An approximative approach to construction of the Glauber dynamics in continuum*

Dmitri Finkelshtein† Yuri Kondratiev‡ Oleksandr Kutoviy§
Elena Zhizhina¶

January 9, 2011

Abstract

We develop a new approach for the construction of the Glauber dynamics in continuum. Existence of the corresponding strongly continuous contraction semigroup in a proper Banach space is shown. Additionally we present the finite- and infinite-volume approximations of the semigroup by families of bounded linear operators.

Keywords. Continuous systems, non-equilibrium Glauber dynamics, spatial birth-and-death processes, semigroup approximation, stochastic evolution

AMS subject classification. 60K35; 41A65; 82C21; 82C22

1 Introduction

The Glauber type stochastic dynamics in continuum are birth-and-death Markov processes on configuration spaces with the given reversible states which are grand canonical Gibbs measures. The corresponding Markov generators are related with the (non-local) Dirichlet forms for the considered Gibbs measures. The latter fact gives a standard way to construct properly associated stationary Markov processes. These processes preserve the initial Gibbs state in the time evolution; they are called the equilibrium Glauber dynamics, see, e.g., [12], [13], [14], [5]. Note that, in applications, the time evolution of initial state is the subject of the primary interest. In what follows, we will try to understand the

*The financial support of DFG through the SFB 701 (Bielefeld University) and German-Ukrainian Project 436 UKR 113/94 and 436 UKR 113/97, and RFFI grant 08-01-00105a is gratefully acknowledged.

†Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine (fdl@imath.kiev.ua).

‡Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany (kondrat@math.uni-bielefeld.de)

§Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany (kutoviy@math.uni-bielefeld.de).

¶Institute for Information Transmission Problems, Moscow, Russia (ejj@iitp.ru).

considered stochastic dynamics as the evolution of initial distributions for the system. Actually, the Markov process itself gives a general technical equipment to study this problem. Let us stress that the transition from the micro-state evolution corresponding to the given initial configuration to the macro-state dynamics is the well developed concept in the theory of infinite particle systems. This point of view appeared initially in the framework of the Hamiltonian dynamics of classical gases, see, e.g., [2].

The study of the non-equilibrium Glauber dynamics needs construction of the time evolution for a wider class of initial measures. The lack of the general Markov processes techniques for the considered systems makes necessary to develop alternative approaches to study the state evolutions in the Glauber dynamics. The approach realized in [10], [11] is probably the only known at the present time. The description of the time evolutions for measures on configuration spaces in terms of an infinite system of evolutional equations for the corresponding correlation functions was used there. The latter system is a Glauber evolution's analog of the famous BBGKY-hierarchy for the Hamiltonian dynamics.

Here we develop another approach to the Glauber dynamics in continuum. This constructive approach was inspired by working up a new algorithm for detection problems in image processing. Object detection, or detecting a configuration of objects from a digital image, is a crucial step in many applications. In paper [1], a new stochastic algorithm to solve object detection problems has been proposed.

The algorithm is based on a continuous time stochastic evolution of macro-objects in a large (but finite) volume in continuum. It was considered a model of possibly partially overlapping discs. Each disc in the final configuration is associated with a given object in the image. The evolution under consideration is a birth-and-death equilibrium dynamics on the configuration space of discs with a given stationary Gibbs measure. In this scheme, the intensity of birth is a constant, whereas intensities of death depend on the energy function and the present configuration. This choice of rates has been made to optimize the convergence speed. Indeed, the volume of the space for birth is much larger than the number of discs in the configuration. To apply the continuous time dynamics to simulation process we have to construct a discretization of this process in time. The resulting discrete time process is a non-homogeneous Markov chain with transition probabilities depending on the energy function and the discretization step. The main point of our approach is that each step in proposed algorithm concerns the whole configuration, so that it is so-called multiple birth-and-death algorithm.

In this paper, we introduce and study the analogous discretization for infinite volume birth and death dynamics of the Glauber type, and prove the convergence to the continuous time process as the step of discretization tends to zero. Furthermore, we use the discretization to construct a non-equilibrium dynamics.

2 Description of model

2.1 General facts and notations

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$, $d \geq 1$. $\mathcal{B}_{\rm b}(\mathbb{R}^d)$ denotes the system of all bounded sets in $\mathcal{B}(\mathbb{R}^d)$.

We define the space of the n-point configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ as

$$\Gamma_Y^{(n)} := \big\{\eta \subset Y \;\big|\; |\eta| = n\big\}, \quad n \in \mathbb{N},$$

where $|\cdot|$ mean the cardinality of a finite set. We put also $\Gamma_Y^{(0)} := \{\emptyset\}$. As a set, $\Gamma_Y^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{Y^n} = \big\{(x_1, \dots, x_n) \in Y^n \;\big|\; x_k \neq x_l \text{ if } k \neq l\big\}.$$

Hence, one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}(\Gamma_Y^{(n)})$. The space of finite configurations in $Y \in \mathcal{B}(\mathbb{R}^d)$ defined as

$$\Gamma_{0,Y} := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_Y^{(n)}$$

is equipped with the topology of disjoint unions. Therefore, one can introduce the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_{0,Y})$. In the case of $Y = \mathbb{R}^d$ we will omit the index Y in the notation, namely, $\Gamma_0 := \Gamma_{0,\mathbb{R}^d}$, $\Gamma^{(n)} := \Gamma_{\mathbb{R}^d}^{(n)}$.

The restriction of the Lebesgue product measure $(dx)^n$ to $\big(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\big)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. Let $\kappa > 0$ be fixed. The Lebesgue–Poisson measure λ_κ on Γ_0 is defined as

$$\lambda_\kappa := \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} m^{(n)}.$$

For any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ the restriction of λ_κ to $\Gamma_\Lambda := \Gamma_{0,\Lambda}$ will be also denoted by λ_κ. We denote also $\lambda = \lambda_1$.

The configuration space over space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$, namely,

$$\Gamma = \Gamma_{\mathbb{R}^d} := \big\{\gamma \subset \mathbb{R}^d \;\big|\; |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)\big\}. \tag{2.1}$$

The space Γ is equipped with the vague topology, i.e., the minimal topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support. Note that $\sum_{x \in \gamma} f(x)$ is always finite in this case since the summation is taken over only finitely many points of γ which belong to the support of f. Γ with the vague topology is a Polish space (see, e.g., [9] and references therein). The corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ appears as the smallest σ-algebra for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are measurable for any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$. Here and below $\gamma_\Lambda := \gamma \cap \Lambda$.

It can be shown that the space $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is the projective limit of the family of spaces $\big\{(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))\big\}_{\Lambda\in\mathcal{B}_{\rm b}(\mathbb{R}^d)}$. The Poisson measure π_κ on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is given as the projective limit of the family of measures $\{\pi_\kappa^\Lambda\}_{\Lambda\in\mathcal{B}_{\rm b}(\mathbb{R}^d)}$, where $\pi_\kappa^\Lambda := e^{-\kappa m(\Lambda)}\lambda_\kappa$ is the probability measure on $\big(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)\big)$. Here $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$.

A function F on Γ is called *cylinder function* if it may be characterized by the following relation:

$$F(\gamma) = F\restriction_{\Gamma_\Lambda}(\gamma_\Lambda) \tag{2.2}$$

for some $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$. The class of all such function will be denoted by $\mathcal{F}_{\rm cyl}(\Gamma)$.

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. The set of bounded measurable functions with bounded support we denote by $B_{\rm bs}(\Gamma_0)$, i.e., $G \in B_{\rm bs}(\Gamma_0)$ if $G\restriction_{\Gamma_0\setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. Note that any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0, in fact, is a sequence of functions $\big\{G^{(n)}\big\}_{n\in\mathbb{N}_0}$, where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$.

The following mapping between $B_{\rm bs}(\Gamma_0)$ and $\mathcal{F}_{\rm cyl}(\Gamma)$ plays the key role in our further considerations:

$$KG(\gamma) := \sum_{\eta\Subset\gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.3}$$

where $G \in B_{\rm bs}(\Gamma_0)$, see, e.g., [8, 15, 16]. The summation in the latter expression is taken over all finite subconfigurations of γ, which is denoted by the symbol $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.4}$$

We denote the restriction of K onto functions on Γ_0 by K_0.

For any fixed $C > 0$ we consider the following (pre-)norm on the space $B_{\rm bs}(\Gamma_0)$

$$\|G\|_C := \int_{\Gamma_0} |G(\eta)| C^{|\eta|} \lambda(d\eta). \tag{2.5}$$

The completion of $B_{\rm bs}(\Gamma_0)$ w.r.t. this pre-norm is the following Banach space of $\mathcal{B}(\Gamma_0)$-measurable functions

$$\mathcal{L}_C := \big\{G : \Gamma_0 \to \mathbb{R} \;\big|\; \|G\|_C < \infty\big\}. \tag{2.6}$$

Let μ be a probability measure on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ such that $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < \infty$ for any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, $n \in \mathbb{N}$. The class of all such measures we denote by $\mathcal{M}^1_{\rm fm}(\Gamma)$. A measure $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$ is called locally absolutely continuous w.r.t. the Poisson measure π if for any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ the projection of μ onto Γ_Λ is absolutely continuous w.r.t. the projection of π onto Γ_Λ. By [8], in this case, there exists a system of measurable symmetric functions $k_\mu^{(n)} : (\mathbb{R}^d)^n \to [0; +\infty)$, $n \in \mathbb{N}$ such

that for any $G \in B_{\rm bs}(\Gamma_0)$ the following identity holds

$$\int_\Gamma (KG^{(n)})(\gamma)\mu(d\gamma) = \frac{1}{n!}\int_{(\mathbb{R}^d)^n} G^{(n)}(x_1,\dots,x_n)k_\mu^{(n)}(x_1,\dots,x_n)dx_1\dots dx_n. \tag{2.7}$$

Functions $k_\mu^{(n)}$ are called the *correlation functions* in mathematical physics as well as functions $\frac{1}{n!}k_\mu^{(n)}$ are called the *factorial moments* in probability theory.

We recall now without a proof the partial case of the well-known technical lemma which plays a very important role in our calculations (cf., [14]).

Lemma 2.1. *For any measurable function* $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \to \mathbb{R}$

$$\int_{\Gamma_0}\sum_{\xi\subset\eta} H\left(\xi, \eta\setminus\xi, \eta\right)\lambda\left(d\eta\right) = \int_{\Gamma_0}\int_{\Gamma_0} H\left(\xi,\eta,\eta\cup\xi\right)\lambda\left(d\xi\right)\lambda\left(d\eta\right) \tag{2.8}$$

if both sides of the equality make sense.

2.2 Glauber dynamics in continuum

Let $\phi : \mathbb{R}^d \to \mathbb{R}_+ := [0;+\infty)$ be even non-negative function which satisfies integrability condition

$$C_\phi = \int_{\mathbb{R}^d}\left(1 - e^{-\phi(x)}\right)dx < +\infty. \tag{2.9}$$

For any $\gamma \in \Gamma$, $x \in \mathbb{R}^d \setminus \gamma$ we set

$$E^\phi(x,\gamma) := \sum_{y\in\gamma}\phi(x-y) \in [0;\infty]. \tag{2.10}$$

Let us define the (pre-) generator of the Glauber dynamics: for any $F \in \mathcal{F}_{\rm cyl}(\Gamma)$ we set

$$\begin{aligned}(LF)(\gamma) :=& \sum_{x\in\gamma}\big[F(\gamma\setminus x) - F(\gamma)\big] \\ &+ z\int_{\mathbb{R}^d}\big[F(\gamma\cup x) - F(\gamma)\big]\exp\big\{-E^\phi(x,\gamma)\big\}dx, \qquad \gamma\in\Gamma.\end{aligned} \tag{2.11}$$

Here $z > 0$ is the *activity* parameter. Note that, because of (2.2), for $F \in \mathcal{F}_{\rm cyl}(\Gamma)$ there exists $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ such that $F(\gamma\setminus x) = F(\gamma)$ for any $x \in \gamma_{\Lambda^c}$ and $F(\gamma\cup x) = F(\gamma)$ for any $x \in \Lambda^c$; note also that $\exp\big\{-E^\phi(x,\gamma)\big\} \le 1$, therefore, the sum and integral in (2.11) are finite.

Using the techniques considered in [7], it is possible to show that there exists a proper subspace $\mathcal{S} \subset \Gamma$ and an $\mathcal{S}$-valued stochastic process with sample paths in the Skorokhod space $D_{\mathcal{S}}[0;\infty)$ associated to the generator L.

This allows us to define the semigroup associated with L in the space of bounded continuous functions on S. This semigroup determines the solution

to the Kolmogorov equation, which formally (only in the sense of action of operator) has the following form:

$$\frac{dF_t}{dt} = LF_t, \qquad F_t \,\big|_{t=0} = F_0. \tag{2.12}$$

However, to show that L is a generator of a semigroup in other functional spaces on Γ seems to be a difficult problem. This difficulty is hidden in the complex structure of the non-linear infinite dimensional space Γ.

In various applications the evolution of the corresponding correlation functions (or measures) helps already to understand the behavior of the process and gives candidates for invariant states. The evolution of correlation functions of the process is related heuristically to the evolution of states of our infinite particle systems. The latter evolution is formally given as a solution to the dual Kolmogorov equation (Fokker–Planck equation):

$$\frac{d\mu_t}{dt} = L^*\mu_t, \qquad \mu_t \,\big|_{t=0} = \mu_0, \tag{2.13}$$

where L^* is the adjoint operator to L on $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, provided, of course, that it exists.

Following the general scheme proposed in [10], we construct the evolution of functions which corresponds to the *symbol* (K-image) $\hat{L} = K^{-1}LK$ of the operator L in L^1-space on Γ_0 w.r.t. the weighted Lebesgue–Poisson measure, namely, in the space $\mathcal{L}_C$, see (2.6).

The evolution equation for *quasi-observables* (functions on Γ_0) corresponding to the Kolmogorov equation (2.12) has the following form

$$\frac{dG_t}{dt} = \widehat{L}G_t, \qquad G_t \,\big|_{t=0} = G_0. \tag{2.14}$$

Then in a way analogous to that in which the corresponding Fokker–Planck equation (2.13) was determined for (2.12) we get the evolution equation for the correlation functions corresponding to the equation (2.14):

$$\frac{dk_t}{dt} = \widehat{L}^*k_t, \qquad k_t \,\big|_{t=0} = k_0, \tag{2.15}$$

where $\widehat{L}^*$ is the mapping dual to $\widehat{L}$ w.r.t. the pairing

$$\langle\langle G, k\rangle\rangle = \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta). \tag{2.16}$$

The existence of the evolution (2.14) in $\mathcal{L}_C$ gives now the following bounds for the solution of (2.15) (if it exists):

$$|k_t(\eta)| \le \mathrm{const} \cdot C^{|\eta|}, \quad \eta \in \Gamma_0. \tag{2.17}$$

The estimate (2.17) is called the Ruelle bound: in [17], [18], it was shown that there is a class of Gibbs measures $\{\mu\}$ whose correlation functions $\{k_\mu > 0\}$

satisfy (2.17) for const = 1. The bound (2.17) is also called sub-Poissonian since $\{C^n\}_{n\geq 0}$ is the system of the correlation functions for the Poisson measure π_C.

In the present paper we obtain the strong solution to the equation (2.14) in $\mathcal{L}_C$. This allows us to solve the equation (2.15) in a weak sense w.r.t. the pairing (2.16). In the forthcoming paper [4] we will consider the strong solution to (2.15) in a proper Banach space. Moreover, we will show that this solution at any moment of time is a correlation function of some state.

3 Construction and properties of the semigroup

3.1 Description of approximation

Let $G \in B_{\rm bs}(\Gamma_0)$ then $F = KG \in \mathcal{F}_{\rm cyl}(\Gamma)$. By [6, 11], we have the following explicit form for the mapping $\hat{L} := K^{-1}LK$ on $B_{\rm bs}(\Gamma_0)$

$$(\hat{L}G)(\eta) = -|\eta|G(\eta) + z\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} e^{-E^{\phi}(x,\xi)}G(\xi\cup x)e_{\lambda}(e^{-\phi(x-\cdot)}-1,\eta\setminus\xi)dx, \quad (3.1)$$

where, by definition, for any $\mathcal{B}(\mathbb{R}^d)$-measurable function f,

$$e_{\lambda}(f,\eta) := \prod_{x\in\eta} f(x), \quad \eta\in\Gamma_0\setminus\{\emptyset\}, \qquad e_{\lambda}(f,\emptyset) := 1. \quad (3.2)$$

Let us denote for any $\eta\in\Gamma_0$

$$(L_0G)(\eta) := -|\eta|G(\eta); \quad (3.3)$$

$$(L_1G)(\eta) := z\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} e^{-E^{\phi}(x,\xi)}G(\xi\cup x)e_{\lambda}(e^{-\phi(x-\cdot)}-1,\eta\setminus\xi)dx. \quad (3.4)$$

Proposition 3.1. *The expression* (3.1) *defines a linear operator* $\hat{L}$ *in* $\mathcal{L}_C$ *with the dense domain* $\mathcal{L}_{2C}\subset\mathcal{L}_C$.

Proof. For any $G\in\mathcal{L}_{2C}$

$$\|L_0G\|_C = \int_{\Gamma_0}|G(\eta)||\eta|C^{|\eta|}\lambda(d\eta) < \int_{\Gamma_0}|G(\eta)|2^{|\eta|}C^{|\eta|}\lambda(d\eta) < \infty$$

and, by Lemma 2.1,

$$\begin{aligned}\|L_1G\|_C &\leq z\int_{\Gamma_0}\sum_{\xi\subset\eta}\int_{\mathbb{R}^d} e^{-E^{\phi}(x,\xi)}\,|G(\xi\cup x)|\,e_{\lambda}\left(\left|e^{-\phi(x-\cdot)}-1\right|,\eta\setminus\xi\right)dxC^{|\eta|}\lambda\left(d\eta\right)\\ &= z\int_{\Gamma_0}\int_{\Gamma_0}\int_{\mathbb{R}^d} e^{-E^{\phi}(x,\xi)}\,|G(\xi\cup x)|\,e_{\lambda}\left(\left|e^{-\phi(x-\cdot)}-1\right|,\eta\right)dxC^{|\eta|}C^{|\xi|}\lambda\left(d\xi\right)\lambda\left(d\eta\right)\\ &\leq \frac{z}{C}\exp\left\{CC_{\phi}\right\}\int_{\Gamma_0}|G\left(\xi\right)|\,|\xi|\,C^{|\xi|}\lambda\left(d\xi\right) < \frac{z}{C}\exp\left\{CC_{\phi}\right\}\int_{\Gamma_0}|G\left(\xi\right)|\,2^{|\xi|}C^{|\xi|}\lambda\left(d\xi\right)\\ &< \infty.\end{aligned}$$

Embedding $\mathcal{L}_{2C}\subset\mathcal{L}_C$ is dense since $B_{\rm bs}(\Gamma_0)\subset\mathcal{L}_{2C}$. □

Let $\delta \in (0;1)$ be arbitrary and fixed. Consider for any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ the following linear mapping on functions $F \in \mathcal{F}_{\rm cyl}(\Gamma_0) := K_0 B_{\rm bs}(\Gamma_0)$

$$\begin{aligned}\left(P_\delta^\Lambda F\right)(\gamma) = &\sum_{\eta \subset \gamma} \delta^{|\eta|} (1-\delta)^{|\gamma \setminus \eta|} \left(\Xi_\delta^\Lambda(\gamma)\right)^{-1} \\ &\times \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\gamma)} F\left((\gamma \setminus \eta) \cup \omega\right) \lambda(d\omega), \quad \gamma \in \Gamma_0,\end{aligned} \tag{3.5}$$

where

$$\Xi_\delta^\Lambda(\gamma) = \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\gamma)} \lambda(d\omega). \tag{3.6}$$

Clearly, P_δ^Λ is a positive preserving mapping and

$$\left(P_\delta^\Lambda 1\right)(\gamma) = \sum_{\eta \subset \gamma} \delta^{|\eta|} (1-\delta)^{|\gamma \setminus \eta|} = 1, \quad \gamma \in \Gamma_0.$$

Operator (3.5) is constructed as a transition operator of a Markov chain, which is a time discretization of a continuous time process with the generator (2.11) and discretization parameter $\delta \in (0;1)$. Roughly speaking, according to the representation (3.5), the probability of transition $\gamma \to (\gamma \setminus \eta) \cup \omega$ (which describes removing of subconfiguration $\eta \subset \gamma$ and birth of a new subconfiguration $\omega \in \Gamma_\Lambda$) after small time δ is equal to

$$\left(\Xi_\delta^\Lambda(\gamma)\right)^{-1} \delta^{|\eta|} (1-\delta)^{|\gamma \setminus \eta|} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\gamma)}.$$

We may rewrite (3.5) in another manner.

Proposition 3.2. *For any $F \in \mathcal{F}_{\rm cyl}(\Gamma_0)$ the following equality holds*

$$\begin{aligned}\left(P_\delta^\Lambda F\right)(\gamma) = &\sum_{\xi \subset \gamma} (1-\delta)^{|\xi|} \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\gamma)} \\ &\times \left(K_0^{-1} F\right)(\xi \cup \omega) \lambda(d\omega).\end{aligned} \tag{3.7}$$

Proof. Let $G := K_0^{-1} F \in B_{\rm bs}(\Gamma_0)$. Since Ξ_δ^Λ doesn't depend on η, for $\gamma \in \Gamma_0$ we have

$$\begin{aligned}\left(P_\delta^\Lambda F\right)(\gamma) = &\left(\Xi_\delta^\Lambda(\gamma)\right)^{-1} \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\gamma)} \\ &\times \sum_{\eta \subset \gamma} \delta^{|\gamma \setminus \eta|} (1-\delta)^{|\eta|} F(\eta \cup \omega) \lambda(d\omega).\end{aligned} \tag{3.8}$$

To rewrite (3.5), we have used also that any $\eta \subset \gamma$ corresponds to a unique $\gamma \setminus \eta \subset \gamma$. Applying the definition of K_0 to $F = K_0 G$ we obtain

$$\begin{aligned}\sum_{\eta \subset \gamma} \delta^{|\gamma \setminus \eta|} (1-\delta)^{|\eta|} F(\eta \cup \omega) &= \sum_{\eta \subset \gamma} \delta^{|\gamma \setminus \eta|} (1-\delta)^{|\eta|} \sum_{\zeta \subset \eta} \sum_{\beta \subset \omega} G(\zeta \cup \beta) \\ &= \sum_{\zeta \subset \gamma} \sum_{\beta \subset \omega} G(\zeta \cup \beta) \sum_{\eta' \subset \gamma \setminus \zeta} \delta^{\left|\gamma \setminus (\eta' \cup \zeta)\right|} (1-\delta)^{\left|\eta' \cup \zeta\right|},\end{aligned} \tag{3.9}$$

where after changing summation over $\eta \subset \gamma$ and $\zeta \subset \eta$ we have used the fact that for any configuration $\eta \subset \gamma$ which contains fixed $\zeta \subset \gamma$ there exists a unique $\eta' \subset \gamma \setminus \zeta$ such that $\eta = \eta' \cup \zeta$. But by the binomial formula

$$\begin{aligned}\sum_{\eta' \subset \gamma \setminus \zeta} \delta^{|\gamma \setminus (\eta' \cup \zeta)|} (1-\delta)^{|\eta' \cup \zeta|} &= (1-\delta)^{|\zeta|} \sum_{\eta' \subset \gamma \setminus \zeta} \delta^{|(\gamma \setminus \zeta) \setminus \eta'|} (1-\delta)^{|\eta'|} \\ &= (1-\delta)^{|\zeta|} (\delta + 1 - \delta)^{|\gamma \setminus \zeta|} = (1-\delta)^{|\zeta|}.\end{aligned} \tag{3.10}$$

Combining (3.8), (3.9), (3.10), we get

$$\begin{aligned}\left(P_\delta^\Lambda F\right)(\gamma) =& \left(\Xi_\delta^\Lambda(\gamma)\right)^{-1} \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\gamma)} \\ &\times \sum_{\zeta \subset \gamma} \sum_{\beta \subset \omega} G(\zeta \cup \beta) (1-\delta)^{|\zeta|} \lambda(d\omega).\end{aligned}$$

Next, Lemma 2.1 yields

$$\begin{aligned}\left(P_\delta^\Lambda F\right)(\gamma) =& \left(\Xi_\delta^\Lambda(\gamma)\right)^{-1} \int_{\Gamma_\Lambda} \int_{\Gamma_\Lambda} (z\delta)^{|\omega \cup \beta|} \prod_{y \in \omega \cup \beta} e^{-E^\phi(y,\gamma)} \\ &\times \sum_{\zeta \subset \gamma} G(\zeta \cup \beta) (1-\delta)^{|\zeta|} \lambda(d\omega) \lambda(d\beta) \\ =& \int_{\Gamma_\Lambda} (z\delta)^{|\beta|} \prod_{y \in \beta} e^{-E^\phi(y,\gamma)} \sum_{\zeta \subset \gamma} G(\zeta \cup \beta) (1-\delta)^{|\zeta|} \lambda(d\beta),\end{aligned}$$

which proves the statement. □

In the next proposition we describe the image of P_δ^Λ under the K_0-transform.

Proposition 3.3. *Let $\hat{P}_\delta^\Lambda = K_0^{-1} P_\delta^\Lambda K_0$. Then for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the following equality holds*

$$\begin{aligned}\left(\hat{P}_\delta^\Lambda G\right)(\eta) =& \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} G(\xi \cup \omega) \\ &\times \prod_{y \in \xi} e^{-E^\phi(y,\omega)} \prod_{y' \in \eta \setminus \xi} \left(e^{-E^\phi(y',\omega)} - 1\right) \lambda(d\omega), \quad \eta \in \Gamma_0.\end{aligned} \tag{3.11}$$

Proof. By (3.7) and the definition of K_0^{-1}, we have

$$\begin{aligned}\left(\hat{P}_\delta^\Lambda G\right)(\eta) &= \sum_{\zeta \subset \eta} (-1)^{|\eta \setminus \zeta|} \sum_{\xi \subset \zeta} (1-\delta)^{|\xi|} \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\zeta)} G(\xi \cup \omega) \lambda(d\omega) \\ &= \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \sum_{\zeta \subset \eta \setminus \xi} (-1)^{|(\eta \setminus \xi) \setminus \zeta|} \int_{\Gamma_\Lambda} (z\delta)^{|\omega|} \prod_{y \in \omega} e^{-E^\phi(y,\zeta \cup \xi)} G(\xi \cup \omega) \lambda(d\omega).\end{aligned}$$

Using the definition (2.10) of the relative energy we obtain

$$\prod_{y\in\omega} e^{-E^{\phi}(y,\zeta\cup\xi)} = \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \prod_{y'\in\zeta} e^{-E^{\phi}\left(y',\omega\right)}.$$

The well-known equality

$$\begin{aligned}\sum_{\zeta\subset\eta\setminus\xi} (-1)^{|(\eta\setminus\xi)\setminus\zeta|} \prod_{y'\in\zeta} e^{-E^{\phi}\left(y',\omega\right)} &= \Big(K_0^{-1} \prod_{y'\in\cdot} e^{-E^{\phi}\left(y',\omega\right)}\Big)(\eta\setminus\xi)\\ &= \prod_{y'\in\eta\setminus\xi} \left(e^{-E^{\phi}\left(y',\omega\right)} - 1\right)\end{aligned}$$

(see, e.g., [6]) completes the proof. □

3.2 Construction of the semigroup on $\mathcal{L}_C$

By analogy with (3.11), we consider the following linear mapping on measurable functions on Γ_0

$$\begin{aligned}\left(\hat{P}_{\delta}G\right)(\eta) :=& \sum_{\xi\subset\eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|}\, G\left(\xi\cup\omega\right) \\ &\times \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \prod_{y'\in\eta\setminus\xi} \left(e^{-E^{\phi}\left(y',\omega\right)} - 1\right) \lambda\left(d\omega\right), \quad \eta\in\Gamma_0.\end{aligned} \tag{3.12}$$

Proposition 3.4. *Let*

$$ze^{CC_{\phi}} \leq C. \tag{3.13}$$

Then $\hat{P}_{\delta}$, given by (3.12)*, is a well defined linear operator in $\mathcal{L}_C$, such that*

$$\left\|\hat{P}_{\delta}\right\| \leq 1. \tag{3.14}$$

Proof. Since $\phi \geq 0$ we have

$$\begin{aligned}\left\|\hat{P}_{\delta}G\right\|_C \leq& \int_{\Gamma_0} \sum_{\xi\subset\eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|}\, |G\left(\xi\cup\omega\right)| \\ &\times \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \prod_{y'\in\eta\setminus\xi} \left|e^{-E^{\phi}\left(y',\omega\right)} - 1\right| \lambda\left(d\omega\right) C^{|\eta|} \lambda\left(d\eta\right) \\ =& \int_{\Gamma_0}\int_{\Gamma_0} (1-\delta)^{|\xi|} \int_{\Gamma_0} (z\delta)^{|\omega|}\, |G\left(\xi\cup\omega\right)| \\ &\times \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \prod_{y'\in\eta} \left|e^{-E^{\phi}\left(y',\omega\right)} - 1\right| \lambda\left(d\omega\right) C^{|\eta|} C^{|\xi|} \lambda\left(d\xi\right) \lambda\left(d\eta\right) \\ =& \int_{\Gamma_0}\int_{\Gamma_0} (1-\delta)^{|\xi|}\, (z\delta)^{|\omega|}\, |G\left(\xi\cup\omega\right)| \\ &\times \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \exp\left\{C\int_{\mathbb{R}^d} \left(1 - e^{-E^{\phi}\left(y',\omega\right)}\right) dy'\right\} \lambda\left(d\omega\right) C^{|\xi|} \lambda\left(d\xi\right).\end{aligned}$$

It is easy to see by the induction principle that for $\phi \geq 0$, $\omega \in \Gamma_0$, $y \notin \omega$

$$1 - e^{-E^{\phi}(y,\omega)} = 1 - \prod_{x \in \omega} e^{-\phi(x-y)} \leq \sum_{x \in \omega} \left(1 - e^{-\phi(x-y)}\right). \tag{3.15}$$

Then

$$\begin{aligned}
\left\|\hat{P}_{\delta} G\right\|_C \leq & \int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} (z\delta)^{|\omega|} |G(\xi \cup \omega)| \\
& \times \exp\left\{ C \sum_{x \in \omega} \int_{\mathbb{R}^d} \left(1 - e^{-\phi(x-y)}\right) dy \right\} \lambda(d\omega)\, C^{|\xi|} \lambda(d\xi) \\
= & \int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} (z\delta)^{|\omega|} |G(\xi \cup \omega)|\, e^{CC_{\phi}|\omega|} C^{|\xi|} \lambda(d\omega)\, \lambda(d\xi) \\
= & \int_{\Gamma_0} \left[(1-\delta) C + z\delta e^{CC_{\phi}}\right]^{|\omega|} |G(\omega)|\, \lambda(d\omega) \leq \|G\|_C .
\end{aligned}$$

For the last inequality we have used that (3.13) implies $(1-\delta) C + z\delta e^{CC_{\phi}} \leq C$. Note that, for λ-a.a. $\eta \in \Gamma_0$

$$\left(\hat{P}_{\delta} G\right)(\eta) < \infty, \tag{3.16}$$

and the statement is proved. □

Proposition 3.5. *Let the inequality* (3.13) *be fulfilled and define*

$$\hat{L}_{\delta} := \frac{1}{\delta}(\hat{P}_{\delta} - \mathbb{1}), \quad \delta \in (0;1),$$

where $\mathbb{1}$ *is the identity operator in* $\mathcal{L}_C$. *Then for any* $G \in \mathcal{L}_{2C}$

$$\left\|(\hat{L}_{\delta} - \hat{L}) G\right\|_C \leq 3\delta \|G\|_{2C}. \tag{3.17}$$

Proof. Let us denote

$$\left(\hat{P}_{\delta}^{(0)} G\right)(\eta) = \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} G(\xi)\, 0^{|\eta \setminus \xi|} = (1-\delta)^{|\eta|} G(\eta); \tag{3.18}$$

$$\left(\hat{P}_{\delta}^{(1)} G\right)(\eta) = z\delta \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\mathbb{R}^d} G(\xi \cup x) \tag{3.19}$$

$$\times \prod_{y \in \xi} e^{-\phi(y-x)} \prod_{y \in \eta \setminus \xi} \left(e^{-\phi(y-x)} - 1\right) dx; \tag{3.20}$$

and

$$\hat{P}_{\delta}^{(\geq 2)} = \hat{P}_{\delta} - \left(\hat{P}_{\delta}^{(0)} + \hat{P}_{\delta}^{(1)}\right). \tag{3.21}$$

Clearly

$$\begin{aligned}
\left\|(\hat{L}_{\delta} - \hat{L}) G\right\|_C &= \left\| \frac{1}{\delta} \left(\hat{P}_{\delta} G - G\right) - \hat{L} G \right\|_C \qquad (3.22) \\
&\leq \left\| \frac{1}{\delta} \left(\hat{P}_{\delta}^{(0)} G - G\right) - L_0 G \right\|_C + \left\| \frac{1}{\delta} \hat{P}_{\delta}^{(1)} G - L_1 G \right\|_C + \frac{1}{\delta} \left\| \hat{P}_{\delta}^{(\geq 2)} G \right\|_C .
\end{aligned}$$

Now we estimate each of the terms in (3.22) separately. By (3.3) and (3.18), we have

$$\left\|\frac{1}{\delta}\left(\hat{P}_\delta^{(0)}G-G\right)-L_0G\right\|_C=\int_{\Gamma_0}\left|\frac{(1-\delta)^{|\eta|}-1}{\delta}+|\eta|\right||G(\eta)|\,C^{|\eta|}\lambda(d\eta).$$

But, for any $|\eta|\geq 2$

$$\begin{aligned}\left|\frac{(1-\delta)^{|\eta|}-1}{\delta}+|\eta|\right|&=\left|\sum_{k=2}^{|\eta|}\binom{|\eta|}{k}(-1)^k\delta^{k-1}\right|\\&=\delta\left|\sum_{k=2}^{|\eta|}\binom{|\eta|}{k}(-1)^k\delta^{k-2}\right|\leq\delta\sum_{k=2}^{|\eta|}\binom{|\eta|}{k}<\delta\cdot 2^{|\eta|}.\end{aligned}$$

Therefore,

$$\left\|\frac{1}{\delta}\left(\hat{P}_\delta^{(0)}G-G\right)-L_0G\right\|_C\leq\delta\|G\|_{2C}. \tag{3.23}$$

Next, by (3.4) and (3.20), one can write

$$\begin{aligned}\left\|\frac{1}{\delta}\hat{P}_\delta^{(1)}G-L_1G\right\|_C=&\,z\int_{\Gamma_0}\left|\sum_{\xi\subset\eta}\left((1-\delta)^{|\xi|}-1\right)\int_{\mathbb{R}^d}G(\xi\cup x)\prod_{y\in\xi}e^{-\phi(y-x)}\right.\\&\times\left.\prod_{y\in\eta\setminus\xi}\left(e^{-\phi(y-x)}-1\right)dx\right|C^{|\eta|}\lambda(d\eta)\\\leq&\,z\int_{\Gamma_0}\int_{\Gamma_0}\left(1-(1-\delta)^{|\xi|}\right)\int_{\mathbb{R}^d}|G(\xi\cup x)|\prod_{y\in\xi}e^{-\phi(y-x)}\\&\times\prod_{y\in\eta}\left(1-e^{-\phi(y-x)}\right)dxC^{|\xi|}C^{|\eta|}\lambda(d\xi)\,\lambda(d\eta),\end{aligned}$$

where we have used Lemma 2.1. Note that for any $|\xi|\geq 1$

$$1-(1-\delta)^{|\xi|}=\delta\sum_{k=0}^{|\xi|-1}(1-\delta)^k\leq\delta\,|\xi|$$

Then, by (3.13) and (2.9), one may estimate

$$\begin{aligned}\left\|\frac{1}{\delta}\hat{P}_\delta^{(1)}G-L_1G\right\|_C&\leq z\delta\int_{\Gamma_0}|\xi|\int_{\mathbb{R}^d}|G(\xi\cup x)|\,dxC^{|\xi|}e^{CC_\phi}\lambda(d\xi)\\&\leq z\delta\int_{\Gamma_0}|\xi|\,(|\xi|-1)\,|G(\xi)|\,C^{|\xi|-1}e^{CC_\phi}\lambda(d\xi).\end{aligned} \tag{3.24}$$

Since $n(n-1)\leq 2^n$, $n\geq 1$ and by (3.13), the latter expression can be bounded by

$$\delta\int_{\Gamma_0}|G(\xi)|\,(2C)^{|\xi|}\,\lambda(d\xi).$$

Finally, Lemma 2.1, (3.15) and bound $e^{-E^{\phi}(y,\omega)} \leq 1$, imply (set $\Gamma_0^{(\geq 2)} := \bigsqcup_{n\geq 2} \Gamma^{(n)}$)

$$\begin{aligned}
\left\| \frac{1}{\delta} \hat{P}_{\delta}^{(\geq 2)} G \right\|_C &\leq \frac{1}{\delta} \int_{\Gamma_0} \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0^{(\geq 2)}} (z\delta)^{|\omega|} \left| G\left(\xi \cup \omega\right) \right| \qquad (3.25) \\
&\quad \times \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \prod_{y\in\eta\setminus\xi} \left(1 - e^{-E^{\phi}(y,\omega)}\right) \lambda\left(d\omega\right) C^{|\eta|} \lambda\left(d\eta\right) \\
&\leq \delta \int_{\Gamma_0} \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0^{(\geq 2)}} z^{|\omega|} \left| G\left(\xi \cup \omega\right) \right| \\
&\quad \times \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \prod_{y\in\eta\setminus\xi} \left(1 - e^{-E^{\phi}(y,\omega)}\right) \lambda\left(d\omega\right) C^{|\eta|} \lambda\left(d\eta\right) \\
&\leq \delta \int_{\Gamma_0} \sum_{\xi \subset \eta} (1-\delta)^{|\xi|} \int_{\Gamma_0} z^{|\omega|} \left| G\left(\xi \cup \omega\right) \right| \\
&\quad \times \prod_{y\in\xi} e^{-E^{\phi}(y,\omega)} \prod_{y\in\eta\setminus\xi} \left(1 - e^{-E^{\phi}(y,\omega)}\right) \lambda\left(d\omega\right) C^{|\eta|} \lambda\left(d\eta\right) \\
&\leq \delta \int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} z^{|\omega|} \left| G\left(\xi \cup \omega\right) \right| \\
&\quad \times \int_{\Gamma_0} \prod_{y\in\eta} \left(1 - e^{-E^{\phi}(y,\omega)}\right) C^{|\eta|} \lambda\left(d\eta\right) \lambda\left(d\omega\right) C^{|\xi|} d\lambda\left(\xi\right) \\
&\leq \delta \int_{\Gamma_0} \int_{\Gamma_0} (1-\delta)^{|\xi|} z^{|\omega|} \left| G\left(\xi \cup \omega\right) \right| e^{CC_{\phi}|\omega|} d\lambda\left(\omega\right) C^{|\xi|} d\lambda\left(\xi\right) \\
&\leq \delta \int_{\Gamma_0} \left[\left(1-\delta\right) C + z e^{CC_{\phi}} \right]^{|\omega|} \left| G\left(\omega\right) \right| d\lambda\left(\omega\right) \\
&\leq \delta \int_{\Gamma_0} \left[\left(2-\delta\right) C \right]^{|\omega|} \left| G\left(\omega\right) \right| d\lambda\left(\omega\right) \leq \delta \int_{\Gamma_0} \left| G\left(\omega\right) \right| \left(2C\right)^{|\omega|} d\lambda\left(\omega\right).
\end{aligned}$$

Combining inequalities (3.23)–(3.25) we obtain the assertion of the proposition. □

We will need the following results in the sequel.

Lemma 3.6 ([3, Corollary 3.8])**.** *Let A be a linear operator on a Banach space L with $D\left(A\right)$ dense in L, and let $\left\|\!\left|\cdot\right|\!\right\|$ be a norm on $D\left(A\right)$ with respect to which $D\left(A\right)$ is a Banach space. For $n \in \mathbb{N}$ let T_n be a linear $\|\cdot\|$-contraction on L such that $T_n : D\left(A\right) \to D\left(A\right)$, and define $A_n = n\left(T_n - 1\right)$. Suppose there exist $\omega \geq 0$ and a sequence $\{\varepsilon_n\} \subset (0; +\infty)$ tending to zero such that for $n \in \mathbb{N}$*

$$\|\left(A_n - A\right) f\| \leq \varepsilon_n \left\|\!\left| f \right|\!\right\|, \ f \in D\left(A\right) \qquad (3.26)$$

and

$$\left\|\!\left| T_n \restriction_{D(A)} \right|\!\right\| \leq 1 + \frac{\omega}{n}. \qquad (3.27)$$

Then A is closable and the closure of A generates a strongly continuous contraction semigroup on L.

Lemma 3.7 (cf. [3, Theorem 6.5]). *Let L, L_n, $n \in \mathbb{N}$ be Banach spaces, and $p_n : L \to L_n$ be bounded linear transformation, such that $\sup_n \|p_n\| < \infty$. For any $n \in \mathbb{N}$, let T_n be a linear contraction on L_n, let $\varepsilon_n > 0$ be such that $\lim_{n\to\infty} \varepsilon_n = 0$, and put $A_n = \varepsilon_n^{-1}(T_n - \mathbb{1})$. Let T_t be a strongly continuous contraction semigroup on L with generator A and let D be a core for A. Then the following are equivalent:*

1. *For each $f \in L$, $T_n^{[t/\varepsilon_n]} p_n f \to p_n T_t f$ in L_n for all $t \geq 0$ uniformly on bounded intervals. Here and below $[\,\cdot\,]$ mean the entire part of a real number.*
2. *For each $f \in D$, there exists $f_n \in L_n$ for each $n \in \mathbb{N}$ such that $f_n \to p_n f$ and $A_n f_n \to p_n A f$ in L_n.*

And now we are able to show the existence of the semigroup on $\mathcal{L}_C$.

Theorem 3.8. *Let*

$$z \leq \min\bigl\{Ce^{-CC_\phi}; 2Ce^{-2CC_\phi}\bigr\}. \tag{3.28}$$

Then $\bigl(\hat{L}, \mathcal{L}_{2C}\bigr)$ from Proposition 3.1 is a closable linear operator in $\mathcal{L}_C$ and its closure $\bigl(\hat{L}, D(\hat{L})\bigr)$ generates a strongly continuous contraction semigroup $\hat{T}_t$ on $\mathcal{L}_C$.

Proof. We apply Lemma 3.6 for $L = \mathcal{L}_C$, $\bigl(A, D(A)\bigr) = \bigl(\hat{L}, \mathcal{L}_{2C}\bigr)$, $\|\!|\cdot|\!\| := \|\cdot\|_{2C}$; $T_n = \hat{P}_\delta$ and $A_n = n\,(T_n - 1) = \frac{1}{\delta}(\hat{P}_\delta - \mathbb{1}) = \hat{L}_\delta$, where $\delta = \frac{1}{n}$, $n \geq 2$.

Condition $ze^{CC_\phi} \leq C$, Proposition 3.4, and Proposition 3.5 provide that T_n, $n \geq 2$ are linear $\|\cdot\|_C$-contractions and (3.26) holds with $\varepsilon_n = \frac{3}{n} = 3\delta$. On the other hand, in addition, Proposition 3.4 applied to the constant $2C$ instead of C gives (3.27) for $\omega = 0$ under condition $ze^{2CC_\phi} \leq 2C$. □

Moreover, since we proved the existence of the semigroup $\hat{T}_t$ on $\mathcal{L}_C$ one can apply contractions $\hat{P}_\delta$ defined above by (3.12) to approximate the semigroup $\hat{T}_t$.

Corollary 3.9. *Let* (3.13) *holds. Then for any $G \in \mathcal{L}_C$*

$$\bigl(\hat{P}_{\frac{1}{n}}\bigr)^{[nt]} G \to \hat{T}_t G, \quad n \to \infty$$

for all $t \geq 0$ uniformly on bounded intervals.

Proof. The statement is a direct consequence of Theorem 3.8, convergence (3.17), and Lemma 3.7 (if we set $L_n = L = \mathcal{L}_C$, $p_n = \mathbb{1}$, $n \in \mathbb{N}$). □

3.3 Finite-volume approximation of $\hat{T}_t$

Note that $\hat{P}_\delta$ defined by (3.12) is a formal point-wise limit of $\hat{P}_\delta^\Lambda$ as $\Lambda \uparrow \mathbb{R}^d$. We have shown in (3.16) that this definition is correct. Corollary 3.9 claims additionally that the linear contractions $\hat{P}_\delta$ approximate the semigroup $\hat{T}_t$, when

$\delta \downarrow 0$. One may also show that mappings $\hat{P}_\delta^\Lambda$ have a similar property when $\Lambda \uparrow \mathbb{R}^d$, $\delta \downarrow 0$.

Let us fix a system $\{\Lambda_n\}_{n\geq 2}$, where $\Lambda_n \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, $\Lambda_n \subset \Lambda_{n+1}$, $\bigcup_n \Lambda_n = \mathbb{R}^d$. We set

$$T_n := \hat{P}_{\frac{1}{n}}^{\Lambda_n}.$$

Note that any T_n is a linear mapping on $B_{\rm bs}(\Gamma_0)$. We consider also the system of Banach spaces of measurable functions on Γ_0

$$\mathcal{L}_{C,n} := \left\{ G : \Gamma_{\Lambda_n} \to \mathbb{R} \;\middle|\; \|G\|_{C,n} := \int_{\Gamma_{\Lambda_n}} |G(\eta)| C^{|\eta|} \lambda(d\eta) < \infty \right\}.$$

Let $p_n : \mathcal{L}_C \to \mathcal{L}_{C,n}$ be a cut-off mapping, namely, for any $G \in \mathcal{L}_C$

$$(p_n G)(\eta) = \mathbb{1}_{\Gamma_{\Lambda_n}}(\eta) G(\eta).$$

Then, obviously, $\|p_n G\|_{C,n} \leq \|G\|_C$. Hence, $p_n : \mathcal{L}_C \to \mathcal{L}_{C,n}$ is a linear bounded transformation with $\|p_n\| = 1$.

Proposition 3.10. *Let* (3.13) *hold. Then for any* $G \in \mathcal{L}_C$

$$\left\| \left(T_n\right)^{[nt]} p_n G - p_n \hat{T}_t G \right\|_{C,n} \to 0, \quad n \to \infty$$

for all $t \geq 0$ *uniformly on bounded intervals.*

Proof. The proof of the proposition is completed by showing that all conditions of Lemma 3.7 hold. Using completely the same arguments as in the proof of Proposition 3.4 one gets that each $T_n = \hat{P}_{\frac{1}{n}}^{\Lambda_n}$ is a linear contraction on $\mathcal{L}_{C,n}$, $n \geq 2$ (note that for any $n \geq 2$, (2.9) implies $\int_{\Lambda_n} \left(1 - e^{-\phi(x)}\right) dx \leq C_\phi < \infty$). Next, we set $A_n = n(T_n - \mathbb{1}_n)$ where $\mathbb{1}_n$ is a unit operator on $\mathcal{L}_{C,n}$ and let us expand T_n in three parts analogously to the proof of Proposition 3.5: $T_n = T_n^{(0)} + T_n^{(1)} + T_n^{(\geq 2)}$. As a result, $A_n = n(T_n^{(0)} - \mathbb{1}_n) + nT_n^{(1)} + nT_n^{(\geq 2)}$. For any $G \in \mathcal{L}_{2C}$ we set $G_n = p_n G \in \mathcal{L}_{2C,n} \subset \mathcal{L}_{C,n}$. To finish the proof we have to verify that for any $G \in \mathcal{L}_{2C}$

$$\|A_n G_n - p_n \hat{L} G\|_{C,n} \to 0, \quad n \to \infty. \tag{3.29}$$

For any $G \in \mathcal{L}_{2C}$

$$\begin{aligned} \|A_n G_n - p_n L G\|_{C,n} \leq {} & \|n(T_n^{(0)} - \mathbb{1}_n) G_n - p_n L_0 G\|_{C,n} \\ & + \|nT_n^{(1)} G_n - p_n L_1 G\|_{C,n} + \|nT_n^{(\geq 2)} G_n\|_{C,n}. \end{aligned} \tag{3.30}$$

Note, that $p_n L_0 G = L_0 G_n$. Using the same arguments as in the proof of Proposition 3.5 we obtain

$$\|n(T_n^{(0)} - \mathbb{1}_n) G_n - p_n L_0 G\|_{C,n} + \|nT_n^{(\geq 2)} G_n\|_{C,n} \leq \frac{2}{n} \|G\|_{2C,n} \leq \frac{2}{n} \|G\|_{2C}.$$

Next,

$$
\begin{aligned}
&\|nT_n^{(1)}G_n - p_n L_1 G\|_{C,n} \\
&\leq z \int_{\Gamma_{\Lambda_n}} \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} \left| \left(1-\frac{1}{n}\right)^{|\xi|} \mathbb{1}_{\Lambda_n}(x) - 1 \right| |G(\xi\cup x)| \\
&\quad \times \prod_{y\in\xi} e^{-\phi(y-x)} \prod_{y\in\eta\setminus\xi} \left(1-e^{-\phi(y-x)}\right) dx C^{|\eta|} d\lambda(\eta) \\
&\leq z \int_{\Gamma_{\Lambda_n}} \int_{\Gamma_{\Lambda_n}} \int_{\mathbb{R}^d} \left[1 - \left(1-\frac{1}{n}\right)^{|\xi|} \mathbb{1}_{\Lambda_n}(x)\right] |G(\xi\cup x)| \\
&\quad \times \prod_{y\in\eta} \left(1-e^{-\phi(y-x)}\right) dx C^{|\eta\cup\xi|} d\lambda(\eta)\, d\lambda(\xi) \\
&\leq C \int_{\Gamma_{\Lambda_n}} \int_{\mathbb{R}^d} \left[1 - \left(1-\frac{1}{n}\right)^{|\xi|} \mathbb{1}_{\Lambda_n}(x)\right] |G(\xi\cup x)|\, dx C^{|\xi|} d\lambda(\xi),
\end{aligned}
$$

where we have used (2.9) and (3.13). Using the same estimates as for (3.24) we may continue

$$
\begin{aligned}
&\leq C \int_{\Gamma_{\Lambda_n}} \int_{\Lambda_n} \left[1 - \left(1-\frac{1}{n}\right)^{|\xi|}\right] |G(\xi\cup x)|\, dx C^{|\xi|} d\lambda(\xi) \\
&\quad + C \int_{\Gamma_{\Lambda_n}} \int_{\Lambda_n^c} |G(\xi\cup x)|\, dx C^{|\xi|} d\lambda(\xi) \\
&\leq \frac{1}{n}\|G\|_{2C,n} + C \int_{\Gamma_0} \int_{\Lambda_n^c} |G(\xi\cup x)|\, dx C^{|\xi|} d\lambda(\xi).
\end{aligned}
$$

But by the Lebesgue dominated convergence theorem,

$$
\int_{\Gamma_0} \int_{\Lambda_n^c} |G(\xi\cup x)|\, dx C^{|\xi|} d\lambda(\xi) \to 0, \quad n\to\infty.
$$

Indeed, $\mathbb{1}_{\Lambda_n^c}(x)|G(\xi\cup x)| \to 0$ point-wisely and may be estimated on $\Gamma_0\times\mathbb{R}^d$ by $|G(\xi\cup x)|$ which is integrable:

$$
C\int_{\Gamma_0}\int_{\mathbb{R}^d} |G(\xi\cup x)|\, dx C^{|\xi|} d\lambda(\xi) = \int_{\Gamma_0} |\xi||G(\xi)|C^{|\xi|} d\lambda(\xi) \leq \|G\|_{2C} < \infty.
$$

Therefore, by (3.30), the convergence (3.29) holds for any $G\in\mathcal{L}_{2C}$, which completes the proof. □

References

[1] X. Descombes, R. Minlos, and E. Zhizhina, Object extraction using a stochastic birth-and-death dynamics in continuum, J. Math. Imaging Vision **33**, no. 3, 347–359 (2009).

[2] R. L. Dobrushin, Ya. G. Sinai, and Yu. M. Sukhov, Dynamical Systems of Statistical Mechanics, in: Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics, Encyclopaedia Math. Sci., Vol. II (Springer, 1989).

[3] S. N. Ethier and T. G. Kurtz, Markov Processes: Characterization and Convergence (Wiley & Sons, 2005).

[4] D. L. Finkelshtein, Yu. G. Kondratiev, and O. V. Kutoviy, Correlation functions evolution for the Glauber dynamics in continuum *In preparation.*

[5] D. L. Finkelshtein, Yu. G. Kondratiev, and E. Lytvynov, Equilibrium Glauber dynamics of continuous particle systems as a scaling limit of Kawasaki dynamics, Random Oper. Stoch. Equations **15**, 105–126 (2007).

[6] D. L. Finkelshtein, Yu. G. Kondratiev, and M. J. Oliveira, Markov evolutions and hierarchical equations in the continuum. I. One-component systems, J. Evol. Equ. **9**, no. 2, 197–233 (2009).

[7] N. L. Garcia and T. G. Kurtz, Spatial birth and death processes as solutions of stochastic equations, Alea **1**, 281–303 (2006).

[8] Yu. G. Kondratiev and T. Kuna, Harmonic analysis on configuration space. I. General theory, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **5**, no. 2, 201–233 (2002).

[9] Yu. G. Kondratiev and O. V. Kutoviy, On the metrical properties of the configuration space, Math. Nachr. **279**, no. 7, 774–783 (2006).

[10] Yu. G. Kondratiev, O. V. Kutoviy, and R. A. Minlos, On non-equilibrium stochastic dynamics for interacting particle systems in continuum, J. Funct. Anal. **255**, no. 1, 200–227 (2008).

[11] Yu. G. Kondratiev, O. V. Kutoviy, and E. Zhizhina, Nonequilibrium Glauber-type dynamics in continuum, J. Math. Phys. **47**, no. 11, 17 pp. (2006).

[12] Yu. G. Kondratiev and E. Lytvynov, Glauber dynamics of continuous particle systems, Ann. Inst. H. Poincaré Probab. Statist. **41**, no. 4, 685–702 (2005).

[13] Yu. G. Kondratiev, E. Lytvynov, and M. Röckner, Equilibrium Kawasaki dynamics of continuous particle systems, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **10**, 185–210 (2007).

[14] Yu. G. Kondratiev, R. A. Minlos, and E. Zhizhina, One-particle subspaces of the generator of Glauber dynamics of continuous particle systems, Rev. Math. Phys. **16**, no. 9, 1–42 (2004).

[15] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. I, Arch. Rational Mech. Anal., **59**, 219–239 (1975).

[16] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. II, Arch. Rational Mech. Anal., **59**, 241–256 (1975).

[17] D. Ruelle, Statistical Mechanics: Rigorous Results (New York, Benjamin, 1969).

[18] D. Ruelle, Superstable interactions in classical statistical mechanics, Commun. Math. Phys. **18**, 127–159 (1970).