Establishment and Fecundity in Spatial Ecological Models: Statistical Approach and Kinetic Equations

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Abstract

We consider spatial population dynamics given by Markov birth-and-death process with constant mortality and birth influenced by establishment or fecundity mechanisms. The independent and density dependent dispersion of spreading are studied. On the base of general methods of [14], we construct the state evolution of considered microscopic ecological systems. We analyze mesoscopic limit for stochastic dynamics under consideration. The corresponding Vlasov-type non-linear kinetic equations are derived and studied.

Keywords: Spatial birth-and-death processes, individual based models, establishment, fecundity, Vlasov-type equation

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1 Introduction

Complex systems theory is a quickly growing interdisciplinary area with a very broad spectrum of motivations and applications. One may characterize complex systems by such properties as diversity and individuality of components, localized interactions among components, and the outcomes of interactions used for replication or enhancement of components. In the study of these systems, proper language and techniques are delivered by the interacting particle models which form a rich and powerful direction in modern stochastic and infinite dimensional analysis. Interacting particle systems are widely used as models in condensed matter physics, chemical kinetics, population biology, ecology, sociology, and economics.

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Mathematical realizations of such models may be considered as a dynamics of points in proper state spaces. In some applications the possible locations for the points of system are structured, e.g., if we consider dynamics on graphs, or, in particular, on lattices. Another class of models can be characterized by the free positions of points in continuum, e.g., in Euclidean space $\mathbb{R}^d$. As it was shown originally in statistical physics, many empirical effects, such as phase transition, are impossible in systems with finite number of points. Therefore, systems with infinite points can be considered as mathematical approximation for realistic systems with huge *but finite* number of elements. Among all infinite systems we will study locally finite ones. Namely, the configuration space over space $\mathbb{R}^d$ consists of all locally finite subsets (configurations) of $\mathbb{R}^d$

$$\Gamma = \Gamma\bigl(\mathbb{R}^d\bigr) := \Bigl\{ \gamma \subset \mathbb{R}^d \Bigm| |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d) \Bigr\}. \tag{1.1}$$

Here $\gamma_\Lambda := \gamma \cap \Lambda$, the symbol $|\cdot|$ stands for the cardinality of a set, and $\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ denotes the class of all bounded Borel sets in $\mathbb{R}^d$. Each configuration may be identified with a Radon measure on $\mathbb{R}^d$ by the relation $\gamma(\Lambda) = |\gamma_\Lambda|$. As a result, Γ can be equipped with the vague topology and the corresponding Borel σ-algebra.

Depending on application the points of system may be interpreted as molecules in physics, plants in ecology, animals in biology, infected people in medicine, companies in economics, market agents in finance, and so on. It is supposed that points of a system evolve in time interacting with each other. In the present paper we focus our attention to the dynamics with birth and death mechanisms.

The spatial birth-and-death dynamics describe an evolution of configurations in $\mathbb{R}^d$, in which points of configurations (particles, individuals, elements) randomly appear (born) and disappear (die) in the space. Heuristically, the corresponding Markov generator has the following form:

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x, \gamma \setminus x) D_x^- F(\gamma) + \int_{\mathbb{R}^d} b(x,\gamma) D_x^+ F(\gamma) dx, \tag{1.2}$$

where for $F : \Gamma \to \mathbb{R}$, $x \notin \gamma$

$$D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma), \qquad D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma). \tag{1.3}$$

Here functions d and b describe rates of death and birth correspondingly (for details see, e.g., [14]).

In the present paper we apply the results of [14] to study the question about the existence of the evolution corresponding to (1.2) for a particular choice of the functions d and b. This question can be answered once we will be able to construct a semigroup associated with L in a proper functional space. This semigroup determines the solution to the Kolmogorov equation, which formally (only in the sense of action of operator) has the following form:

$$\frac{dF_t}{dt} = LF_t, \qquad F_t \bigm|_{t=0} = F_0.$$

To show directly that L is a generator of a semigroup in some reasonable functional spaces on Γ seems to be difficult problem. This difficulty is hidden in the complex structure of non-linear infinite dimensional space Γ. However, in various applications the corresponding evolution of states (measures on configuration space) helps already to understand the behavior of the process and makes possible to predict the equilibrium states of our system. In fact, properties of such an evolution itself are very important for application. The evolution of states is heuristically given as a solution to the dual Kolmogorov equation (Fokker–Planck equation):

$$\frac{d\mu_t}{dt} = L^*\mu_t, \qquad \mu_t \big|_{t=0} = \mu_0, \tag{1.4}$$

where L^* is an adjoint operator to L defined on some space of measures on Γ, provided, of course, that it exists.

Technically, we will study solutions of (1.4) in terms of correlations functions, $k_t^{(n)}$, $n \geq 0$ which are symmetric functions on $(\mathbb{R}^d)^n$ and related to a density of distribution for each n points of our system (rigorous definition will be given in Section 2).

Among all birth-and-death processes we will consider only those in which new particles appear from existing ones. These processes correspond to the models of the spatial ecology. In the recent paper [12], we studied Bolker–Dieckmann–Law–Pacala ecological model, which corresponds to the following mechanism of evolution. Each existing individual can give birth to the new one independently of all other individuals of the system. It may also die influenced by the global regulation (mortality) again independently of all other members of the population or it dies because of the interaction with the rest of the population (local regulation). The latter mechanism may be described as a competition (e.g., for resources) between individuals in the population. Heuristically, the corresponding Markov generator has the form (1.2) with

$$d(x,\gamma) = m + \varkappa^- \sum_{y\in\gamma} a^-(x-y), \tag{1.5}$$

$$b(x,\gamma) = \varkappa^+ \sum_{y\in\gamma} a^+(x-y), \tag{1.6}$$

Here a^+, a^- are probability densities, and constants $m, \varkappa^+, \varkappa^- \geq 0$. In population ecology, the constant m is called mortality and the functions a^+, a^- are known as dispersion and competition kernel, respectively.

By [12], if $m = \varkappa^- = 0$ (free growth model) then the first correlation function (density of the system) grows exponentially in time. To suppress this growth we may consider the case $m > \varkappa^- = 0$ (contact model, see also [18,20]). Then for $m \geq \varkappa^+$ we obtain globally bounded density (even decaying in time for $m > \varkappa^+$). Nevertheless, locally the system will show clustering. Namely, $k_t^{(n)} \sim n!$ on a small regions for $t \geq 0$ (see [12] for details). The main result of [12] may be informally stated in the following way: if the mortality m and the competition kernel $\varkappa^- a^-$ are large enough, then the dynamics of correlation functions associated with the pre-generator (1.2) preserves (sub-)Poissonian bound for correlation functions for all times, i.e., $k_t^{(n)} \leq C^n$, $C > 0$, $n \geq 1$.

In the present article we introduce new mechanisms of local regulation in the corresponding system, alternatively to (1.5). Namely, we set $\varkappa^- = 0$ in (1.5) and consider two different modifications of (1.6). The first one includes the influence of the whole system on the reproduction (fertility, fecundity) of each single individual. The second modification of (1.6) contains a mechanism which shows establishment of each individual in the system. The precise descriptions are given in the next section. Such models have been actively studied in modern ecological literature, see e.g. [8] and references therein. Here, for the first time, we present a rigorous mathematical description for these evolutions.

This article is organized in the following way. In Section 2, we describe the model rigorously providing the proper spaces for the corresponding functional evolutions. In Section 3 we apply general results about birth-and-death dynamics on configuration spaces obtained in [14]. Informally, the main results state that if mortality m is big enough and negative influence of establishment or fecundity is dominated by dispersion then the corresponding evolution exist. In Section 4, we study the mesoscopic description of our model in terms of Vlasov scaling.

It should be noted also, that the Vlasov-type scalings for some Markov processes on finite configuration spaces were considered in [2–6]. Note that the corresponding limiting hierarchy was obtained at the heuristic level. In the present paper, we prove a weak convergence to the limiting hierarchy in the case of infinite continuous systems for bounded but non-integrable densities.

It is worth pointing out that the necessity of a big mortality is a result of perturbation theory for linear operators which gives the existence of the corresponding dynamics for the infinite time interval. However, with the help of another technique considered in [7, 10], we are able to show the existence of the dynamics with any mortality but only on finite interval of time. This result will be presented in the forthcoming paper.

2 Description of model

We recall that the configuration space Γ is given by (1.1). It is equipped with the vague topology, i.e., the weakest topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support. The space Γ with the vague topology is a Polish space (see, e.g., [16] and references therein). The corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ will be the smallest σ-algebra for which all mappings $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ are measurable for any $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$, see, e.g., [1]. We set $\mathcal{F}_\mathrm{cyl}(\Gamma)$ for the class of all *cylinder functions* on Γ. Each $F \in \mathcal{F}_\mathrm{cyl}(\Gamma)$ is characterized by the following relation: $F(\gamma) = F(\gamma_\Lambda)$ for some $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$.

Let $0 \le \phi \in L^1(\mathbb{R}^d)$ be given even function such that

$$c_\phi := \int_{\mathbb{R}^d} \Big(1 - e^{-\phi(x)}\Big) dx \in (0; +\infty). \tag{2.1}$$

For any even $0 \le f \in L^1(\mathbb{R}^d)$ we denote

$$\begin{aligned} E^f(\eta) &:= \sum_{x\in\eta} \sum_{y\in\eta\setminus x} f(x-y), \qquad \eta \in \Gamma_0 \\ E^f(x,\gamma) &:= \sum_{y\in\gamma\setminus x} f(x-y), \qquad \gamma \in \Gamma,\, x \in \mathbb{R}^d, \\ \langle f \rangle &:= \int_{\mathbb{R}^d} f(x) dx. \end{aligned}$$

As it was already mentioned in the Introduction we would like to study two classes of the interacting particle systems (IPS), whose mechanisms of evolution are described by the corresponding heuristically given Markov generators:

$$\begin{aligned} (L_{\mathrm{est}}F)(\gamma) &:= m \sum_{x\in\gamma} \big[F(\gamma\setminus x) - F(\gamma)\big] \\ &\quad + \sum_{y\in\gamma} \int_{\mathbb{R}^d} b_0(x,y,\gamma\setminus y) e^{-E^\phi(x,\gamma)} \big[F(\gamma\cup x) - F(\gamma)\big] dx \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} (L_{\mathrm{fec}}F)(\gamma) &:= m \sum_{x\in\gamma} \big[F(\gamma\setminus x) - F(\gamma)\big] \\ &\quad + \sum_{y\in\gamma} e^{-E^\phi(y,\gamma\setminus y)} \int_{\mathbb{R}^d} b_0(x,y,\gamma\setminus y) \big[F(\gamma\cup x) - F(\gamma)\big] dx \end{aligned} \tag{2.3}$$

The first model shows the influence of establishment in the system and the second one presents fecundity. Here and in the sequel the mortality m is always supposed to be strictly positive. One can see that the establishment rate $e^{-E^\phi(x,\gamma)}$ will be smaller if x will be inside or close to the dense region of the configuration γ. In its turn the fecundity rate $e^{-E^\phi(y,\gamma\setminus y)}$ would be also smaller if y is situated in the dense area of γ. The non-negative measurable rate b_0 represents the dispersion of the model. Let $0 \le a^+, b^+ \in L^1(\mathbb{R}^d)$ be given even functions, and $\langle a^+ \rangle = 1$. We consider two types of the dispersion:

- *density independent dispersion*

$$b_0(x,y,\gamma\setminus y) = \varkappa^+ a^+(x-y),$$

- *density dependent dispersion*

$$b_0(x,y,\gamma\setminus y) = a^+(x-y)\Bigg(\varkappa^+ + \sum_{y'\in\gamma\setminus y} b^+(y-y')\Bigg).$$

As it was mentioned above, we will study evolution of our model in terms of its correlation functions. Below we introduce some basic notions needed to describe the corresponding evolution.

The space of n-point configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma^{(n)}(Y) := \Big\{ \eta \subset Y \;\Big|\; |\eta| = n \Big\}, \quad n \in \mathbb{N}.$$

By definition we take $\Gamma^{(0)}(Y) := \{\emptyset\}$. As a set, $\Gamma^{(n)}(Y)$ may be identified with the symmetrization of $\widetilde{Y^n} = \big\{(x_1, \ldots, x_n) \in Y^n \;\big|\; x_k \neq x_l \text{ if } k \neq l\big\}$. Hence one can introduce the corresponding Borel σ-algebra, which we denote by $\mathcal{B}\big(\Gamma^{(n)}(Y)\big)$. The space of finite configurations in an arbitrary $Y \in \mathcal{B}(\mathbb{R}^d)$ is defined by

$$\Gamma_0(Y) := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}(Y).$$

This space is equipped with the topology of the disjoint union. On $\Gamma_0(Y)$ we consider the corresponding Borel σ-algebra denoted by $\mathcal{B}\big(\Gamma_0(Y)\big)$. In the case of $Y = \mathbb{R}^d$ we will omit Y in the notation. Namely, $\Gamma_0 := \Gamma_0(\mathbb{R}^d)$, $\Gamma^{(n)} := \Gamma^{(n)}(\mathbb{R}^d)$.

The restriction of the Lebesgue product measure $(dx)^n$ to $\big(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)})\big)$ we denote by $m^{(n)}$. We set $m^{(0)} := \delta_{\{\emptyset\}}$. The Lebesgue–Poisson measure λ on Γ_0 is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}.$$

For any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the restriction of λ to $\Gamma(\Lambda) := \Gamma_0(\Lambda)$ will be also denoted by λ. The space $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ can be obtained as the projective limit of the family of spaces $\big\{\big(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))\big)\big\}_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$, see, e.g., [1]. The Poisson measure π on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ is given as the projective limit of the family of measures $\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)}$, where $\pi^\Lambda := e^{-m(\Lambda)}\lambda$ is the probability measure on $\big(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda))\big)$ and $m(\Lambda)$ is the Lebesgue measure of $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$; see, e.g., [1].

A set $M \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $M \subset \bigsqcup_{n=0}^{N} \Gamma^{(n)}(\Lambda)$. The set of bounded measurable functions with bounded support we denote by $B_{\mathrm{bs}}(\Gamma_0)$, i.e., $G \in B_{\mathrm{bs}}(\Gamma_0)$ if $G \restriction_{\Gamma_0 \setminus M} = 0$ for some bounded $M \in \mathcal{B}(\Gamma_0)$. Any $\mathcal{B}(\Gamma_0)$-measurable function G on Γ_0, in fact, is defined by a sequence of functions $\big\{G^{(n)}\big\}_{n \in \mathbb{N}_0}$ where $G^{(n)}$ is a $\mathcal{B}(\Gamma^{(n)})$-measurable function on $\Gamma^{(n)}$. As usual, functions on Γ are called *observables* and functions on Γ_0 are called *quasi-observables*.

There exists a mapping from $B_{\mathrm{bs}}(\Gamma_0)$ into $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$, which plays the key role in our further considerations. It has the following form

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.4}$$

where $G \in B_{\mathrm{bs}}(\Gamma_0)$, see, e.g., [15,21,22]. The summation in (2.4) is taken over all finite subconfigurations $\eta \in \Gamma_0$ of the (infinite) configuration $\gamma \in \Gamma$; we denote this by the symbol, $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta}(-1)^{|\eta\setminus\xi|}F(\xi), \quad \eta \in \Gamma_0. \tag{2.5}$$

Note that if function F has special form

$$F(\gamma) = \sum_{x\in\gamma} H(x, \gamma \setminus x),$$

where $H(x, \cdot)$ is defined point-wisely at least on Γ_0, then, by direct computation,

$$\big(K^{-1}F\big)(\eta) = \sum_{x\in\eta}\big(K^{-1}H(x,\cdot)\big)(\eta \setminus x), \quad \eta \in \Gamma_0. \tag{2.6}$$

We set also

$$(K_0G)(\eta) := (KG)(\eta), \qquad \eta \in \Gamma_0.$$

Let us define, for any $\mathcal{B}(\mathbb{R}^d)$-measurable function f, the so-called coherent state

$$e_\lambda(f, \eta) := \prod_{x\in\eta} f(x), \ \eta \in \Gamma_0\setminus\{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1.$$

Then

$$(K_0 e_\lambda(f))(\eta) = e_\lambda(f + 1, \eta), \quad \eta \in \Gamma_0 \tag{2.7}$$

and for any $f \in L^1(\mathbb{R}^d, dx)$

$$\int_{\Gamma_0} e_\lambda(f, \eta) d\lambda(\eta) = \exp\Big\{\int_{\mathbb{R}^d} f(x)dx\Big\}. \tag{2.8}$$

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous with respect to the Poisson measure π if for any $\Lambda \in \mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$ the projection of μ onto $\Gamma(\Lambda)$ is absolutely continuous with respect to the projection of π onto $\Gamma(\Lambda)$. By [15], in this case, there exists a *correlation functional* $k_\mu : \Gamma_0 \to \mathbb{R}_+$ such that for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ the following equality holds

$$\int_\Gamma (KG)(\gamma)d\mu(\gamma) = \int_{\Gamma_0} G(\eta)k_\mu(\eta)d\lambda(\eta).$$

The restrictions $k_\mu^{(n)}$ of this functional on $\Gamma_0^{(n)}$, $n \in \mathbb{N}_0$ are called *correlation functions* of the measure μ. Note that $k_\mu^{(0)} = k_\mu(\emptyset) = 1$.

We recall now without a proof the partial case of the well-known technical lemma (see e.g. [19]) which plays very important role in our calculations.

Lemma 2.1. *For any measurable function $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \to \mathbb{R}$*

$$\int_{\Gamma_0} \sum_{\xi\subset\eta} H(\xi, \eta \setminus \xi, \eta)\, d\lambda(\eta) = \int_{\Gamma_0}\int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi)\, d\lambda(\xi)\, d\lambda(\eta) \tag{2.9}$$

if both sides of the equality make sense.

For arbitrary and fixed $C > 1$ we consider the functional Banach space

$$\mathcal{L}_C := L^1(\Gamma_0, C^{|\eta|}\lambda(d\eta)). \tag{2.10}$$

In the sequel, symbol $\|\cdot\|_C$ stands for the norm of the space (2.10).

Let $d\lambda_C := C^{|\cdot|}d\lambda$, then the dual space

$$(\mathcal{L}_C)' = \big(L^1(\Gamma_0, d\lambda_C)\big)' = L^\infty(\Gamma_0, d\lambda_C).$$

The space $(\mathcal{L}_C)'$ is isometrically isomorphic to the Banach space

$$\mathcal{K}_C := \Big\{k : \Gamma_0 \to \mathbb{R} \;\Big|\; k\, C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda)\Big\}$$

with the norm

$$\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|}k(\cdot)\|_{L^\infty(\Gamma_0,\lambda)}$$

where the isomorphism is provided by the isometry R_C

$$(\mathcal{L}_C)' \ni k \longmapsto R_C k := k\, C^{|\cdot|} \in \mathcal{K}_C.$$

In fact, one may consider the duality between the Banach spaces $\mathcal{L}_C$ and $\mathcal{K}_C$ given by the following expression

$$\langle\!\langle G, k\rangle\!\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda, \quad G \in \mathcal{L}_C,\ k \in \mathcal{K}_C \tag{2.11}$$

with $|\langle\!\langle G, k\rangle\!\rangle| \le \|G\|_C \cdot \|k\|_{\mathcal{K}_C}$. It is clear that $k \in \mathcal{K}_C$ implies

$$|k(\eta)| \le \|k\|_{\mathcal{K}_C}\, C^{|\eta|} \qquad \text{for } \lambda\text{−a.a. } \eta \in \Gamma_0.$$

In the paper [17], it was proposed the analytic approach for the construction of non-equilibrium dynamics on Γ, which uses deeply the harmonic analysis on configuration spaces. By this approach the dynamics of correlation functions corresponding to (1.4) is given by the evolutional equation

$$\frac{dk_t}{dt} = L^\triangle k_t, \qquad k_t \,\big|_{t=0} = k_0, \tag{2.12}$$

where $L^\triangle$ is a dual operator to the K-image of L defined by the expression

$$\hat{L} := K^{-1}LK$$

with respect to the duality (2.11). Hence, $L^\triangle = \hat{L}^*$. In order to construct the evolution of correlation functions we are going to follow such a scheme: we show that $\hat{L}$ is a generator of a C_0-semigroup in the certain Banach space and after consider the dual semigroup which solves the Cauchy problem (2.12).

3 Functional evolutions

Let
$$\mathcal{D} := \big\{ G \in \mathcal{L}_C \,\big|\, |\cdot| G(\cdot) \in \mathcal{L}_C \big\}.$$
Note that $B_{\mathrm{bs}}(\Gamma_0) \subset \mathcal{D}$. In particular, $\mathcal{D}$ is a dense set in $\mathcal{L}_C$.

In [14], we have found sufficient conditions for operator $(\hat{L}, \mathcal{D})$ to be a generator of a semigroup in $\mathcal{L}_C$. In the case of Markov generators (2.2) or (2.3), this result may be formulated in the following way.

Lemma 3.1 (Theorem 3.2 of [14]). *Suppose there exists $0 < a < \frac{C}{2}$ such that*

$$\sum_{x\in\xi} \int_{\Gamma_0} \big| \big(K_0^{-1} b\left(x, (\xi \setminus x) \cup \cdot \right)\big)(\eta) \big| \, C^{|\eta|} d\lambda(\eta) \le am|\xi|, \tag{3.1}$$

where $b(x,\eta)$ is equal either

$$e^{-E^{\phi}(x,\eta)} \sum_{y\in\eta} b_0(x,y,\eta\setminus y) \quad \textit{or} \quad \sum_{y\in\eta} e^{-E^{\phi}(y,\eta\setminus y)} b_0(x,y,\eta\setminus y).$$

Then $(\hat{L}, \mathcal{D})$ is the generator of a holomorphic semigroup in $\mathcal{L}_C$.

It is worth noting that if (3.1) is valid, then for any $G \in \mathcal{D}$

$$\big(\hat{L}G\big)(\eta) = -m|\eta| G(\eta) + \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} G(\xi\cup x) \big(K_0^{-1} b(x, \cdot \cup \xi) \big)(\eta \setminus \xi) dx. \tag{3.2}$$

Theorem 3.2. *Let $0 \le a^+, b^+, \phi \in L^1(\mathbb{R}^d)$ be even functions such that (2.1) holds and $\langle a^+ \rangle = 1$, $B := \langle b^+ \rangle \ge 0$. Suppose, additionally, that there exist constants $A_1, A_2 \ge 0$ such that*

$$\begin{aligned} 0 \le a^+(x) &\le A_1 \phi(x), \qquad x \in \mathbb{R}^d, && (3.3)\\ a^+(x-y) b^+(y-y') &\le A_2 \phi(x-y)\phi(x-y'), \qquad x,y,y' \in \mathbb{R}^d, && (3.4)\\ \frac{A_1 \varkappa^+}{eC} + \frac{4A_2}{e^2 C} &+ \frac{A_1 B}{e} + \varkappa^+ + \frac{A_2 \langle\phi\rangle}{e} + CB < \frac{m}{2} e^{-c_\phi C}. && (3.5) \end{aligned}$$

Then (3.1) holds and $\big(\hat{L}_{\mathrm{est}} = K^{-1} L_{\mathrm{est}} K, \mathcal{D}\big)$ is the generator of a holomorphic semigroup $\hat{U}_{\mathrm{est}}(t)$ in $\mathcal{L}_C$.

Remark 3.3. In the density independent case, $b^+ \equiv 0$, the assumption (3.4) holds with $A_2 = 0$. Moreover, since $B = 0$, the condition (3.5) will have the following form
$$\frac{A_1 \varkappa^+}{eC} + \varkappa^+ < \frac{m}{2} e^{-c_\phi C}.$$

Before proof of Theorem 3.2, we give an example of a^+, b^+ which satisfy (3.4) in the Lemma below.

Lemma 3.4. *Suppose that there exist constants $E_1, E_2 > 0$ and $\delta > d$ such that*

$$a^+(x) \le \frac{E_1}{(1+|x|)^{2\delta}}, \qquad b^+(x) \le \frac{E_1}{(1+|x|)^{\delta}} \le E_2\phi(x), \qquad x \in \mathbb{R}^d.$$

Then (3.4) *holds with $A_2 = E_2^2$.*

Proof of Lemma 3.4. Using obvious inequality

$$1 + |x - y'| \le 1 + |x - y| + |y - y'| \le (1 + |x - y|)\,(1 + |y - y'|)$$

we obtain that

$$a^+(x-y)b^+(y-y') \le \frac{E_1}{(1+|x-y|)^{\delta}} \frac{E_1}{(1+|x-y'|)^{\delta}} \le E_2^2\phi(x-y)\phi(x-y'),$$

that proves the statement. □

Proof of Theorem 3.2. Let us set

$$b_{\mathrm{est}}(x,\gamma) = e^{-E^{\phi}(x,\gamma)} \sum_{y\in\gamma} a^+(x-y)\Bigg(\varkappa^+ + \sum_{y'\in\gamma\setminus y} b^+(y-y')\Bigg). \tag{3.6}$$

To check (3.1), we will try to estimate the integral

$$\int_{\Gamma_0} \left|\left(K_0^{-1} b_{\mathrm{est}}\,(x, \xi\cup\cdot)\right)(\eta)\right| C^{|\eta|} d\lambda\,(\eta)\,, \qquad \xi\in\Gamma_0$$

uniformly in $x \in \mathbb{R}^d$ and $\xi \in \Gamma_0$.

In view of (3.6), one has

$$\begin{aligned} b_{\mathrm{est}}\,(x,\xi\cup\eta) \;=\;& e^{-E^{\phi}(x,\xi)} e^{-E^{\phi}(x,\eta)} \sum_{y\in\xi} a^+(x-y)\Bigg(\varkappa^+ + \sum_{y'\in\xi\setminus y} b^+(y-y')\Bigg) \\ &+ e^{-E^{\phi}(x,\xi)} e^{-E^{\phi}(x,\eta)} \sum_{y'\in\eta}\sum_{y\in\xi} a^+(x-y) b^+(y-y') \\ &+ e^{-E^{\phi}(x,\xi)} e^{-E^{\phi}(x,\eta)} \sum_{y'\in\eta} a^+(x-y')\Bigg(\varkappa^+ + \sum_{y\in\xi} b^+(y-y')\Bigg) \\ &+ e^{-E^{\phi}(x,\xi)} e^{-E^{\phi}(x,\eta)} \sum_{y\in\eta} a^+(x-y) \sum_{y'\in\eta\setminus y} b^+(y-y'). \end{aligned}$$

Using (2.5)–(2.7), we obtain

$$
\begin{aligned}
& \left(K_0^{-1} b_{\mathrm{est}}\left(x, \xi \cup \cdot\right)\right)(\eta) \qquad (3.7) \\
= \; & e_\lambda\left(e^{-\phi(x-\cdot)}-1, \eta\right) b_{\mathrm{est}}(x, \xi) \\
& +e^{-E^\phi(x, \xi)} \sum_{y' \in \eta} \sum_{y \in \xi} a^{+}(x-y) b^{+}(y-y') e^{-\phi(x-y')} e_\lambda\left(e^{-\phi(x-\cdot)}-1, \eta \setminus y'\right) \\
& +e^{-E^\phi(x, \xi)} \sum_{y' \in \eta} e_\lambda\left(e^{-\phi(x-\cdot)}-1, \eta \setminus y'\right) a^{+}(x-y') e^{-\phi(x-y')} \\
& \qquad \times\left(\varkappa^{+}+\sum_{y \in \xi} b^{+}(y-y')\right) \\
& +e^{-E^\phi(x, \xi)} \sum_{y \in \eta} \sum_{y' \in \eta \setminus y} a^{+}(x-y) b^{+}(y-y') e^{-\phi(x-y)} e^{-\phi(x-y')} \\
& \qquad \times e_\lambda\left(e^{-\phi(x-\cdot)}-1, \eta \setminus\{y, y'\}\right).
\end{aligned}
$$

Next, let $\kappa=e^{c_\phi C}$ then, by (2.8),

$$
\begin{aligned}
& \int_{\Gamma_0}\left|\left(K_0^{-1} b_{\mathrm{est}}\left(x, \xi \cup \cdot\right)\right)(\eta)\right| C^{|\eta|} d \lambda(\eta) \\
\leq \; & \kappa b_{\mathrm{est}}(x, \xi)+C e^{-E^\phi(x, \xi)} \int_{\Gamma_0} \int_{\mathbb{R}^d} \sum_{y \in \xi} a^{+}(x-y) b^{+}(y-y') e^{-\phi(x-y')} \\
& \qquad \times e_\lambda\left(\left|e^{-\phi(x-\cdot)}-1\right|, \eta\right) C^{|\eta|} d y' d \lambda(\eta) \\
& +C e^{-E^\phi(x, \xi)} \int_{\Gamma_0} \int_{\mathbb{R}^d} e_\lambda\left(\left|e^{-\phi(x-\cdot)}-1\right|, \eta\right) a^{+}(x-y') e^{-\phi(x-y')} \\
& \qquad \times\left(\varkappa^{+}+\sum_{y \in \xi} b^{+}(y-y')\right) C^{|\eta|} d y' d \lambda(\eta) \\
& +C^2 e^{-E^\phi(x, \xi)} \int_{\Gamma_0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^{+}(x-y) b^{+}(y-y') e^{-\phi(x-y)} e^{-\phi(x-y')} \\
& \qquad \times e_\lambda\left(\left|e^{-\phi(x-\cdot)}-1\right|, \eta\right) C^{|\eta|} d y' d y d \lambda(\eta) \\
\leq \; & \kappa b_{\mathrm{est}}(x, \xi)+\kappa B C e^{-E^\phi(x, \xi)} \sum_{y \in \xi} a^{+}(x-y) \\
& +\kappa C e^{-E^\phi(x, \xi)} \varkappa^{+}\left\langle a^{+} e^{-\phi}\right\rangle \\
& +\kappa C e^{-E^\phi(x, \xi)} \sum_{y \in \xi} \int_{\mathbb{R}^d} a^{+}(x-y') e^{-\phi(x-y')} b^{+}(y-y') d y' \\
& +\kappa C^2 e^{-E^\phi(x, \xi)} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} a^{+}(x-y) b^{+}(y-y') e^{-\phi(x-y)} e^{-\phi(x-y')} d y' d y .
\end{aligned}
$$

By (3.3), one has

$$e^{-E^\phi(x,\xi)}\sum_{y\in\xi}a^+(x-y)\le A_1e^{-E^\phi(x,\xi)}E^\phi(x,\xi)\le\frac{A_1}{e},$$

where we used the elementary inequality $xe^{-x}\le e^{-1}$, $x\ge 0$. Next, by (3.4), we may estimate

$$\begin{aligned}&e^{-E^\phi(x,\xi)}\sum_{y\in\xi}\int_{\mathbb{R}^d}a^+(x-y')e^{-\phi(x-y')}b^+(y-y')dy'\\ \le\;& A_2e^{-E^\phi(x,\xi)}\sum_{y\in\xi}\int_{\mathbb{R}^d}\phi(x-y)\phi(x-y')dy'\le\frac{A_2\langle\phi\rangle}{e}.\end{aligned}$$

Moreover, (3.3), (3.4) yield

$$\begin{aligned}b_{\rm est}(x,\xi)\;\le\;& A_1\varkappa^+e^{-E^\phi(x,\xi)}\sum_{y\in\xi}\phi(x-y)\\ &+A_2e^{-E^\phi(x,\xi)}\sum_{y\in\xi}\phi(x-y)\sum_{y'\in\xi\setminus y}\phi(x-y')\\ \le\;&\frac{A_1\varkappa^+}{e}+A_2e^{-E^\phi(x,\xi)}\bigl((E^\phi(x,\xi)\bigr)^2\le\frac{A_1\varkappa^+}{e}+\frac{4A_2}{e^2},\end{aligned}$$

since $x^2e^{-x}\le 4e^{-2}$, $x\ge0$.

Therefore, we have

$$\begin{aligned}&\int_{\Gamma_0}\left|\left(K_0^{-1}b_{\rm est}\left(x,\xi\cup\cdot\right)\right)(\eta)\right|C^{|\eta|}d\lambda\left(\eta\right)\\ \le\;&\kappa\left(\frac{A_1\varkappa^+}{e}+\frac{4A_2}{e^2}\right)+\kappa CB\frac{A_1}{e}+\kappa C\varkappa^++\kappa C\frac{A_2\langle\phi\rangle}{e}+\kappa C^2B=:D.\end{aligned}$$

To obtain (3.1), it is enough to suppose that $D\le am$, where $\frac{a}{C}<\frac12$. Hence, we need that $m>\frac{2D}{C}$ only, that is (3.5). The theorem is proved. $\square$

Theorem 3.5. *Let $0\le a^+,b^+,\phi\in L^1(\mathbb{R}^d)$ be even functions such that* (2.1) *holds and $\langle a^+\rangle=1$, $B=\langle b^+\rangle\ge0$. Suppose, additionally, that there exists constants $A_1,A_2\ge0$ such that for a.a. $x,y,y'\in\mathbb{R}^d$*

$$\begin{aligned}0\le a^+(x)\;&\le\;A_1\phi(x)e^{-\phi(x)}, &(3.8)\\ b^+(x)\;&\le\;A_2\phi(x), &(3.9)\\ \varkappa^++\frac{A_2}{e}\;&+\;CB+\Bigl(\frac{\varkappa^+}{C}+B\Bigr)\frac{A_1}{e}+\frac{4A_1A_2}{e^2}C<\frac{m}{2}e^{-c_\phi C}. &(3.10)\end{aligned}$$

Then (3.1) *holds and $\bigl(\hat L_{\rm fec}=K^{-1}L_{\rm fec}K,\mathcal{D})$ is the generator of a holomorphic semigroup $\hat U_{\rm fec}(t)$ in $\mathcal{L}_C$.*

Remark 3.6. In the density independent case, $A_2 = B = 0$, and (3.10) may be rewritten in the form:

$$\varkappa^+\Big(1 + \frac{A_1}{eC}\Big) < \frac{m}{2} e^{-c_\phi C}.$$

Proof. Set

$$b_{\rm fec}(x,\gamma) = \sum_{y\in\gamma} e^{-E^\phi(y,\gamma\setminus y)} b_0(x,y,\gamma\setminus y).$$

Then, one has

$$\begin{aligned}
b_{\rm fec}\left(x,\eta\cup\xi\right) \;=\; & \sum_{y\in\eta} e^{-E^\phi(y,\xi)} e^{-E^\phi(y,\eta\setminus y)} a^+(x-y)\Big(\varkappa^+ + \sum_{y'\in\xi} b^+(y-y')\Big) \\
& + \sum_{y\in\eta} e^{-E^\phi(y,\xi)} e^{-E^\phi(y,\eta\setminus y)} a^+(x-y) \sum_{y'\in\eta\setminus y} b^+(y-y') \\
& + \sum_{y\in\xi} e^{-E^\phi(y,\eta)} e^{-E^\phi(y,\xi\setminus y)} a^+(x-y)\Big(\varkappa^+ + \sum_{y'\in\xi\setminus y} b^+(y-y')\Big) \\
& + \sum_{y'\in\eta}\sum_{y\in\xi} e^{-E^\phi(y,\eta)} e^{-E^\phi(y,\xi\setminus y)} a^+(x-y) b^+(y-y'),
\end{aligned}$$

and, using (2.5)–(2.7), we obtain

$$\begin{aligned}
& \left(K_0^{-1} b_{\rm fec}\left(x,\xi\cup\cdot\right)\right)(\eta) \qquad (3.11)\\
= \; & \sum_{y\in\eta} e^{-E^\phi(y,\xi)} e_\lambda\left(e^{-\phi(y-\cdot)} - 1, \eta\setminus y\right) a^+(x-y)\Big(\varkappa^+ + \sum_{y'\in\xi} b^+(y-y')\Big) \\
& + \sum_{y\in\eta} e^{-E^\phi(y,\xi)} a^+(x-y) \sum_{y'\in\eta\setminus y} b^+(y-y') e^{-\phi\left(y-y'\right)} e_\lambda\left(e^{-\phi(y-\cdot)} - 1, \eta\setminus y\setminus y'\right) \\
& + \sum_{y\in\xi} e_\lambda\left(e^{-\phi(y-\cdot)} - 1, \eta\right) e^{-E^\phi(y,\xi\setminus y)} a^+(x-y)\Big(\varkappa^+ + \sum_{y'\in\xi\setminus y} b^+(y-y')\Big) \\
& + \sum_{y'\in\eta} e_\lambda\left(e^{-\phi(y-\cdot)} - 1, \eta\setminus y'\right) e^{-\phi\left(y-y'\right)} \sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)} a^+(x-y) b^+(y-y').
\end{aligned}$$

Therefore, for $\kappa = e^{c_\phi C}$ we have, by (2.8),

$$\begin{aligned}
& \int_{\Gamma_0} \left|\left(K_0^{-1} b_{\rm fec}\left(x,\xi\cup\cdot\right)\right)(\eta)\right| C^{|\eta|} d\lambda\left(\eta\right) \\
\le \;\; & \kappa C \int_{\mathbb{R}^d} e^{-E^\phi(y,\xi)} a^+(x-y)\Big(\varkappa^+ + \sum_{y'\in\xi} b^+(y-y')\Big) dy \\
& + \kappa C^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-E^\phi(y,\xi)} a^+(x-y) b^+(y-y') e^{-\phi\left(y-y'\right)} dy dy' \\
& + \kappa \sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)} a^+(x-y)\Big(\varkappa^+ + \sum_{y'\in\xi\setminus y} b^+(y-y')\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\kappa C\int_{\mathbb{R}^d} e^{-\phi(y-y')}\sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)}a^+(x-y)b^+(y-y')dy'\\
\le\quad &\kappa C\varkappa^+ + \kappa C\int_{\mathbb{R}^d} a^+(x-y)e^{-E^\phi(y,\xi)}\sum_{y'\in\xi} b^+(y-y')dy\\
&+\kappa C^2B + (\kappa\varkappa^+ + \kappa CB)\sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)}a^+(x-y)\\
&+\kappa\sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)}a^+(x-y)\sum_{y'\in\xi\setminus y} b^+(y-y').
\end{aligned}
$$

By (3.9),

$$
\kappa C\int_{\mathbb{R}^d} a^+(x-y)e^{-E^\phi(y,\xi)}\sum_{y'\in\xi} b^+(y-y')dy \le \frac{\kappa C A_2}{e}.
$$

Now we will verify the sufficient condition (3.1). By (3.8), one has

$$
\begin{aligned}
&\sum_{x\in\xi}\sum_{y\in\xi\setminus x} e^{-E^\phi(y,(\xi\setminus x)\setminus y)}a^+(x-y) = \sum_{y\in\xi}\sum_{x\in\xi\setminus y} e^{-E^\phi(y,(\xi\setminus y)\setminus x)}a^+(x-y)\\
=\quad &\sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)}\sum_{x\in\xi\setminus y} e^{\phi(x-y)}a^+(x-y) \le \frac{A_1}{e}|\xi|,
\end{aligned}
$$

and, by (3.8) and (3.9) we have

$$
\begin{aligned}
&\sum_{x\in\xi}\sum_{y\in\xi\setminus x} e^{-E^\phi(y,(\xi\setminus x)\setminus y)}a^+(x-y)\sum_{y'\in(\xi\setminus x)\setminus y} b^+(y-y')\\
=\quad &\sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)}\sum_{x\in\xi\setminus y} e^{\phi(x-y)}a^+(x-y)\sum_{y'\in(\xi\setminus y)\setminus x} b^+(y-y')\\
\le\quad &A_1A_2\sum_{y\in\xi} e^{-E^\phi(y,\xi\setminus y)}\sum_{x\in\xi\setminus y}\phi(x-y)\sum_{y'\in\xi\setminus y}\phi(y-y') \le \frac{4A_1A_2}{e^2}|\xi|.
\end{aligned}
$$

Hence, by (3.10), we obtain (3.1). □

Let $\big(\hat{L}', \mathrm{Dom}(\hat{L}')\big)$ be an operator in $(\mathcal{L}_C)'$ which is dual to the closed operator $\big(\hat{L}, \mathcal{D}\big)$. Here and below $\hat{L}$ means either $\hat{L}_{\mathrm{est}}$ or $\hat{L}_{\mathrm{fec}}$. We consider also its image on $\mathcal{K}_C$ under the isometry R_C, namely, let $\hat{L}^* = R_C\hat{L}'R_{C^{-1}}$ with the domain $\mathrm{Dom}(\hat{L}^*) = R_C\mathrm{Dom}(\hat{L}')$.

By Proposition 3.5 of [14], for any $\alpha\in(0;1)$

$$
\mathcal{K}_{\alpha C}\subset \mathrm{Dom}(\hat{L}^*).
$$

Under the conditions of Theorem 3.2 or Theorem 3.5, there exists $a\in\big(0;\frac{C}{2}\big)$ such that (3.1) holds. In the following let $\hat{T}(t)$ denotes either $\hat{U}_{\mathrm{est}}(t)$ or $\hat{U}_{\mathrm{fec}}(t)$. One can consider the adjoint semigroup $\hat{T}'(t)$ in $(\mathcal{L}_C)'$ and its image $\hat{T}^*(t)$ in

$\mathcal{K}_C$. By, e.g., Subsection II.2.6 of [9], the restriction $\hat{T}^{\odot}(t)$ of the semigroup $\hat{T}^*(t)$ onto its invariant Banach subspace $\overline{\mathrm{Dom}(\hat{L}^*)}$ (here and below all closures are in the norm of the space $\mathcal{K}_C$) is a strongly continuous semigroup. Moreover, its generator $\hat{L}^{\odot}$ will be part of $\hat{L}^*$, namely,

$$\mathrm{Dom}(\hat{L}^{\odot}) = \Big\{ k \in \mathrm{Dom}(\hat{L}^*) \;\Big|\; \hat{L}^* k \in \overline{\mathrm{Dom}(\hat{L}^*)} \Big\}$$

and $\hat{L}^* k = \hat{L}^{\odot} k$ for any $k \in \mathrm{Dom}(\hat{L}^{\odot})$.

Theorem 3.7 (Theorem 3.8 of [14])**.** *For any $\alpha \in \big(\frac{2a}{C}; 1\big)$ the set $\overline{\mathcal{K}_{\alpha C}}$ is a $\hat{T}^{\odot}(t)$-invariant Banach subspace of $\mathcal{K}_C$.*

Therefore, for $\alpha \in \big(\frac{2a}{C}; 1\big)$, one can consider the restriction $\hat{T}^{\odot\alpha}$ of the semigroup $\hat{T}^{\odot}$ onto $\overline{\mathcal{K}_{\alpha C}}$. This restriction will be strongly continuous semigroup with generator $\hat{L}^{\odot\alpha}$ which is restriction of $\hat{L}^{\odot}$ onto $\overline{\mathcal{K}_{\alpha C}}$ (see, e.g., Subsection II.2.3 of [9]). Therefore,

$$\mathrm{Dom}(\hat{L}^{\odot\alpha}) = \Big\{ k \in \overline{\mathcal{K}_{\alpha C}} \;\Big|\; \hat{L}^* k \in \overline{\mathcal{K}_{\alpha C}} \Big\},$$

and $\hat{L}^{\odot\alpha}$ coincides with $\hat{L}^*$ on $\mathrm{Dom}(\hat{L}^{\odot\alpha})$. Note that for any $k \in \mathcal{K}_{\alpha C} \subset D(\hat{L}^*)$

$$\big(\hat{L}^* k\big)(\eta) = -m|\eta| k(\eta)| + \sum_{x\in\eta} \int_{\Gamma_0} k(\xi \cup (\eta \setminus x)) \big(K_0^{-1} b(x, \cdot \cup \eta \setminus x)\big)(\xi) d\lambda(\xi).$$

The explicit expressions can be found using (3.7) or (3.11).

Hence, we have the strong solution (in the sense of the norm in $\mathcal{K}_C$) of the evolution equation

$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t \tag{3.12}$$

at least on the subspace $\mathcal{K}_{\alpha C}$.

Remark 3.8. To study stationary equation $\hat{L}^* k = 0$ corresponding to (3.12) on the set of functions $k \in \mathcal{K}_{\alpha C}$ such that $k(\emptyset) = 1$, one may consider even weaker assumptions without denominator 2 in (3.5) or (3.10). However, by Proposition 3.9 of [14], a unique solution of this equation will satisfy $k(\eta) = 0$ for all $|\eta| \neq 0$.

4 Vlasov scaling

To begin with, we would like to explain the idea of the Vlasov-type scaling. The general scheme describing this scaling for the birth-and-death dynamics as well as for the conservative ones may be found in [13]. This approach was successfully realized for the Bolker–Dieckmann–Law–Pacala model (1.2)–(1.6) in [11].

Let us now detail how we proceed to organize the Vlasov-type scaling. We will initially scale the generator L by the scaling parameter $\varepsilon > 0$, in such a way that the following holds. First of all the K-image $\hat{L}_\varepsilon$ of the rescaled operator L_ε

has to be a generator of a semigroup on some $\mathcal{L}_{C_\varepsilon}$. Consider the corresponding dual semigroup $\hat{T}^*_\varepsilon(t)$. Let us choose an initial function of the corresponding Cauchy problem depending on ε in such a way that $k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta)$, $\varepsilon \to 0$, $\eta \in \Gamma_0$ with some function r_0, independent of ε. Secondly, the scaling $L \mapsto L_\varepsilon$ has to be performed to assure that the semigroup $\hat{T}^*_\varepsilon(t)$ preserves the order of the singularity:

$$(\hat{T}^*_\varepsilon(t) k_0^{(\varepsilon)})(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0.$$

Moreover, the dynamics $r_0 \mapsto r_t$ should preserve coherent states. Namely, if $r_0(\eta) = e_\lambda(\rho_0, \eta)$, then $r_t(\eta) = e_\lambda(\rho_t, \eta)$ and there exists explicit (nonlinear, in general) differential equation for ρ_t:

$$\frac{\partial}{\partial t} \rho_t(x) = v(\rho_t)(x)$$

which is called the Vlasov-type equation.

Below we realize this approach for the case of

$$(LF)(\gamma) = m \sum_{x \in \gamma} D_x^- F(\gamma) + \int_{\mathbb{R}^d} b(x, \gamma) D_x^+ F(\gamma) dx,$$

where $b = b(a^+, b^+, \phi)$ is either birth rate with establishment (see (2.2)) or the one corresponding to the fecundity mechanism. Let us consider for any $\varepsilon \in (0; 1]$ the following scaling

$$(L_\varepsilon F)(\gamma) = m \sum_{x \in \gamma} D_x^- F(\gamma) + \varepsilon^{-1} \int_{\mathbb{R}^d} b_\varepsilon(x, \gamma) D_x^+ F(\gamma) dx,$$

with $b_\varepsilon = b(\varepsilon a^+, \varepsilon b^+, \varepsilon \phi)$. Here $D_x^\pm$ are given by (1.3). We denote by $b_{\varepsilon,\mathrm{est}}$ and $b_{\varepsilon,\mathrm{fec}}$ the scaled rates for the corresponding models. We define also the renormalized operator (see [13], [11] for details)

$$\hat{L}_{\varepsilon,\mathrm{ren}} := R_{\varepsilon^{-1}} K^{-1} L_\varepsilon K R_\varepsilon,$$

where $(R_\sigma G)(\eta) = \sigma^{|\eta|} G(\eta)$ for arbitrary $\sigma > 0$.

Lemma 4.1. *Suppose that the conditions of Theorem 3.2 (or Theorem 3.5) are satisfied with $\langle \phi \rangle$ instead of c_ϕ in* (3.5) *(in* (3.10)*, correspondingly). Then there exists $a \in \left(0; \frac{C}{2}\right)$ such that*

$$\sum_{x \in \xi} \int_{\Gamma_0} \left| \left(K_0^{-1} b_\varepsilon \left(x, (\xi \setminus x) \cup \cdot \right) \right) (\eta) \right| \varepsilon^{-|\eta|} C^{|\eta|} d\lambda(\eta) \le am|\xi|, \tag{4.1}$$

where $b_\varepsilon = b_{\varepsilon,\mathrm{est}}$ (or $b_\varepsilon = b_{\varepsilon,\mathrm{fec}}$, correspondingly).

Proof. We begin with the establishment case. Set

$$\psi_\varepsilon(x) = \varepsilon^{-1}\big(e^{-\varepsilon\phi(x)} - 1\big), \quad x \in \mathbb{R}^d.$$

By (3.7), we have

$$\begin{aligned}
&\varepsilon^{-|\eta|}\left(K_0^{-1} b_{\varepsilon,\mathrm{est}}(x, \xi \cup \cdot)\right)(\eta) \qquad (4.2)\\
= \quad & \varepsilon e_\lambda\left(\psi_\varepsilon(x-\cdot), \eta\right) e^{-\varepsilon E^\phi(x,\xi)} \sum_{y\in\xi} a^+(x-y)\Bigg(\varkappa^+ + \sum_{y'\in\xi\setminus y} \varepsilon b^+(y-y')\Bigg)\\
&+\varepsilon e^{-\varepsilon E^\phi(x,\xi)} \sum_{y'\in\eta}\sum_{y\in\xi} a^+(x-y) b^+(y-y') e^{-\varepsilon\phi(x-y')} e_\lambda\left(\psi_\varepsilon(x-\cdot), \eta\setminus y'\right)\\
&+e^{-\varepsilon E^\phi(x,\xi)} \sum_{y'\in\eta} e_\lambda\left(\psi_\varepsilon(x-\cdot), \eta\setminus y'\right) a^+(x-y') e^{-\varepsilon\phi(x-y')}\\
&\qquad\times\Bigg(\varkappa^+ + \varepsilon\sum_{y\in\xi} b^+(y-y')\Bigg)\\
&+e^{-\varepsilon E^\phi(x,\xi)} \sum_{y\in\eta}\sum_{y'\in\eta\setminus y} a^+(x-y) b^+(y-y') e^{-\varepsilon\phi(x-y)} e^{-\varepsilon\phi(x-y')}\\
&\qquad\times e_\lambda\left(\psi_\varepsilon(x-\cdot), \eta\setminus\{y,y'\}\right).
\end{aligned}$$

Since $\varepsilon \in (0;1]$ and

$$|\psi_\varepsilon(x)| \le \phi(x), \qquad x \in \mathbb{R}^d,$$

the estimate for $\varepsilon^{-|\eta|}\left|K_0^{-1} b_{\varepsilon,\mathrm{est}}(x, \xi\cup\cdot)\right|(\eta)$ will be almost the same as for $\left|K_0^{-1} b(x,\xi\cup\cdot)\right|(\eta)$ in the proof of Theorem 3.2. The changes will concern the term $|e^{-\phi} - 1|$ which will be substitute by ϕ. This leads to the new constant $\langle\phi\rangle$ instead of c_ϕ in further estimates. The rest part of the proof is the same as for the non-scaled case.

The same approach may be used for the case of fecundity. Indeed,

$$\begin{aligned}
&\varepsilon^{-|\eta|}\left(K_0^{-1} b_{\varepsilon,\mathrm{fec}}(x, \xi \cup \cdot)\right)(\eta) \qquad (4.3)\\
= \quad & \sum_{y\in\eta} e^{-\varepsilon E^\phi(y,\xi)} e_\lambda\left(\psi_\varepsilon(y-\cdot), \eta\setminus y\right) a^+(x-y)\Bigg(\varkappa^+ + \sum_{y'\in\xi} \varepsilon b^+(y-y')\Bigg)\\
&+\sum_{y\in\eta} e^{-\varepsilon E^\phi(y,\xi)} a^+(x-y) \sum_{y'\in\eta\setminus y} b^+(y-y') e^{-\varepsilon\phi(y-y')} e_\lambda\left(\psi_\varepsilon(y-\cdot), \eta\setminus y\setminus y'\right)\\
&+\varepsilon\sum_{y\in\xi} e_\lambda\left(\psi_\varepsilon(y-\cdot), \eta\right) e^{-\varepsilon E^\phi(y,\xi\setminus y)} a^+(x-y)\Bigg(\varkappa^+ + \sum_{y'\in\xi\setminus y} \varepsilon b^+(y-y')\Bigg)\\
&+\varepsilon\sum_{y'\in\eta} e_\lambda\left(\psi_\varepsilon(y-\cdot), \eta\setminus y'\right) e^{-\varepsilon\phi(y-y')} \sum_{y\in\xi} e^{-\varepsilon E^\phi(y,\xi\setminus y)} a^+(x-y) b^+(y-y').
\end{aligned}$$

The analogous arguments to establishment case complete the proof. $\square$

Under conditions of Lemma 4.1 we have the following result about the renormalized semigroups in $\mathcal{L}_C$ and $\mathcal{K}_C$.

Proposition 4.2 (Proposition 4.1 of [14]). *Let the conditions of Lemma 4.1 hold. Then for any $\varepsilon \in (0;1]$, $\big(\hat{L}_{\varepsilon,\mathrm{est},\mathrm{ren}}, \mathcal{D}\big)$ and $\big(\hat{L}_{\varepsilon,\mathrm{fec},\mathrm{ren}}, \mathcal{D}\big)$ are the generators of holomorphic semigroups $\hat{U}_{\varepsilon,\mathrm{est}}(t)$ and $\hat{U}_{\varepsilon,\mathrm{fec}}(t)$ on $\mathcal{L}_C$, correspondingly. Moreover, there exists $\alpha_0 \in (0;\frac{1}{\nu})$ such that for any $\alpha \in (\alpha_0;\frac{1}{\nu})$ and $\varepsilon \in (0;1]$ there exist a strongly continuous semigroups $\hat{U}^{\odot\alpha}_{\varepsilon,\sharp}(t)$ on the space $\overline{\mathcal{K}_{\alpha C}}$ with generator $\hat{L}^{\odot\alpha}_{\varepsilon,\sharp} = \hat{L}^*_{\varepsilon,\sharp,\mathrm{ren}}$ on the domain*

$$\mathrm{Dom}\big(L^{\odot\alpha}_{\varepsilon,\sharp}\big) = \big\{k \in \overline{\mathcal{K}_{\alpha C}} \;\big|\; \hat{L}^*_{\varepsilon,\sharp,\mathrm{ren}} k \in \overline{\mathcal{K}_{\alpha C}}\big\}.$$

*Here and below '$\sharp$' means '*est*' or '*fec*', correspondingly. Note that, for $k \in \mathcal{K}_{\alpha C}$*

$$\begin{aligned}(\hat{L}^*_{\varepsilon,\sharp,\mathrm{ren}} k)(\eta) &= -m|\eta|k(\eta) \\ &\quad+\sum_{x\in\eta}\int_{\Gamma_0} k(\xi\cup(\eta\setminus x))\varepsilon^{-|\xi|}\big(K_0^{-1}b_{\varepsilon,\sharp}(x,\cdot\cup\eta\setminus x)\big)(\xi)d\lambda(\xi).\end{aligned} \tag{4.4}$$

By (4.2), (4.3), there exist the following point-wise limits

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-|\eta|}\left(K_0^{-1}b_{\varepsilon,\mathrm{est}}\left(x,\xi\cup\cdot\right)\right)(\eta) \\ =\;& \varkappa^+\sum_{y'\in\eta} e_\lambda\left(-\phi(x-\cdot),\eta\setminus y'\right)a^+(x-y') \\ &+\sum_{y\in\eta} a^+(x-y)\sum_{y'\in\eta\setminus y} b^+(y-y')e_\lambda\left(-\phi(x-\cdot),\eta\setminus\{y,y'\}\right) =: B_x^{V,\mathrm{est}}(\eta)\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-|\eta|}\left(K_0^{-1}b_{\varepsilon,\mathrm{fec}}\left(x,\xi\cup\cdot\right)\right)(\eta) \\ =\;& \varkappa^+\sum_{y\in\eta} e_\lambda\left(-\phi(y-\cdot),\eta\setminus y\right)a^+(x-y) \\ &+\sum_{y\in\eta} a^+(x-y)\sum_{y'\in\eta\setminus y} b^+(y-y')e_\lambda\left(-\phi(y-\cdot),\eta\setminus y\setminus y'\right) =: B_x^{V,\mathrm{fec}}(\eta).\end{aligned} \tag{4.6}$$

It is worth pointing out that these limits do not depend on ξ. Hence, we have point-wise limits for $\hat{L}_{\varepsilon,\sharp,\mathrm{ren}}$:

$$(\hat{L}_{V,\sharp}G)(\eta) := -m|\eta|G(\eta) + \sum_{\xi\subset\eta}\int_{\mathbb{R}^d} G(\xi\cup x)B_x^{V,\sharp}(\eta\setminus\xi)dx. \tag{4.7}$$

The convergences (4.5) and (4.6) in the space $\mathcal{L}_C$ are established by our next Lemma.

Lemma 4.3. *Let conditions of Lemma 4.1 hold. Then, for a.a. $x \in \mathbb{R}^d$ and for λ-a.a. $\xi \in \Gamma_0$, the convergence* (4.5) *and* (4.6) *hold in the sense of norm of $\mathcal{L}_C$.*

Proof. By (4.2) and (4.3), it is easy to see that $\varepsilon^{-|\eta|}\left(K_0^{-1}b_{\varepsilon,\sharp}\left(x,\xi\cup\cdot\right)\right)(\eta)$ has the form $A_\varepsilon(\eta)+\varepsilon B_\varepsilon(\eta)$. Moreover, the proof of Lemma 4.1 assures that B_ε has an integrable majorant. Hence, by the dominated convergence theorem, $\varepsilon B_\varepsilon \to 0$ in $\mathcal{L}_C$. Next, using again the dominated convergence theorem and taking into account (4.5) and (4.6), we will be able to show convergence of A_ε to $B_x^{V,\sharp}$ in $\mathcal{L}_C$ once we find uniform in ε integrable estimate for the corresponding differences $|A_\varepsilon - B_x^{V,\sharp}|$.

Since $e^{-\varepsilon\phi}\le 1$ and $\psi_\varepsilon(x)<\phi(x)$, for the establishment case, we have

$$\begin{aligned}
&\varkappa^+\sum_{y'\in\eta}a^+(x-y')\\
&\qquad\times\left|e^{-\varepsilon E^\phi(x,\xi)}e^{-\varepsilon\phi(x-y')}e_\lambda\left(\psi_\varepsilon(x-\cdot),\eta\setminus y'\right)-e_\lambda\left(-\phi(x-\cdot),\eta\setminus y'\right)\right|\\
&+\sum_{y\in\eta}\sum_{y'\in\eta\setminus y}a^+(x-y)b^+(y-y')\\
&\qquad\times\left|e^{-\varepsilon E^\phi(x,\xi)}e^{-\varepsilon\phi(x-y)}e^{-\varepsilon\phi(x-y')}e_\lambda\left(\psi_\varepsilon(x-\cdot),\eta\setminus\{y,y'\}\right)\right.\\
&\qquad\qquad\left.-e_\lambda\left(-\phi(x-\cdot),\eta\setminus\{y,y'\}\right)\right|\\
\le\ &2\varkappa^+\sum_{y'\in\eta}a^+(x-y')e_\lambda\left(\phi(x-\cdot),\eta\setminus y'\right)\\
&+2\sum_{y\in\eta}\sum_{y'\in\eta\setminus y}a^+(x-y)b^+(y-y')e_\lambda\left(\phi(x-\cdot),\eta\setminus\{y,y'\}\right).
\end{aligned}$$

The last expression is an element of $\mathcal{L}_C$, in view of (2.9) and (2.8). Indeed,

$$\int_{\Gamma_0}\sum_{y'\in\eta}a^+(x-y')e_\lambda\left(\phi(x-\cdot),\eta\setminus y'\right)C^{|\eta|}d\lambda(\eta)=e^{C\langle\phi\rangle},$$

and, a similar equality holds for the second term.

One can get the same result for the fecundity case in a similar way. □

Let us denote by $\bar{B}_c^\infty$ the closed ball of radius $c>0$ in the Banach space $L^\infty(\mathbb{R}^d)$.

Using Lemma 4.3 one can easily pass to the limit in (4.1). Therefore, in view of the general results presented in [14] we are able to state now the main theorem of this section.

Theorem 4.4 (Proposition 4.2, Theorem 4.4. of [14])**.** *Let the conditions of Lemma 4.1 hold. Then*

1. $\left(\hat{L}_{V,\sharp},\mathcal{D}\right)$ *are generators of a holomorphic semigroups* $\hat{U}_{V,\sharp}(t)$ *on* $\mathcal{L}_C$.
2. $\hat{U}_{\varepsilon,\sharp}(t)\longrightarrow\hat{U}_{V,\sharp}(t)$ *strongly in* $\mathcal{L}_C$ *uniformly on finite time intervals.*

3. *There exists $\alpha_0 \in (0;1)$ such that for any $\alpha \in (\alpha_0;1)$ the operator $\hat{L}_{V,\sharp}^{\odot\alpha} = \hat{L}_{V,\sharp}^{*}$ with the domain*

$$\mathrm{Dom}\big(L_{V,\sharp}^{\odot\alpha}\big) = \big\{k \in \overline{\mathcal{K}_{\alpha C}} \;\big|\; \hat{L}_{V,\sharp}^{*} k \in \overline{\mathcal{K}_{\alpha C}}\big\}.$$

will be a generator of a strongly continuous semigroup $\hat{U}_{V,\sharp}^{\odot\alpha}(t)$ on the space $\overline{\mathcal{K}_{\alpha C}}$. Moreover, for $k \in \mathcal{K}_{\alpha C}$

$$(\hat{L}_{V,\sharp}^{*} k)(\eta) = -m|\eta|k(\eta) + \sum_{x\in\eta} \int_{\Gamma_0} k(\xi \cup (\eta \setminus x)) B_x^{V,\sharp}(\xi) d\lambda(\xi).$$

4. *Let $\alpha \in (\alpha_0;1)$, $\rho_0 \in \bar{B}_{\alpha C}^{\infty}$. Then the evolution equation*

$$\begin{cases} \dfrac{\partial}{\partial t} k_t = \hat{L}_V^{*} k_t \\ k_t\big|_{t=0} = e_\lambda(\rho_0, \eta) \end{cases}$$

has a unique solution $k_t = e_\lambda(\rho_t)$ in $\overline{\mathcal{K}_{\alpha C}}$ provided ρ_t belongs to $\bar{B}_{\alpha C}^{\infty}$ and satisfies the Vlasov-type equation

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + \int_{\Gamma_0} e_\lambda(\rho_t, \xi) B_x^{V,\sharp}(\xi) d\lambda(\xi). \tag{4.8}$$

Taking into account the explicit expressions for $B_x^{V,\sharp}$, one can rewrite (4.8) in more simple form. Namely, using (2.9), for the establishment case we obtain

$$\frac{\partial}{\partial t}\rho_t(x) = -m\rho_t(x) + \int_{\Gamma_0}\int_{\mathbb{R}^d} e_\lambda(\rho_t, \eta\cup y)\varkappa^+ e_\lambda\left(-\phi(x-\cdot),\eta\right) a^+(x-y) dy d\lambda(\eta)$$
$$+ \int_{\Gamma_0}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e_\lambda(\rho_t, \eta\cup\{y,y'\}) a^+(x-y) b^+(y-y') e_\lambda\left(-\phi(x-\cdot),\eta\right) dy dy' d\lambda(\eta),$$

and, by (2.8), we will have

$$\begin{aligned} \frac{\partial}{\partial t}\rho_t(x) &= -m\rho_t(x) + \varkappa^+(\rho_t * a^+)(x)\exp\big\{-(\rho_t*\phi)(x)\big\} \\ &\quad + \big(\{(\rho_t * b^+)\rho_t\} * a^+\big)(x)\exp\big\{-(\rho_t*\phi)(x)\big\}. \end{aligned} \tag{4.9}$$

Here and below $*$ means usual convolutions of functions on $\mathbb{R}^d$.

Analogously, for the fecundity case, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\rho_t(x) &= -m\rho_t(x) + \varkappa^+\big(\{\rho_t \exp(-\rho_t*\phi)\} * a^+\big)(x) \\ &\quad + \big(\{(\rho_t * b^+)\rho_t \exp(-\rho_t*\phi)\} * a^+\big)(x). \end{aligned} \tag{4.10}$$

Of course, we are mostly interesting in nonnegative solution of Vlasov equation to have $k_t = e_\lambda(\rho_t)$ is a correlation function of Poisson non-homogeneous measure with intensity ρ_t. The existence and uniqueness of such solution we establishes by the following propositions.

Proposition 4.5. *Suppose there exists $A>0$ such that $0\le \max\{a^+(x),b^+(x)\}\le A\phi(x)$, $x\in\mathbb{R}^d$. Let $c>0$ and*

$$\begin{aligned} \varkappa^+\Bigl(1+\frac{A}{e}\langle\phi\rangle\Bigr)+c\langle b^+\rangle\Bigl(2+\frac{A}{e}\langle\phi\rangle\Bigr) &< m, & (4.11)\\ \frac{A}{e}\bigl(\varkappa^+ +\langle b^+\rangle\bigr) &\le m. & (4.12)\end{aligned}$$

Then the equation (4.9) with initial $0\le\rho_0\in\bar{B}_c^\infty$ has a non-negative solution ρ_t. Moreover, $\rho_t\in\bar{B}_c^\infty$ and it is a unique solution from $\bar{B}_c^\infty$.

Proof. Let us fix some $T>0$ and consider the Banach space $X_T=C([0;T],L^\infty(\mathbb{R}^d))$ of all continuous functions on $[0;T]$ with values in $L^\infty(\mathbb{R}^d)$; the norm on X_T is given by

$$\|u\|_T:=\max_{t\in[0;T]}\|u_t\|_{L^\infty(\mathbb{R}^d)}.$$

We denote by X_T^+ the cone of all nonnegative functions from X_T. Denote also by $B_{T,c}^+$ the set of all functions u from X_T^+ with $\|u\|_T\le c$. This set form a complete metric space with a metric induced by the norm in X_T.

Let Φ be a mapping which assign to any $v\in X_T$ the solution u_t of the linear Cauchy problem

$$\left\{\begin{aligned} \frac{\partial}{\partial t}u_t(x) &= -mu_t(x)+\varkappa^+(v_t*a^+)(x)\exp\bigl\{-(v_t*\phi)(x)\bigr\}\\ &\quad+\bigl(\{(v_t*b^+)v_t\}*a^+\bigr)(x)\exp\bigl\{-(v_t*\phi)(x)\bigr\},\\ u_t\big|_{t=0}(x) &= \rho_0(x),\end{aligned}\right.$$

for a.a. $x\in\mathbb{R}^d$. Therefore,

$$\begin{aligned}(\Phi v)_t(x) &= e^{-mt}\rho_0(x) & (4.13)\\ &\quad+\int_0^t e^{-m(t-s)}\varkappa^+(v_s*a^+)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}ds\\ &\quad+\int_0^t e^{-m(t-s)}\bigl(\{(v_s*b^+)v_s\}*a^+\bigr)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}ds.\end{aligned}$$

It is easy to see that $\Phi v\in X_T$. Indeed, one can estimate

$$\begin{aligned}\bigl|(\Phi v)_t(x)\bigr| &\le |\rho_0(x)|+\bigl(\varkappa^+\|v\|_T+\langle b^+\rangle\|v\|_T^2\bigr)\int_0^t e^{-(t-s)m}ds\\ &\le c+\frac{\varkappa^+\|v\|_T+\langle b^+\rangle\|v\|_T^2}{m},\end{aligned}$$

where we have used the trivial inequality

$$\|f*g\|_{L^\infty(\mathbb{R}^d)}\le\|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)},\quad f\in L^1(\mathbb{R}^d),\ g\in L^\infty(\mathbb{R}^d). \qquad (4.14)$$

Clearly, u_t solves (4.9) if and only if u is a fixed point of the mapping $\Phi: X_T\to X_T$.

We have that $v\in X_T^+$ implies $\Phi v\in X_T^+$. Next, for any $v,w\in X_T^+$

$$\begin{aligned}
&\bigl|(\Phi v)_t(x)-(\Phi w)_t(x)\bigr|\\
\leq\;& \varkappa^+\int_0^t e^{-m(t-s)}\Bigl|(v_s*a^+)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}\\
&\qquad -(w_s*a^+)(x)\exp\bigl\{-(w_s*\phi)(x)\bigr\}\Bigr|ds\\
&+\int_0^t e^{-m(t-s)}\Bigl|\bigl(\{(v_s*b^+)v_s\}*a^+\bigr)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}\\
&\qquad -\bigl(\{(w_s*b^+)w_s\}*a^+\bigr)(x)\exp\bigl\{-(w_s*\phi)(x)\bigr\}\Bigr|ds.
\end{aligned}$$

In addition to (4.14), we will obvious inequalities $e^{-x}x\leq e^{-1}$ for $x\geq 0$, $|e^{-a}-e^{-b}|\leq|a-b|$ for $a,b\geq 0$, and, moreover,

$$|pe^{-a}-qe^{-b}|\leq e^{-a}|p-q|+qe^{-b}|e^{-(a-b)}-1|\leq e^{-a}|p-q|+qe^{-b}|a-b|,$$

for any $a\geq b\geq 0$, $p,q\geq 0$.

By the latter inequality, we obtain, for any $x\in\mathbb{R}^d$, $s\in[0,t]$ such that

$$(w_s*\phi)(x)\leq(v_s*\phi)(x), \tag{4.15}$$

that

$$\begin{aligned}
&\Bigl|(v_s*a^+)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}-(w_s*a^+)(x)\exp\bigl\{-(w_s*\phi)(x)\bigr\}\Bigr|\\
\leq\;&\Bigl((|v_s-w_s|*a^+)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}\\
&+(w_s*a^+)(x)\exp\bigl\{-(w_s*\phi)(x)\bigr\}(|v_s-w_s|*\phi)(x)\Bigr)\\
\leq\;&\|v-w\|_T\Bigl(1+\frac{A}{e}\langle\phi\rangle\Bigr),
\end{aligned}$$

and the same inequality holds for any $x\in\mathbb{R}^d$, $s\in[0,t]$, such that the inequality opposite to (4.15) holds. As a result,

$$\begin{aligned}
&\varkappa^+\int_0^t e^{-m(t-s)}\Bigl|(v_s*a^+)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}\\
&\qquad-(w_s*a^+)(x)\exp\bigl\{-(w_s*\phi)(x)\bigr\}\Bigr|ds\\
\leq\;&\varkappa^+\|v-w\|_T\Bigl(1+\frac{A}{e}\langle\phi\rangle\Bigr)\int_0^t e^{-m(t-s)}ds\leq\|v-w\|_T\frac{\varkappa^+}{m}\Bigl(1+\frac{A}{e}\langle\phi\rangle\Bigr).
\end{aligned}$$

Similarly, under condition (4.15),

$$\begin{aligned}
&\Bigl|\bigl(\{(v_s*b^+)v_s\}*a^+\bigr)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}\\
&-\bigl(\{(w_s*b^+)w_s\}*a^+\bigr)(x)\exp\bigl\{-(w_s*\phi)(x)\bigr\}\Bigr|\\
\leq\;&\Bigl(\bigl(\bigl|(v_s*b^+)v_s-(w_s*b^+)w_s\bigr|*a^+\bigr)(x)\exp\bigl\{-(v_s*\phi)(x)\bigr\}\\
&+\bigl(\{(w_s*b^+)w_s\}*a^+\bigr)(x)\exp\bigl\{-(w_s*\phi)(x)\bigr\}(|v_s-w_s|*\phi)(x)\Bigr)
\end{aligned}\tag{4.16}$$

Using the bound

$$
\begin{aligned}
& \big(\{(w_s*b^+)w_s\}*a^+\big)(x)\exp\big\{-(w_s*\phi)(x)\big\} \\
\le \quad & \|w_s*b^+\|_{L^\infty(\mathbb{R}^d)}(w_s*a^+)(x)\exp\big\{-(w_s*\phi)(x)\big\},
\end{aligned}
$$

we may continue to estimate (4.16) as follows

$$
\begin{aligned}
\le \quad & \Big\|(v_s*b^+)v_s-(v_s*b^+)w_s\Big\|_{L^\infty(\mathbb{R}^d)}+\Big\|(v_s*b^+)w_s-(w_s*b^+)w_s\Big\|_{L^\infty(\mathbb{R}^d)} \\
& +\big\|w_s\big\|_{L^\infty(\mathbb{R}^d)}\langle b^+\rangle\frac{A}{e}\|v-w\|_T\langle\phi\rangle.
\end{aligned}
$$

For $\|v\|_T\le c$, $\|w\|_T\le c$ one can estimate this expression by

$$
Q:=2c\|v-w\|_T\langle b^+\rangle+c\langle b^+\rangle\frac{A}{e}\|v-w\|_T\langle\phi\rangle,
$$

and, clearly, the same estimation will be for $x\in\mathbb{R}^d$, $s\in[0,t]$ such that (4.15) does not holds. As a result,

$$
\begin{aligned}
& \int_0^t e^{-m(t-s)}\Big|\big(\{(v_s*b^+)v_s\}*a^+\big)(x)\exp\big\{-(v_s*\phi)(x)\big\} \\
& -\big(\{(w_s*b^+)w_s\}*a^+\big)(x)\exp\big\{-(w_s*\phi)(x)\big\}\Big|ds \\
\le \quad & Q\int_0^t e^{-m(t-s)}ds\le\frac{Q}{m}.
\end{aligned}
\tag{4.17}
$$

Therefore, for $v,w\in X_T^+$, $\|v\|_T\le c$, $\|w\|_T\le c$

$$
\|\Phi v-\Phi w\|_T\le\frac{\varkappa^+}{m}\Big(1+\frac{A}{e}\langle\phi\rangle\Big)\|v-w\|_T+\frac{c\langle b^+\rangle}{m}\Big(2+\frac{A}{e}\langle\phi\rangle\Big)\|v-w\|_T.
$$

Moreover, if $\rho_0\in\bar{B}_c^\infty$ and $v\in B_{T,c}^+$ then, by (4.13),

$$
\begin{aligned}
|(\Phi v)_t(x)| \quad & \le \quad e^{-mt}c+\frac{A\varkappa^+}{me}\big(1-e^{-mt}\big)c+c\langle b^+\rangle\frac{A}{me}\big(1-e^{-mt}\big) \\
& = \quad \frac{cA}{me}\big(\varkappa^++\langle b^+\rangle\big)+e^{-mt}c\Big(1-\frac{A}{me}\big(\varkappa^++\langle b^+\rangle\big)\Big)\le c,
\end{aligned}
$$

provided (4.12) holds.

As a result, by (4.11), (4.12), Φ is a contraction mapping on the complete metric space $B_{T,c}^+$. Therefore, according to the classical Banach fixed point theorem, there exists a unique fixed point on $B_{T,c}^+$. $\square$

The same considerations may be applied to the Vlasov equation (4.10). To combine these results with statement of Theorem 4.4 we need additionally that (4.11), (4.12) hold with $c=\alpha C$.

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References

[1] S. Albeverio, Y. Kondratiev, and M. Röckner, Analysis and geometry on configuration spaces, *J. Funct. Anal.* **154** (1998), 444–500.

[2] V. P. Belavkin, Quantum branching processes and the nonlinear dynamics of multiquantum systems, *Dokl. Akad. Nauk SSSR* **301** (1988), 1348–1352.

[3] V. P. Belavkin, Multiquantum systems and point processes. I. Generating functionals and nonlinear semigroups, *Rep. Math. Phys.* **28** (1989), 57–90.

[4] V. P. Belavkin and V. N. Kolokol'tsov, On a general kinetic equation for many-particle systems with interaction, fragmentation and coagulation, *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.* **459** (2003), 727–748.

[5] V. P. Belavkin and V. P. Maslov, The uniformization method in the theory of nonlinear Hamiltonian systems of Vlasov and Hartree type, *Teoret. Mat. Fiz.* **33** (1977), 17–31.

[6] V. P. Belavkin, V. P. Maslov, and S. È. Tariverdiev, The asymptotic dynamics of a system with a large number of particles described by Kolmogorov-Feller equations, *Teoret. Mat. Fiz.* **49** (1981), 298–306.

[7] C. Berns, Y. Kondratiev, Y. Kozitsky, and O. Kutoviy, Kawasaki dynamics in continuum: micro- and mesoscopic descriptions. *To appear:* Journal of Dynamics and Differential Equations (2013).

[8] B. M. Bolker and S. W. Pacala, *Spatial moment equations for plant competitions: Understanding Spatial Strategies and the Advantages of Short Dispersal*, The American Naturalist, **153**(6) (1999), 575–602.

[9] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[10] D. Finkelshtein, Y. Kondratiev, and Y. Kozitsky, Glauber dynamics in continuum: a constructive approach to evolution of states, Discrete and Cont. Dynam. Syst. - Ser. A., **33**(4) 2013, 1431–1450.

[11] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, Operator approach to Vlasov scaling for some models of spatial ecology. *To appear:* Meth. Funct. Anal. Topology (2013).

[12] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, Individual based model with competition in spatial ecology, *SIAM J. Math. Anal.* **41** (2009), 297–317.

[13] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, Vlasov scaling for stochastic dynamics of continuous systems, *J. Stat. Phys.* **141** (2010), 158–178.

[14] D. Finkelshtein, Y. Kondratiev, and O. Kutoviy, Semigroup approach to non-equilibrium birth-and-death stochastic dynamics in continuum, *J. of Funct. Anal.* **262** (2012), 1274–1308.

[15] Y. Kondratiev and T. Kuna, Harmonic analysis on configuration space. I. General theory, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **5** (2002), 201–233.

[16] Y. Kondratiev and O. Kutoviy, On the metrical properties of the configuration space, *Math. Nachr.* **279** (2006), 774–783.

[17] Y. Kondratiev, O. Kutoviy, and R. Minlos, On non-equilibrium stochastic dynamics for interacting particle systems in continuum, *J. Funct. Anal.* **255** (2008), 200–227.

[18] Y. Kondratiev, O. Kutoviy, and S. Pirogov, Correlation functions and invariant measures in continuous contact model, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **11** (2008), 231–258.

[19] Y. Kondratiev, R. Minlos, and E. Zhizhina, One-particle subspace of the Glauber dynamics generator for continuous particle systems, *Rev. Math. Phys.* **16** (2004), 1073–1114.

[20] Y. Kondratiev and A. Skorokhod, On contact processes in continuum, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **9** (2006), 187–198.

[21] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. I, *Arch. Rational Mech. Anal.* **59** (1975), 219–239.

[22] A. Lenard, States of classical statistical mechanical systems of infinitely many particles. II. Characterization of correlation measures, *Arch. Rational Mech. Anal.* **59** (1975), 241–256.