Glauber Dynamics in Continuum: A Constructive Approach to Evolution of States*

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Abstract

The evolutions of states is described corresponding to the Glauber dynamics of an infinite system of interacting particles in continuum. The description is conducted on both micro- and mesoscopic levels. The microscopic description is based on solving linear equations for correlation functions by means of an Ovsjannikov-type technique, which yields the evolution in a scale of Banach spaces. The mesoscopic description is performed by means of the Vlasov scaling, which yields a linear infinite chain of equations obtained from those for the correlation function. Its main peculiarity is that, for the initial correlation function of the inhomogeneous Poisson measure, the solution is the correlation function of such a measure with density which solves a nonlinear differential equation of convolution type.

1 Introduction

In the statistical theory of large systems [2], the system states are described as probability measures on the corresponding phase space rather than pointwise, which is typical for the standard theory of dynamical systems. For such large systems, in order to obtain the description independent of the system size one employs the models where the system is infinite and distributed over a non-compact manifold with positive density. A particular case constitute models of interacting point particles distributed over $\mathbb{R}^d$, which are widely used in mathematical physics, ecology, sociology, etc, see [3, 6, 8–10, 13–15]. Here the states

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are probability measures on the space of the particle configurations

$$\Gamma \equiv \Gamma(\mathbb{R}^d) := \big\{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d \ \big\}, \tag{1.1}$$

where $|A|$ denotes the cardinality of A. The system is characterized by a collection of appropriate functions $F : \Gamma \to \mathbb{R}$, called *observables*. For a state μ, the quantity

$$\langle\!\langle F, \mu \rangle\!\rangle = \int_\Gamma F(\gamma)\mu(d\gamma)$$

is called the mean value of observable F in state μ. Then the system evolution is described as the evolution of observables obtained from the Kolmogorov equation

$$\frac{d}{dt}F_t = LF_t, \qquad F_t|_{t=0} = F_0, \qquad t > 0, \tag{1.2}$$

where the 'generator' L is specified within the choice of the model. The evolution of states is obtained from the Fokker–Planck equation

$$\frac{d}{dt}\mu_t = L^*\mu_t, \qquad \mu_t|_{t=0} = \mu_0, \tag{1.3}$$

related to (1.2) by the duality

$$\langle\!\langle F_0, \mu_t \rangle\!\rangle = \langle\!\langle F_t, \mu_0 \rangle\!\rangle. \tag{1.4}$$

Note that L ought to be Markovian in order that the solutions of (1.3) be probability measures. One of the possibilities here is to describe the evolution pathwise—by constructing a stochastic Markov process $X_t^{\mu_0}$, corresponding to the 'generator' L and to the initial state μ_0. Then the state μ_t is just the distribution law of $X_t^{\mu_0}$. However, for a number of important models this way encounters serious problems and hence is rather unrealistic. Moreover, the mere existence of the process tells not too much about the properties of the system evolution. Thus, the main idea which we realize in this work is to describe the system evolution as the Markov evolution of states $\mu_0 \mapsto \mu_t$, not necessarily based on the pathwise description, and accompanied with a more detailed study of its properties. In a sense, our approach is suggested by classical works on the Hamiltonian dynamics where the system evolution is described as the evolution of the corresponding correlation functions obtained by solving the equation

$$\frac{d}{dt}k_t = L^\Delta k_t, \qquad k_t|_{t=0} = k_0. \tag{1.5}$$

In the Hamiltonian dynamics, the analog of (1.5) is the BBGKY hierarchy. As mentioned in [2], kinetic equations of the Hamiltonian dynamics allow one to describe the evolution approximately but in more detail and in simpler terms. Such kinetic equations can be obtained from equations like (1.5), provided all necessary information about their solutions is available, see the corresponding discussion in [2, section 6].

In the present article, we describe the Markov evolution of states in terms of the correlation functions on both microscopic, obtained from (1.5), and mesoscopic levels. The latter will be done by means of a nonlinear (kinetic) equation obtained from (1.5) in the Vlasov scaling limit. As in [8, 15], our object is the Glauber dynamics described by the 'generator' (1.2) having the form

$$(LF)(\gamma) = \sum_{x\in\gamma} [F(\gamma \setminus x) - F(\gamma)] + \varkappa \int_{\mathbb{R}^d} \exp\Big(-\sum_{y\in\gamma} \phi(x-y)\Big) [F(\gamma \cup x) - F(\gamma)]\, dx. \tag{1.6}$$

Here the first term describes the particle death with constant rate, whereas the second one is the birth term with activity $\varkappa > 0$ and an interaction potential $\phi \geq 0$, which is supposed to obey a natural integrability condition only. In contrast to [8, 15], where $\varkappa$ and ϕ were subject to a certain constraint, here we obtain the evolution $k_0 \mapsto k_t$ for all $\varkappa$ and ϕ, which, however, is restricted to a limited time interval $[0, T_*)$. Instead of the semigroup techniques used in [8, 15], we apply Picard-like approximations and a method suggested in [4, pp. 94, 95], which allows for constructing classical solutions in a scale of Banach spaces[1] $\{\mathcal{K}_\alpha\}_{\alpha\in\mathcal{I}\subset\mathbb{R}}$, $\mathcal{K}_{\alpha'} \subset \mathcal{K}_{\alpha''}$ for $\alpha'' < \alpha'$. Namely, in Theorem 3.6 we show that, for any $\alpha_0 \in \mathbb{R}$ and any $\alpha < \alpha_0$, there exists $T(\alpha_0, \alpha) > 0$ such that, for any $t \in [0, T(\alpha_0, \alpha))$, there exists $\alpha_t \in (\alpha, \alpha_0)$ such that the problem (1.5) with $k_0 \in \mathcal{K}_{\alpha_0}$ has a classical solution $k_t \in \mathcal{K}_{\alpha_t}$ being the correlation function of a certain μ_t. The latter fact is obtained by means of the corresponding result of [3]. This yields the evolution $\mu_0 \mapsto \mu_t$. In addition, in Theorem 3.9 we show that, for $k_0(\eta) \leq \varkappa^{|\eta|}$, the solution obeys $k_t(\eta) \leq \varkappa^{|\eta|}$ and hence can be continued in time to the whole $\mathbb{R}_+$. These are the main results of Section 3. In Section 4, we perform the Vlasov scaling and obtain the Vlasov hierarchy—a linear evolution equation $(d/dt)r_t = L_V r_t$, which we study in the same scale of Banach spaces where the correlation functions evolve. Its main peculiarity is the fact that if r_0 is the correlation function of a nonhomogeneous Poisson measure π_{ϱ_0} with density ϱ_0, then the solution r_t is the correlation function for π_{ϱ_t} with ϱ_t satisfying a nonlinear nonlocal equation, see Lemma 4.2 and Theorem 4.4. Finally, in Theorem 4.5 we show that the rescaled correlation functions converge in the scaling limit to the corresponding r_t. In Section 5, we briefly summarize and compare with each other the results of Sections 3 and 4.

2 The basic notions and the model

2.1 The notions

All the details of the framework used in this paper can be found in [6, 8–11, 14]. We consider an infinite system of point particles located in $\mathbb{R}^d$, $d \geq 1$. By $\mathcal{B}(\mathbb{R}^d)$

[1] Further developments are known under the name *Ovsjannikov's method*, see e.g. [18].

and $\mathcal{B}_{\rm b}(\mathbb{R}^d)$ we denote the set of all Borel and the set of all bounded Borel subsets of $\mathbb{R}^d$, respectively. For $X \in \mathcal{B}(\mathbb{R}^d)$, the set of n-particle configurations in X is

$$\Gamma_X^{(0)} = \{\emptyset\}, \qquad \Gamma_X^{(n)} = \big\{\eta \subset X : |\eta| = n\big\}, \quad n \in \mathbb{N},$$

where $|\cdot|$ denotes cardinality. $\Gamma_X^{(n)}$ can be identified with the symmetrization of $\{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j, \text{ for } i \neq j\}$, which allows one to introduce the corresponding topology and hence the Borel σ-algebra $\mathcal{B}(\Gamma_X^{(n)})$. The set of finite configurations in X is

$$\Gamma_{0,X} = \bigsqcup_{n \in \mathbb{N}_0} \Gamma_X^{(n)}.$$

We equip it with the topology of the disjoint union and hence with the Borel σ-algebra $\mathcal{B}(\Gamma_{0,X})$. For $X = \mathbb{R}^d$, we write $\Gamma^{(n)}$ and Γ_0 meaning $\Gamma_X^{(n)}$ and $\Gamma_{0,X}$, respectively. The restriction of the Lebesgue product measure $dx_1 dx_2 \cdots dx_n$ to $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$ is denoted by $m^{(n)}$. The Lebesgue-Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is given by

$$\lambda = \delta_\emptyset + \sum_{n=1}^{\infty} \frac{1}{n!} m^{(n)}. \tag{2.1}$$

For any $X \in \mathcal{B}(\mathbb{R}^d)$, the restriction of λ to $\Gamma_{0,X}$ will also be denoted by λ.

The set of all configurations in $\mathbb{R}^d$ is

$$\Gamma = \{\gamma \subset \mathbb{R}^d : |\gamma \cap \Lambda| < \infty \ \text{ for all } \ \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)\}. \tag{2.2}$$

We equip it with the vague topology—the weakest topology in which all the maps

$$\Gamma \ni \gamma \mapsto \langle \gamma, f \rangle = \sum_{x \in \gamma} f(x), \quad f \in C_0(\mathbb{R}^d),$$

are continuous. Here $C_0(\mathbb{R}^d)$ stands for the set of all continuous $f : \mathbb{R}^d \to \mathbb{R}$, which have compact supports. The vague topology on Γ admits a metrization, which turns it into a complete and separable metric (Polish) space, see e.g. [12]. By $\mathcal{B}(\Gamma)$ we denote the corresponding Borel σ-algebra. It turns out that the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the family $\{(\Gamma_{0,\Lambda}, \mathcal{B}(\Gamma_{0,\Lambda}))\}_{\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)}$. Then the Poisson measure π on $(\Gamma, \mathcal{B}(\Gamma))$ is defined as the projective limit of the family $\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)}$, where

$$\pi^\Lambda = \exp(-m(\Lambda))\lambda, \tag{2.3}$$

$m(\Lambda)$ being the Lebesgue measure of Λ. The Poisson measure π_ϱ corresponding to the density $\varrho : \mathbb{R} \to \mathbb{R}_+$ is introduced by means of the measure λ_ϱ, defined as in (2.1) with m replaced by m_ϱ, where, for $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$,

$$m_\varrho(\Lambda) = \int_\Lambda \varrho(x) dx, \tag{2.4}$$

which is supposed to be finite. Then π_ϱ is defined by its projections

$$\pi_\varrho^\Lambda = \exp(-m_\varrho(\Lambda))\lambda_\varrho^\Lambda. \tag{2.5}$$

For a measurable $f : \mathbb{R}^d \to \mathbb{R}$ and $\eta \in \Gamma_0$, the Lebesgue-Poisson exponent is

$$e(f,\eta) := \prod_{x\in\eta} f(x), \qquad e(f,\emptyset) := 1. \tag{2.6}$$

Clearly $e(f,\cdot) \in L^1(\Gamma_0, d\lambda)$ for any $f \in L^1(\mathbb{R}^d)$, and

$$\int_{\Gamma_0} e(f,\eta)\lambda(d\eta) = \exp\Big\{\int_{\mathbb{R}^d} f(x)dx\Big\}. \tag{2.7}$$

A set $M \in \mathcal{B}(\Gamma_0)$ is said to be bounded if

$$M \subset \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)} \tag{2.8}$$

for some $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $N \in \mathbb{N}$. By $B_{\rm bs}(\Gamma_0)$ we denote the set of all bounded measurable functions $G : \Gamma_0 \to \mathbb{R}$, which have bounded supports. That is, each such G is the zero function on $\Gamma_0 \setminus M$ for some bounded M. Noteworthy, any measurable $G : \Gamma_0 \to \mathbb{R}$ is in fact a sequence of measurable symmetric functions $G^{(n)} : (\mathbb{R}^d)^n \to \mathbb{R}$.

For $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and $\gamma \in \Gamma$, by γ_Λ we denote $\gamma \cap \Lambda$; thus, $\gamma_\Lambda \in \Gamma_{0,\Lambda}$. A measurable function $F : \Gamma \to \mathbb{R}$ is called a *cylinder* function if there exist $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ and a measurable $G : \Gamma_{0,\Lambda} \to \mathbb{R}$ such that $F(\gamma) = G(\gamma_\Lambda)$ for all $\gamma \in \Gamma$. By $\mathcal{F}_{\rm cyl}(\Gamma)$ we denote the set of all cylinder functions. For $\gamma \in \Gamma$, by writing $\eta \Subset \gamma$ we mean that $\eta \subset \gamma$ and η is finite, i.e., $\eta \in \Gamma_0$. For $G \in B_{\rm bs}(\Gamma_0)$, we set

$$(KG)(\gamma) = \sum_{\eta\Subset\gamma} G(\eta), \qquad \gamma \in \Gamma. \tag{2.9}$$

It is known that K is linear and positivity preserving, and maps $B_{\rm bs}(\Gamma_0)$ into $\mathcal{F}_{\rm cyl}(\Gamma)$ (see e.g. [11]).

By $\mathcal{M}^1_{\rm fm}(\Gamma)$ we denote the set of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ which have finite local moments, that is, for which

$$\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < \infty \qquad \text{for all } n \in \mathbb{N} \text{ and } \Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d). \tag{2.10}$$

A measure ρ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is said to be *locally finite* if $\rho(M) < \infty$ for every bounded $M \subset \Gamma_0$. By $\mathcal{M}_{\rm lf}(\Gamma_0)$ we denote the set of all such measures. For $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, by p_Λ we denote the map $\Gamma \ni \gamma \mapsto p_\Lambda(\gamma) = \gamma_\Lambda$. Then, for $A \subset \Gamma_{0,\Lambda}$, we write $p_\Lambda^{-1}(A) = \{\gamma \in \Gamma : p_\Lambda(\gamma) \in A\}$. A measure $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$ is said to be *locally absolutely continuous* with respect to the Poisson measure π if, for every $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$, $\mu^\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to π^Λ, see (2.3).

Let $M \subset \Gamma_0$ be bounded, and let $\mathbb{I}_M$ be its indicator function on Γ_0. Then $\mathbb{I}_M$ is in $B_{\rm bs}(\Gamma_0)$ and hence one can apply (2.9). For $\mu \in \mathcal{M}^1_{\rm fm}(\Gamma)$, let

$$\rho_\mu(M) = \int_\Gamma (K\mathbb{I}_M)(\gamma)\mu(d\gamma), \tag{2.11}$$

which uniquely determines a measure $\rho_\mu \in \mathcal{M}_{\rm lf}(\Gamma_0))$. It is called the *correlation measure* for μ. This defines the map $K^* : \mathcal{M}^1_{\rm fm}(\Gamma) \to \mathcal{M}_{\rm lf}(\Gamma_0))$ such that $K^*\mu = \rho_\mu$. In particular, $K^*\pi = \lambda$. It is known that, see [11, Proposition 4.14], ρ_μ is absolutely continuous with respect to λ if μ is locally absolutely continuous with respect to π. In this case, we have that for any $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$,

$$k_\mu(\eta) = \frac{d\rho_\mu}{d\lambda}(\eta) = \int_{\Gamma_{0,\Lambda}} \frac{d\mu^\Lambda}{d\pi^\Lambda}(\eta \cup \gamma)\pi^\Lambda(d\gamma). \tag{2.12}$$

The Radon–Nikodym derivative k_μ is called the *correlation function* corresponding to the measure μ.

Finally, we mention the following integration rule, c.f. [8, Lemma 2.1],

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta)\lambda(d\eta) = \int_{\Gamma_0}\int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi)\lambda(d\xi)\lambda(d\eta), \tag{2.13}$$

which holds for any appropriate function H if both parts of (2.13) are finite.

2.2 The model

Let $\phi : \mathbb{R}^d \to \mathbb{R}_+ := [0, +\infty)$ be such that $\phi(x) = \phi(-x)$ and be integrable in the following sense

$$c_\phi := \int_{\mathbb{R}^d} \Big(1 - e^{-\phi(x)}\Big)dx < \infty. \tag{2.14}$$

For $\gamma \in \Gamma$, we set

$$E^\phi(x, \gamma) = \sum_{y \in \gamma} \phi(x - y), \tag{2.15}$$

with the possibility that $E^\phi(x, \gamma) = +\infty$ for some γ. In the model we consider, the dynamics of the observables is defined by the 'generator' (1.6) with ϕ just mentioned and $\varkappa > 0$ being the birth *activity* parameter. The action of the 'generator' (1.6) on $F \in \mathcal{F}_{\rm cyl}(\Gamma)$ is well-defined. Indeed, for any $F \in \mathcal{F}_{\rm cyl}(\Gamma)$, one finds $\Lambda \in \mathcal{B}_{\rm b}(\mathbb{R}^d)$ such that $F(\gamma \setminus x) = F(\gamma \cup x) = F(\gamma)$ for any $x \in \Lambda^c := \mathbb{R}^d \setminus \Lambda$. Thus, the sum and the integral in (1.6) are finite.

Following the general scheme developed in [13] one constructs the evolution of the *quasi-observables*, which are functions on Γ_0. This evolution is obtained as a solution to the following Cauchy problem

$$\frac{dG_t}{dt} = \hat{L}G_t, \qquad G_t|_{t=0} = G_0, \tag{2.16}$$

where $\hat{L} = K^{-1}LK$ is the so called *symbol* of L, which has the form

$$(\hat{L}G)(\eta) = -|\eta|G(\eta) + \varkappa \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e(t_x, \eta \setminus \xi)e(\tau_x, \xi)G(\xi \cup x)dx, \tag{2.17}$$

where $e(t_x, \cdot)$ is defined in (2.6), and

$$\tau_x(y) = e^{-\phi(x-y)}, \qquad t_x(y) = \tau_x(y) - 1, \tag{2.18}$$

see, e.g., [6, 15]. Clearly, the action of $\hat{L}$ on $G \in B_{\rm bs}(\Gamma_0)$ is well-defined. Its extension to wider classes of G will be done in a while.

For a measurable function $k : \Gamma_0 \to \mathbb{R}$ and $G \in B_{\rm bs}(\Gamma_0)$, we define

$$\langle\langle G, k\rangle\rangle = \int_{\Gamma_0} G(\eta)k(\eta)\lambda(d\eta). \tag{2.19}$$

This pairing can be extended to the corresponding classes of G and k. Then (2.16) and (2.19) lead to the following (dual) Cauchy problem

$$\frac{dk_t}{dt} = L^{\Delta}k_t, \qquad k_t|_{t=0} = k_0. \tag{2.20}$$

The action of L^{Δ} is obtained by means of (2.13) from

$$\langle\langle \hat{L}G, k\rangle\rangle = \langle\langle G, L^{\Delta}k\rangle\rangle,$$

and from (2.19) and (2.17). It thus has the form (see, e.g., [6, 15])

$$\begin{aligned}(L^{\Delta}k)(\eta) = &- |\eta|k(\eta) \\ &+ \varkappa \sum_{x\in\eta} e(\tau_x, \eta \setminus x) \int_{\Gamma_0} e(t_x, \xi)k(\eta \setminus x \cup \xi)\lambda(d\xi).\end{aligned} \tag{2.21}$$

Of course, the case of a special interest in (2.20) is where k_0 is the correlation function of a certain $\mu_0 \in \mathcal{M}^1_{\rm fm}(\Gamma)$, see (2.12). However, the mere existence of the solution k_t does not guarantee that this k_t is a correlation function.

In [13–15], the solution $G_0 \mapsto G_t$ of (2.16), for all $t \geq 0$ and 'small' $\varkappa$ and c_ϕ, was obtained in a certain Banach space by means of the construction of a C_0-semigroup based on perturbation methods. Then the evolution of the correlation functions $k_0 \mapsto k_t$ was obtained in the weak sense, in which k_t is defined by k_0 via the relation

$$\langle\langle G_0, k_t\rangle\rangle = \langle\langle G_t, k_0\rangle\rangle. \tag{2.22}$$

Regarding the problems (2.16) and (2.20), in the present article we realize the following program:

- Show that (2.16) has a unique classical solution for all $\varkappa > 0$ and c_ϕ, which we do in Theorem 3.1 for t belonging to a bounded interval.
- Show that the solution of (2.16) exists for all $t \geq 0$ if $\varkappa c_\phi < 1/e$, which we do in Theorem 3.2.
- Show that (2.20) has a unique classical solution k_t for all $\varkappa > 0$ and c_ϕ, being the correlation function of a certain $\mu_t \in \mathcal{M}^1_{\rm fm}(\Gamma)$, which yields the evolution of states $\mu_0 \mapsto \mu_t$. We do this in Theorem 3.6 for t belonging to a bounded interval.
- Show that the solution of (2.16) exists for all $t \geq 0$ if $k_0(\eta) \leq \varkappa^{|\eta|}$, which we do in Theorem 3.9.

These results give the microscopic evolution of states corresponding to (1.6). A similar program concerning the mesoscopic evolution is formulated and realized in Section 4 below.

3 The microscopic description

3.1 The evolution of quasi-observables

First we study the problem (2.16), (2.17). For $\alpha \in \mathbb{R}$, we consider the Banach space

$$\mathcal{G}_\alpha = L^1(\Gamma_0, e^{-\alpha|\cdot|}d\lambda), \tag{3.1}$$

that is, $G \in \mathcal{G}_\alpha$ if

$$\|G\|_\alpha := \int_{\Gamma_0} \exp(-\alpha|\eta|)\,|G(\eta)|\,\lambda(d\eta) < \infty. \tag{3.2}$$

We will seek the solution of (2.16), (2.17) as the limit of $\{G_t^{(n)}\}_{n\in\mathbb{N}_0} \subset \mathcal{G}_\alpha$, where $G_t^{(0)} = G_0$ and

$$G_t^{(n)} = G_0 + \int_0^t \hat{L}G_s^{(n-1)}ds, \quad n \in \mathbb{N}. \tag{3.3}$$

The latter can be iterated to give

$$G_t^{(n)} = G_0 + \sum_{m=1}^{n} \frac{1}{m!} t^m \hat{L}^m G_0. \tag{3.4}$$

We have $\tau_x(y) \leq 1$ since $\phi \geq 0$, see (2.18); hence, from (2.17)

$$\begin{aligned}\left|\hat{L}G(\eta)\right| &\leq |\eta||G(\eta)| + \varkappa \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} e(|t_x|, \eta\setminus\xi)|G(\xi\cup x)|dx \\ &:= H_1(\eta) + H_2(\eta).\end{aligned}$$

For any α', α'' such that $\alpha' < \alpha''$, we have

$$\begin{aligned}\|H_1\|_{\alpha''} &= \int_{\Gamma_0} |\eta| \exp\left(-(\alpha''-\alpha')|\eta|\right)|G(\eta)| \exp\left(-\alpha'|\eta|\right)\lambda(d\eta) \\ &\leq \frac{\|G\|_{\alpha'}}{(\alpha''-\alpha')e},\end{aligned} \tag{3.5}$$

where we have used the following obvious estimate

$$|\xi| \exp(-(\alpha''-\alpha')|\xi|) \leq \frac{1}{(\alpha''-\alpha')e}.$$

Furthermore,

$$
\begin{aligned}
\|H_2\|_{\alpha''} &= \varkappa \int_{\Gamma_0} \sum_{\xi\subset\eta} \int_{\mathbb{R}^d} e(|t_x|;\eta\setminus\xi)\exp(-\alpha''|\eta\setminus\xi| - \alpha''|\xi|)|G(\xi\cup x)|dx\lambda(d\eta) \\
&= \varkappa \int_{\Gamma_0} \int_{\Gamma_0} \int_{\mathbb{R}^d} e(|t_x|;\eta)\exp(-\alpha''|\eta| - \alpha''|\xi|)|G(\xi\cup x)|dx\lambda(d\xi)\lambda(d\eta) \\
&\leq \varkappa \int_{\mathbb{R}^d} \int_{\Gamma_0} \exp(-\alpha''|\xi|)|G(\xi\cup x)|dx\lambda(d\xi) \\
&\qquad \times \sup_{y\in\mathbb{R}} \left\{ \int_{\Gamma_0} e(|t_y|;\eta)\exp(-\alpha''|\eta|)\lambda(d\eta) \right\} \\
&= \varkappa \exp\left(c_\phi e^{-\alpha''}\right) \int_{\Gamma_0} \sum_{x\in\xi} \exp\left(-\alpha''|\xi\setminus x|\right)|G(\xi)|\lambda(d\xi) \\
&= \varkappa \exp\left(\alpha'' + c_\phi e^{-\alpha''}\right) \int_{\Gamma_0} |\xi| \exp\left(-\alpha''|\xi\setminus x|\right)|G(\xi)|\lambda(d\xi) \\
&= \varkappa \exp\left(\alpha'' + c_\phi e^{-\alpha''}\right) \frac{\|G\|_{\alpha'}}{(\alpha''-\alpha')e},
\end{aligned}
$$

where we have used also (2.7) and (2.13). By means of the latter estimate and (3.5) we finally get

$$
\|\hat{L}G\|_{\alpha''} \leq \frac{\|G\|_{\alpha'}}{(\alpha''-\alpha')e}\left[1 + \varkappa\exp\left(\alpha'' + c_\phi e^{-\alpha''}\right)\right]. \tag{3.6}
$$

From (3.6) we see that $\hat{L}$ can be defined as a bounded linear operator $\hat{L} : \mathcal{G}_{\alpha'} \to \mathcal{G}_{\alpha''}$ with the norm

$$
\|\hat{L}\|_{\alpha'\alpha''} \leq \frac{1}{(\alpha''-\alpha')e}\left[1 + \varkappa\exp\left(\alpha'' + c_\phi e^{-\alpha''}\right)\right]. \tag{3.7}
$$

Given $\alpha_0 \in \mathbb{R}$, $\alpha > \alpha_0$, $m \in \mathbb{N}$, and $l = 0, \ldots, m$, we take $\alpha_l = \alpha_0 + l\epsilon$, $\epsilon = (\alpha - \alpha_0)/m$. Then by (3.7) we get

$$
\|\hat{L}^m\|_{\alpha_0\alpha} \leq \|\hat{L}\|_{\alpha_0\alpha_1} \cdots \|\hat{L}\|_{\alpha_{m-1}\alpha} \leq (mM)^m, \tag{3.8}
$$

where

$$
M = \frac{1}{(\alpha-\alpha_0)e}\left[1 + \varkappa\exp\left(\alpha + c_\phi e^{-\alpha_0}\right)\right].
$$

Put

$$
T(\alpha,\alpha_0) = \frac{\alpha-\alpha_0}{1 + \varkappa\exp\left(\alpha + c_\phi e^{-\alpha_0}\right)}. \tag{3.9}
$$

Note that

$$
\begin{aligned}
T(\alpha,\alpha_0) &< \frac{1}{\varkappa}\exp\left(\log(\alpha-\alpha_0) - (\alpha-\alpha_0) - \alpha_0 - c_\phi e^{-\alpha_0}\right) \\
&\leq \frac{1}{\varkappa}\exp\left(-1 - \log c_\phi - 1\right) = \frac{1}{e^2\varkappa c_\phi}.
\end{aligned} \tag{3.10}
$$

Hence, we can set

$$T_* := \sup_{\alpha\in\mathbb{R}} \sup_{\alpha_0<\alpha} T(\alpha,\alpha_0) < \infty. \tag{3.11}$$

Our main result concerning the problem (2.16), (2.17) is the following statement.

Theorem 3.1. *Let α_0 and α be any real numbers such that $\alpha_0 < \alpha$. Then the problem (2.16), (2.17) with $G_0 \in \mathcal{G}_{\alpha_0}$ has a unique classical solution $G_t \in \mathcal{G}_\alpha$ on the time interval $t \in [0, T(\alpha,\alpha_0))$.*

Proof. Applying (3.8) in (3.4) we get that the sequence $\{G_t^{(n)}\}_{n\in\mathbb{N}_0}$ converges in $\mathcal{G}_\alpha$ uniformly on any $[0,T] \subset [0,T(\alpha,\alpha_0))$. In fact, one can show that for any $\alpha_1 \in (\alpha_0,\alpha)$ this sequence converges in $\mathcal{G}_{\alpha_1}$ to some $\tilde{G}_t$ uniformly on any $[0,T] \subset [0,T(\alpha_1,\alpha_0))$. Note that $T(\alpha,\alpha_0)$ continuously depend on α, therefore, we may consider any $[0,T] \subset [0,T(\alpha,\alpha_0))$. Since $\|\cdot\|_\alpha \le \|\cdot\|_{\alpha'}$, $\tilde{G}_t$ coincide with G_t in $\mathcal{G}_\alpha$. Hence, G_t belongs to any $\mathcal{G}_{\alpha_1}$, in particular, G_t is in the domain of the (unbounded) operator $\hat{L}$ in the space $\mathcal{G}_\alpha$. By (3.8), the sequence $\{\frac{d}{dt}G_t^{(n)}\}_{n\in\mathbb{N}_0}$ also converges in $\mathcal{G}_\alpha$ uniformly on any $[0,T] \subset [0,T(\alpha,\alpha_0))$. Therefore, G_t is a solution of (2.16), (2.17). The uniqueness can be shown in the same way as in [18, p. 16, 17]. □

The statement just proven describes systems with any $\varkappa$ and c_ϕ. It is, however, possible to get more if one imposes appropriate restrictions on these parameters. Namely, the dynamics in this case is described by a C_0-semigroup $S(t) : \mathcal{G}_\alpha \to \mathcal{G}_\alpha$, where the space is the same as in (3.1). Set

$$\mathcal{G}_\alpha^+ = \{G \in \mathcal{G}_\alpha \ : \ G \ge 0\}, \quad \mathcal{H}_\alpha = \{G \in \mathcal{G}_\alpha \ : \ |\cdot|G \in \mathcal{G}_\alpha\}, \tag{3.12}$$

and also

$$\mathcal{H}_\alpha^+ = \mathcal{H}_\alpha \cap \mathcal{G}_\alpha^+. \tag{3.13}$$

Theorem 3.2. *Assume that*

$$\varkappa c_\phi < 1/e, \tag{3.14}$$

and let $\alpha_\phi = \log c_\phi$. Then, for every $G_0 \in \mathcal{H}_{\alpha_\phi}$, the problem (2.16), (2.17) has a unique classical solution $G_t \in \mathcal{G}_{\alpha_\phi}$, $t \ge 0$, given by $G_t = S(t)G_0$, where $\{S(t)\}_{t\ge 0}$ is a C_0-semigroup on $\mathcal{G}_{\alpha_\phi}$.

Remark 3.3. (a) The above theorem is true not only for $\alpha = \alpha_\phi$ but also for $\alpha \in (\alpha_\phi - \delta, \alpha_\phi + \delta)$ for some $\delta > 0$, see the proof below. (b) The condition (3.14) is well-known, see [16, Chapter 4]. It provides the convergence of cluster expansions for the gas of classical particles with the pair-wise repulsion $U(x,y) = \phi(x-y)$. (c) As a matter of fact, the condition (3.14), in fact, coincides with the conditions obtained in [8] where the dynamics defined by a semigroup has been constructed by the completely different method.

In the proof of Theorem 3.2 we use two statements. The first one is an adaptation of [1, Corollary 5.16].

Proposition 3.4. *Given $\alpha \in \mathbb{R}$, assume that A is the generator of a positive C_0-semigroup in $\mathcal{G}_\alpha$ and $B = B_1 - B_2$ be such that $A + B_1 + B_2$ is also the generator of a positive C_0-semigroup. Then $A + B$ generates a C_0-semigroup in $\mathcal{G}_\alpha$.*

For $\eta \in \Gamma_0$, we set $\Xi_e(\eta) = \{\xi \subset \eta \ : \ |\eta \setminus \xi| \text{ is even}\}$ and $\Xi_o(\eta) = \{\xi \subset \eta \ : \ |\eta \setminus \xi| \text{ is odd}\}$, and thereby

$$(B_+G)(\eta) = \varkappa \sum_{\xi \in \Xi_e(\eta)} \int_{\mathbb{R}^d} e(\tau_x, \xi) e(t_x, \eta \setminus \xi) G(\xi \cup x) dx, \tag{3.15}$$

$$(B_-G)(\eta) = -\varkappa \sum_{\xi \in \Xi_0(\eta)} \int_{\mathbb{R}^d} e(\tau_x, \xi) e(t_x, \eta \setminus \xi) G(\xi \cup x) dx. \tag{3.16}$$

We also set

$$(AG)(\eta) = -|\eta| G(\eta), \tag{3.17}$$

and

$$\begin{aligned} (\hat{L}^+G)(\eta) &= ((A + B_+ + B_-)G)(\eta) \\ &= -|\eta| G(\eta) + \varkappa \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} e(\tau_x; \xi) e(|t_x|; \eta \setminus \xi) G(\xi \cup x) dx. \end{aligned} \tag{3.18}$$

Clearly, for any α, A with $Dom(A) = \mathcal{H}_\alpha$ generates a positive semigroup of contractions in $\mathcal{G}_\alpha$. All the three operators $-A$, $B_\pm$ are positive. The second statement we need to prove Theorem 3.2 is an adaptation to our case of [17, Theorem 2.7].

Proposition 3.5. *Let A and $B_\pm$ be as above. Suppose that there exist real α and α', $\alpha' < \alpha$, such that*

$$\forall G \in \mathcal{H}_\alpha^+ : \quad \int_{\Gamma_0} (\hat{L}^+G)(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) \leq 0, \tag{3.19}$$

$$\begin{aligned} \forall G \in \mathcal{H}_{\alpha'}^+ : \quad & \int_{\Gamma_0} (\hat{L}^+G)(\eta) \exp(-\alpha'|\eta|) \lambda(d\eta) \\ & \leq C \int_{\Gamma_0} G(\eta) \exp(-\alpha'|\eta|) \lambda(d\eta) - \varepsilon \int_{\Gamma_0} |\eta| G(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) \end{aligned} \tag{3.20}$$

for some positive C and ε. Then the closure of $\hat{L}^+$ in $\mathcal{G}_\alpha$ generates a positive C_0-semigroup in this space.

Proof of Theorem 3.2. Given α, for $G \in \mathcal{H}_\alpha^+$, similarly as in (3.7) we get

$$\begin{aligned} & \int_{\Gamma_0} (\hat{L}^+G)(\eta) \exp(-\alpha|\eta|) \lambda(d\eta) \\ & \leq -\left(1 - \varkappa \exp(\alpha + c_\phi e^{-\alpha})\right) \int_{\Gamma_0} |\eta| G(|\eta|) \exp(-\alpha|\eta|) \lambda(d\eta). \end{aligned} \tag{3.21}$$

By (3.14), we have $\varkappa \exp\left(\alpha_\phi + c_\phi e^{-\alpha_\phi}\right) < 1$; hence, one can pick small enough $\delta > 0$ such that $\varkappa \exp\left(\alpha + c_\phi e^{-\alpha}\right) < 1$ for any $\alpha \in (\alpha_\phi - \delta, \alpha_\phi + \delta)$. Then we fix such α and obtain (3.19) as the coefficient in the last line in (3.21) is negative. Next we pick $\sigma > 0$ such that $\alpha' := \alpha - \sigma$ is also in $(\alpha_\phi - \delta, \alpha_\phi + \delta)$. For this α', LHS(3.20) ≤ 0 for $G \in \mathcal{H}^+_{\alpha'}$. On the other hand,

$$\begin{aligned}\int_{\Gamma_0} |\eta| G(|\eta|) \exp(-\alpha|\eta|)\lambda(d\eta) &= \int_{\Gamma_0} |\eta| e^{-\sigma|\eta|} G(|\eta|) \exp(-\alpha'|\eta|)\lambda(d\eta) \\ &\leq \frac{1}{e\sigma} \int_{\Gamma_0} G(|\eta|) \exp(-\alpha'|\eta|)\lambda(d\eta),\end{aligned}$$

which yields that RHS(3.20) ≥ 0 for any C and sufficiently small ε. Hence, we can apply Proposition 3.5, by which the closure of $\hat{L}^+$ in $\mathcal{G}_\alpha$ generates a positive C_0-semigroup. But, under the condition (3.14), $\hat{L}^+$ is closed as A is closed. Thus, we are able now to apply Proposition 3.4 and complete the proof. □

3.2 The evolution of correlation functions and states

In this subsection, the problem (2.20), (2.21) will be studied in the Banach space, cf. (3.1),

$$\mathcal{K}_\alpha = L^\infty(\Gamma_0, e^{\alpha|\cdot|} d\lambda) \tag{3.22}$$

which we equip with the norm

$$\|k\|_\alpha = \operatorname{ess\,sup}\left\{|k(\eta)| \exp(\alpha|\eta|) : \eta \in \Gamma_0\right\}. \tag{3.23}$$

For the sake of convenience, here we use the same notation $\|\cdot\|_\alpha$ as in (3.2), which, however, should not cause any ambiguity as it will always be clear from the context which norm is meant.

Recall that $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ stands for the set of probability measures on Γ obeying (2.10). Given $\alpha \in \mathbb{R}$, by $\mathcal{M}_\alpha$ we denote the set of all those $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ whose correlation functions, defined in (2.12), belong to $\mathcal{K}_\alpha$. The main result of this section is contained in the following statement.

Theorem 3.6. *Fix any $\alpha_0 \in \mathbb{R}$ and any $\alpha < \alpha_0$, and let $T(\alpha_0, \alpha)$ be as in (3.9). Then, for every $t \in \big(0, T(\alpha_0, \alpha)\big)$, there exists $\alpha_t \in (\alpha, \alpha_0)$ such that the problem (2.20) with $k_0 \in \mathcal{K}_{\alpha_0}$ has a unique classical solution $k_t \in \mathcal{K}_{\alpha_t}$. This solution is the correlation function of a certain (unique) $\mu_t \in \mathcal{M}_{\alpha_t}$, which yields the evolution of states $\mu_0 \mapsto \mu_t$ of the considered model in the scale $\{\mathcal{M}_{\alpha_t}\}_{t \in [0, T(\alpha_0, \alpha))}$.*

Proof. First we prove the existence of the solution k_t in the Banach space $\mathcal{K}_{\alpha_t}$ and then show that it is a correlation function.

As in (3.3), we seek the solution of (2.20) as the limit of the sequence $\{k_t^{(n)}\}_{n \in \mathbb{N}_0}$, where

$$k_t^{(n)} = k_0 + \int_0^t L^\Delta k_s^{(n-1)} ds, \quad k_t^{(0)} = k_0, \tag{3.24}$$

which yields, cf. (3.4),

$$k_t^{(n)} = k_0 + \sum_{m=1}^{n} \frac{1}{m!} t^m \left(L^\Delta\right)^m k_0. \tag{3.25}$$

By (3.23), we have

$$|k(\eta)| \leq \|k\|_\alpha \exp(-\alpha|\eta|).$$

Thus, for α' and α'' as in (3.6), we get from (2.21)

$$\begin{aligned}\left|(L^\Delta k)(\eta)\right| \leq & \|k\|_{\alpha''} \exp(-\alpha'|\eta|)\Big\{ |\eta| \exp(-(\alpha''-\alpha')|\eta|) \\ & + \varkappa \exp\left(\alpha'' - (\alpha''-\alpha')|\eta|\right) \sum_{x\in\eta} \int_{\Gamma_0} e(|t_x|,\xi) \exp(-\alpha''|\xi|)\lambda(d\xi)\Big\}.\end{aligned}$$

Similarly as in producing (3.6) we then get from the latter

$$\|L^\Delta k\|_{\alpha'} \leq \frac{\|k\|_{\alpha''}}{(\alpha''-\alpha')e} \left[1 + \varkappa \exp\left(\alpha'' + c_\phi e^{-\alpha''}\right)\right]. \tag{3.26}$$

Hence, L^Δ can be defined as a bounded linear operator $L^\Delta : \mathcal{K}_{\alpha''} \to \mathcal{K}_{\alpha'}$ with the norm

$$\|L^\Delta\|_{\alpha''\alpha'} \leq \frac{1}{(\alpha''-\alpha')e} \left[1 + \varkappa \exp\left(\alpha_0 + c_\phi e^{-\alpha}\right)\right]. \tag{3.27}$$

Now we fix $t \in (0, T(\alpha_0,\alpha))$, and for $\tilde{\alpha} \in (\alpha, \alpha_0)$ and a given $m \in \mathbb{N}$, set $\alpha_l = \alpha_0 - l\epsilon$, $\epsilon = (\alpha_0 - \tilde{\alpha})/m$. Then, by (3.27),

$$\begin{aligned}\| \left(L^\Delta\right)^m \|_{\alpha_0\tilde{\alpha}} &\leq m^m \left[\frac{1 + \varkappa \exp\left(\alpha_0 + c_\phi e^{-\alpha}\right)}{(\alpha_0 - \tilde{\alpha})e}\right]^m \\ &= \left(\frac{m}{e}\right)^m \left(\frac{\alpha_0 - \alpha}{\alpha_0 - \tilde{\alpha}}\right)^m \frac{1}{[T(\alpha_0,\alpha)]^m},\end{aligned} \tag{3.28}$$

see (3.9). Pick $\delta > 0$ such that $t + \delta < T(\alpha_0, \alpha)$. Then for

$$\alpha < \alpha_t \leq \alpha_0 \left(1 - \frac{t+\delta}{T(\alpha_0,\alpha)}\right) + \alpha \frac{t+\delta}{T(\alpha_0,\alpha)}, \tag{3.29}$$

we have from (3.28)

$$\| \left(L^\Delta\right)^m \|_{\alpha_0\alpha_t} \leq \left(\frac{m}{(t+\delta)e}\right)^m.$$

Applying the last estimate in (3.24) we obtain the convergence of the sequence $\{k_t^{(n)}\}_{n\in\mathbb{N}_0}$ in $\mathcal{K}_{\alpha_t}$, which yields the existence and uniqueness of the solution k_t similarly to that in the proof of Theorem 3.1.

Now let us show that the solution just constructed is such that, for any $t \in (0, T(\alpha_0, \alpha))$, there exists $\mu_t \in \mathcal{M}^1_{\rm fm}(\Gamma)$ such that, cf. (2.12),

$$k_t(\eta) = \frac{d(K^*\mu_t)}{d\lambda}(\eta), \tag{3.30}$$

if $k_0 \in \mathcal{K}_{\alpha_0}$ is the correlation function of the corresponding $\mu_0 \in \mathcal{M}^1_{\rm fm}(\Gamma)$. Under the condition (2.14), there exists a proper $\mathcal{S} \subset \Gamma$ and an $\mathcal{S}$-valued process with sample paths in the Skorokhod space $D_{\mathcal{S}}(\mathbb{R}_+)$ associated with L, see [3, Theorem 2.13]. This yields the evolution $\mu_0 \mapsto \mu_t$, where μ_t is the law of the process, and hence the evolution of the corresponding correlation functions $k_{\mu_0} \mapsto k_{\mu_t}$, which satisfy (2.20). By the uniqueness just established, $k_t = k_{\mu_t}$ for all $t \in [0, T(\alpha_0, \alpha))$, which yields (3.30) and hence completes the proof. □

Remark 3.7. Theorem 3.6 establishes the evolution $k_0 \mapsto k_t$ which takes places in the scale of spaces $\mathcal{K}_{\alpha_t}$. It is clear from the proof that all such spaces are contained in $\mathcal{K}_\alpha$, that is, the mentioned theorem can be formulated similarly as Theorem 3.1.

Another our remark addresses the regularity of the solutions k_t. Instead of (3.22) let us consider

$$\widetilde{\mathcal{K}}_\alpha = \{k \in C(\Gamma_0 \to \mathbb{R}) : \|k\|_\alpha < \infty\}, \tag{3.31}$$

where this time

$$\|k\|_\alpha = \sup\left\{|k(\eta)| \exp(\alpha|\eta|) : \eta \in \Gamma_0\right\}. \tag{3.32}$$

Remark 3.8. Let α_0, α, and $T(\alpha_0, \alpha)$ be as in Theorem 3.6. Suppose in addition that the function ϕ is continuous. Then the problem (2.20) with $k_0 \in \widetilde{\mathcal{K}}_{\alpha_0}$ has a unique classical solution $k_t \in \widetilde{\mathcal{K}}_{\alpha_t}$ with $t \in [0, T(\alpha_0, \alpha))$, where α_t is the same as in Theorem 3.6.

Now we show that the evolution of k_t obtained in Theorems 3.1 and 3.6 can be continued in time to the whole $\mathbb{R}_+$. Recall that the space $\mathcal{K}_\alpha$ was defined in (3.22) and $\varkappa > 0$ is the birth activity parameter, see (1.6). Set

$$\alpha_\varkappa = -\log \varkappa, \qquad K_\varkappa := \{k \in \mathcal{K}_{\alpha_\varkappa} : \|k\|_{\alpha_\varkappa} \leq 1\}. \tag{3.33}$$

Theorem 3.9. *The solution of the problem (2.20) with $k_0 \in K_\varkappa$ can be continued to any positive t. Moreover, for every $t \geq 0$, the solution k_t is also in $K_\varkappa$.*

Note that a correlation function k is in $K_\varkappa$ if and only if $k(\eta) \leq \varkappa^{|\eta|}$ for λ-a.a. η. Thus, we state that $k_0(\eta) \leq \varkappa^{|\eta|}$ implies $k_t(\eta) \leq \varkappa^{|\eta|}$ for all $t \geq 0$.

The proof of Theorem 3.9 is based on the following estimate.

Lemma 3.10. *Suppose that $k_0 \in K_\varkappa$. For $\alpha < \alpha_\varkappa$, let k_t be the solution described by Theorem 3.6. Then, for all $t \in [0, T(\alpha_\varkappa, \alpha))$, we have that k_t is also in $K_\varkappa$.*

Proof. For $G \in B_{\mathrm{bs}}(\Gamma_0)$ and k_t as in Theorem 3.6, we set

$$\varphi(t) = \langle\langle G, k_t \rangle\rangle = \int_\Gamma (KG)(\gamma)\mu_t(d\gamma), \tag{3.34}$$

where μ_t is the corresponding state. According to (1.6), we have that

$$\begin{aligned}\frac{d}{dt}\varphi(t) &= \int_\Gamma (LKG)(\gamma)\mu_t(d\gamma) \\ &= \int_\Gamma \sum_{x\in\gamma} \left[(KG)(\gamma\setminus x) - (KG)(\gamma)\right]\mu_t(d\gamma) \\ &\quad + \varkappa \int_\Gamma \int_{\mathbb{R}^d} \left[(KG)(\gamma\cup x) - (KG)(\gamma)\right] \exp\left(-E^\phi(x,\gamma)\right)\mu_t(d\gamma).\end{aligned} \tag{3.35}$$

By (2.9),

$$\begin{aligned}(KG)(\gamma\setminus x) - (KG)(\gamma) &= \sum_{\xi\Subset\gamma\setminus x} G(\xi) - \sum_{\xi\Subset\gamma\setminus x} G(\xi) - \sum_{\xi\Subset\gamma\setminus x} G(\xi\cup x) \\ &= -\sum_{\xi\Subset\gamma\setminus x} G(\xi\cup x),\end{aligned}$$

and

$$(KG)(\gamma\cup x) - (KG)(\gamma) = \sum_{\xi\Subset\gamma} G(\xi\cup x) + \sum_{\xi\Subset\gamma} G(\xi) - \sum_{\xi\Subset\gamma} G(\xi) = \sum_{\xi\Subset\gamma} G(\xi\cup x).$$

Here all sums are finite since $G \in B_{\mathrm{bs}}(\Gamma_0)$.

Applying these equalities in (3.35) together with the following

$$\sum_{x\in\gamma}\sum_{\xi\Subset\gamma\setminus x} G(\xi\cup x) = \sum_{\xi\Subset\gamma}\sum_{x\in\xi} G(\xi) = \sum_{\xi\Subset\gamma} |\xi| G(\xi),$$

we arrive at

$$\begin{aligned}\frac{d}{dt}\varphi(t) &= -\int_{\Gamma_0} |\eta| G(\eta) k_t(\eta)\lambda(d\eta) \\ &\quad + \varkappa \int_\Gamma \left(\int_{\mathbb{R}^d} \sum_{\xi\Subset\gamma} G(\xi\cup x)dx\right) \exp\left(-E^\phi(x,\gamma)\right)\mu_t(d\gamma).\end{aligned} \tag{3.36}$$

Assume now that G is a positive element of $L^1(\Gamma_0, d\lambda)$. As ϕ in (2.15) is also positive, the second line in (3.36) can be estimated

$$\begin{aligned}&\varkappa \int_\Gamma \left(\int_{\mathbb{R}^d} \sum_{\xi\Subset\gamma} G(\xi\cup x)dx\right) \exp\left(-E^\phi(x,\gamma)\right)\mu_t(d\gamma) \\ &\le \varkappa \int_{\mathbb{R}^d} \left(\int_\Gamma \sum_{\xi\Subset\gamma} G(\xi\cup x)\mu_t(d\gamma)\right)dx = \varkappa \int_{\mathbb{R}^d} \left(\int_{\Gamma_0} G(\eta\cup x)k_t(\eta)\lambda(d\eta)\right)dx \\ &= \varkappa \int_{\Gamma_0} G(\eta) \sum_{x\in\eta} k_t(\eta\setminus x)\lambda(d\eta).\end{aligned}$$

In obtaining the latter equality we have used (2.13). Applying this estimate in (3.36) we arrive at

$$\frac{d}{dt}k_t(\eta) \leq -|\eta|k_t(\eta) + \varkappa \sum_{x\in\eta} k_t(\eta \setminus x), \tag{3.37}$$

which holds for λ-almost all $\eta \in \Gamma_0$. By standard methods we get from this

$$k_t(\eta) \leq k_0(\eta)e^{-|\eta|t} + \varkappa \int_0^t e^{-(t-s)|\eta|} \sum_{x\in\eta} k_s(\eta \setminus x)ds. \tag{3.38}$$

As a correlation function, $k_t(\eta) \geq 0$ for λ-almost all η. For a fixed t and $|\eta| = n$, assume that $k_s(\xi) \leq \varkappa^{|\xi|}$ for all $s \in [0,t]$ and all $|\xi| = n-1$. As we also assume $k_0(\eta) \leq \varkappa^{|\eta|}$, by (3.38) $k_s(\eta) \leq \varkappa^{|\eta|}$, also for all $s \in [0,t]$. Thus, $k_t \in K_\varkappa$. □

Proof of Theorem 3.9. Take any $\alpha < \alpha_\varkappa$. Then the solution k_t exists in $\mathcal{K}_\alpha$ for $t \in [0, T(\alpha_\varkappa, \alpha))$. If k_0 is in $K_\varkappa$, by Lemma 3.10 k_t is also in $K_\varkappa$. We take any $\tau \in [0, T(\alpha_\varkappa, \alpha))$ and consider the problem (2.20) for $\tilde{k}_t$ with the initial condition $\tilde{k}_0 = k_\tau \in K_\varkappa$. This gives the continuation in question to $[0, \tau + T(\alpha_\varkappa, \alpha))$. Then we repeat the above arguments. □

4 The mesoscopic description

The mesoscopic description of the dynamics of our model will be conducted in the Vlasov scaling framework, see [5] where the detailed presentation of this approach and the most updated related bibliography can be found. Our program in this section is as follows:

- Derive the Vlasov hierarchy as the (Vlasov) scaling limit of (2.20) and prove the existence of its solutions, see (4.12) and Proposition 4.1.
- Then derive the Vlasov equation from the Vlasov hierarchy and prove the existence of its solutions, see (4.15) and Theorem 4.4.
- Prove the convergence of the rescaled correlation functions to the solutions of the Vlasov hierarchy, see Theorem 4.5.

4.1 The Vlasov hierarchy

In the Vlasov scaling limit, which is achieved by letting $\varepsilon \to 0$, the particle density is supposed to diverge whereas the interaction gets weak and of long range. Thus, we assume that the correlation function $k^{(\varepsilon)}$ of the particle system we consider depends of the scaling parameter and diverges in such a way that the renormalized function

$$k^{(\varepsilon)}_{\text{ren}}(\eta) = \varepsilon^{|\eta|}k^{(\varepsilon)}(\eta), \tag{4.1}$$

has a finite limit, which we denote by r. The evolution of yet not rescaled functions is described by the equation (2.20) in which the generator L_ε^Δ contains $\varepsilon\phi$ and $\varepsilon^{-1}\varkappa$ in place of ϕ and $\varkappa$, respectively. Thus, the evolution of the rescaled functions (4.1) is described by the equation

$$\frac{d}{dt}k_{t,\mathrm{ren}}^{(\varepsilon)} = L_{\varepsilon,\mathrm{ren}}k_{t,\mathrm{ren}}^{(\varepsilon)}, \qquad k_{t,\mathrm{ren}}^{(\varepsilon)}|_{t=0} = k_{0,\mathrm{ren}}^{(\varepsilon)}, \tag{4.2}$$

where

$$L_{\varepsilon,\mathrm{ren}} = R_\varepsilon L_\varepsilon^\Delta R_\varepsilon^{-1}, \qquad (R_\varepsilon k)(\eta) := \varepsilon^{|\eta|}k(\eta). \tag{4.3}$$

By (2.21), we thus have

$$\begin{aligned}(L_{\varepsilon,\mathrm{ren}}k)(\eta) = &-|\eta|k(\eta) \\ &+\varkappa\sum_{x\in\eta} e(\tau_x^{(\varepsilon)},\eta\setminus x)\int_{\Gamma_0} e(\varepsilon^{-1}t_x^{(\varepsilon)},\xi)k(\eta\setminus x\cup\xi)\lambda(d\xi),\end{aligned} \tag{4.4}$$

with

$$\tau_x^{(\varepsilon)}(y) := \exp\left[-\varepsilon\phi(x-y)\right], \qquad t_x^{(\varepsilon)} := \tau_x^{(\varepsilon)} - 1. \tag{4.5}$$

Note that, for small enough ε, $R_\varepsilon^{-1}k$ might not be a correlation function, even if k is, see (4.3).

For any α_0, α', α'', and α such that $\alpha < \alpha' < \alpha'' < \alpha_0$, as in (3.27) we get

$$\|L_{\varepsilon,\mathrm{ren}}\|_{\alpha''\alpha'} \le \frac{1}{(\alpha''-\alpha')e}\left[1+\varkappa\exp\left(\alpha_0 + c_\phi^{(\varepsilon)}e^{-\alpha}\right)\right], \tag{4.6}$$

where, cf. (2.14),

$$c_\phi^{(\varepsilon)} = \varepsilon^{-1}\int_{\mathbb{R}^d}\left(1-e^{-\varepsilon\phi(x)}\right)dx. \tag{4.7}$$

Suppose now that ϕ is in $L^1(\mathbb{R}^d)$ and set

$$\langle\phi\rangle = \int_{\mathbb{R}^d}\phi(x)dx. \tag{4.8}$$

Recall that we still assume $\phi \ge 0$. Then

$$\begin{aligned}\|L_{\varepsilon,\mathrm{ren}}\|_{\alpha''\alpha'} &\le \sup_{\varepsilon>0}\{\mathrm{RHS}(4.6)\} \\ &= \frac{1}{(\alpha''-\alpha')e}\left[1+\varkappa\exp\left(\alpha_0+\langle\phi\rangle e^{-\alpha}\right)\right].\end{aligned} \tag{4.9}$$

Now let us informally pass in (4.4) to the limit $\varepsilon \to 0$. We then obtain the following operator

$$\begin{aligned}(L_V k)(\eta) = &-|\eta|k(\eta) \\ &+\varkappa\sum_{x\in\eta}\int_{\Gamma_0} e(-\phi(x-\cdot),\xi)k(\eta\setminus x\cup\xi)\lambda(d\xi).\end{aligned} \tag{4.10}$$

It certainly obeys

$$\|L_V\|_{\alpha''\alpha'} \leq \frac{1}{(\alpha''-\alpha')e}\left[1+\varkappa\exp\left(\alpha_0+\langle\phi\rangle e^{-\alpha}\right)\right], \tag{4.11}$$

and hence along with the problem (4.2) we can consider

$$\frac{d}{dt}r_t = L_V r_t, \qquad r_t|_{t=0} = r_0, \tag{4.12}$$

which is called the *Vlasov hierarchy* for the Glauber dynamic we consider. Set, cf. (3.9),

$$\widetilde{T}(\alpha_0,\alpha) := \frac{\alpha_0-\alpha}{1+\varkappa\exp\left(\alpha_0+\langle\phi\rangle e^{-\alpha}\right)} \leq T(\alpha_0,\alpha). \tag{4.13}$$

The latter inequality holds since $c_\phi^{(\varepsilon)} \leq \langle\phi\rangle$, see (4.7) and (4.8). Repeating the arguments used in the proof of Theorem 3.6 we obtain the following

Proposition 4.1. *Let ϕ, α_0, α, be as in Theorem 3.6 and $\widetilde{T}(\alpha_0,\alpha)$ be as in (4.13). Then the problem (4.2) (resp. (4.12)) with any $\varepsilon > 0$ and $k_{0,\mathrm{ren}}^{(\varepsilon)} \in \mathcal{K}_{\alpha_0}$ (resp. $r_0 \in \mathcal{K}_{\alpha_0}$) has a unique classical solution $k_{t,\mathrm{ren}}^{(\varepsilon)} \in \mathcal{K}_\alpha$ (resp. $r_t \in \mathcal{K}_\alpha$) with $t \in \left[0,\widetilde{T}(\alpha_0,\alpha)\right)$.*

Note that the passage from (4.4) to (4.10) was only 'informal', so we have no information how 'close' is r_t to $k_{t,\mathrm{ren}}^{(\varepsilon)}$. Another observation is that (4.12) has a very special solution, which we obtain now.

4.2 The Vlasov equation

For the potential ϕ and an appropriate function g, we write

$$(\phi * g)(x) = \int_{\mathbb{R}^d} \phi(x-y)g(y)dy. \tag{4.14}$$

Let us consider in $L^\infty(\mathbb{R}^d)$ the following problem, cf. [5, Example 8],

$$\frac{d}{dt}\varrho_t(x) = -\varrho_t(x) + \varkappa\exp\left(-(\phi*\varrho_t)(x)\right), \quad \varrho_t|_{t=0} = \varrho_0. \tag{4.15}$$

Given $\alpha \in \mathbb{R}$, we denote

$$\begin{aligned} \Delta_\alpha &= \left\{\varrho \in L^\infty(\mathbb{R}^d) : \|\varrho\|_{L^\infty(\mathbb{R}^d)} \leq e^{-\alpha}\right\}, \\ \Delta_\alpha^+ &= \left\{\varrho \in \Delta_\alpha : \varrho(x) \geq 0 \text{ a. e.}\right\}. \end{aligned} \tag{4.16}$$

Lemma 4.2. *Suppose that, for some $\alpha_0 \in \mathbb{R}$ and $T > 0$, the problem (4.15) with $\varrho_0 \in \Delta_{\alpha_0}^+$ has a unique classical solution $\varrho_t \in \Delta_{\alpha_0}^+$ on the time interval $[0,T)$. Then the solution $r_t \in \mathcal{K}_\alpha$, $\alpha < \alpha_0$, of the problem (4.12), as in Proposition 4.1, with $r_0(\eta) = e(\varrho_0,\eta) \in \mathcal{K}_{\alpha_0}$ has the form*

$$r_t(\eta) = e(\varrho_t,\eta) = \prod_{x\in\eta}\varrho_t(x), \tag{4.17}$$

and hence remains in $\mathcal{K}_{\alpha_0}$.

Proof. First of all we note that $e(\varrho,\cdot)\in\mathcal{K}_{\alpha_0}$ if and only if $\varrho\in\Delta_{\alpha_0}$, see (3.23). Now set $\tilde{r}_t=e(\varrho_t,\cdot)$ with ϱ_t solving (4.15). This $\tilde{r}_t$ solves (4.12), which can easily be checked by computing d/dt and employing (4.15). In view of the uniqueness as in Proposition 4.1, we then have $\tilde{r}_t=r_t$ on the time interval where both solutions exist. □

Remark 4.3. As (4.17) is the correlation function for the Poisson measure π_{ϱ_t}, see (2.4) and (2.5), the property established by the above lemma can be called the *chaos preservation*. Indeed, the most chaotic state of the system is the free state described by a Poisson measure.

Let us show now that the problem (4.15) does have the solution we need (cf. [7, Theorem 3.3]). In a standard way, this problem can be transformed into the following integral equation

$$\varrho_t(x)=\varrho_0(x)e^{-t}+\varkappa\int_0^t e^{-(t-s)}\exp\left(-(\phi*\varrho_s)(x)\right)ds. \tag{4.18}$$

Following classical Picard's scheme we seek the solution as the limit of the iterative sequence $\{\varrho_t^{(n)}\}_{n\in\mathbb{N}_0}$, defined as

$$\varrho_t^{(n)}(x)=\varrho_0(x)e^{-t}+\varkappa\int_0^t e^{-(t-s)}\exp\big(-(\phi*\varrho_s^{(n-1)})(x)\big)ds,\quad n\in\mathbb{N}, \tag{4.19}$$

and $\varrho_t^{(0)}=\varrho_0$. Clearly $\varrho_t^{(n)}\geq 0$ for all $n\in\mathbb{N}_0$. Thus, we have to show that $\varrho_t^{(n)}(x)\leq e^{-\alpha_0}$, at least for some $t>0$. By the induction over n, we see that this holds, for all $t>0$, if

$$\varkappa\leq e^{-\alpha_0}. \tag{4.20}$$

Now let us show that $\{\varrho_t^{(n)}\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence in $L^\infty(\mathbb{R}^d)$, assuming $\varrho_s^{(n)}\in\Delta_{\alpha_0}$, for all $n\in\mathbb{N}_0$ and $s\leq t$. From (4.19), using an elementary inequality

$$|e^{-a}-e^{-b}|\leq|a-b|,\qquad a,b\geq 0,$$

we get

$$\|\varrho_t^{(n)}-\varrho_t^{(n-1)}\|_{L^\infty(\mathbb{R}^d)}\leq q(t)\sup_{s\in[0,t]}\|\varrho_s^{(n-1)}-\varrho_s^{(n-2)}\|_{L^\infty(\mathbb{R}^d)},$$

where

$$q(t):=\varkappa\langle\phi\rangle\left(1-e^{-t}\right).$$

Now we take $T>0$ such that $q(T)<1$. Then the latter estimate yields

$$\sup_{t\in[0,T]}\|\varrho_t^{(n)}-\varrho_t^{(n-1)}\|_{L^\infty(\mathbb{R}^d)}\leq q(T)\sup_{t\in[0,T]}\|\varrho_t^{(n-1)}-\varrho_t^{(n-2)}\|_{L^\infty(\mathbb{R}^d)} \tag{4.21}$$

Therefore, the sequence $\{\varrho_t^{(n)}\}_{n\in\mathbb{N}_0}$ converges in $L^\infty(\mathbb{R}^d)$, uniformly on $[0,T]$. Thus, its limit is the unique classical solution of (4.15). Since this limit is still in $\Delta_{\alpha_0}^+$, the evolution can be continued. Taking into account Lemma 4.2 we come to the following conclusion.

Theorem 4.4. *Given $\varkappa > 0$, let α_0 be as in (4.20). Then the unique classical solution of (4.12) with $r_0 = e(\varrho_0, \cdot)$, $\varrho_0 \in \Delta^+_{\alpha_0}$, exists for all $t > 0$ and is given by (4.17) with $\varrho_t \in \Delta^+_{\alpha_0}$ being the solution of (4.15).*

4.3 The scaling limit $\varepsilon \to 0$

Our final task in this work is to show that the solution of (4.2) $k_t^{(\varepsilon)}$ converges in $\mathcal{K}_{\alpha_0}$ uniformly on $[0, T]$, $T < T(\alpha_0, \alpha)$, to the solution of (4.12), see Proposition 4.1. Here we should impose an additional condition on the potential ϕ, which, however, seems to be quite natural. Recall that in this section we suppose $\phi \in L^1(\mathbb{R}^d)$.

Theorem 4.5. *Let ϕ, α_0, α, and $\widetilde{T}(\alpha_0, \alpha)$ be as in Proposition 4.1. Assume also that $\phi \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and consider the problems (4.2) and (4.12) with $k^{(\varepsilon)}_{0,\mathrm{ren}} = r_0 \in \mathcal{K}_{\alpha_0}$. For their solutions $k^{(\varepsilon)}_{t,\mathrm{ren}}$ and r_t, it follows that $k^{(\varepsilon)}_{t,\mathrm{ren}} \to r_t$ in $\mathcal{K}_\alpha$, as $\varepsilon \to 0$, uniformly on every $[0, T]$, $T < \widetilde{T}(\alpha_0, \alpha)$.*

Proof. Given $n \in \mathbb{N}$, let $k^{(\varepsilon)}_{t,n}$ and $r_{t,n}$ be defined as in (3.24) with $L_{\varepsilon,\mathrm{ren}}$ and L_V, respectively. Like in the proof of Theorem 3.6, one can show that the sequences of $k^{(\varepsilon)}_{t,n}$ and $r_{t,n}$ converge in $\mathcal{K}_\alpha$ to $k^{(\varepsilon)}_t$ and r_t, respectively, uniformly on every $[0, T]$, $T < T(\alpha_0, \alpha)$. Then, for $\delta > 0$, one finds $n \in \mathbb{N}$ such that, for all $t \in [0, T]$,

$$\|k^{(\varepsilon)}_{t,n} - k^{(\varepsilon)}_{t,\mathrm{ren}}\|_\alpha + \|r_{t,n} - r_t\|_\alpha < \delta/2. \tag{4.22}$$

In view of (3.24),

$$\begin{aligned} \|k^{(\varepsilon)}_{t,\mathrm{ren}} - r_t\|_\alpha &\leq \left\| \sum_{m=1}^{n} \frac{1}{m!} t^m \left(L^m_{\varepsilon,\mathrm{ren}} - L^m_V\right) r_0 \right\|_\alpha + \frac{\delta}{2} \\ &\leq \|L_{\varepsilon,\mathrm{ren}} - L_V\|_{\alpha_0\alpha} \|r_0\|_{\alpha_0} T \exp\left(T b(\alpha_0, \alpha)\right) + \frac{\delta}{2}, \end{aligned} \tag{4.23}$$

where, see (3.27),

$$b(\alpha_0, \alpha) := \frac{1}{(\alpha_0 - \alpha)e} \left[1 + \varkappa \exp\left(\alpha_0 + \langle \phi \rangle e^{-\alpha}\right)\right].$$

Here we used the following representation

$$\begin{aligned} L^m_{\varepsilon,\mathrm{ren}} - L^m_V = &\left(L_{\varepsilon,\mathrm{ren}} - L_V\right) L^{m-1}_{\varepsilon,\mathrm{ren}} + L_V \left(L_{\varepsilon,\mathrm{ren}} - L_V\right) L^{m-2}_{\varepsilon,\mathrm{ren}} \\ &+ \cdots + L^{m-2}_V \left(L_{\varepsilon,\mathrm{ren}} - L_V\right) L_{\varepsilon,\mathrm{ren}} + L^{m-1}_V \left(L_{\varepsilon,\mathrm{ren}} - L_V\right). \end{aligned} \tag{4.24}$$

Thus, we have to show that

$$\|L_{\varepsilon,\mathrm{ren}} - L_V\|_{\alpha_0\alpha} \to 0, \quad \text{as } \varepsilon \to 0, \tag{4.25}$$

which will allow us to make the first summand in the right-hand side of (4.23) also smaller than $\delta/2$ and thereby to complete the proof.

Subtracting (4.10) from (4.4) we get

$$(L_{\varepsilon,\mathrm{ren}} - L_V)\, k(\eta) = \varkappa \sum_{x\in\eta} \int_{\Gamma_0} Q_\varepsilon(x, \eta\setminus x, \xi) k(\eta\setminus x\cup\xi)\lambda(d\xi) \tag{4.26}$$

where

$$\begin{aligned} Q_\varepsilon(x,\eta\setminus x,\xi) :&= e(\tau_x^{(\varepsilon)}, \eta\setminus x) e(\varepsilon^{-1} t_x^{(\varepsilon)}, \xi) - e(-\phi(x-\cdot),\xi) \\ &= e(\varepsilon^{-1} t_x^{(\varepsilon)}, \xi) - e(-\phi(x-\cdot) - \big[1 - e(\tau_x^{(\varepsilon)}, \eta\setminus x)\big] e(\varepsilon^{-1} t_x^{(\varepsilon)}, \xi). \end{aligned} \tag{4.27}$$

For $t>0$, the function $e^{-t}-1+t$ takes positive values only; hence,

$$\Psi(t) := (e^{-t}-1+t)/t^2, \quad t>0,$$

is positive and bounded, say by $C>0$. Then by means of the following elementary analog of (4.24)

$$b_1\cdots b_n - a_1\cdots a_n \le \sum_{i=1}^n (b_i - a_i) b_1\cdots b_{i-1} b_{i+1}\cdots b_n, \quad b_i \ge a_i > 0,$$

we obtain

$$\begin{aligned} \Big| e(\varepsilon^{-1} t_x^{(\varepsilon)}, \xi) - e(-\phi(x-\cdot)\Big| &\le \sum_{y\in\xi} \varepsilon [\phi(x-y)]^2 \Psi\left(\varepsilon\phi(x-y)\right) \prod_{z\in\xi\setminus y} \phi(x-z) \\ &\le \varepsilon C \sum_{y\in\xi} [\phi(x-y)]^2 e(\phi(x-\cdot), \xi\setminus y), \end{aligned}$$

and

$$\Big| \big[1 - e(\tau_x^{(\varepsilon)}, \eta\setminus x)\big] e(\varepsilon^{-1} t_x^{(\varepsilon)}, \xi) \Big| \le \varepsilon \sum_{y\in\eta\setminus x} \phi(x-y) e(\phi(x-\cdot),\xi).$$

Then from (4.26) for λ-almost all η we have, see (3.23),

$$\begin{aligned} &|(L_{\varepsilon,\mathrm{ren}} - L_V) k(\eta)| \\ &\le \varkappa \|k\|_{\alpha_0} e^{-\alpha_0|\eta|} \sum_{x\in\eta} \int_{\Gamma_0} \exp(-\alpha_0|\xi| + \alpha_0) \\ &\qquad \times \Big\{ \varepsilon C \sum_{y\in\xi} [\phi(x-y)]^2 e(\phi(x-\cdot), \xi\setminus y) + \varepsilon \sum_{y\in\eta\setminus x} \phi(x-y) e(\phi(x-\cdot),\xi) \Big\} \lambda(d\xi) \\ &\le \varkappa\varepsilon \|k\|_{\alpha_0} e^{-\alpha_0|\eta|} \sum_{x\in\eta} \int_{\Gamma_0} e^{-\alpha_0|\xi|} e(\phi(x-\cdot),\xi) \\ &\qquad \times \Big\{ C \int_{\mathbb{R}^d} [\phi(x-y)]^2 dy + e^{\alpha_0} \sum_{y\in\eta\setminus x} \phi(x-y) \Big\} \lambda(d\xi) \\ &\le \varkappa\varepsilon \|k\|_{\alpha_0} \|\phi\|_{L^\infty(\mathbb{R}^d)} \exp\left(\langle\phi\rangle e^{-\alpha_0}\right) \left[C\langle\phi\rangle|\eta| + e^{\alpha_0}|\eta|(|\eta|-1)\right] e^{-\alpha_0|\eta|}. \end{aligned}$$

This yields

$$\|L_{\varepsilon,\mathrm{ren}} - L_V\|_{\alpha_0\alpha} \leq \varepsilon\varkappa\|\phi\|_{L^\infty(\mathbb{R}^d)} \exp\left(\langle\phi\rangle e^{-\alpha_0}\right) \times \left[\frac{C\langle\phi\rangle}{(\alpha_0-\alpha)e} + \frac{4e^{\alpha_0}}{((\alpha_0-\alpha)e)^2}\right], \tag{4.28}$$

and thereby (4.25). □

5 Concluding remarks

Regarding the evolution of quasi-observables, in Theorem 3.1 we have proven its existence in $\mathcal{G}_\alpha$ if $G_0 \in \mathcal{G}_{\alpha_0}$, for any α_0 and any $\alpha > \alpha_0$, and for all values of the model parameters c_ϕ and $\varkappa$, however, on a bounded time interval. Note that the bound $T(\alpha, \alpha_0)$ is small for big $c_\phi\varkappa$, see (3.10). Note also that there exists the scale of spaces $\mathcal{G}_{\alpha_t} \subset \mathcal{G}_\alpha$ such that $G_t \in \mathcal{G}_{\alpha_t}$ for $t \in [0, T(\alpha, \alpha_0))$, similarly to Theorem 3.6. For $c_\phi\varkappa < 1/e$, the evolution $G_0 \mapsto G_t$ is described by a C_0-semigroup, and hence has no time bounds, see Theorem 3.2.

Let us turn now to the evolution of states and correlation functions. The main peculiarity of Theorem 3.6 is that, in contrast to the results of [13–15], here we (a) impose no restrictions on c_ϕ and $\varkappa$; (b) describe the evolution directly, not as a weak evolution via (2.22). The price is the time restriction, similar as in Theorem 3.1. Again, we can start in $\mathcal{K}_{\alpha_0}$ with any $\alpha_0 \in \mathbb{R}$, and obtain that $k_t \in \mathcal{K}_{\alpha_t} \subset \mathcal{K}_\alpha$, also for any $\alpha < \alpha_0$. The time bound $T(\alpha_0, \alpha)$ depends on the choice of α_0 and α. If the initial states is dominated by the Poisson measure with intensity $\varkappa$, that is, if $k_0(\eta) \leq \varkappa^{|\eta|}$, then the solution described by Theorem 3.6 has also the property $k_t(\eta) \leq \varkappa^{|\eta|}$, and hence can be continued in time ad infinitum, see Theorem 3.9. Of course, in this case α_0 should obey (4.20). The main aim of using the Vlasov hierarchy (4.12) is obtaining the scaling limit of the rescaled correlation functions $k^{(\varepsilon)}_{t,\mathrm{ren}}$. For any $\alpha_0 \in \mathbb{R}$ and $r_0 \in \mathcal{K}_{\alpha_0}$, this hierarchy has a unique classical solution r_t in any $\mathcal{K}_\alpha$, $\alpha < \alpha_0$, with $t \in [0, \widetilde{T}(\alpha_0, \alpha))$, see Proposition 4.1. Here, however, for general r_0 we have no tools for continuing r_t, like we did in Theorem 3.9 where we used the connection of L^Δ with L given by (1.6), since neither Markov operator corresponds to L_V. But if r_0 is Poissonian, i.e., $r_0 = e(\varrho_0, \cdot)$, then (4.12) has the solution $r_t = e(\varrho_t, \cdot)$ with infinite time lives in 'sufficiently large' $\mathcal{K}_{\alpha_0}$, see Theorem 4.4. The latter means that the Poissonian correlation function $k(\eta) = \varkappa^{|\eta|}$ belongs to this $\mathcal{K}_{\alpha_0}$, see (4.20).

Note also that in the recent paper [7] it was shown the existence and strong convergence in the Vlasov scaling for the classical solution in one space $\mathcal{K}_\alpha$ but again under the condition $c_\phi\varkappa < 1/e$.

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