Equilibrium Glauber dynamics of continuous particle systems as a scaling limit of Kawasaki dynamics

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Abstract

A Kawasaki dynamics in continuum is a dynamics of an infinite system of interacting particles in $\mathbb{R}^d$ which randomly hop over the space. In this paper, we deal with an equilibrium Kawasaki dynamics which has a Gibbs measure μ as invariant measure. We study a scaling limit of such a dynamics, where the scaling is of Kac type. Informally, we expect that, in the limit, only jumps of "infinite length" will survive, i.e., we expect to arrive at a Glauber dynamics in continuum (a birth-and-death process in $\mathbb{R}^d$). We prove that, in the low activity-high temperature regime, the generators of the Kawasaki dynamics converge to the generator of a Glauber dynamics. The convergence is on the set of exponential functions, in the $L^2(\mu)$-norm. Furthermore, additionally assuming that the potential of pair interaction is positive, we prove the weak convergence of the finite-dimensional distributions of the processes.

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1 Introduction

A Kawasaki dynamics in continuum is a dynamics of an infinite system of interacting particles in $\mathbb{R}^d$ which randomly hop over the space. In this paper, we deal with an equilibrium Kawasaki dynamics which has a Gibbs measure μ as invariant measure.

About μ we assume that it corresponds to an activity parameter $z > 0$ and a potential of pair interaction ϕ. The generator of the Kawasaki dynamics is given, on an appropriate set of cylinder functions, by

$$(HF)(\gamma) = -\sum_{x \in \gamma} \int_{\mathbb{R}^d} dy \, a(x-y) \exp\left[-\sum_{u \in \gamma \setminus x} \phi(u-y)\right]$$
$$\times (F(\gamma \setminus x \cup y) - F(\gamma)), \quad \gamma \in \Gamma. \tag{1.1}$$

Here, Γ denotes the configuration space over $\mathbb{R}^d$, i.e., the space of all locally finite subsets of $\mathbb{R}^d$, and, for simplicity of notations, we just write x instead of $\{x\}$. About the function $a(\cdot)$ in (1.1) we assume that it is non-negative, integrable and symmetric with respect to the origin. The factor $a(x-y) \exp\left[-\sum_{u \in \gamma \setminus x} \phi(u-y)\right]$ in (1.1) describes the rate with which, given a configuration $\gamma \in \Gamma$, a particle $x \in \gamma$ jumps to y.

Under very mild assumptions on the Gibbs measure μ, it was proved in [10] that there indeed exists a Markov process on Γ with *cadlag* paths whose generator is given by (1.1). We assume that the initial distribution of this dynamics is μ, and perform the following scaling of this dynamics. For each $\varepsilon > 0$, we consider the equilibrium Kawasaki dynamics whose generator is given by formula (1.1) in which $a(\cdot)$ is replace by the function

$$a_\varepsilon(\cdot) := \varepsilon^d a(\varepsilon \cdot). \tag{1.2}$$

We denote this generator by H_ε, and study the limit of the corresponding dynamics as $\varepsilon \to 0$ (Kac-type limit).

Informally, we expect that, in the limit, only jumps of infinite length will survive, i.e., jumps from a point to 'infinity' and from 'infinity' to a point. Thus, we expect to arrive at a Glauber dynamics in continuum, i.e., a birth-and-death process in $\mathbb{R}^d$, cf. [9, 10]. In fact, heuristic calculations show that the limiting Glauber dynamics has the generator

$$(H_0 F)(\gamma) = -\alpha \sum_{x \in \gamma} (F(\gamma \setminus x) - F(\gamma))$$
$$- \alpha \int_{\mathbb{R}^d} z \, dx \, \exp\left[-\sum_{u \in \gamma} \phi(u-x)\right] (F(\gamma \cup x) - F(\gamma)), \tag{1.3}$$

where

$$\alpha = z^{-1} k_\mu^{(1)} \int_{\mathbb{R}^d} a(x) \, dx, \tag{1.4}$$

$k_\mu^{(1)}$ being the first correlation function of the measure μ. Thus, α describes the rate with which a particle $x \in \gamma$ dies, whereas $\alpha z \exp\left[-\sum_{u \in \gamma} \phi(u-x)\right]$ describes the rate with which, given a configuration γ, a new particle is born at $x \in \mathbb{R}^d \setminus \gamma$. The existence

of a Markov process on Γ with *cadlag* paths, whose generator is given by (1.3), was proved in [9] (see also [10]).

The main results of this paper are as follows:

- For any stable potential ϕ in the low activity-high temperature regime, the generators H_ε converge to the generator H_0. The convergence is on the set of exponential functions, in the $L^2(\Gamma, \mu)$-norm.

- For any positive potential ϕ in the low activity-high temperature regime, the finite-dimensional distributions of the Kawasaki dynamics with generator H_ε and initial distribution μ weakly converge to the finite-dimensional distributions of the Glauber dynamics with generator H_0 and initial distribution μ.

To prove the first main result, we essentially use the Ruelle bound on the correlation functions of the measure μ, as well as the integrability of the Ursell (cluster) functions of μ, proved by Brox [3]. To derive from here the convergence of the finite-dimensional distributions of the dynamics, we additionally need that the set of finite sums of exponential functions forms a core for the generator of the limiting dynamics, H_0. For this, we use a result from [9] on a core for H_0, which holds under the assumptions of positivity of the potential ϕ.

We note that the generator of the Kawasaki dynamics is independent of the activity parameter $z > 0$. Hence, at least heuristically, the Kawasaki dynamics has a continuum of symmetrizing Gibbs measures, indexed by the activity $z > 0$. On the other hand, the limiting Glauber dynamics has only one of these measures as the symmetrizing one. Thus, the result of the scaling essentially depends on the initial distribution of the dynamics.

In the case of no interaction between particles, $\phi = 0$, it is also possible to prove the 'Kawasaki to Glauber' convergence for a non-equilibrium dynamics whose initial distribution has Ursell functions decaying at infinity, see [11] for details.

The paper is organized as follows. In Section 2, we recall some known facts about Gibbs measures on the configuration space Γ. In Section 3, we recall a rigorous construction of the equilibrium Kawasaki and Glauber dynamics. Our two main results are proved in Sections 4 and 5, respectively. Finally, in Section 6, we make remarks on the results obtained, and discuss some related open problems.

2 Gibbs measures in the low activity-high temperature regime

The configuration space over $\mathbb{R}^d$, $d \in \mathbb{N}$, is defined by

$$\Gamma := \{\gamma \subset \mathbb{R}^d : |\gamma_\Lambda| < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d\},$$

where $|\cdot|$ denotes the cardinality of a set and $\gamma_\Lambda := \gamma \cap \Lambda$. One can identify any $\gamma \in \Gamma$ with the positive Radon measure $\sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(\mathbb{R}^d)$, where ε_x is the Dirac measure with mass at x, $\sum_{x \in \varnothing} \varepsilon_x :=$ zero measure, and $\mathcal{M}(\mathbb{R}^d)$ stands for the set of all positive Radon measures on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$. The space Γ can be endowed with the relative topology as a subset of the space $\mathcal{M}(\mathbb{R}^d)$ with the vague topology, i.e., the weakest topology on Γ with respect to which all maps

$$\Gamma \ni \gamma \mapsto \langle f, \gamma \rangle := \int_{\mathbb{R}^d} f(x)\gamma(dx) = \sum_{x \in \gamma} f(x), \quad f \in C_0(\mathbb{R}^d),$$

are continuous. Here, $C_0(\mathbb{R}^d)$ is the space of all continuous real-valued functions on $\mathbb{R}^d$ with compact support. We will denote by $\mathcal{B}(\Gamma)$ the Borel σ-algebra on Γ. We note that Γ endowed with the vague topology is a Polish space, see e.g. [14].

A pair potential is a Borel-measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x) \in \mathbb{R}$ for all $x \in \mathbb{R}^d \setminus \{0\}$. For $\gamma \in \Gamma$ and $x \in \mathbb{R}^d \setminus \gamma$, we define a relative energy of interaction between a particle at x and the configuration γ as follows:

$$E(x, \gamma) := \begin{cases} \sum_{y \in \gamma} \phi(x - y), & \text{if } \sum_{y \in \gamma} |\phi(x - y)| < +\infty, \\ +\infty, & \text{otherwise.} \end{cases}$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a (grand canonical) Gibbs measure corresponding to the pair potential ϕ and activity $z > 0$ if it satisfies the Georgii–Nguyen–Zessin identity ([17, Theorem 2], see also [12, Theorem 2.2.4]):

$$\int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) F(\gamma, x) = \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z \, dx \exp\left[-E(x, \gamma)\right] F(\gamma \cup x, x) \qquad (2.1)$$

for any measurable function $F : \Gamma \times \mathbb{R}^d \to [0; +\infty]$. We denote the set of all such measures μ by $\mathcal{G}(z, \phi)$.

Let us formulate conditions on the pair potential ϕ.

(S) (Stability) There exists $B \geq 0$ such that, for any $\gamma \in \Gamma$, $|\gamma| < \infty$,

$$\sum_{\{x,y\} \subset \gamma} \phi(x - y) \geq -B|\gamma|.$$

In particular, condition (S) implies that $\phi(x) \geq -2B$, $x \in \mathbb{R}^d$.

(P) (Positivity) We have
$$\phi(x) \geq 0, \quad x \in \mathbb{R}^d.$$

The condition (P) is stronger than (S). More precisely, if (P) holds, then we can choose $B = 0$ in (S).

(LA-HT) (Low activity-high temperature regime) We have:

$$\int_{\mathbb{R}^d} \left| e^{-\phi(x)} - 1 \right| z \, dx < \left(2e^{1+2B} \right)^{-1},$$

where B is as in (S).

In particular, if (P) holds, then (LA-HT) means:

$$\int_{\mathbb{R}^d} \left| e^{-\phi(x)} - 1 \right| z \, dx < (2e)^{-1}.$$

Let $\mu \in \mathcal{G}(z, \phi)$. Assume that, for any $n \in \mathbb{N}$, there exists a non-negative, measurable symmetric function $k_\mu^{(n)}$ on $\left(\mathbb{R}^d \right)^n$ such that, for any measurable symmetric function $f^{(n)} : \left(\mathbb{R}^d \right)^n \to [0, +\infty]$

$$\int_\Gamma \left\langle f^{(n)}, : \gamma^{\otimes n} : \right\rangle \mu(d\gamma) = \frac{1}{n!} \int_{\left(\mathbb{R}^d \right)^n} f^{(n)}(x_1, \ldots, x_n) k_\mu^{(n)}(x_1, \ldots, x_n) \, dx_1 \cdots dx_n.$$

Here

$$\left\langle f^{(n)}, : \gamma^{\otimes n} : \right\rangle := \sum_{\{x_1, \ldots, x_n\} \subset \gamma} f^{(n)}(x_1, \ldots, x_n).$$

The functions $k_\mu^{(n)}$ are called correlation functions of the measure μ. If there exists a constant $\xi > 0$ such that

$$\forall (x_1, \ldots, x_n) \in \left(\mathbb{R}^d \right)^n : \quad k_\mu^{(n)}(x_1, \ldots, x_n) \leq \xi^n, \tag{2.2}$$

then we say that the correlation functions $k_\mu^{(n)}$ satisfy the Ruelle bound.

Under the conditions (S) and (LA-HT), there exists a Gibbs measure $\mu \in \mathcal{G}(z, \phi)$ which has correlation functions satisfying the Ruelle bound, see e.g. [20]. This measure μ is constructed as a weak limit of finite volume Gibbs measures with empty boundary condition, see [16] for details. We will call this measure the Gibbs measure corresponding to (z, ϕ) and the construction with empty boundary condition.

In what follows, we will always assume that (S) and (LA-HT) are satisfied and the Gibbs measure μ as discussed above is fixed. We note that, if the condition (P) is satisfied, then this measure μ is unique in the set $\mathcal{G}(z, \phi)$, see [20] and [13, Theorem 6.2]. We also note that the relative energy $E(x, \gamma)$ is finite $dx \, \mu(d\gamma)$-a.e. on $\mathbb{R}^d \times \Gamma$.

Via a recursion formula, one can transform the correlation functions $k_\mu^{(n)}$ into the Ursell functions $u_\mu^{(n)}$ and vice versa, see e.g. [20]. Their relation is given by

$$k_\mu(\eta) = \sum u_\mu(\eta_1) \cdots u_\mu(\eta_j), \quad \eta \in \Gamma_0, \ \eta \neq \varnothing, \tag{2.3}$$

where

$$\Gamma_0 := \{ \gamma \in \Gamma : |\gamma| < \infty \},$$

for any $\eta = \{x_1, \ldots, x_n\} \in \Gamma_0$

$$k_\mu(\eta) := k_\mu^{(n)}(x_1, \ldots, x_n), \qquad u_\mu(\eta) := u_\mu^{(n)}(x_1, \ldots, x_n),$$

and the summation in (2.3) is over all partitions of the set η into nonempty mutually disjoint subsets $\eta_1, \ldots, \eta_j \subset \eta$ such that $\eta_1 \cup \cdots \cup \eta_j = \eta$, $j \in \mathbb{N}$. For example,

$$k_\mu^{(1)}(x) = u_\mu^{(1)}(x),$$
$$k_\mu^{(2)}(x_1, x_2) = u_\mu^{(2)}(x_1, x_2) + u_\mu^{(1)}(x_1)u_\mu^{(1)}(x_2).$$

For our fixed Gibbs measure μ, both the correlation functions and the Ursell functions of μ are translation invariant. In particular, the first correlation function $k_\mu^{(1)}(\cdot)$ is a constant, which we denote by $k_\mu^{(1)}$.

Furthermore, for any $n \in \mathbb{N}$,

$$U_\mu^{(n+1)} \in L^1\big(\big(\mathbb{R}^d\big)^n, dx_1 \cdots dx_n\big), \tag{2.4}$$

where

$$U_\mu^{(n+1)}(x_1, \ldots, x_n) := u_\mu^{(n+1)}(x_1, \ldots, x_n, 0), \qquad (x_1, \ldots, x_n) \in \big(\mathbb{R}^d\big)^n, \tag{2.5}$$

see [3, Theorem 4.5].

As a straightforward corollary of the Georgii–Nguyen–Zessin identity (2.1), we get the following equality:

$$\int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx_1) \int_{\mathbb{R}^d} \gamma(dx_2) F(\gamma, x_1, x_2)$$
$$= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z \, dx_1 \int_{\mathbb{R}^d} z \, dx_2 \exp\left[-E(x_1, \gamma) - E(x_2, \gamma) - \phi(x_1 - x_2)\right]$$
$$\times F(\gamma \cup \{x_1, x_2\}, x_1, x_2)$$
$$+ \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z \, dx \exp\left[-E(x, \gamma)\right] F(\gamma \cup x, x, x) \tag{2.6}$$

for any measurable function $F : \Gamma \times \mathbb{R}^d \times \mathbb{R}^d \to [0, +\infty]$.

Let $f : \mathbb{R}^d \to \mathbb{R}$ be such that $e^f - 1 \in L^1\big(\mathbb{R}^d, dx\big)$. Then, using the representation

$$e^{\langle f, \gamma \rangle} = 1 + \sum_{n=1}^{\infty} \left\langle \big(e^f - 1\big)^{\otimes n}, : \gamma^{\otimes n} : \right\rangle,$$

we get

$$\int_\Gamma e^{\langle f, \gamma \rangle} \mu(d\gamma)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \big(e^f - 1\big)^{\otimes n}(x_1, \ldots, x_n) k_\mu^{(n)}(x_1, \ldots, x_n) \, dx_1 \cdots dx_n. \tag{2.7}$$

Hence, by using the Ruelle bound, we conclude that $e^{\langle f, \cdot \rangle} \in L^1(\Gamma, \mu)$. Furthermore, if $e^{2f} - 1 \in L^1(\mathbb{R}^d, dx)$, then $e^{\langle f, \cdot \rangle} \in L^2(\Gamma, \mu)$.

3   Kawasaki and Glauber dynamics

We introduce the set $\mathcal{F}C_{\mathrm{b}}\big(C_0\big(\mathbb{R}^d\big),\Gamma\big)$ of all functions of the form

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle),$$

where $N \in \mathbb{N}$, $\varphi_1,\ldots,\varphi_N \in C_0\big(\mathbb{R}^d\big)$, and $g_F \in C_{\mathrm{b}}\big(\mathbb{R}^d\big)$, where $C_{\mathrm{b}}\big(\mathbb{R}^d\big)$ denotes the set of all continuous bounded functions on $\mathbb{R}^N$.

For each function $F : \Gamma \to \mathbb{R}$, $\gamma \in \Gamma$, and $x,y \in \mathbb{R}^d$, we denote

$$\big(D_x^- F\big)(\gamma) := F(\gamma \setminus x) - F(\gamma),$$
$$\big(D_{xy}^{-+} F\big)(\gamma) := F(\gamma \setminus x \cup y) - F(\gamma).$$

We fix a function $a : \mathbb{R}^d \to [0,+\infty)$ such that $a(-x) = a(x)$, $x \in \mathbb{R}^d$, and $a \in L^1\big(\mathbb{R}^d, dx\big)$. We define bilinear forms

$$\mathcal{E}_\varepsilon(F,G) := \frac{1}{2} \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y)$$
$$\times \exp\left[-E(y, \gamma \setminus x)\right] \big(D_{xy}^{-+} F\big)(\gamma)\big(D_{xy}^{-+} G\big)(\gamma), \qquad \varepsilon > 0,$$
$$\mathcal{E}_0(F,G) := \alpha \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx)\big(D_x^- F\big)(\gamma)\big(D_x^- G\big)(\gamma),$$

where $F,G \in \mathcal{F}C_{\mathrm{b}}\big(C_0\big(\mathbb{R}^d\big),\Gamma\big)$, $a_\varepsilon(\cdot)$ is defined by (1.2), and α is given by (1.4).

The next theorem follows from [9, Proposition 3.1 and Theorem 3.1] and [10, Proposition 4.3].

Theorem 3.1. i) *For each $\varepsilon \geq 0$, the bilinear form $\big(\mathcal{E}_\varepsilon, \mathcal{F}C_{\mathrm{b}}\big(C_0\big(\mathbb{R}^d\big),\Gamma\big)\big)$ is closable on $L^2(\Gamma,\mu)$ and its closure will be denoted by $(\mathcal{E}_\varepsilon, \mathrm{Dom}\,(\mathcal{E}_\varepsilon))$.*

ii) *Denote by $(H_\varepsilon, \mathrm{Dom}(H_\varepsilon))$, $\varepsilon \geq 0$, the generator of $(\mathcal{E}_\varepsilon, \mathrm{Dom}\,(\mathcal{E}_\varepsilon))$. Then*

$$\mathcal{F}C_{\mathrm{b}}\big(C_0\big(\mathbb{R}^d\big),\Gamma\big) \subset \bigcap_{\varepsilon \geq 0} \mathrm{Dom}\,(H_\varepsilon),$$

and for any $F \in \mathcal{F}C_{\mathrm{b}}\big(C_0\big(\mathbb{R}^d\big),\Gamma\big)$

$$(H_\varepsilon F)(\gamma) = - \int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y)$$
$$\times \exp\left[-E(y, \gamma \setminus x)\right] \big(D_{xy}^{-+} F\big)(\gamma), \quad \varepsilon > 0, \qquad (3.1)$$
$$(H_0 F)(\gamma) = -\alpha \int_{\mathbb{R}^d} \gamma(dx)\big(D_x^- F\big)(\gamma) - \alpha \int_{\mathbb{R}^d} z\, dx \exp\left[-E(x,\gamma)\right]\big(D_x^+ F\big)(\gamma). \qquad (3.2)$$

iii) *For each $\varepsilon \geq 0$, there exists a conservative Hunt process*

$$\boldsymbol{M}^\varepsilon = \left(\boldsymbol{\Omega}^\varepsilon,\ \boldsymbol{F}^\varepsilon,\ (\boldsymbol{F}_t^\varepsilon)_{t \geq 0},\ (\boldsymbol{\Theta}_t^\varepsilon)_{t \geq 0},\ (\boldsymbol{X}^\varepsilon(t))_{t \geq 0},\ (\boldsymbol{P}_\gamma^\varepsilon)_{\gamma \in \Gamma}\right)$$

on Γ (*see e.g.* [15, p. 92]) *which is properly associated with* $(\mathcal{E}_\varepsilon, \mathrm{Dom}\,(\mathcal{E}_\varepsilon))$, *i.e., for all* ($\mu$-*versions of*) $F \in L^2(\Gamma, \mu)$ *and all* $t > 0$ *the function*

$$\Gamma \ni \gamma \mapsto (p_t^\varepsilon F)(\gamma) := \int_\Omega F(\boldsymbol{X}^\varepsilon(t)) d\boldsymbol{P}_\gamma^\varepsilon$$

is an $\mathcal{E}_\varepsilon$-*quasi-continuous version of* $\exp\left[-t H_\varepsilon\right] F$. $\boldsymbol{M}^\varepsilon$ *is up to* μ-*equivalence unique* (*cf.* [15, Chap. IV, Sect. 6]). *In particular,* $\boldsymbol{M}^\varepsilon$ *has* μ *as invariant measure.*

Remark 3.1. In Theorem 3.1, $\boldsymbol{M}^\varepsilon$ can be taken canonical, i.e., $\boldsymbol{\Omega}^\varepsilon$ is the set $D([0, +\infty), \Gamma)$ of all *cadlag* functions $\omega : [0, +\infty) \to \Gamma$ (i.e., ω is right continuous on $[0, +\infty)$ and has left limits on $(0, +\infty)$), $\boldsymbol{X}^\varepsilon(t)(\omega) = \omega(t), t \geq 0$, $\omega \in \boldsymbol{\Omega}^\varepsilon$, $(\boldsymbol{F}_t^\varepsilon)_{t \geq 0}$ together with $\boldsymbol{F}^\varepsilon$ is the correponding minimum completed admissible family (cf. [6, Section 4.1]) and $\boldsymbol{\Theta}_t^\varepsilon$, $t \geq 0$, are the corresponding natural time shifts.

4 Convergence of the generators

We will now study the limiting behavior of the generators of the Kawasaki dynamics, H_ε, as $\varepsilon \to 0$. We start with the following

Lemma 4.1. *For any* $\varepsilon \geq 0$ *and any* $\varphi \in C_0\big(\mathbb{R}^d\big)$, *the function* $F(\gamma) := e^{\langle \varphi, \gamma \rangle}$ *belongs to* $\mathrm{Dom}\,(H_\varepsilon)$ *and the action of* H_ε *on* F *is given by formula* (3.1) *for* $\varepsilon > 0$ *and by* (3.2) *for* $\varepsilon = 0$.

Proof. We first note that since $e^{2\varphi} - 1 \in L^1\big(\mathbb{R}^d, dx\big)$, we have $e^{\langle \varphi, \cdot \rangle} \in L^2(\Gamma, \mu)$.

Assume that $\varepsilon > 0$. For each $n \in \mathbb{N}$, we define $g_n \in C_\mathrm{b}(\mathbb{R})$ by

$$g_n(u) = \begin{cases} e^u, & u \leq n, \\ e^n, & u > n. \end{cases} \tag{4.1}$$

Then $g_n(\langle \varphi, \cdot \rangle) \in \mathcal{F}C_\mathrm{b}\big(C_0\big(\mathbb{R}^d\big), \Gamma\big)$. Since

$$g_n(\langle \varphi, \gamma \rangle) \leq e^{\langle \varphi, \gamma \rangle}, \quad \gamma \in \Gamma,$$

by the majorized convergence theorem, we have

$$g_n(\langle \varphi, \cdot \rangle) \to e^{\langle \varphi, \cdot \rangle} \text{ in } L^2(\Gamma, \mu) \text{ as } n \to \infty.$$

Next, by (2.6), (2.7), and the Ruelle bound,

$$
\int_{\Gamma} \mu(d\gamma) \left(- \int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y)\ \exp\left[-E(y, \gamma \setminus x) + \langle \varphi, \gamma \rangle \right] \left(e^{-\varphi(x) + \varphi(y)} - 1 \right) \right)^2
$$

$$
= \int_{\mathbb{R}^d} z\, dx_1 \int_{\mathbb{R}^d} dy_1 \int_{\mathbb{R}^d} z\, dx_2 \int_{\mathbb{R}^d} dy_2\, a_\varepsilon(x_1 - y_1) a_\varepsilon(x_2 - y_2)
$$

$$
\times \left(e^{-\varphi(x_1)} \left(e^{\varphi(y_1)} - 1 \right) + \left(e^{-\varphi(x_1)} - 1 \right) \right) \left(e^{-\varphi(x_2)} \left(e^{\varphi(y_2)} - 1 \right) + \left(e^{-\varphi(x_2)} - 1 \right) \right)
$$

$$
\times \exp\left[-\phi(x_1 - x_2) - \phi(y_1 - x_2) - \phi(y_2 - x_1) \right]
$$

$$
\times \int_{\Gamma} \mu(d\gamma) \exp\left[-\sum_{u \in \gamma} \left(\phi(u - x_1) + \phi(u - x_2) + \phi(u - y_1) + \phi(u - y_2) \right) \right]
$$

$$
+ \int_{\mathbb{R}^d} z\, dx \int_{\mathbb{R}^d} dy_1 \int_{\mathbb{R}^d} dy_2\, a_\varepsilon(x - y_1) a_\varepsilon(x - y_2)
$$

$$
\times \left(e^{-\varphi(x)} \left(e^{\varphi(y_1)} - 1 \right) + \left(e^{-\varphi(x)} - 1 \right) \right) \left(e^{-\varphi(x)} \left(e^{\varphi(y_2)} - 1 \right) + \left(e^{-\varphi(x)} - 1 \right) \right)
$$

$$
\times \int_{\Gamma} \mu(d\gamma) \exp\left[-\sum_{u \in \gamma} \left(\phi(u - x) + \phi(u - y_1) + \phi(u - y_2) \right) \right]
$$

$$
< \infty.
$$

Here, we used the following estimate: for any $x, y_1, y_2 \in \mathbb{R}^d$

$$
\int_{\mathbb{R}^d} \left| e^{-\phi(u-x) - \phi(u-y_1) - \phi(u-y_2)} - 1 \right| du
$$

$$
= \int_{\mathbb{R}^d} \left| e^{-\phi(u-x) - \phi(u-y_1)} \left(e^{-\phi(u-y_2)} - 1 \right) \right.
$$

$$
\left. + e^{-\phi(u-x)} \left(e^{-\phi(u-y_1)} - 1 \right) + \left(e^{-\phi(u-x)} - 1 \right) \right| du
$$

$$
\leq \left(e^{4B} + e^{2B} + 1 \right) \int_{\mathbb{R}^d} \left| e^{-\phi(u)} - 1 \right| du < \infty, \tag{4.2}
$$

where B is as in (S), and an analogous estimate for the function

$$
e^{-\phi(u-x_1) - \phi(u-x_2) - \phi(u-y_1) - \phi(u-y_2)} - 1.
$$

By using (4.1), we get, for any $\gamma \in \Gamma$, $x \in \gamma$, and $y \in \mathbb{R}^d$:

$$
|g_n(\langle \varphi, \gamma \setminus x \cup y \rangle) - g_n(\langle \varphi, \gamma \rangle)|
$$

$$
= |g_n(\langle \varphi, \gamma \rangle - \varphi(x) + \varphi(y)) - g_n(\langle \varphi, \gamma \rangle)|
$$

$$
\leq \exp\left[\max\left\{ \langle \varphi, \gamma \rangle - \varphi(x) + \varphi(y), \langle \varphi, \gamma \rangle \right\} \right] (-\varphi(x) + \varphi(y))
$$

$$
\leq \exp\left[\langle |\varphi|, \gamma \rangle + |\varphi(x)| + |\varphi(y)| \right] (|\varphi(x)| + |\varphi(y)|),
$$

and, hence, for any $\gamma \in \Gamma$,

$$\left| -\int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y) \exp\left[-E(y, \gamma \setminus x)\right] D_{xy}^{-+} g_n(\langle \varphi, \gamma \rangle) \right|$$
$$\leq \int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y) \exp\left[-E(y, \gamma \setminus x) + \langle |\varphi|, \gamma \rangle + |\varphi(x)| + |\varphi(y)|\right]$$
$$\times (|\varphi(x)| + |\varphi(y)|). \tag{4.3}$$

Analogously to (4.2), we conclude that the right hand side of (4.3), as a function of $\gamma \in \Gamma$, belongs to $L^2(\Gamma, d\mu)$. Therefore, by the majorized convergence theorem,

$$-\int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y) \exp\left[-E(y, \gamma \setminus x)\right] D_{xy}^{-+} g_n(\langle \varphi, \gamma \rangle)$$
$$\rightarrow -\int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, a_\varepsilon(x-y) \exp\left[-E(y, \gamma \setminus x)\right] D_{xy}^{-+} e^{\langle \varphi, \gamma \rangle}$$

in $L^2(\Gamma, \mu)$ as $n \to \infty$. From here, the statement of the lemma follows in the case $\varepsilon > 0$. The case $\varepsilon = 0$ can be treated analogously. $\square$

We may rewrite $H_\varepsilon = H_\varepsilon^+ + H_\varepsilon^-, \varepsilon \geq 0$, where

$$\left(H_\varepsilon^- F\right)(\gamma) = -\int_{\mathbb{R}^d} \gamma(dx) \left(D_x^- F\right)(\gamma) \int_{\mathbb{R}^d} dy\, e^{-E(y, \gamma \setminus x)} a_\varepsilon(x-y),$$
$$\left(H_\varepsilon^+ F\right)(\gamma) = -\int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} \gamma(dx)\, e^{-E(y, \gamma \setminus x)} a_\varepsilon(x-y)[F(\gamma \setminus x \cup y) - F(\gamma \setminus x)],$$
$$\left(H_0^- F\right)(\gamma) = -\alpha \int_{\mathbb{R}^d} \gamma(dx) \left(D_x^- F\right)(\gamma),$$
$$\left(H_0^+ F\right)(\gamma) = -\alpha \int_{\mathbb{R}^d} dy\, \exp\left[-E(y, \gamma)\right] \left(D_y^+ F\right)(\gamma).$$

Theorem 4.1. *Assume that the pair potential ϕ and activity $z > 0$ satisfy the conditions* (S) *and* (LA-HT). *Let μ be the Gibbs measure from $\mathcal{G}(z, \phi)$ which corresponds to the construction with empty boundary condition. Then, for any $\varphi \in C_0(\mathbb{R}^d)$,*

$$H_\varepsilon^\pm e^{\langle \varphi, \cdot \rangle} \to H_0^\pm e^{\langle \varphi, \cdot \rangle} \text{ in } L^2(\Gamma, \mu) \text{ as } \varepsilon \to 0,$$

so that

$$H_\varepsilon e^{\langle \varphi, \cdot \rangle} \to H_0 e^{\langle \varphi, \cdot \rangle} \text{ in } L^2(\Gamma, \mu) \text{ as } \varepsilon \to 0,$$

Proof. We fix any $\varphi \in C_0(\mathbb{R}^d)$ and denote $F(\gamma) := e^{\langle \varphi, \gamma \rangle}$. We need to prove that

$$\int_\Gamma \left(H_\varepsilon^\pm F\right)^2(\gamma)\, \mu(d\gamma) \to \int_\Gamma \left(H_0^\pm F\right)^2(\gamma)\, \mu(d\gamma) \text{ as } \varepsilon \to 0, \tag{4.4}$$

$$\int_\Gamma \left(H_\varepsilon^\pm F\right)(\gamma) \left(H_0^\pm F\right)(\gamma)\, \mu(d\gamma) \to \int_\Gamma \left(H_0^\pm F\right)^2(\gamma)\, \mu(d\gamma) \text{ as } \varepsilon \to 0. \tag{4.5}$$

10

Using (2.1) and (2.6), we get

$$\int_\Gamma \left(H_0^- F\right)^2(\gamma)\,\mu(d\gamma)$$

$$= \alpha^2 \int_{\mathbb{R}^d} z\,dx\left(e^{\varphi(x)}-1\right)^2 \int_\Gamma \mu(d\gamma)\exp\left[-E(x,\gamma)+\langle 2\varphi,\gamma\rangle\right]$$

$$+ \alpha^2 \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} z\,dx'\left(e^{\varphi(x)}-1\right)e^{\varphi(x)}\left(e^{\varphi(x')}-1\right)e^{\varphi(x')}e^{-\phi(x-x')}$$

$$\times \int_\Gamma \mu(d\gamma)\exp\left[-E(x,\gamma)-E(x',\gamma)+\langle 2\varphi,\gamma\rangle\right] \qquad (4.6)$$

and

$$\int_\Gamma \left(H_0^+ F\right)^2(\gamma)\,\mu(d\gamma)$$

$$= \alpha^2 \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} z\,dx'\left(e^{\varphi(x)}-1\right)\left(e^{\varphi(x')}-1\right)$$

$$\times \int_\Gamma \mu(d\gamma)\exp\left[-E(x,\gamma)-E(x',\gamma)+\langle 2\varphi,\gamma\rangle\right]. \qquad (4.7)$$

Using the same arguments and changing the variables, we have

$$\int_\Gamma \left(H_\varepsilon^- F\right)^2(\gamma)\,\mu(d\gamma)$$

$$= \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy'\left(e^{\varphi(x)}-1\right)^2 a_\varepsilon(x-y)a_\varepsilon(x-y')$$

$$\times \int_\Gamma \mu(d\gamma)\,\exp\left[-E(x,\gamma)-E(y,\gamma)-E(y',\gamma)+\langle 2\varphi,\gamma\rangle\right]$$

$$+ \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} z\,dx' \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy'\left(e^{\varphi(x)}-1\right)e^{\varphi(x)}\left(e^{\varphi(x')}-1\right)e^{\varphi(x')}$$

$$\times a_\varepsilon(x-y)a_\varepsilon(x'-y')e^{-\phi(x-x')}e^{-\phi(x-y')}e^{-\phi(y-x')}$$

$$\times \int_\Gamma \mu(d\gamma)\exp\left[-E(x,\gamma)-E(x',\gamma)-E(y,\gamma)-E(y',\gamma)+\langle 2\varphi,\gamma\rangle\right]$$

$$= \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy'\left(e^{\varphi(x)}-1\right)^2 a(y)a(y')$$

$$\times \int_\Gamma \mu(d\gamma)\,\exp\left[-E(x,\gamma)-E\left(\frac{y}{\varepsilon}+x,\gamma\right)-E\left(\frac{y'}{\varepsilon}+x,\gamma\right)+\langle 2\varphi,\gamma\rangle\right]$$

$$+ \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} z\,dx' \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy'\left(e^{\varphi(x)}-1\right)e^{\varphi(x)}\left(e^{\varphi(x')}-1\right)e^{\varphi(x')}$$

$$\times a(y)a(y')e^{-\phi(x-x')}e^{-\phi\left(\frac{y'}{\varepsilon}+x'-x\right)}e^{-\phi\left(\frac{y}{\varepsilon}+x-x'\right)}$$

11

$$\times \int_\Gamma \mu(d\gamma) \exp\Big[-E(x,\gamma) - E(x',\gamma)$$
$$- E\Big(\frac{y}{\varepsilon} + x, \gamma\Big) - E\Big(\frac{y'}{\varepsilon} + x', \gamma\Big) + \langle 2\varphi, \gamma\rangle\Big]. \tag{4.8}$$

Analogously,

$$\int_\Gamma \big(H_\varepsilon^+ F\big)^2(\gamma)\,\mu(d\gamma)$$
$$= \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy' \,\big(e^{\varphi(x)} - 1\big)\Big(e^{\varphi\big(\frac{y'-y}{\varepsilon}+x\big)} - 1\Big)a(y)a(y')$$
$$\times \int_\Gamma \mu(d\gamma)\,\exp\Big[-E\Big(x - \frac{y}{\varepsilon}, \gamma\Big) - E(x,\gamma) - E\Big(\frac{y'-y}{\varepsilon} + x, \gamma\Big) + \langle 2\varphi, \gamma\rangle\Big]$$
$$+ \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} z\,dx' \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy' \,\big(e^{\varphi(y)} - 1\big)e^{\varphi\big(\frac{x}{\varepsilon}+y\big)}\Big(e^{\varphi(y')} - 1\Big)e^{\varphi\big(\frac{x'}{\varepsilon}+y'\big)}$$
$$\times a(x)a(x')e^{-\phi\big(\frac{x-x'}{\varepsilon}+y-y'\big)}e^{-\phi\big(\frac{x}{\varepsilon}+y-y'\big)}e^{-\phi\big(y-\frac{x'}{\varepsilon}-y'\big)}$$
$$\times \int_\Gamma \mu(d\gamma)\,\exp\Big[-E\Big(\frac{x}{\varepsilon} + y, \gamma\Big)$$
$$- E\Big(\frac{x'}{\varepsilon} + y', \gamma\Big) - E(y,\gamma) - E(y',\gamma) + \langle 2\varphi, \gamma\rangle\Big]. \tag{4.9}$$

In the same way,

$$\int_\Gamma \big(H_\varepsilon^- F\big)(\gamma)\big(H_0^- F\big)(\gamma)\,\mu(d\gamma)$$
$$= \alpha \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} dy\,\big(e^{\varphi(x)} - 1\big)^2 a(y)$$
$$\times \int_\Gamma \mu(d\gamma)\,\exp\Big[-E(x,\gamma) - E\Big(\frac{y}{\varepsilon} + x, \gamma\Big) + \langle 2\varphi, \gamma\rangle\Big]$$
$$+ \alpha \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} z\,dx' \int_{\mathbb{R}^d} dy\,\big(e^{\varphi(x)} - 1\big)e^{\varphi(x)}\Big(e^{\varphi(x')} - 1\Big)e^{\varphi(x')}$$
$$\times a(y)e^{-\phi(x-x')}e^{-\phi\big(\frac{y}{\varepsilon}+x-x'\big)}$$
$$\times \int_\Gamma \mu(d\gamma)\,\exp\Big[-E(x,\gamma) - E(x',\gamma) - E\Big(\frac{y}{\varepsilon} + x, \gamma\Big) + \langle 2\varphi, \gamma\rangle\Big], \tag{4.10}$$

and finally,

$$\int_\Gamma \big(H_\varepsilon^+ F\big)(\gamma)\big(H_0^+ F\big)(\gamma)\,\mu(d\gamma)$$
$$= \alpha \int_{\mathbb{R}^d} z\,dx \int_{\mathbb{R}^d} dy \int_{\mathbb{R}^d} dy' \,\big(e^{\varphi(y)} - 1\big)\Big(e^{\varphi(y')} - 1\Big)e^{\varphi\big(\frac{x}{\varepsilon}+y\big)}e^{-\phi\big(\frac{x}{\varepsilon}+y-y'\big)}a(x)$$

$$\times \int_{\Gamma} \mu(d\gamma) \, \exp\left[-E\left(\frac{x}{\varepsilon} + y, \gamma\right) - E(y, \gamma) - E(y', \gamma) + \langle 2\varphi, \gamma \rangle\right]. \quad (4.11)$$

Since the functions $e^{\varphi} - 1, e^{-\phi} - 1$ are bounded and integrable on $\mathbb{R}^d$, from the Ruelle bound and (2.7) we conclude that all the integrals over Γ in the right hand sides of the equalities (4.8)–(4.9) are bounded by constants. Therefore, by the majorized convergence theorem, to find the limit of these expressions as $\varepsilon \to 0$, it suffices to find the limit of the corresponding integrals over Γ for fixed variables x, x', y, y'. Therefore, due to (4.6) and (4.7), formulas (4.4), (4.5) will immediately follow from the following

Lemma 4.2. *Let a function $\psi : \mathbb{R}^d \to \mathbb{R}$ be such that $e^{\psi} - 1$ is bounded and integrable. Suppose that $x, y, x', y' \in \mathbb{R}^d$ and $x \neq y$. Then*

$$\int_{\Gamma} \exp\left[-E\left(\frac{x}{\varepsilon} + x', \gamma\right) - E\left(\frac{y}{\varepsilon} + y', \gamma\right) + \langle \psi, \gamma \rangle\right] \mu(d\gamma) \to \frac{\left(k_{\mu}^{(1)}\right)^2}{z^2} \int_{\Gamma} \exp\left[\langle \psi, \gamma \rangle\right] \mu(d\gamma),$$
$$(4.12)$$

$$\int_{\Gamma} \exp\left[-E\left(\frac{x}{\varepsilon} + x', \gamma\right) + \langle \psi, \gamma \rangle\right] \mu(d\gamma) \to \frac{k_{\mu}^{(1)}}{z} \int_{\Gamma} \exp\left[\langle \psi, \gamma \rangle\right] \mu(d\gamma)$$
$$(4.13)$$

as $\varepsilon \to 0$.

Proof. By (2.7),

$$\int_{\Gamma} \exp\left[-E\left(\frac{x}{\varepsilon} + x', \gamma\right) - E\left(\frac{y}{\varepsilon} + y', \gamma\right) + \langle \psi, \gamma \rangle\right] \mu(d\gamma)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \left(\exp\left[-\phi(\cdot - x(\varepsilon)) - \phi(\cdot - y(\varepsilon)) + \psi(\cdot)\right] - 1\right)^{\otimes n} (u_1, \ldots, u_n)$$
$$\times k_{\mu}^{(n)}(u_1, \ldots, u_n) \, du_1 \cdots du_n, \quad (4.14)$$

where $x(\varepsilon) := \frac{x}{\varepsilon} + x'$, $y(\varepsilon) := \frac{y}{\varepsilon} + y'$.

Using the Ruelle bound, (S), and (LA-HT), we conclude that, in order to find the limit of the right hand side of (4.14) as $\varepsilon \to 0$, it suffices to find the limit of each term

$$C_{\varepsilon}^{(n)} := \int_{(\mathbb{R}^d)^n} \left(\exp\left[-\phi(\cdot - x(\varepsilon)) - \phi(\cdot - y(\varepsilon)) + \psi(\cdot)\right] - 1\right)^{\otimes n} (u_1, \ldots, u_n)$$
$$\times k_{\mu}^{(n)}(u_1, \ldots, u_n) \, du_1 \cdots du_n$$
$$= \sum_{n_1 + n_2 + n_3 = n} \binom{n}{n_1 \, n_2 \, n_3} \int_{(\mathbb{R}^d)^n} \left(f_{1,\varepsilon}^{\otimes n_1} \otimes f_{2,\varepsilon}^{\otimes n_2} \otimes f_{3,\varepsilon}^{\otimes n_3}\right)(u_1, \ldots, u_n)$$
$$\times k_{\mu}^{(n)}(u_1, \ldots, u_n) \, du_1 \cdots du_n, \quad (4.15)$$

13

where

$$f_{1,\varepsilon}(u) := (\exp\left[-\psi(u)\right] - 1)\exp\left[-\phi(u - x(\varepsilon)) - \phi(u - y(\varepsilon))\right],$$
$$f_{2,\varepsilon}(u) := (\exp\left[-\phi(u - x(\varepsilon))\right] - 1)\exp\left[-\phi(u - y(\varepsilon))\right],$$
$$f_{3,\varepsilon}(u) := \exp\left[-\phi(u - y(\varepsilon))\right] - 1, \qquad u \in \mathbb{R}.$$

Using (2.3), we see that

$$\int_{\left(\mathbb{R}^d\right)^n} \left(f_{1,\varepsilon}^{\otimes n_1} \otimes f_{2,\varepsilon}^{\otimes n_2} \otimes f_{3,\varepsilon}^{\otimes n_3}\right)(u_1,\ldots,u_n)k_\mu^{(n)}(u_1,\ldots,u_n)\,du_1\cdots du_n$$
$$= \sum \int_{\left(\mathbb{R}^d\right)^n} \left(f_{1,\varepsilon}^{\otimes n_1} \otimes f_{2,\varepsilon}^{\otimes n_2} \otimes f_{3,\varepsilon}^{\otimes n_3}\right)(u_1,\ldots,u_n)u_\mu(\eta_1)\cdots u_\mu(\eta_j)\,du_1\cdots du_n,$$

where the summation is over all partitions of $\eta = \{u_1,\ldots,u_n\}$. We now have to distinguish the three following cases.

Case 1: Each element η_i of the partition is either a subset of $\{u_1,\ldots,u_{n_1}\}$, or a subset of $\{u_{n_1+1},\ldots,u_{n_1+n_2}\}$, or a subset of $\{u_{n_1+n_2+1},\ldots,u_n\}$.

By using the translation invariance of the Ursell functions, we get that the corresponding term is equal to

$$\int_{\left(\mathbb{R}^d\right)^n} \left(f_{1,\varepsilon}^{\otimes n_1} \otimes g_{2,\varepsilon}^{\otimes n_2} \otimes g_{3,\varepsilon}^{\otimes n_3}\right)(u_1,\ldots,u_n)u_\mu(\eta_1)\cdots u_\mu(\eta_j)\,du_1\cdots du_n, \qquad (4.16)$$

where

$$g_{2,\varepsilon}(u) := (\exp\left[-\phi(u)\right] - 1)\exp\left[-\phi\left(u + \frac{x - y}{\varepsilon} + x' - y'\right)\right],$$
$$g_{3,\varepsilon}(u) := \exp\left[-\phi(u)\right] - 1, \qquad u \in \mathbb{R}.$$

Since $e^{-\phi} - 1 \in L^1(\mathbb{R}^d, dx)$, for each $\delta > 0$ there exists $r > 0$ such that

$$\int_{\mathbb{R}^d} I(|x| \geq r, |\phi(x)| \geq \delta)\,dx < \delta, \qquad (4.17)$$

where $I(\cdot)$ denotes the indicator function. Hence, by the majorized convergence theorem, since $x \neq y$, (4.16) converges to

$$\int_{\left(\mathbb{R}^d\right)^n} (\exp\left[\psi(\cdot)\right] - 1)^{\otimes n_1}$$
$$\otimes (\exp\left[-\phi(\cdot)\right] - 1)^{\otimes(n_2+n_3)}(u_1,\ldots,u_n)u_\mu(\eta_1)\ldots u_\mu(\eta_j)\,du_1\cdots du_n.$$

Case 2: There is an element of the partition which has non-empty intersections with both sets $\{u_1,\ldots,u_{n_1}\}$ and $\{u_{n_1+1},\ldots,u_n\}$.

14

By using (2.4), (2.5), (4.17), and the majorized convergence theorem, we conclude that the term converges to zero as $\varepsilon \to 0$.

Case 3: Case 2 is not satisfied, but there is an element η_l of the partition which has non-empty intersections with both sets $\{u_{n_1+1}, \ldots, u_{n_1+n_2}\}$ and $\{u_{n_1+n_2+1}, \ldots, u_n\}$.

Shift all the variables entering η_l by $y(\varepsilon)$. Now, analogously to Case 2, the term converges to zero as $\varepsilon \to 0$.

Thus, again using (2.3), for each $n \in \mathbb{N}$,

$$
\begin{aligned}
C_\varepsilon^{(n)} \to &\sum_{n_1+n_2+n_3=n} \binom{n}{n_1\, n_2\, n_3} \\
&\times \int_{(\mathbb{R}^d)^{n_1}} \left(\exp[\psi(\cdot)] - 1\right)^{\otimes n_1} (u_1, \ldots, u_{n_1}) \\
&\qquad \times k_\mu^{(n_1)}(u_1, \ldots, u_{n_1})\, du_1 \cdots du_{n_1} \\
&\times \int_{(\mathbb{R}^d)^{n_2}} \left(\exp[-\phi(\cdot)] - 1\right)^{\otimes n_2} (u_{n_1+1}, \ldots, u_{n_1+n_2}) \\
&\qquad \times k_\mu^{(n_2)}(u_{n_1+1}, \ldots, u_{n_1+n_2})\, du_{n_1+1} \cdots du_{n_1+n_2} \\
&\times \int_{(\mathbb{R}^d)^{n_3}} \left(\exp[-\phi(\cdot)] - 1\right)^{\otimes n_3} (u_{n_1+n_2+1}, \ldots, u_n) \\
&\qquad \times k_\mu^{(n_3)}(u_{n_1+n_2+1}, \ldots, u_n)\, du_{n_1+n_2+1} \cdots du_n.
\end{aligned}
$$

Therefore, the right hand side of (4.14) converges to

$$
\left(\int_\Gamma \exp\left[-\sum_{u\in\gamma} \phi(u)\right] \mu(d\gamma)\right)^2 \int_\Gamma \mu(d\gamma) \exp[\langle \psi, \gamma\rangle].
$$

Let $f \in C_0(\mathbb{R}^d)$, $f \geq 0$, $f \neq 0$. Then

$$
\begin{aligned}
k_\mu^{(1)} \int_{\mathbb{R}^d} f(x)\, dx &= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) f(x) \\
&= \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} z\, dx \exp\left[-\sum_{u\in\gamma} \phi(u - x)\right] f(x) \\
&= \int_{\mathbb{R}^d} z\, dx\, f(x) \int_\Gamma \mu(d\gamma) \exp\left[-\sum_{u\in\gamma} \phi(u - x)\right] \\
&= \int_{\mathbb{R}^d} z\, dx\, f(x) \int_\Gamma \mu(d\gamma) \exp\left[-\sum_{u\in\gamma} \phi(u)\right].
\end{aligned}
$$

Hence,

$$\int_\Gamma \exp\left[-\sum_{u\in\gamma}\phi(u)\right]\mu(d\gamma) = \frac{k_\mu^{(1)}}{z},$$

which proves (4.12). The proof of (4.13) is analogous. □

5 Convergence of the processes

For each $\varepsilon \geq 0$, we take the canonical version of the process $\boldsymbol{M}^\varepsilon$ from Theorem 3.1, and define a stochastic process $\boldsymbol{Y}^\varepsilon = (\boldsymbol{Y}^\varepsilon(t))_{t\geq 0}$ whose law is the probability measure on $D\left([0,+\infty),\Gamma\right)$ given by

$$\boldsymbol{Q}^\varepsilon := \int_\Gamma \boldsymbol{P}_\gamma^\varepsilon \, \mu(d\gamma).$$

By virtue of Theorem 3.1, the process $\boldsymbol{Y}^\varepsilon$ has μ as invariant measure.

Theorem 5.1. *Let* (P) *and* (LA-HT), *be satisfied. Then the finite-dimensional distributions of the process* $\boldsymbol{Y}^\varepsilon$ *weakly converge to the finite-dimensional distributions of* $\boldsymbol{Y}^0$ *as* $\varepsilon \to 0$.

Proof. Fix any $0 \leq t_1 < t_2 < \cdots < t_n$, $n \in \mathbb{N}$. For $\varepsilon \geq 0$, denote by $\mu_{t_1,\ldots,t_n}^\varepsilon$ the finite-dimensional distribution of the process $\boldsymbol{Y}^\varepsilon$ at times $t_1,\ldots,t_n$, which is a probability measure on Γ^n. Since Γ is a Polish space, by [18, Chapter II, Theorem 3.2], the measure μ is tight on Γ. Since all the marginal distributions of the measure $\mu_{t_1,\ldots,t_n}^\varepsilon$ are μ, we therefore conclude that the set $\{\mu_{t_1,\ldots,t_n}^\varepsilon \mid \varepsilon > 0\}$ is pre-compact in the space $\mathcal{M}(\Gamma^n)$ of the probability measures on Γ^n with respect to the weak topology, see e.g. [18, Chapter II, Section 6]. Hence, by [5, Chapter 3, Theorem 3.17], the statement of the theorem will follow from Theorem 4.1 if we show that the set of all finite linear combinations of the functions of the form $e^{\langle\varphi,\cdot\rangle}$, $\varphi \in C_0(\mathbb{R})$, constitutes a core of $(H_0, \mathrm{Dom}\,(H_0))$.

Under the assumptions of the theorem, it follows from the proof of [9, Theorem 4.1] that the set of all finite sums of the functions of the form $\prod_{i=1}^n \langle f_i, \cdot\rangle$, $f_i \in C_0(\mathbb{R}^d)$, $i = 1,\ldots,n$, and constants forms a core of $(H_0, \mathrm{Dom}\,(H_0))$. Therefore, by the polarization identity (see e.g. [1, Chapter 2, formula (2.7)]) the set of all finite linear combinations of the functions of the form $\langle f, \cdot\rangle^n$, $f \in C_0(\mathbb{R}^d)$, $n = 0, 1, 2, \ldots$, forms a core of $(H_0, \mathrm{Dom}\,(H_0))$. Hence, to prove the theorem, it suffices to show that, for each $n \in \mathbb{N}$ and $f \in C_0(\mathbb{R}^d)$, the function $\langle f, \cdot\rangle^n$ can be approximated by finite linear combinations of functions $e^{\langle\varphi,\cdot\rangle}$, $\varphi \in C_0(\mathbb{R}^d)$, in the graph norm of the operator $(H_0, \mathrm{Dom}\,(H_0))$. This statement will follows from the two following lemmas.

Lemma 5.1. *For any* $\varphi, \psi \in C_0(\mathbb{R}^d)$ *and* $n \in \mathbb{N}$, $\langle\psi,\cdot\rangle^n e^{\langle\varphi,\cdot\rangle} \in \mathrm{Dom}\,(H_0)$.

Proof. The proof of this lemma is absolutely analogous to the proof of Lemma 4.1. □

Lemma 5.2. *For any $\varphi \in C_0(\mathbb{R}^d)$, $t \in \mathbb{R}$, and $n = 0, 1, 2, \ldots$,*

$$\langle \varphi, \cdot \rangle^n \frac{1}{u}\left(e^{(t+u)\langle \varphi, \cdot \rangle} - e^{t\langle \varphi, \cdot \rangle}\right) \to \langle \varphi, \cdot \rangle^{n+1} e^{t\langle \varphi, \cdot \rangle} \quad \text{as } u \to 0, \tag{5.1}$$

where convergence is in the sense of the graph norm of the operator $(H_0, \mathrm{Dom}\,(H_0))$.

Proof. Using the estimate

$$\left| \frac{1}{u}\left(e^{u\langle \varphi, \gamma \rangle} - 1\right) \right| \le e^{\langle |\varphi|, \gamma \rangle} \langle |\varphi|, \gamma \rangle, \quad |u| \le 1, \ \varphi \in C_0(\mathbb{R}^d),$$

and the majorized convergence theorem, we easily get the convergence (5.1) in $L^2(\Gamma, \mu)$.

Next, we have the estimate, for $|u| \le 1$

$$\left| \int_{\mathbb{R}^d} \gamma(dx)\left(D_x^- \langle \varphi, \cdot \rangle^n \frac{1}{u}\left(e^{(t+u)\langle \varphi, \cdot \rangle} - e^{t\langle \varphi, \cdot \rangle}\right)\right)(\gamma) \right|$$

$$= \left| \int_{\mathbb{R}^d} \gamma(dx)\left[\left(\langle \varphi, \gamma \setminus x \rangle^n - \langle \varphi, \gamma \rangle^n\right)\frac{1}{u}\left(e^{(t+u)\langle \varphi, \gamma \setminus x \rangle} - e^{t\langle \varphi, \gamma \setminus x \rangle}\right) \right. \right.$$

$$+ \langle \varphi, \gamma \rangle^n \left(e^{t\langle \varphi, \gamma \setminus x \rangle} - e^{t\langle \varphi, \gamma \rangle}\right)\frac{1}{u}\left(e^{u\langle \varphi, \gamma \setminus x \rangle} - 1\right)$$

$$\left. \left. + \langle \varphi, \gamma \rangle^n e^{t\langle \varphi, \gamma \rangle}\frac{1}{u}\left(e^{u\langle \varphi, \gamma \setminus x \rangle} - e^{u\langle \varphi, \gamma \rangle}\right)\right] \right|$$

$$\le \int_{\mathbb{R}^d} \gamma(dx)\left[\sum_{k=0}^{n-1} \langle |\varphi|, \gamma \rangle^k |\varphi(x)|^{n-k} e^{(|t|+1)\langle |\varphi|, \gamma \rangle} \langle |\varphi|, \gamma \rangle \right.$$

$$+ \langle |\varphi|, \gamma \rangle^n e^{t\langle |\varphi|, \gamma \rangle} t |\varphi(x)| e^{\langle |\varphi|, \gamma \rangle} \langle |\varphi|, \gamma \rangle$$

$$\left. + \langle |\varphi|, \gamma \rangle^n e^{t\langle |\varphi|, \gamma \rangle} e^{\langle |\varphi|, \gamma \rangle} |\varphi(x)| \right].$$

Therefore, by the majorized convergence theorem

$$\int_{\mathbb{R}^d} \gamma(dx)\left(D_x^- \langle \varphi, \cdot \rangle^n \frac{1}{u}\left(e^{(t+u)\langle \varphi, \cdot \rangle} - e^{t\langle \varphi, \cdot \rangle}\right)\right)(\gamma)$$

$$\to \int_{\mathbb{R}^d} \gamma(dx)\left(D_x^- \langle \varphi, \cdot \rangle^{n+1} e^{t\langle \varphi, \cdot \rangle}\right)(\gamma) \quad \text{as } u \to 0$$

in $L^2(\Gamma, \mu)$.

Finally, noticing that $\exp[-E(x, \gamma)] \le 1$ due to (P), we conclude, analogously to the above, that

$$\int_{\mathbb{R}^d} z\, dx \exp[-E(x, \gamma)]\left(D_x^+ \langle \varphi, \cdot \rangle^n \frac{1}{u}\left(e^{(t+u)\langle \varphi, \cdot \rangle} - e^{t\langle \varphi, \cdot \rangle}\right)\right)(\gamma)$$

$$\to \int_{\mathbb{R}^d} z\, dx \exp[-E(x, \gamma)]\left(D_x^+ \langle \varphi, \cdot \rangle^{n+1} e^{t\langle \varphi, \cdot \rangle}\right)(\gamma) \quad \text{as } u \to 0$$

in $L^2(\Gamma, \mu)$. From here, the statement of the lemma follows. $\qquad \square$

Using the induction in $n = 0, 1, 2, \ldots$, we conclude from Lemma 5.2 that any function of the form $\langle f, \cdot \rangle^{n+1} e^{t\langle f, \cdot \rangle}$, $f \in C_0(\mathbb{R}^d)$, may be approximated by finite linear combinations of the functions $e^{\langle \varphi, \cdot \rangle}$, $\varphi \in C_0(\mathbb{R}^d)$, in the graph norm of $(H_0, \mathrm{Dom}\,(H_0))$. Letting $t = 0$, we get the needed statement. $\square$

6 Concluding remarks and open problems

It is known (cf. [19]) that any Gibbs measure of Ruelle type, corresponding to a superstable potential ϕ (see [21]), satisfies the generalized Ruelle bound:

$$k_\mu(x_1, \ldots, x_n) \leq C^n \exp\left[-\sum_{1 \leq i < j \leq n} \phi(x_i - x_j) \right] \tag{6.1}$$

for some $C > 0$ independent of $n \in \mathbb{N}$ (compare with the usual Ruelle bound (2.2)). Using (6.1) and harmonic analysis on the configuration space (cf. [8]), it is possible to derive the convergence of the generators, as in Theorem 4.1, under the following assumption on a decay of correlations of the measure μ: For each $n \in \mathbb{N}$ and for $dx_1 \cdots dx_n$-a.e. $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$,

$$u_\mu^{(n+1)}\left(x_1, \ldots, x_n, \frac{y}{\varepsilon} \right) \to 0,$$
$$u_\mu^{(n+2)}\left(x_1, \ldots, x_n, \frac{y}{\varepsilon}, \frac{y'}{\varepsilon} \right) \to 0 \quad \text{as } \varepsilon \to 0, \tag{6.2}$$

where the convergence is in the $dy\, dy'$-measure on each compact set in $(\mathbb{R}^d)^2$. By (2.4), (2.5), this assumption is satisfied in the low activity-high temperature regime. It is sill an open problem whether other Gibbs measures of Ruelle type satisfy (6.2). We believe that (6.2) indeed holds for any Gibbs measure of Ruelle type which is a pure phase. Note that, if this were so, we would derive the convergence of the processes, as in Theorem 5.1, assuming additionally (P).

Next, let us consider the following generalization of the Kawasaki and Glauber dynamics. For any $s \in [0, 1]$, let us define bilinear forms

$$\mathcal{E}_\varepsilon^s(F, G) := \frac{1}{2} \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) \int_{\mathbb{R}^d} dy\, c_\varepsilon^s(x, y, \gamma \setminus x)\big(D_{xy}^{-+} F\big)(\gamma)\big(D_{xy}^{-+} G\big)(\gamma), \quad \varepsilon > 0,$$

$$\mathcal{E}_0^s(F, G) := \alpha \int_\Gamma \mu(d\gamma) \int_{\mathbb{R}^d} \gamma(dx) c_0^s(x, \gamma \setminus x)\big(D_x^- F\big)(\gamma)\big(D_x^- G\big)(\gamma),$$

where $F, G \in \mathcal{F}C_{\mathrm{b}}\big(C_0(\mathbb{R}^d), \Gamma\big)$ and

$$c_\varepsilon^s(x, y, \gamma) := a_\varepsilon(x - y) \exp\left[sE(x, \gamma) - (1 - s)E(y, \gamma) \right],$$
$$c_0^s(x, \gamma) := \exp\left[sE(x, \gamma) \right].$$

18

In particular, for $s = 0$, $\mathcal{E}_\varepsilon^0 = \mathcal{E}_\varepsilon$ and $\mathcal{E}_0^0 = \mathcal{E}_0$. By [10], each of these bilinear forms leads to an equilibrium Markov processes on Γ, which is a Kawasaki dynamics for $\varepsilon > 0$, and a Glauber dynamics for $\varepsilon = 0$.

Under the same assumptions as in Theorem 4.1, it can proved that, for any $F(\gamma) := e^{\langle \varphi, \gamma \rangle}$ with $\varphi \in C_0(\mathbb{R}^d)$,

$$\mathcal{E}_\varepsilon^s(F, F) \to \mathcal{E}_0^s(F, F) \quad \text{as } \varepsilon \to 0.$$

The idea of proof is the same as before, we only use the second statement of Lemma 4.2. Moreover, we expect that (under an additional assumption if $s \in (1/2, 1]$) an analog of Theorem 4.1 is also true in this case. However, to derive from here the convergence of the corresponding processes, as in Theorem 5.1, we are still missing a theorem on a core for the generator of the closure of the bilinear form $\mathcal{E}_0^s$.

Another version of our results may be applied to the dynamics considered in [2, Section 5]. There, the 'Kawasaki' dynamics corresponding to the following bilinear form was studied:

$$\mathcal{E}_\Lambda^N(F, G) := \frac{1}{2|\Lambda|} \int_{\Gamma_\Lambda^N} \mu_\Lambda^N(d\gamma) \int_\Lambda \gamma(dx) \int_\Lambda dy \left(D_{xy}^{-+} F\right)(\gamma) \left(D_{xy}^{-+} G\right)(\gamma). \tag{6.3}$$

Here, Λ is a measurable bounded domain in $\mathbb{R}^d$, by $|\Lambda|$ we denote the volume of Λ, $\Gamma_\Lambda^N \subset \Gamma$ is the set of all N-point subsets of Λ and μ_Λ^N is the canonical Gibbs measure on Γ_Λ^N corresponding to the potential ϕ (see [2] for detail, and note that we have chosen empty boundary condition).

Now, let $\Lambda \nearrow \mathbb{R}^d$, $N \to \infty$ and $\frac{N}{|\Lambda|} \to \rho = \text{const}$ (the so-called N/V limit). Then, by [7], the measures μ_Λ^N have a limiting point in the weak topology of probability measures on Γ. This limiting point is a (grand canonical) Gibbs measure μ corresponding to the potential ϕ. A heuristic calculation shows that the Kawasaki dynamics corresponding to (6.3) converges to the Glauber dynamics with equilibrium measure μ. In this way, we obtain an N/V-approximation of the Glauber dynamics on Γ.

It was shown in [2] that, under (P), the generator of (6.3) has a spectral gap which is $\geq$

$$1 - \frac{3(N-1)}{|\Lambda|} \int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) \, dx,$$

provided that the above value is positive. Hence, we should expect that the generator of the limiting Glauber dynamics has a spectral gap which is $\geq$

$$1 - 3\rho \int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) \, dx.$$

This can be compared with the lower bound of the spectral gap

$$1 - z \int_{\mathbb{R}^d} (1 - e^{-\phi(x)}) \, dx,$$

obtained in [9].

There is still an open problem whether an equilibrium Glauber dynamics in infinite volume can have a spectral gap if the pair potential ϕ has a negative part. We hope that the finite-volume approximations as discussed above should give an insight into this problem.

It should also be possible to approximate the Glauber dynamics in infinite volume by Glauber dynamics in a finite volume which have a grand canonical Gibbs measure as invariant measure.

There is another interesting open problem in this direction: approximation of the Kawasaki dynamics in infinite volume by finite-volume Kawasaki dynamics. Consider e.g. the bilinear form $\mathcal{E}_1$ as in Section 3. This form can be approximated by the forms

$$
\mathcal{E}_\Lambda(F, F) = \frac{1}{2} \int_{\Gamma_\Lambda} \mu_\Lambda(d\gamma) \int_\Lambda \gamma(dx) \int_\Lambda dy \, a(x - y) \exp\left[-E(y, \gamma \setminus x)\right] \left|\left(D_{xy}^{-+} F\right)(\gamma)\right|^2.
$$

Here, Γ_Λ is the set of all finite subsets of Λ and μ_Λ is the grand canonical Gibbs measure on Γ_Λ corresponding to ϕ and empty boundary condition. The problems are: 1) prove that the generator of $\mathcal{E}_\Lambda$ has a spectral gap, 2) estimate how quickly it shrinks as $\Lambda \nearrow \mathbb{R}^d$. Note that problems of such type have been studied in the lattice case, see e.g. [4].

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