AN OPERATOR APPROACH TO VLASOV SCALING FOR SOME MODELS OF SPATIAL ECOLOGY

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Abstract. We consider Vlasov-type scaling for Markov evolution of birth-and-death type in continuum, which is based on a proper scaling of corresponding Markov generators and has an algorithmic realization in terms of related hierarchical chains of correlation functions equations. The existence of rescaled and limiting evolutions of correlation functions and convergence to the limiting evolution are shown. The obtained results enable us to derive a non-linear Vlasov-type equation for the density of the limiting system.

1. Introduction

The Vlasov equation is a famous example of a kinetic equation which describes the dynamical behavior of a many-body system. In physics, it characterizes the Hamiltonian motion of an infinite particle system influenced by weak long-range forces in the mean field scaling limit. The detailed exposition of the Vlasov scaling for the Hamiltonian dynamics was given by W. Braun and K. Hepp [3] and later by R. L. Dobrushin [5] for more general deterministic dynamical systems. The limiting Vlasov-type equations for particle densities in both papers are considered in classes of integrable functions (or finite measures in the weak form). This actually corresponds to the situation of finite volume systems or systems with zero mean density in an infinite volume. The Vlasov equation for the integrable functions was investigated in details by V. V. Kozlov [17]. An excellent review about kinetic equations which describe dynamical multi-body systems was given by H. Spohn [25], [26]. Note that in the framework of interacting diffusions a similar problem is known as the McKean–Vlasov limit.

Motivated by the study of Vlasov scaling for some classes of stochastic evolutions in continuum for which the use of the mentioned above approaches breaks down (even in the finite volumes), we developed the general approach to study the Vlasov-type dynamics (see [9]). It is based on a proper scaling of the hierarchical equations for the evolution of correlation functions and can be interpreted in terms of the rescaled Markov generators. To the best of our knowledge presently it is only this technique that may give a possibility to control the convergence in the Vlasov limit in the case of non-integrable densities which is generic for infinite volume infinite particle systems. Speaking about the evolutions whose kinetic equations can not be studied by the classical techniques described in [3] and [5], we have in mind, first of all, spatial birth-and-death Markov processes (e.g., continuous Glauber dynamics, spatial ecological models) and hopping particles Markov evolutions (e.g., Kawasaki dynamics in continuum). The main difficulty to carry out the approach proposed by W. Braun, K. Hepp [3] and R. L. Dobrushin [5] for such models is the absence of a proper description in terms of stochastic evolutional equations. Another problem concerns the possible variation of particles number in the

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evolution. The important point to note also is that an application of the technique proposed in [9] leads to a limiting hierarchy which posses a chaos preservation property.

The aim of this paper is to study the Vlasov scaling for the individual based model (IBM) in spatial ecology introduced by B. Bolker and S. Pacala [1, 2], U. Dieckmann and R. Law [4] (BDLP model) using the scheme developed in [9]. A population in this model is represented by a configuration of motionless organisms (plants) located in an infinite habitat (a Euclidean space in our considerations). The unbounded habitat is taken to avoid boundary effects in the population evolution.

The evolution equation for the correlation functions of the BDLP model was studied in details in [8]. In [1, 2, 4] this system was called the system of spatial moment equations for plant competition and, actually, this system itself was taken as a definition of the dynamics in the BDLP model. The mathematical structure of the correlation functions evolution equation is close to other well-known hierarchical systems in mathematical physics, e.g., BBGKY hierarchy for the Hamiltonian dynamics (see, e.g. [6]). As in all hierarchical chains of equations, we can not expect an explicit form of the solution, and furthermore, the existence problem for these equations is a highly delicate question.

According to the general scheme (see [9]), we state conditions on structural coefficients of the BDLP Markov generator, which give a weak convergence of the rescaled generator to the limiting generator of the related Vlasov hierarchy. Next, we may compute limiting Vlasov type equation for the BDLP model leaving the question about the strong convergence of the hierarchy solutions for a separate analysis. A control of the strong convergence of the rescaled hierarchy is, in general, a difficult technical problem. In particular, this problem remains open for BBGKY hierarchy for the case of Hamiltonian dynamics as well as for Bogoliubov–Streltsova hierarchy corresponding to the gradient diffusion model. In the present paper we show the existence of the rescaled and limiting evolutions of correlation functions related to the Vlasov scaling of the BDLP model and the convergence to the limiting evolution. With this evolution for a special class of initial conditions is related a non-linear equation for the density, which is called Vlasov equation for the considered stochastic dynamics.

Let us mention that a version of the BDLP model for the case of finite populations was studied in the paper [12]. In this work the authors developed a probabilistic representation for the finite BDLP process and applied this technique to analyze a mean-field limit in the spirit of classical Dobrushin or McKean–Vlasov schemes. They obtained an integro-differential equation for the limiting deterministic process corresponding to an integrable initial condition. The latter equation coincides with the Vlasov equation for the BDLP model derived below in our approach.

The present paper is organized in the following way. Section 2 is devoted to the general settings required for the description of the model which we study. In Subsection 3.1 we discuss the general Vlasov scaling approach for spatial continuos models. Subsection 3.2 is devoted to the abstract convergence result for semigroups in Banach spaces which will be crucial to prove the main statements of the paper presented in Subsection 3.3. The corresponding proofs are given in Subsection 3.4.

We would like to note also about our recent paper [10]. In that paper we realized some parts of scheme mentioned above for general birth-and-death operators in continuum. In principle, the convergence result for the rescaled hierarchy in the present paper may be obtained after proper computations from the general scheme. We include this result in the present paper by the several reasons. First of all, the rescaled generator for the BDLP model has very specific structure (see Proposition 3.2). This form is quite typical for spatial ecological models and it is very important for modeling and computer simulations in applications to ecological models (see, e.g., [20] and references therein). On the other hand, the use of this structure allows us to make the proof of convergence more clear and natural. Moreover, it yields explicit expressions for resolvents of generators (see Remark 3.18) as well as estimates for these resolvents (see Subsection 3.4). It is clear now that the technique considered in the present paper might be easily applied for many other cases of stochastic dynamics of spatial ecology, in particular, for various multi-types dynamics. It is worth noting also that the nonlinear Vlasov (a.k.a. mesoscopic) equation for BDLP model which we derive in Theorem 3.11 is the generalization of the equation considered in [21]. In this theorem we prove the existence and uniqueness of a nonnegative bounded solution to this equation from the class of integrable densities, which is very important for real-life applications.

2. Basic facts and description of model

2.1. **General facts and notations.** Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$ and let $\mathcal{B}_b(\mathbb{R}^d)$ denote the system of all bounded sets in $\mathcal{B}(\mathbb{R}^d)$.

The space of n-point configuration is

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \left\{ \eta \subset \mathbb{R}^d \big| \ |\eta| = n \right\}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $|A|$ denotes the cardinality of the set A. The space $\Gamma_\Lambda^{(n)} := \Gamma_{0,\Lambda}^{(n)}$ for $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is defined analogously to the space $\Gamma_0^{(n)}$. As a set, $\Gamma_0^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \big| \ x_k \neq x_l \text{ if } k \neq l \right\},$$

i.e. $\widetilde{(\mathbb{R}^d)^n}/S_n$, where S_n is the permutation group over $\{1, \ldots, n\}$ acting naturally on $\widetilde{(\mathbb{R}^d)^n}$. Hence one can introduce the corresponding topology and Borel σ-algebra, which we denote by $O(\Gamma_0^{(n)})$ and $\mathcal{B}(\Gamma_0^{(n)})$, respectively. Also one can define a measure $m^{(n)}$ as the image of the product of Lebesgue measures $dm(x) = dx$ on $\big(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\big)$.

The space of finite configurations

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

is equipped with the topology which has the structure of disjoint union. Therefore, one can define the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0)$.

A set $B \in \mathcal{B}(\Gamma_0)$ is called bounded if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. The Lebesgue–Poisson measure λ_z on Γ_0 is defined as

$$\lambda_z := \sum_{n=0}^{\infty} \frac{z^n}{n!} m^{(n)}.$$

Here $z > 0$ is the so called activity parameter. The restriction of λ_z to Γ_Λ will be also denoted by λ_z.

The configuration space

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \ \big| \ |\gamma \cap \Lambda| < \infty \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}$$

is equipped with the vague topology. It is a Polish space (see e.g. [15]). The corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ is defined as the smallest σ-algebra for which all mappings $N_\Lambda : \Gamma \to \mathbb{N}_0$, $N_\Lambda(\gamma) := |\gamma \cap \Lambda|$ are measurable, i.e.,

$$\mathcal{B}(\Gamma) = \sigma \left(N_\Lambda \big| \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right).$$

One can also show that Γ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$ w.r.t. the projections $p_\Lambda : \Gamma \to \Gamma_\Lambda$, $p_\Lambda(\gamma) := \gamma_\Lambda$, $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$.

The Poisson measure π_z on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_z^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$, where π_z^Λ is the measure on Γ_Λ defined by $\pi_z^\Lambda := e^{-zm(\Lambda)} \lambda_z$.

We will use the following classes of functions: $L^0_{\mathrm{ls}}(\Gamma_0)$ is the set of all measurable functions on Γ_0 which have a local support, i.e. $G \in L^0_{\mathrm{ls}}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G\restriction_{\Gamma_0\setminus\Gamma_\Lambda}= 0$; $B_{\mathrm{bs}}(\Gamma_0)$ is the set of bounded measurable functions with bounded support, i.e. $G\restriction_{\Gamma_0\setminus B}= 0$ for some bounded $B \in \mathcal{B}(\Gamma_0)$.

On Γ we consider the set of cylinder functions $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$, i.e. the set of all measurable functions G on $\big(\Gamma, \mathcal{B}(\Gamma))\big)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. These functions are characterized by the following relation: $F(\gamma) = F\restriction_{\Gamma_\Lambda}(\gamma_\Lambda)$.

The following mapping between measurable functions on Γ_0, e.g. $L^0_{\mathrm{ls}}(\Gamma_0)$, and measurable functions on Γ, e.g. $\mathcal{F}_{\mathrm{cyl}}(\Gamma)$, plays the key role in our further considerations

$$KG(\gamma) := \sum_{\eta\Subset\gamma} G(\eta), \quad \gamma \in \Gamma, \tag{2.1}$$

where $G \in L^0_{\mathrm{ls}}(\Gamma_0)$, see e.g. [14, 18, 19]. The summation in the latter expression is taken over all finite subconfigurations of γ, which is denoted by the symbol $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad \eta \in \Gamma_0. \tag{2.2}$$

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ on $\big(\Gamma, \mathcal{B}(\Gamma)\big)$ which have finite local moments of all orders, i.e. $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}_0$. A measure ρ on $\big(\Gamma_0, \mathcal{B}(\Gamma_0)\big)$ is called locally finite iff $\rho(A) < \infty$ for all bounded sets A from $\mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$.

One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the K-transform, i.e., for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

The measure $\rho_\mu := K^*\mu$ is called the correlation measure of μ.

As shown in [14] for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G \in L^1(\Gamma_0, \rho_\mu)$ the series (2.1) is μ-a.s. absolutely convergent. Furthermore, $KG \in L^1(\Gamma, \mu)$ and

$$\int_{\Gamma_0} G(\eta)\,\rho_\mu(d\eta) = \int_\Gamma (KG)(\gamma)\,\mu(d\gamma). \tag{2.3}$$

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called locally absolutely continuous w.r.t. π_z iff $\mu_\Lambda := \mu\circ p_\Lambda^{-1}$ is absolutely continuous with respect to π^Λ_z for all $\Lambda \in \mathcal{B}_\Lambda(\mathbb{R}^d)$. In this case $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t λ_z. We denote

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_z}(\eta), \quad \eta \in \Gamma_0.$$

The functions

$$k^{(n)}_\mu : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+ \tag{2.4}$$

$$k^{(n)}_\mu(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are the correlation functions well known in statistical physics, see e.g. [23], [24].

2.2. **Description of model.** We consider a system of interacting individuals (particles) in the space $\mathbb{R}^d$ which evolves in time. The state of the system at a fixed moment of time $t > 0$ is described by a random configuration γ_t from Γ. Heuristically, the mechanism of the evolution is given by a Markov generator which has the following form

$$L := L^- + L^+,$$

where

$$
(2.5)\quad \begin{aligned}
(L^{-}F)(\gamma) &:= (L^{-}(m,\varkappa^{-},a^{-})F)(\gamma) := \sum_{x\in\gamma}\Big[m+\varkappa^{-}\sum_{y\in\gamma\setminus x}a^{-}(x-y)\Big]D_{x}^{-}F(\gamma),\\
(L^{+}F)(\gamma) &:= (L^{+}(\varkappa^{+},a^{-})F)(\gamma) := \varkappa^{+}\int_{\mathbb{R}^d}\sum_{y\in\gamma}a^{+}(x-y)D_{x}^{+}F(\gamma)\,dx.
\end{aligned}
$$

Here $0 \leq a^{-}$, $a^{+} \in L^{1}(\mathbb{R}^{d})$ are arbitrary, even functions such that

$$
\int_{\mathbb{R}^d} a^{-}(x)\,dx = \int_{\mathbb{R}^d} a^{+}(x)\,dx = 1
$$

(in other words, a^{-}, a^{+} are probability densities) and m, $\varkappa^{-}$, $\varkappa^{+} > 0$ are some positive constants.

The pre-generator L describes the Bolker–Dieckmann–Law–Pacala BDLP model, which was introduced in [1, 2, 4]. During the corresponding stochastic evolution, the birth of individuals occurs independently and the death is ruled not only by the global regulation (mortality m) but also by the local regulation with the kernel $\varkappa^{-}a^{-}$. This regulation may be described as a competition (e.g., for resources) between individuals in the population.

The evolution of the one-dimensional distribution for such systems can be expressed in terms of their characteristics, e.g. the correlation functions (see (2.4)). The dynamics of correlation functions for the BDLP model was studied in [8]. The main result of this paper informally says the following:

If the mortality m and the competition kernel $\varkappa^{-}a^{-}$ are large enough, then the dynamics of correlation functions associated with the pre-generator (2.5) *exists and preserves (sub-)Poissonian bound.*

For the reader's convenience we repeat below the relevant material from [8] without proofs.

Let $\hat{L}^{\pm} := K^{-1}L^{\pm}K$ be the K-image of $L^{\pm}$, which can be initially defined on functions from $B_{\mathrm{bs}}(\Gamma_0)$. For arbitrary and fixed $C > 0$ we consider the operator $\hat{L} := \hat{L}^{+} + \hat{L}^{-}$ in the functional space

$$
\mathcal{L}_C = L^{1}\left(\Gamma_0, C^{|\eta|}d\lambda\left(\eta\right)\right).
$$

Below, symbol $\|\cdot\|_C$ stands for the norm of this space.

For any $\omega > 0$ we define $\mathcal{H}(\omega)$ to be the set of all densely defined closed operators T on $\mathcal{L}_C$, the resolvent set $\rho(T)$ of which contains the sector

$$
\mathrm{Sect}\left(\frac{\pi}{2}+\omega\right) := \left\{\zeta\in\mathbb{C}\;\middle|\;|\arg\zeta| < \frac{\pi}{2}+\omega\right\},
$$

and for any $\varepsilon > 0$

$$
\|(T-\zeta 1\!\!1)^{-1}\| \leq \frac{M_\varepsilon}{|\zeta|},\quad |\arg\zeta| \leq \frac{\pi}{2}+\omega-\varepsilon,
$$

where M_ε does not depend on ζ.

The first non-trivial result, which is based on the perturbation theory, says that the operator $\hat{L}$ with the domain

$$
D(\hat{L}) := \left\{G\in\mathcal{L}_C\;\middle|\;|\cdot|\,G(\cdot)\in\mathcal{L}_C,\; E^{a^{-}}(\cdot)G(\cdot)\in\mathcal{L}_C\right\}
$$

is a generator of a holomorphic C_0-semigroup $\hat{U}_t$ on $\mathcal{L}_C$.

To construct the corresponding evolution of correlation functions we note that the dual space $(\mathcal{L}_C)' = \big(L^{1}(\Gamma_0, d\lambda_C)\big)' = L^{\infty}(\Gamma_0, d\lambda_C)$, where $d\lambda_C := C^{|\cdot|}d\lambda$. The space $(\mathcal{L}_C)'$ is isometrically isomorphic to the Banach space

$$
\mathcal{K}_C := \left\{k:\Gamma_0\to\mathbb{R}\;\middle|\;k(\cdot)C^{-|\cdot|}\in L^{\infty}(\Gamma_0,\lambda)\right\}
$$

with the norm

$$\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|}k(\cdot)\|_{L^\infty(\Gamma_0,\lambda)},$$

where the isomorphism is provided by the isometry R_C

$$(\mathcal{L}_C)' \ni k \longmapsto R_C k := k(\cdot)C^{|\cdot|} \in \mathcal{K}_C. \tag{2.6}$$

In fact, we have duality between Banach spaces $\mathcal{L}_C$ and $\mathcal{K}_C$ given by the following expression:

$$\langle\!\langle G, k\rangle\!\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda, \quad G \in \mathcal{L}_C, \quad k \in \mathcal{K}_C \tag{2.7}$$

with

$$|\langle\!\langle G, k\rangle\!\rangle| \le \|G\|_C \cdot \|k\|_{\mathcal{K}_C}. \tag{2.8}$$

It is clear that for any $k \in \mathcal{K}_C$

$$|k(\eta)| \le \|k\|_{\mathcal{K}_C} C^{|\eta|} \quad \text{for} \quad \lambda\text{-a.a. } \eta \in \Gamma_0. \tag{2.9}$$

Let $\hat{L}'$ be the adjoint operator to $\hat{L}$ in $(\mathcal{L}_C)'$ with domain $D(\hat{L}')$. Its image in $\mathcal{K}_C$ under the isometry R_C we denote by $\hat{L}^* = R_C \hat{L}' R_{C^{-1}}$. It is evident that the domain of $\hat{L}^*$ will be $D(\hat{L}^*) = R_C D(\hat{L}')$, correspondingly. Then, for any $G \in \mathcal{L}_C$, $k \in D(\hat{L}^*)$

$$\begin{aligned}\int_{\Gamma_0} G \cdot \hat{L}^* k \, d\lambda &= \int_{\Gamma_0} G \cdot R_C \hat{L}' R_{C^{-1}} k \, d\lambda = \int_{\Gamma_0} G \cdot \hat{L}' R_{C^{-1}} k \, d\lambda_C \\ &= \int_{\Gamma_0} \hat{L}G \cdot R_{C^{-1}} k \, d\lambda_C = \int_{\Gamma_0} \hat{L}G \cdot k \, d\lambda,\end{aligned}$$

therefore, $\hat{L}^*$ is the dual operator to $\hat{L}$ w.r.t. the duality (2.7). By [11], we have the precise form of $\hat{L}^*$

$$\begin{aligned}(\hat{L}^* k)(\eta) = &- \Bigl(m|\eta| + \varkappa^- E^{a^-}(\eta) \Bigr) k(\eta) \\ &+ \varkappa^+ \sum_{x\in\eta} \sum_{y \in \eta\setminus x} a^+(x-y) k(\eta \setminus x) \\ &+ \varkappa^+ \int_{\mathbb{R}^d} \sum_{y\in\eta} a^+(x-y) k((\eta\setminus y)\cup x)\, dx \\ &+ \varkappa^- \int_{\mathbb{R}^d} \sum_{y\in\eta} a^-(x-y) k(\eta\cup x)\, dx.\end{aligned} \tag{2.10}$$

Now we consider the adjoint semigroup $\hat{T}'(t)$ on $(\mathcal{L}_C)'$ and its image $\hat{T}^*(t)$ in $\mathcal{K}_C$. The latter one describes the evolution of correlation functions. Transferring the general results about adjoint semigroups (see, e.g., [7]) onto semigroup $\hat{T}^*(t)$ we deduce that it will be weak*-continuous and weak*-differentiable at 0. Moreover, $\hat{L}^*$ will be the weak*-generator of $\hat{T}^*(t)$. Here and subsequently we mean "weak*-properties" w.r.t. the duality (2.7).

3. Vlasov scaling

3.1. **Description of Vlasov scaling.** We begin with a general idea of the Vlasov-type scaling. It is of interest to construct some scaling L_ε, $\varepsilon > 0$ of the generator L such that the following scheme is realized.

Suppose that we know a proper scaling of L and we are able to prove the existence of the semigroup $\hat{T}_\varepsilon(t)$ with the generator $\hat{L}_\varepsilon := K^{-1} L_\varepsilon K$ in the space $\mathcal{L}_C$ for some $C > 0$.

Let us consider the Cauchy problem corresponding to the adjoint operator $\hat{L}^*$ and take an initial function with a strong singularity in ε. Namely,

$$k_0^{(\varepsilon)}(\eta) \sim \varepsilon^{-|\eta|} r_0(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0,$$

where the function r_0 is independent of ε. The solution to this problem is described by the dual semigroup $\hat{T}_\varepsilon^*(t)$. The scaling $L \mapsto L_\varepsilon$ has to be chosen in such a way that $\hat{T}_\varepsilon^*(t)$ preserves the order of the singularity:

$$(\hat{T}_\varepsilon^*(t) k_0^{(\varepsilon)})(\eta) \sim \varepsilon^{-|\eta|} r_t(\eta), \quad \varepsilon \to 0, \quad \eta \in \Gamma_0.$$

Another important requirement on a proper scaling concerns the dynamics $r_0 \mapsto r_t$. It should preserve the so-called Lebesgue–Poisson exponents: if

$$r_0(\eta) = e_\lambda(\rho_0, \eta) := \prod_{x \in \eta} \rho_0(x)$$

then

$$r_t(\eta) = e_\lambda(\rho_t, \eta) := \prod_{x \in \eta} \rho_t(x)$$

and there exists an explicit (nonlinear, in general) differential equation for ρ_t

$$\frac{\partial}{\partial t} \rho_t(x) = \upsilon(\rho_t(x)), \tag{3.1}$$

which will be called a Vlasov-type equation.

Now let us explain the main technical steps to realize a Vlasov-type scaling. Let us consider for any $\varepsilon > 0$ the following mapping (cf. (2.6)) on functions on Γ_0

$$(R_\varepsilon r)(\eta) := \varepsilon^{|\eta|} r(\eta). \tag{3.2}$$

This mapping is "self-dual" w.r.t. the duality (2.7), moreover, $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$. Then we have $k_0^{(\varepsilon)} \sim R_{\varepsilon^{-1}} r_0$, and we need $r_t \sim R_\varepsilon \hat{T}_\varepsilon^*(t) k_0^{(\varepsilon)} \sim R_\varepsilon \hat{T}_\varepsilon^*(t) R_{\varepsilon^{-1}} r_0$. Therefore, we have to show that for any $t \geq 0$ the operator family $R_\varepsilon \hat{T}_\varepsilon^*(t) R_{\varepsilon^{-1}}$, $\varepsilon > 0$ has limiting (in a proper sense) operator $U(t)$ and

$$U(t) e_\lambda(\rho_0) = e_\lambda(\rho_t). \tag{3.3}$$

But, informally, $\hat{T}_\varepsilon^*(t) = \exp\{t \hat{L}_\varepsilon^*\}$ and $R_\varepsilon \hat{T}_\varepsilon^*(t) R_{\varepsilon^{-1}} = \exp\{t R_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}\}$. Let us consider the "renormalized" operator

$$\hat{L}_{\varepsilon,\,\mathrm{ren}}^* := R_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}. \tag{3.4}$$

In fact, we need that there exists an operator $\hat{V}^*$ (called Vlasov operator) such that $\exp\{t R_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}\} \to \exp\{t \hat{V}^*\} =: U(t)$ for which (3.3) holds. Hence, heuristic way to produce the scaling $L \mapsto L_\varepsilon$ is to demand that

$$\lim_{\varepsilon \to 0} \left(\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) - \hat{L}_{\varepsilon,\,\mathrm{ren}}^* e_\lambda(\rho_t, \eta) \right) = 0, \quad \eta \in \Gamma_0,$$

if ρ_t satisfies (3.1). The point-wise limit of $\hat{L}_{\varepsilon,\,\mathrm{ren}}^*$ will be a natural candidate for $\hat{V}^*$. Having chosen a proper scaling we proceed to the following technical steps which give a rigorous meaning to the idea introduced above. Note that definition (3.4) implies $\hat{L}_{\varepsilon,\,\mathrm{ren}} = R_{\varepsilon^{-1}} \hat{L}_\varepsilon R_\varepsilon$. We prove that "renormalized" operator $\hat{L}_{\varepsilon,\,\mathrm{ren}}$ is the generator of a contraction semigroup $\hat{T}_{\varepsilon,\,\mathrm{ren}}(t)$ on $\mathcal{L}_C$. Next we show that this semigroup converges strongly to some semigroup $\hat{T}_V(t)$ with the generator $\hat{V}$. This limiting semigroup leads us directly to the solution for the Vlasov-type equation. Below we show how to realize this scheme in details.

3.2. **Approximation in Banach space.** In this subsection we study general question about the strong convergence of semigroups in Banach spaces. The obtained results will be crucial in the realization of the Vlasov-type scaling for the BDLP model.

Let $\{U_t^\varepsilon,\, t\geq 0\}$, $\varepsilon\geq 0$ be a family of semigroups on a Banach space E. We set $(L_\varepsilon,\, D(L_\varepsilon))$ to be the generator of $\{U_t^\varepsilon,\, t\geq 0\}$ for each $\varepsilon\geq 0$. Our purpose now is to describe the strong convergence of semigroups $\{U_t^\varepsilon,\, t\geq 0\}$, $\varepsilon\geq 0$ in terms of the corresponding generators as ε tends to 0. According to the classical result (see e.g. [13]), it is enough to show that there exists $\beta>0$ and $\lambda:\ \operatorname{Re}\lambda>\beta$ such that

$$(L_\varepsilon-\lambda 1\!\!1)^{-1}\xrightarrow{\ s\ }(L_0-\lambda 1\!\!1)^{-1},\quad \varepsilon\to 0, \tag{3.5}$$

where $1\!\!1$ is the identical operator. In this subsection we show how to prove (3.5) under the following assumptions on the family $(L_\varepsilon,\, D(L_\varepsilon))$, $\varepsilon\geq 0$:

Assumptions (A):

(1) For any $\varepsilon\geq 0$, the operator $(L_\varepsilon,\, D(L_\varepsilon))$ admits a representation
$$L_\varepsilon=A_1(\varepsilon)+A_2(\varepsilon),$$
where $A_1(\varepsilon)$ is a closed operator and $D(A_1(\varepsilon))=D(A_2(\varepsilon)):=D(L_\varepsilon)$.

(2) There exist $\beta>0$ and λ, $\operatorname{Re}\lambda>\beta$, such that
  (a) λ belongs to the resolvent set of $A_1(\varepsilon)$ for any $\varepsilon\geq 0$ and
  $$(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\xrightarrow{\ s\ }(A_1(0)-\lambda 1\!\!1)^{-1},\quad \varepsilon\to 0,$$
  (b) the family of resolvents $(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}$, $\varepsilon>0$, is uniformly bounded, moreover,
  $$\sup_{\varepsilon>0}\left\|(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\right\|\leq\left\|(A_1(0)-\lambda 1\!\!1)^{-1}\right\|,$$
  (c) for any $\varepsilon\geq 0$
  $$\left\|A_2(\varepsilon)\,(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\right\|<1,$$
  (d) the operator $\left(A_2(\varepsilon)\,(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}+1\!\!1\right)^{-1}$ converges strongly to the operator $\left(A_2(0)\,(A_1(0)-\lambda 1\!\!1)^{-1}+1\!\!1\right)^{-1}$ as $\varepsilon\to 0$.

The strong convergence result for the family $\{U_t^\varepsilon,\, t\geq 0\}$, $\varepsilon\geq 0$ is established by our next theorem.

Theorem 3.1. *Let $(L_\varepsilon,\, D(L_\varepsilon))$, $\varepsilon\geq 0$ be the family of generators corresponding to a family of C_0-semigroups $\{U_t^\varepsilon,\, t\geq 0\}$, $\varepsilon\geq 0$. Then, U_t^ε converges strongly to U_t^0 as $\varepsilon\to 0$ uniformly on each finite interval of time, provided assumptions* **(A)** *are satisfied.*

Proof. The proof is completed by showing (3.5). For any $\varepsilon\geq 0$ and λ from the resolvent set of $A_1(\varepsilon)$ we have

$$\operatorname{Ran}\left((A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\right)=D\left(A_1(\varepsilon)\right)=D\left(A_2(\varepsilon)\right)=D(L_\varepsilon).$$

Hence,

$$\begin{aligned}L_\varepsilon-\lambda 1\!\!1&=A_1(\varepsilon)+A_2(\varepsilon)-\lambda 1\!\!1\\&=\left(A_2(\varepsilon)\,(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}+1\!\!1\right)(A_1(\varepsilon)-\lambda 1\!\!1).\end{aligned}\tag{3.6}$$

Combining (3.6) with the assumption 2(c) of **(A)** we get the following representations for the resolvent:

$$\begin{aligned}(L_\varepsilon-\lambda 1\!\!1)^{-1}&=(A_1(\varepsilon)+A_2(\varepsilon)-\lambda 1\!\!1)^{-1}\\&=(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}\left(A_2(\varepsilon)\,(A_1(\varepsilon)-\lambda 1\!\!1)^{-1}+1\!\!1\right)^{-1}.\end{aligned}\tag{3.7}$$

From this formula, triangle inequality and assumptions 2(a), 2(b) and 2(d) of **(A)** we conclude the assertion of the theorem. □

3.3. **Main results.** We check at once that a proper scaling for the BDLP pre-generator is the following one:

$$
\begin{aligned}
(L_\varepsilon F)(\gamma) :=& \sum_{x\in\gamma}\Big[m+\varepsilon\varkappa^- \sum_{y\in\gamma\setminus x} a^-(x-y)\Big] D_x^- F(\gamma) \\
&+\varkappa^+ \int_{\mathbb{R}^d} \sum_{y\in\gamma} a^+(x-y) D_x^+ F(\gamma)\, dx, \quad \varepsilon>0.
\end{aligned} \tag{3.8}
$$

Next we consider the formal K-image of L_ε and the corresponding renormalized operator on $B_{\mathrm{bs}}(\Gamma_0)$:

$$
\hat{L}_\varepsilon G := K^{-1} L_\varepsilon K G; \quad \hat{L}_{\varepsilon,\,\mathrm{ren}} G := R_{\varepsilon^{-1}} \hat{L}_\varepsilon R_\varepsilon G.
$$

In the proposition below we calculate the precise form of the operator $\hat{L}_{\varepsilon,\,\mathrm{ren}}$ for the BDLP model.

Proposition 3.2. *For any $\varepsilon>0$ and any $G\in B_{\mathrm{bs}}(\Gamma_0)$*

$$
\hat{L}_{\varepsilon,\mathrm{ren}} G = A_1 G + A_2 G + \varepsilon\left(B_1 G + B_2 G\right),
$$

where

$$
\begin{aligned}
(A_1 G)(\eta) =& -m\,|\eta|\, G(\eta), \\
(A_2 G)(\eta) =& -\varkappa^- \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a^-(x-y)\, G(\eta\setminus x) \\
&+\varkappa^+ \sum_{y\in\eta}\int_{\mathbb{R}^d} a^+(x-y)\, G(\eta\setminus y\cup x)\, dx, \\
(B_1 G)(\eta) =& -\varkappa^- E^{a^-}(\eta)\, G(\eta), \\
(B_2 G)(\eta) =& \varkappa^+ \sum_{y\in\eta}\int_{\mathbb{R}^d} a^+(x-y)\, G(\eta\cup x)\, dx.
\end{aligned}
$$

Proof. According to the definition, we have $\hat{L}_{\varepsilon,\mathrm{ren}} = R_{\varepsilon^{-1}} \hat{L}_\varepsilon R_\varepsilon$, where

$$
\hat{L}_\varepsilon = \hat{L}^-\left(m, \varepsilon\varkappa^- a^-\right) + \varepsilon^{-1}\hat{L}^+\left(\varepsilon\varkappa^+ a^+\right).
$$

As a result,

$$
\begin{aligned}
(\hat{L}_\varepsilon G)(\eta) =& (A_1 G)(\eta) + \varepsilon (B_1 G)(\eta) + (B_2 G)(\eta) \\
&- \varepsilon\varkappa^- \sum_{x\in\eta}\sum_{y\in\eta\setminus x} a^-(x-y)\, G(\eta\setminus x) \\
&+ \varkappa^+ \sum_{y\in\eta}\int_{\mathbb{R}^d} a^+(x-y)\, G(\eta\setminus y\cup x)\, dx
\end{aligned}
$$

and hence

$$
(\hat{L}_{\varepsilon,\mathrm{ren}} G)(\eta) = (A_1 G)(\eta) + (A_2 G)(\eta) + \varepsilon((B_1 + B_2)\, G)(\eta),
$$

which completes the proof. □

Remark 3.3. It is easily seen that the operator $\hat{V} := A_1 + A_2$ will be the point-wise limit of $\hat{L}_{\varepsilon,\,\mathrm{ren}}$ as ε tends to 0. Therefore, the adjoint operator to $\hat{V}$ w.r.t. to the duality (2.7) (if it exists) can be considered as a candidate for the Vlasov operator in our model.

Below we give a rigorous meaning to the operator $\hat{L}_{\varepsilon,\mathrm{ren}}$. Let us define the set

$$D_1 := \Big\{ G \in \mathcal{L}_C \;\Big|\; E^{a^-}(\cdot)\, G(\cdot) \in \mathcal{L}_C,\ |\cdot|\, G(\cdot) \in \mathcal{L}_C \Big\}.$$

Proposition 3.4. *For any ε, m, $\varkappa^-, C > 0$ the operator*

$$A_1(\varepsilon) := A_1 + \varepsilon B_1 \tag{3.9}$$

with the domain D_1 is the generator of a contraction C_0-semigroup on $\mathcal{L}_C$. Moreover, $A_1(\varepsilon) \in \mathcal{H}(\omega)$ for all $\omega \in \big(0; \frac{\pi}{2}\big)$.

Proof. See the proof of Proposition 4.2 in [8]. □

Remark 3.5. It is a simple matter to check that Proposition 3.4 holds also in the case $\varepsilon = 0$, provided the domain of the operator $A_1(0) := A_1$ is changed to

$$D_0 := \{ G \in \mathcal{L}_C \mid \ |\cdot|\, G \in \mathcal{L}_C \} \supset D_1.$$

The next task is to show that for any $\varepsilon > 0$ the operator

$$A_2(\varepsilon) := \hat{L}_{\varepsilon,\mathrm{ren}} - A_1(\varepsilon) = A_2 + \varepsilon B_2 \tag{3.10}$$

with the domain D_1 as well as the operator $A_2(0) := A_2$ with the domain D_0 are relatively bounded w.r.t. the operator $(A_1(\varepsilon),\, D_1)$ and $(A_1,\, D_0)$, correspondingly. This is demonstrated in Propositions 3.6 and 3.7, which can be proved similarly to Lemmas 4.4 and 4.5 in [8].

Proposition 3.6. *For any $\delta > 0$ and any $\varkappa^-, \varkappa^+, m, C > 0$ such that*

$$\frac{\varkappa^- C}{m} + \frac{\varkappa^+}{m} \le \delta$$

the following estimate holds:

$$\|A_2 G\|_C \le \delta \, \|A_1 G\|_C\,, \quad G \in D_0.$$

Moreover, for all $\varepsilon > 0$

$$\|A_2 G\|_C \le \delta \, \|A_1(\varepsilon) G\|_C\,, \quad G \in D_1.$$

Now, the operator (A_2, D_0) is well-defined on $\mathcal{L}_C$.

Proposition 3.7. *For any $\varepsilon, \delta > 0$ and any $\varkappa^-, \varkappa^+, m, C > 0$ such that*

$$\varepsilon \varkappa^+ E^{a^+}(\eta) < \delta C \Big(\varepsilon \varkappa^- E^{a^-}(\eta) + m\,|\eta| \Big), \quad \eta \neq \emptyset$$

the following estimate holds:

$$\|\varepsilon B_2 G\|_C \le a \, \|A_1(\varepsilon) G\|_C\,, \quad G \in D_1$$

with $a < \delta$.

Remark 3.8. Proposition 3.7 enables us to take $D(B_2) = D_1$. As a result, Remark 3.5 shows that the domain of the operator $A_2(\varepsilon)$ will be $D_0 \cap D_1 = D_1$.

We are now in position to show that the operator $(\hat{L}_{\varepsilon,\mathrm{ren}},\, D_1)$ generates a semigroup on $\mathcal{L}_C$. To this end we use the classical result about a perturbation of a holomorphic semigroup (see, e.g. [13]). For the convenience of the reader we formulate below the main statement without proof:

For any $T \in \mathcal{H}(\omega)$, $\omega \in (0; \frac{\pi}{2})$ and for any $\epsilon > 0$ there exist positive constants α, δ such that if the operator A satisfies

$$||Au|| \le a||Tu|| + b||u||, \quad u \in D(T) \subset D(A),$$

with $a < \delta$, $b < \delta$, then $T + A$ is the generator of a holomorphic semigroup. In particular, if $b = 0$, then $T + A \in \mathcal{H}(\omega - \epsilon)$.

Theorem 3.9. *Let the functions* a^-, a^+ *and the constants* $m, \varkappa^-, \varkappa^+, C > 0$ *satisfy*

$$m > 4\left(\varkappa^- C + \varkappa^+\right), \tag{3.11}$$

$$C\varkappa^- a^-(x) \geq 4\varkappa^+ a^+(x), \ x \in \mathbb{R}^d. \tag{3.12}$$

Then, for any $\varepsilon > 0$ *the operator* $(\hat{L}_{\varepsilon,\mathrm{ren}}, D_1)$ *is the generator of a holomorphic semigroup* $\hat{U}_{t,\varepsilon}$, $t \geq 0$ *on* $\mathcal{L}_C$.

Proof. Let $\varepsilon > 0$ be arbitrary and fixed. By definition,

$$\hat{L}_{\varepsilon,\mathrm{ren}} = A_1(\varepsilon) + A_2(\varepsilon).$$

The direct application of the theorem about perturbation of holomorphic semigroups (see the formulation above the assertion of Theorem 3.9) to $T = A_1(\varepsilon)$ and $A = A_2(\varepsilon)$ gives now the desired claim. It is important to note that Proposition 3.4 enables us to consider δ equal to $\frac{1}{2}$ in the formulation of the classical theorem introduced above. The appearance of the multiplicand 4 on the left-hand side of both assumptions in assertion of Theorem 3.9 is motivated exactly by the latter fact. □

Theorem 3.10. *Assume that the constants* $m, \varkappa^-, \varkappa^+, C > 0$ *satisfy*

$$m > 2\left(\varkappa^- C + \varkappa^+\right).$$

Then, the operator $\hat{V} = A_1 + A_2$ *with the domain* D_0 *is the generator of a holomorphic semigroup* $\hat{U}_t^V$, $t \geq 0$ *on* $\mathcal{L}_C$.

Proof. We use the same classical result as for Theorem 3.9 in the case: A_1 is the generator of a holomorphic semigroup, A_2 is relatively bounded w.r.t. A_1 with boundary less than $\frac{1}{2}$. □

Now we may repeat the same considerations as in the end of Section 2. Namely, applying the general results about adjoint semigroups (see, e.g., [7]) to the semigroup $(\hat{U}_t^V)^*$ in $\mathcal{K}_C$, we deduce that it will be weak*-continuous and weak*-differentiable at 0. Moreover, $\hat{V}^*$ will be the weak*-generator of $\hat{T}^*(t)$. This means, in particular, that for any $G \in D(\hat{V}) \subset \mathcal{L}_C$, $k \in D(\hat{V}^*) \subset \mathcal{K}_C$

$$\frac{d}{dt}\left\langle\!\!\left\langle G, (\hat{U}_t^V)^* k \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle G, \hat{V}^*(\hat{U}_t^V)^* k \right\rangle\!\!\right\rangle. \tag{3.13}$$

An explicit form of $\hat{V}^*$ follows from (2.10), namely, for any $k \in D(\hat{V}^*)$

$$\begin{aligned} \hat{V}^* k(\eta) = &- m|\eta|k(\eta) - \varkappa^- \int_{\mathbb{R}^d} \sum_{x\in\eta} a^-(x-y) k(\eta \cup y)\, dy \\ &+ \varkappa^+ \sum_{x\in\eta} \int_{\mathbb{R}^d} a^+(x-y) k(\eta \setminus x \cup y)\, dy. \end{aligned} \tag{3.14}$$

As a result, we have that for any $k_0 \in D(\hat{V}^*)$ the function $k_t = (\hat{U}_t^V)^* k_0$ provides a weak* solution of the following Cauchy problem:

$$\begin{cases} \dfrac{\partial}{\partial t} k_t &= \hat{V}^* k_t, \\ k_t\big|_{t=0} &= k_0. \end{cases} \tag{3.15}$$

In the next theorem we show that the limiting Vlasov dynamics has chaos preservation property, i.e. preserves the Lebesgue–Poisson exponents.

Theorem 3.11. *Let the conditions of Theorem 3.9 be satisfied and, additionally,* $C \geq \dfrac{4}{16e-1}$*. Let* $\rho_0 \geq 0$ *be a measurable nonnegative function on* $\mathbb{R}^d$ *such that* $\operatorname{ess\,sup}_{x\in\mathbb{R}^d} \rho_0(x) \leq C$*. Then the Cauchy problem* (3.15) *with* $k_0 = e_\lambda(\rho_0)$ *has a weak* solution* $k_t = e_\lambda(\rho_t) \in \mathcal{K}_C$*, where* ρ_t *is a unique nonnegative solution to the Cauchy problem*

$$\begin{cases} \dfrac{\partial}{\partial t}\rho_t(x) = \varkappa^+(a^+ * \rho_t)(x) - \varkappa^-\rho_t(x)(a^- * \rho_t)(x) - m\rho_t(x), \\ \rho_t\big|_{t=0}(x) = \rho_0(x), \end{cases} \tag{3.16}$$

and $\operatorname{ess\,sup}_{x\in\mathbb{R}^d} \rho_t(x) \leq C$*,* $t \geq 0$*.*

Proof. First of all, if (3.16) has a solution $\rho_t(x) \geq 0$ then

$$\frac{\partial}{\partial t}\rho_t(x) \leq \varkappa^+(a^+ * \rho_t)(x) - m\rho_t(x)$$

and, therefore, $\rho_t(x) \leq r_t(x)$ where $r_t(x)$ is a solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}r_t(x) = \varkappa^+(a^+ * r_t)(x) - mr_t(x), \\ r_t\big|_{t=0}(x) = \rho_0(x) \geq 0, \end{cases}$$

for a.a. $x \in \mathbb{R}^d$. Hence,

$$r_t(x) = e^{-(m-\varkappa^+)t}e^{\varkappa^+ tL_{a^+}}\rho_0(x),$$

where

$$(L_{a^+}f)(x) := \int_{\mathbb{R}^d} a^+(x-y)[f(y)-f(x)]dy.$$

Since for $f \in L^\infty(\mathbb{R}^d)$ we have $\big|(L_{a^+}f)(x)\big| \leq 2\|f\|_{L^\infty(\mathbb{R}^d)}$ then, by (3.11),

$$r_t(x) \leq Ce^{-(m-\varkappa^+)t}e^{2\varkappa^+ t} \leq C,$$

which yields $0 \leq \rho_t(x) \leq C$.

To prove the existence and uniqueness of the solution of (3.16) let us fix some $T > 0$ and define the Banach space $X_T = C([0;T], L^\infty(\mathbb{R}^d))$ of all continuous functions on $[0;T]$ with values in $L^\infty(\mathbb{R}^d)$; the norm on X_T is given by $\|u\|_T := \max\limits_{t\in[0;T]} \|u_t\|_{L^\infty(\mathbb{R}^d)}$. We denote by X_T^+ the cone of all nonnegative functions from X_T.

Let Φ be a mapping which assigns to any $v \in X_T$ the solution u_t of the linear Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}u_t(x) = \varkappa^+(a^+ * v_t)(x) - \varkappa^-u_t(x)(a^- * v_t)(x) - mu_t(x), \\ u_t\big|_{t=0}(x) = \rho_0(x), \end{cases} \tag{3.17}$$

for a.a. $x \in \mathbb{R}^d$. Therefore,

$$\begin{aligned} (\Phi v)_t(x) = &\exp\left\{-\int_0^t \big(m + \varkappa^-(a^- * v_s)(x)\big)\,ds\right\}\rho_0(x) \\ &+ \int_0^t \exp\left\{-\int_s^t \big(m + \varkappa^-(a^- * v_\tau)(x)\big)\,d\tau\right\}\varkappa^+(a^+ * v_s)(x)\,ds. \end{aligned} \tag{3.18}$$

We have that $v \in X_T^+$ implies $\Phi v \geq 0$ as well as the estimate

$$(\Phi v)_t(x) \leq \rho_0(x) + \varkappa^+\|v\|_T\int_0^t e^{-(t-s)m}ds \leq C + \frac{\varkappa^+}{m}\|v\|_T,$$

where we use the trivial inequality

$$\|f*g\|_{L^\infty(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)}\|g\|_{L^\infty(\mathbb{R}^d)}, \quad f\in L^1(\mathbb{R}^d), \quad g\in L^\infty(\mathbb{R}^d). \tag{3.19}$$

Therefore, $\Phi v \in X_T^+$. For simplicity of notations we denote for $v\in X_T^+$

$$(Bv)(t,x) = m + \varkappa^-(a^- * v_t)(x) \ge m > 0.$$

Then, for any $v,w\in X_T^+$

$$\begin{aligned}
&\bigl|(\Phi v)_t(x) - (\Phi w)_t(x)\bigr|\\
&\quad\le \left|\exp\left\{-\int_0^t (Bv)(s,x)\,ds\right\} - \exp\left\{-\int_0^t (Bw)(s,x)\,ds\right\}\right| \rho_0(x)\\
&\qquad + \int_0^t \left|\exp\left\{-\int_s^t (Bv)(\tau,x)\,d\tau\right\}\varkappa^+(a^+ * v_s)(x)\right.\\
&\qquad \left. - \exp\left\{-\int_s^t (Bw)(\tau,x)\,d\tau\right\}\varkappa^+(a^+ * w_s)(x)\right| ds.
\end{aligned}$$

We have

$$\begin{aligned}
&\left|\exp\left\{-\int_0^t (Bv)(s,x)\,ds\right\} - \exp\left\{-\int_0^t (Bw)(s,x)\,ds\right\}\right|\\
&\quad\le e^{-mt}\left|\exp\left\{-\int_0^t \varkappa^-(a^- * v_s)(x)\,ds\right\} - \exp\left\{-\int_0^t \varkappa^-(a^- * w_s)(x)\,ds\right\}\right|\\
&\quad\le e^{-mt}\left|\int_0^t \varkappa^-(a^- * v_s)(x)\,ds - \int_0^t \varkappa^-(a^- * w_s)(x)\,ds\right|\\
&\quad\le e^{-mt}\varkappa^-\|v-w\|_T\cdot t \le \frac{\varkappa^-}{em}\|v-w\|_T,
\end{aligned}$$

where we used (3.19) and obvious inequalities $|e^{-a}-e^{-b}|\le |a-b|$ for $a,b\ge 0$; $e^{-x}x\le e^{-1}$ for $x\ge 0$.

Next, using another simple inequalities for any $a,b,p,q\ge 0$, $a\ge b$,

$$|pe^{-a}-qe^{-b}| \le e^{-a}|p-q| + qe^{-b}|e^{-(a-b)}-1| \le e^{-a}|p-q| + qe^{-b}|a-b|,$$

it is easy to verify that

$$\begin{aligned}
&\int_0^t \left|\exp\left\{-\int_s^t (Bv)(\tau,x)\,d\tau\right\}\varkappa^+(a^+ * v_s)(x)\right.\\
&\qquad\left. - \exp\left\{-\int_s^t (Bw)(\tau,x)\,d\tau\right\}\varkappa^+(a^+ * w_s)(x)\right| ds\\
&\quad\le \varkappa^+\int_0^t \exp\left\{-\int_s^t (Bv)(\tau,x)\,d\tau\right\}\bigl|a^+ * (v_s-w_s)\bigr|(x)\,ds\\
&\qquad + \int_0^t \exp\left\{-\int_s^t (Bw)(\tau,x)\,d\tau\right\}(\varkappa^+ a^+ * w_s)(x)\\
&\qquad \times \left|\int_s^t (Bv)(\tau,x)\,d\tau - \int_s^t (Bw)(\tau,x)\,d\tau\right| ds\\
&\quad\le \varkappa^+\|v-w\|_T\int_0^t e^{-m(t-s)}ds\\
&\qquad + \int_0^t \exp\left\{-\int_s^t \varkappa^-(a^- * w_\tau)(x)\,d\tau\right\}(\varkappa^+ a^+ * w_s)(x)\\
&\qquad \times e^{-m(t-s)}\int_s^t \varkappa^-(a^- * |v_\tau - w_\tau|)(x)\,d\tau ds
\end{aligned}$$

and, using (3.12) and the inequalities above, one can continue

$$\begin{aligned}
&\leq \frac{\varkappa^+}{m}\|v-w\|_T + \frac{C}{4}\frac{\varkappa^-}{em}\|v-w\|_T \\
&\quad\times \int_0^t \exp\left\{-\int_s^t \varkappa^-(a^- * w_\tau)(x)\,d\tau\right\}\varkappa^-(a^- * w_s)(x)\,ds \\
&= \frac{\varkappa^+}{m}\|v-w\|_T + \frac{C}{4}\frac{\varkappa^-}{em}\|v-w\|_T \\
&\quad\times \int_0^t \frac{\partial}{\partial s}\exp\left\{-\int_s^t \varkappa^-(a^- * w_\tau)(x)\,d\tau\right\}ds \\
&\leq \left(\frac{\varkappa^+}{m} + \frac{C}{4}\frac{\varkappa^-}{em}\right)\|v-w\|_T.
\end{aligned}$$

Therefore, for $v, w \in X_T^+$

$$\|\Phi v - \Phi w\|_T \leq \left(\frac{\varkappa^+}{m} + \left(1+\frac{C}{4}\right)\frac{\varkappa^-}{em}\right)\|v-w\|_T \leq \frac{4(\varkappa^+ + C\varkappa^-)}{m}\|v-w\|_T,$$

if, e.g., $1+\frac{C}{4} \leq 4Ce$, that means $C \geq \frac{4}{16e-1}$.

As a result, by (3.11), Φ is a contraction mapping on the cone X_T^+. Taking, as usual, $v^{(n)} = \Phi^n v^{(0)}$, $n \geq 1$ for $v^{(0)} \in X_T^+$ we obtain that $\{v^{(n)}\} \subset X_T^+$ is a Cauchy sequence in X_T which has, therefore, a unique limit point $v \in X_T$. Since X_T^+ is a closed cone we have that $v \in X_T^+$. Then, identically to the classical Banach fixed point theorem, v will be a fixed point of Φ on X_T and a unique fixed point on X_T^+. Then, this v is the nonnegative solution of (3.16) on the interval $[0;T]$. By the note above, $v_t(x) \leq C$. Changing initial value in (3.16) to $\rho_t\big|_{t=T}(x) = v_T(x)$ we may repeat all our considerations on the time-interval $[T;2T]$ with the same estimate $v_t(x) \leq C$; and so on. As a a result, (3.16) has a unique global bounded non-negative solution $\rho_t(x)$ on $\mathbb{R}_+$.

Consider now

$$k_t(\eta) = e_\lambda(\rho_t, \eta) \in \mathcal{K}_C,$$

then

$$\frac{\partial}{\partial t}e_\lambda(\rho_t,\eta) = \sum_{x\in\eta}\frac{\partial \rho_t}{\partial t}(x)e_\lambda(\rho_t, \eta\setminus x).$$

Using (3.16) and (3.14), we immediately conclude that $k_t(\eta) = e_\lambda(\rho_t,\eta)$ is a solution to (3.15). □

The main result of the paper is formulated in the next theorem. Its proof will be given in Subsection 3.4.

Theorem 3.12. *Under conditions of Theorem 3.9 the semigroup $\hat{U}_{t,\varepsilon}$ converges strongly to the semigroup $\hat{U}_t^V$ as $\varepsilon \to 0$ uniformly on any finite intervals of time.*

3.4. **Proofs.** According to Theorem 3.1, the statement of Theorem 3.12 will be proved once we verify Assumptions **(A)** for the operators $(A_1(\varepsilon), D_1)$, $(A_2(\varepsilon), D_1)$, $\varepsilon > 0$, defined in the previous subsection. Note, that $A_1(0) = A_1$ and $A_2(0) = A_2$ are defined on the domain D_0.

In the following proposition we verify Assumption 2(a) of **(A)**.

Proposition 3.13. *Let $\lambda > 0$ then*

$$(A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \xrightarrow{s} (A_1 - \lambda 1\!\!1)^{-1}, \quad \varepsilon \to 0.$$

Proof. For any $G \in \mathcal{L}_C$

$$\begin{aligned}
&\left\| (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} G - (A_1 - \lambda 1\!\!1)^{-1} G \right\|_C \\
&\quad = \int_{\Gamma_0} \left| G(\eta) \left(\frac{1}{-m|\eta| - \varepsilon \varkappa^- E^{a^-}(\eta) - \lambda} - \frac{1}{-m|\eta| - \lambda} \right) \right| C^{|\eta|} d\lambda(\eta) \\
&\quad = \int_{\Gamma_0} |G(\eta)| F_\varepsilon(\eta) C^{|\eta|} d\lambda(\eta),
\end{aligned}$$

where

$$F_\varepsilon(\eta) := \frac{\varepsilon \varkappa^- E^{a^-}(\eta)}{\left(m|\eta| + \varepsilon \varkappa^- E^{a^-}(\eta) + \lambda\right)(m|\eta| + \lambda)}, \quad \eta \in \Gamma_0.$$

Since $0 \le F_\varepsilon(\eta) < 1/\lambda$ and $F_\varepsilon(\eta) \to 0$ as $\varepsilon \to 0$ for any $\eta \in \Gamma_0$, we get the desired statement. □

Next we check Assumption 2(b) of **(A)**.

Proposition 3.14. *Let $\lambda > 0$ be arbitrary and fixed. Then*

$$\sup_{\varepsilon \ge 0} \left\| (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \right\| \le \left\| (A_1 - \lambda 1\!\!1)^{-1} \right\|.$$

Proof. For any $G \in \mathcal{L}_C$ and any $\varepsilon > 0$

$$\begin{aligned}
&\left\| (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} G \right\|_C \\
&\quad = \int_{\Gamma_0} |G(\eta)| \frac{1}{m|\eta| + \varepsilon \varkappa^- E^{a^-}(\eta) + \lambda} C^{|\eta|} d\lambda(\eta) \\
&\quad \le \int_{\Gamma_0} |G(\eta)| \frac{1}{m|\eta| + \lambda} C^{|\eta|} d\lambda(\eta) = \left\| (A_1 - \lambda 1\!\!1)^{-1} G \right\|_C \\
&\quad \le \left\| (A_1 - \lambda 1\!\!1)^{-1} \right\| \cdot \|G\|_C.
\end{aligned}$$

This finishes the proof. □

Assumption 2(c) of **(A)** is proved in the next Proposition.

Proposition 3.15. *Let the conditions of Theorem 3.9 be satisfied. Then, for any $\lambda > 0$*

$$\sup_{\varepsilon \ge 0} \left\| A_2(\varepsilon) (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \right\| < \frac{1}{2} \tag{3.20}$$

Proof. First we prove the assertion for $\varepsilon = 0$. Since $D(A_1) = D(A_2) = D_0$ and $\mathrm{Ran}\left((A_1 - \lambda 1\!\!1)^{-1}\right) = D(A_1)$, the operator $A_2 (A_1 - \lambda 1\!\!1)^{-1}$ is well defined. Next, inequality (3.11) and Proposition 3.6 yield

$$\left\| A_2 (A_1 - \lambda 1\!\!1)^{-1} \right\| < \frac{1}{4}. \tag{3.21}$$

Indeed,

$$\|A_2 G\|_C \le a \|A_1 G\|_C < a \|(A_1 - \lambda 1\!\!1) G\|_C$$

with $a < \frac{1}{4}$. Therefore,

$$\left\| A_2 (A_1 - \lambda 1\!\!1)^{-1} G \right\|_C < \frac{1}{4} \|G\|_C,$$

and (3.21) is proved.

Now, let $\varepsilon > 0$ be arbitrary and fixed. The main arguments we use to show

$$\left\| A_2(\varepsilon) (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \right\| < \frac{1}{2}$$

are the following:

1) $D(A_1(\varepsilon)) = D_1 \subset D_0 = D(A_2)$. Hence, $A_2 (A_1(\varepsilon) - \lambda 1\!\!1)^{-1}$ is well-defined on $\mathcal{L}_C$. Moreover, Proposition 3.6 implies

$$\left\| A_2 (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \right\| < \frac{1}{4}, \quad \varepsilon > 0.$$

2) $D(B_2) = D(A_1(\varepsilon)) = D_1$ and for any $\varepsilon > 0$

$$\left\| \varepsilon B_2 (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \right\| < \frac{1}{4},$$

which follows from Proposition 3.7.

3) Since $A_2(\varepsilon) := A_2 + \varepsilon B_2$, we have

$$\left\| A_2(\varepsilon) (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \right\| < \frac{1}{2}. \tag{3.22}$$

The latter concludes the proof. □

We set

$$Q_\varepsilon = \left(A_2(\varepsilon) (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} + 1 \right)^{-1}, \quad Q = \left(A_2 (A_1 - \lambda 1\!\!1)^{-1} + 1\!\!1 \right)^{-1}.$$

The latter convergence is nothing else but Assumption 2(d) of **(A)** in our notations. In order to verify Assumption 2(d) of **(A)** we have to show that $Q_\varepsilon \xrightarrow{s} Q$ as $\varepsilon \to 0$.

Suppose that we can show that

$$\begin{aligned} A_2 (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} &\xrightarrow{s} A_2 (A_1 - \lambda 1\!\!1)^{-1}, \quad && \varepsilon \to 0. \\ \varepsilon B_2 (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} &\xrightarrow{s} 0, && \varepsilon \to 0. \end{aligned} \tag{3.23}$$

Then,

$$\begin{aligned} C_\varepsilon :=& A_2(\varepsilon) (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \\ =& A_2 (A_1 + \varepsilon B_1 - \lambda 1\!\!1)^{-1} + \varepsilon B_2 (A_1 + \varepsilon B_1 - \lambda 1\!\!1)^{-1} \xrightarrow{s} A_2 (A_1 - \lambda 1\!\!1)^{-1} \end{aligned}$$

To check

$$Q_\varepsilon = (C_\varepsilon + 1\!\!1)^{-1} \xrightarrow{s} Q \tag{3.24}$$

we proceed as follows:

$$\begin{aligned} &(C_\varepsilon + 1\!\!1)^{-1} - Q \\ &\quad = (C_\varepsilon + 1\!\!1)^{-1} - \left(A_2 (A_1 - \lambda 1\!\!1)^{-1} + 1\!\!1 \right)^{-1} \\ &\quad = (C_\varepsilon + 1\!\!1)^{-1} \left(A_2 (A_1 - \lambda 1\!\!1)^{-1} + 1\!\!1 - C_\varepsilon - 1\!\!1 \right) \left(A_2 (A_1 - \lambda 1\!\!1)^{-1} + 1\!\!1 \right)^{-1} \\ &\quad = (C_\varepsilon + 1\!\!1)^{-1} \left(A_2 (A_1 - \lambda 1\!\!1)^{-1} - C_\varepsilon \right) \left(A_2 (A_1 - \lambda 1\!\!1)^{-1} + 1\!\!1 \right)^{-1}. \end{aligned}$$

Assuming (3.23) it is obvious now that convergence (3.24) is equivalent to

$$\sup_{\varepsilon > 0} \left\| (C_\varepsilon + 1\!\!1)^{-1} \right\| < \infty,$$

which is clear from

$$\left\| (C_\varepsilon + 1\!\!1)^{-1} \right\| \le \frac{1}{1 - \|C_\varepsilon\|} \quad \text{and} \quad \|C_\varepsilon\| < \frac{1}{2}.$$

The last bound we conclude from (3.22). As a result we shall have established Theorem 3.12 if we show (3.23).

Lemma 3.16. $A_2 (A_1(\varepsilon) - \lambda 1\!\!1)^{-1} \xrightarrow{s} A_2 (A_1 - \lambda 1\!\!1)^{-1}$, as $\varepsilon \to 0$.

Proof. Proposition 3.6 and

$$D\left(A_1(\varepsilon)\right) = D_1 \subset D\left(A_1\right) = D\left(A_2\right) = D_0$$

leads to the following formula:

$$A_2\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1} = A_2\left(A_1 - \lambda 1\!\!1\right)^{-1}\left(A_1 - \lambda 1\!\!1\right)\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1}.$$

Now, we are left with the task to show that

$$\left(A_1 - \lambda 1\!\!1\right)\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1} \xrightarrow{s} 1, \quad \text{as} \quad \varepsilon \to 0.$$

But, for any $G \in \mathcal{L}_C$

$$\begin{aligned}
&\left\|\left(\left(A_1 - \lambda 1\!\!1\right)\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1} - 1\!\!1\right) G\right\|_C \\
&\quad = \int_{\Gamma_0} \left|\frac{m\left|\eta\right| + \lambda}{m\left|\eta\right| + \varepsilon\varkappa^- E^{a^-}\left(\eta\right) + \lambda} - 1\right| \left|G\left(\eta\right)\right| C^{|\eta|} d\lambda\left(\eta\right) \\
&\quad = \int_{\Gamma_0} \frac{\varepsilon\varkappa^- E^{a^-}\left(\eta\right)}{m\left|\eta\right| + \varepsilon\varkappa^- E^{a^-}\left(\eta\right) + \lambda} \left|G\left(\eta\right)\right| C^{|\eta|} d\lambda\left(\eta\right) \to 0, \quad \text{as} \quad \varepsilon \to 0
\end{aligned}$$

due to Lebesgue's dominated convergence theorem. □

Lemma 3.17. $\varepsilon B_2\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1} \xrightarrow{s} 0$, as $\varepsilon \to 0$.

Proof. Since $\left\|B_2 G\right\|_C \le \frac{1}{4}\left\|B_1 G\right\|_C$, we have to show that

$$\left\|\varepsilon B_1\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1} G\right\|_C \to 0, \quad \text{as} \quad \varepsilon \to 0.$$

But,

$$\begin{aligned}
&\left\|\varepsilon B_1\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1} G\right\|_C \\
&\quad = \int_{\Gamma_0} \frac{\varepsilon\varkappa^- E^{a^-}\left(\eta\right)}{m\left|\eta\right| + \varepsilon\varkappa^- E^{a^-}\left(\eta\right) + \lambda} \left|G\left(\eta\right)\right| C^{|\eta|} d\lambda\left(\eta\right) \to 0, \quad \varepsilon \to 0.
\end{aligned}$$

□

The last two lemmas conclude the proof of the main Theorem.

Remark 3.18. Under assumptions of Proposition 3.15 we get the following representation for the resolvents of $\hat{V}$ and $\hat{L}_{\varepsilon,\mathrm{ren}}$:

$$\begin{aligned}
\left(\hat{V} - \lambda 1\!\!1\right)^{-1} &= \left(A_1 + A_2 - \lambda 1\!\!1\right)^{-1} = \left(A_1 - \lambda 1\!\!1\right)^{-1}\left(A_2\left(A_1 - \lambda 1\!\!1\right)^{-1} + 1\!\!1\right)^{-1}, \\
\left(\hat{L}_{\varepsilon,\mathrm{ren}} - \lambda 1\!\!1\right)^{-1} &= \left(A_1(\varepsilon) + A_2(\varepsilon) - \lambda 1\!\!1\right)^{-1} \\
&= \left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1}\left(A_2(\varepsilon)\left(A_1(\varepsilon) - \lambda 1\!\!1\right)^{-1} + 1\!\!1\right)^{-1}, \quad \lambda > 0.
\end{aligned} \tag{3.25}$$

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