\alpha(t)$ is the rigorous analog of $\hat{T}_\varepsilon^*$ in (3.1).

Let now $\{k_0, k_0^{(\varepsilon)} | \varepsilon > 0\} \subset K_{\alpha C}$. Then, by (3.50),

$$\begin{aligned}
&\big\|U_\varepsilon^\alpha(t) R_{\varepsilon^{-1}} k_0^{(\varepsilon)} - R_{\varepsilon^{-1}} \hat{T}_V^{\odot\alpha}(t) k_0\big\|_{\mathcal{K}_{\varepsilon^{-1}C}} \\
=&\big\|R_\varepsilon\big(U_\varepsilon^\alpha(t) R_{\varepsilon^{-1}} k_0^{(\varepsilon)} - R_{\varepsilon^{-1}} \hat{T}_V^{\odot\alpha}(t) k_0\big)\big\|_{\mathcal{K}_C} \\
=&\big\|\hat{T}_{\varepsilon,\,\mathrm{ren}}^{\odot\alpha}(t) k_0^{(\varepsilon)} - \hat{T}_V^{\odot\alpha}(t) k_0\big\|_{\mathcal{K}_C} \le A\varepsilon t \|k_0\|_{\mathcal{K}_{\alpha C}} + \|k_0^{(\varepsilon)} - k_0\|_{\mathcal{K}_C}.
\end{aligned} \tag{3.56}$$

On the other hand,

$$\begin{aligned}
&\big\|U_\varepsilon^a(t) R_{\varepsilon^{-1}} k_0^{(\varepsilon)} - R_{\varepsilon^{-1}} \hat{T}_V^{\odot\alpha}(t) k_0\big\|_{\mathcal{K}_{\varepsilon^{-1}C}} \\
=&\operatorname*{ess\,sup}_{\eta\in\Gamma_0} \left\{ (\varepsilon^{-1}C)^{-|\eta|} \big|R_{\varepsilon^{-1}} \hat{T}_V^{\odot\alpha}(t) k_0(\eta)\big| \left| \frac{U_\varepsilon^a(t) R_{\varepsilon^{-1}} k_0^{(\varepsilon)}(\eta)}{R_{\varepsilon^{-1}} \hat{T}_V^{\odot\alpha}(t) k_0(\eta)} - 1 \right| \right\} \\
=&\operatorname*{ess\,sup}_{\eta\in\Gamma_0} \left\{ C^{-|\eta|} \big|\hat{T}_V^{\odot\alpha}(t) k_0(\eta)\big| \left| \frac{U_\varepsilon^a(t) R_{\varepsilon^{-1}} k_0^{(\varepsilon)}(\eta)}{R_{\varepsilon^{-1}} \hat{T}_V^{\odot\alpha}(t) k_0(\eta)} - 1 \right| \right\}.
\end{aligned} \tag{3.57}$$

In particular, if

$$\lim_{\varepsilon\to 0}\|k_0^{(\varepsilon)}-k_0\|_{\mathcal{K}_C}=0 \tag{3.58}$$

then (3.56), (3.57) imply

$$\lim_{\varepsilon\to 0}\frac{U_\varepsilon^a(t)R_{\varepsilon^{-1}}k_0^{(\varepsilon)}(\eta)}{R_{\varepsilon^{-1}}\hat{T}_V^{\odot\alpha}(t)k_0(\eta)}=1 \quad \text{for } \lambda\text{–a.a. } \eta\in\Gamma_0. \tag{3.59}$$

The equality (3.59) is a rigorous realization of the equivalence (3.1) (with changes $k_0^{(\varepsilon)}$ onto $R_{\varepsilon^{-1}}k_0^{(\varepsilon)}$).

Moreover, let $T>0$ and suppose that there exists a function $c:\Gamma_0\to(0;+\infty)$ such that

$$q(\alpha,T):=\sup_{t\in[0;T]}\operatorname*{ess\,sup}_{\eta\in\Gamma_0}\frac{c(\eta)}{\hat{T}_V^{\odot\alpha}(t)k_0(\eta)}<+\infty. \tag{3.60}$$

Then, using the equality

$$\begin{aligned}
&c(\eta)C^{-|\eta|}\left|\frac{U_\varepsilon^a(t)R_{\varepsilon^{-1}}k_0^{(\varepsilon)}(\eta)}{R_{\varepsilon^{-1}}\hat{T}_V^{\odot\alpha}(t)k(\eta)}-1\right|\\
=&C^{-|\eta|}\left|\hat{T}_V^{\odot\alpha}(t)k_0(\eta)\right|\left|\frac{U_\varepsilon^a(t)R_{\varepsilon^{-1}}k_0^{(\varepsilon)}(\eta)}{R_{\varepsilon^{-1}}\hat{T}_V^{\odot\alpha}(t)k_0(\eta)}-1\right|\frac{c(\eta)}{\left|\hat{T}_V^{\odot\alpha}(t)k_0(\eta)\right|},
\end{aligned}$$

we obtain that for such k_0 and for any $t\in[0;T]$

$$\left\|\frac{U_\varepsilon^a(t)R_{\varepsilon^{-1}}k_0^{(\varepsilon)}(\eta)}{R_{\varepsilon^{-1}}\hat{T}_V^{\odot\alpha}(t)k_0(\eta)}-1\right\|_{C,c}\le q(\alpha,T)A\varepsilon t\|k_0\|_{\mathcal{K}_{\alpha C}}+\|k_0^{(\varepsilon)}-k_0\|_{\mathcal{K}_C}, \tag{3.61}$$

where

$$\|k\|_{C,c}=\operatorname*{ess\,sup}_{\eta\in\Gamma_0}\frac{|k(\eta)|}{C^{|\eta|}c^{-1}(\eta)}.$$

This gives that the equivalence (3.1) may be shown in a proper Banach space which is independent on ε.

Remark 3.16. The condition (3.60) on k_0 is reasonable: for example, for $k_0=e_\lambda(\rho_0)$, since, by the Theorem 3.14, we have $\hat{T}_V^{\odot\alpha}(t)k_0(\eta)=e_\lambda(\rho_t,\eta)$, where ρ_t satisfies (3.52); therefore, (3.60) holds for any $|\rho_0(x)|\le\alpha C$ such that

$$\sup_{t\in[0;T]}\inf_{x\in\mathbb{R}^d}|\rho_t(x)|\ge\rho_{\min}>0$$

if we set $c(\eta)=e_\lambda(\rho_{\min},\eta)=\rho_{\min}^{|\eta|}$. Moreover, we obtain that $|\rho_t(x)|\le\alpha C$. The following example shows which function $k_0^{(\varepsilon)}$ one can choose in this case.

Example 3.17. Let $k_0(\eta) = \rho_0^{|\eta|}$, $\rho_0 \in (0;\alpha C)$. Let us consider the scaled Lebesgue–Poisson exponent $k_0^{(\varepsilon)}(\eta) = e_\lambda\big(\rho_0(1+\varepsilon u(\cdot)),\eta\big)$, where $\sup_{x\in\mathbb{R}^d}|u(x)| = \bar{u} < \infty$, $\varepsilon > 0$. Then for any $\varepsilon < \frac{\alpha C-\rho_0}{\rho_0\bar{u}}$ we have $|k_0^{(\varepsilon)}(\eta)| < (\alpha C)^{|\eta|}$. Moreover,

$$
\begin{aligned}
C^{-|\eta|}\big|k_0^{(\varepsilon)}(\eta) - k_0(\eta)\big| &= \Big(\frac{\rho_0}{C}\Big)^{|\eta|}\big|e_\lambda\big(1+\varepsilon u(\cdot),\eta\big) - 1\big|\\
\leq&\Big(\frac{\rho_0}{C}\Big)^{|\eta|}\varepsilon \sup_{s\in(0;\varepsilon)}\left|\frac{d}{ds}e_\lambda\big(1+su(\cdot),\eta\big)\right|\\
=&\Big(\frac{\rho_0}{C}\Big)^{|\eta|}\varepsilon \sup_{s\in(0;\varepsilon)}\left|\sum_{x\in\eta}u(x)e_\lambda\big(1+su(\cdot),\eta\setminus x\big)\right|\\
\leq&\Big(\frac{\rho_0}{C}\Big)^{|\eta|}\varepsilon\sum_{x\in\eta}\bar{u}e_\lambda\big(1+\varepsilon\bar{u},\eta\setminus x\big)\\
\leq&\Big(\frac{\rho_0}{C}\Big)^{|\eta|}\varepsilon|\eta|\bar{u}\left(1+\frac{\alpha C-\rho_0}{\rho_0\bar{u}}\bar{u}\right)^{|\eta|-1}\\
=&\varepsilon\frac{\rho_0}{\alpha C}|\eta|\alpha^{|\eta|} \leq \varepsilon\frac{\rho_0}{\alpha C}\frac{-1}{e\ln\alpha}.
\end{aligned}
$$

As a result, $\|k_0^{(\varepsilon)} - k_0\|_{\mathcal{K}_C} \to 0$ as $\varepsilon \to 0$.

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