

# Semigroup approach to birth-and-death stochastic dynamics in continuum

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## Abstract

We describe a general approach to the construction of a state evolution corresponding to the Markov generator of a spatial birth-and-death dynamics in  $\mathbb{R}^d$ . We present conditions on the birth-and-death intensities which are sufficient for the existence of an evolution as a strongly continuous semigroup in a proper Banach space of correlation functions satisfying the Ruelle bound. The convergence of a Vlasov-type scaling for the corresponding stochastic dynamics is considered.

**Key words.**  $C_0$ -semigroups, continuous systems, Markov evolution, spatial birth-and-death dynamics, correlation functions, evolution equations, Vlasov scaling, Vlasov equation, scaling limits

**AMS subject classification.** 46E30, 47D06, 82C21, 35Q83

## 1 Introduction

Spatial Markov processes in  $\mathbb{R}^d$  may be described as stochastic evolutions of locally finite subsets (configurations)  $\gamma \subset \mathbb{R}^d$ , i.e., any  $\gamma$  has a finite number of points in an arbitrary ball in  $\mathbb{R}^d$ . One of the most important classes of such stochastic dynamics is given by the birth-and-death Markov processes in the space  $\Gamma$  of all configurations from  $\mathbb{R}^d$ . These are processes in which an infinite number of individuals exist at each instant, and the rates at which new individuals appear and some old ones disappear depend on the instantaneous configuration of existing individuals [19]. The corresponding Markov generators have a natural heuristic representation in terms of birth and death intensities. The birth intensity  $b(x, \gamma) \geq 0$  characterizes the appearance of a new point at  $x \in \mathbb{R}^d$  in the presence of a given configuration  $\gamma \in \Gamma$ . The death intensity

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$d(x, \gamma) \geq 0$  characterizes the probability of the event that the point  $x$  of the configuration  $\gamma$  disappears, depending on the location of the remaining points of the configuration,  $\gamma \setminus x$ . Here and below, for simplicity of notation, we write  $x$  instead of  $\{x\}$ . Heuristically, the corresponding Markov generator is described by the following expression

$$(LF)(\gamma) := \sum_{x \in \gamma} d(x, \gamma \setminus x) [F(\gamma \setminus x) - F(\gamma)] + \int_{\mathbb{R}^d} b(x, \gamma) [F(\gamma \cup x) - F(\gamma)] dx, \quad (1.1)$$

for proper functions  $F : \Gamma \rightarrow \mathbb{R}$ .

The study of spatial birth-and-death processes was initiated by C. Preston [31]. This paper dealt with a solution of the backward Kolmogorov equation

$$\frac{\partial}{\partial t} F_t = LF_t \quad (1.2)$$

under the restriction that only a finite number of individuals are alive at each moment of time. Under certain conditions, corresponding processes exist and are temporally ergodic, that is, there exists a unique stationary distribution. Note that a more general setting for birth-and-death processes only requires that the number of points in any compact set remains finite at all times. A further progress in the study of these processes was achieved by R. Holley and D. Stroock in [19]. They described in detail an analytic framework for birth-and-death dynamics. In particular, they analyzed the case of a birth-and-death process in a bounded region.

Stochastic equations for spatial birth-and-death processes were formulated in [17], through a spatial version of the time-change approach. Further, in [18], these processes were represented as solutions to a system of stochastic equations, and conditions for the existence and uniqueness of solutions to these equations, as well as for the corresponding martingale problems, were given. Unfortunately, quite restrictive assumptions on the birth and death rates in [18] do not allow an application of these results to several particular models that are interesting for applications (see e.g. Examples 1–3 below).

A growing interest to the study of spatial birth-and-death processes, which we have recently observed, is stimulated by (among others) an important role which these processes play in several applications. For example, in spatial plant ecology, a general approach to the so-called individual based models was developed in a series of works, see e.g. [3, 4, 6, 30] and the references therein. These models are described as birth-and-death Markov processes in the configuration space  $\Gamma$  with specific rates  $b$  and  $d$  which reflect biological notions such as competition, establishment, fecundity etc. Other examples of birth-and-death processes may be found in mathematical physics. In particular, the Glauber-type stochastic dynamics in  $\Gamma$  is properly associated with the grand canonical Gibbs measures for classical gases. This gives a possibility to study these Gibbs measures as equilibrium states for specific birth-and-death Markov

evolutions [2]. Starting with a Dirichlet form for a given Gibbs measure, one can consider an equilibrium stochastic dynamics [24]. However, these dynamics give the time evolution of initial distributions from a quite narrow class. Namely, the class of admissible initial distributions is essentially reduced to the states which are absolutely continuous with respect to the invariant measure. In the present paper we construct non-equilibrium stochastic dynamics which may have a much wider class of initial states.

Concerning the study of particular birth-and-death models, let us stress that, on the one hand, for most cases appearing in applications, the existence problem for a corresponding Markov process is still open. On the other hand, the evolution of a state in the course of a stochastic dynamics is an important question in its own right. A mathematical formulation of this question may be realized through the forward Kolmogorov equation for probability measures (states) on the configuration space  $\Gamma$ :

$$\frac{\partial}{\partial t} \mu_t = L^* \mu_t. \quad (1.3)$$

Here  $L^*$  is the (informally) adjoint operator of  $L$  with respect to the pairing

$$\langle F, \mu \rangle := \int_{\Gamma} F(\gamma) d\mu(\gamma). \quad (1.4)$$

In the physical literature, (1.3) is known as the Fokker–Planck equation. However, the mere existence of the corresponding Markov process will not give us much information about properties of the solution to (1.3).

An important technical observation concerns the possibility to reformulate the equations for states in terms of time evolutions for corresponding correlation functions, see e.g. [16] and references therein. Namely, a probability measure  $\mu$  on  $\Gamma$  may be characterized by a sequence  $\{k^{(n)}(x_1, \dots, x_n)\}_{n=0}^{\infty}$  of symmetric non-negative functions on  $(\mathbb{R}^d)^n$ . Then, (1.3) may be rewritten in the form

$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t, \quad (1.5)$$

where  $\hat{L}^*$  is the corresponding image of the operator  $L^*$  from (1.3) acting on sequences of functions  $k_t = \{k_t^{(n)}\}_{n=0}^{\infty}$ .

In various applications, correlation functions satisfy the so-called Ruelle bound

$$|k^{(n)}(x_1, \dots, x_n)| \leq C^n, \quad x_1, \dots, x_n \in \mathbb{R}^d, \quad n \in \mathbb{N} \quad (1.6)$$

for some  $C > 0$ . For example, for the correlation functions of the Gibbs measure mentioned above, such inequalities hold true, see e.g. [33]. Hence, it is rather natural to study the solutions to the equation (1.5) in weighted  $L^\infty$ -type space of functions with the Ruelle bound. However, analysis of the existence problem in such a class of correlation functions meets essential difficulties related to the use of non-separable  $L^\infty$  spaces and properties of strongly continuous semigroups

acting in these spaces. One of technical possibilities to study such semigroups is based on the use of the pre-dual evolution equations in some  $L^1$  spaces.

Namely, we will exploit the duality

$$\langle\langle G, k \rangle\rangle := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} G^{(n)}(x_1, \dots, x_n) k^{(n)}(x_1, \dots, x_n) dx_1 \dots dx_n, \quad (1.7)$$

which is a pairing between a sequence  $k = \{k^{(n)}\}_{n=0}^{\infty}$  of functions which satisfy (1.6) and a sequence  $G = \{G^{(n)}\}_{n=0}^{\infty}$  of the so-called quasi-observables. The latter are integrable functions satisfying

$$\sum_{n=0}^{\infty} \frac{C^n}{n!} \int_{(\mathbb{R}^d)^n} |G^{(n)}(x_1, \dots, x_n)| dx_1 \dots dx_n < \infty. \quad (1.8)$$

Then, the equation (1.5) may be rewritten as follows

$$\frac{\partial}{\partial t} G_t = \hat{L} G_t, \quad (1.9)$$

with the corresponding operator  $\hat{L}$  acting on sequences  $G_t = \{G_t^{(n)}\}_{n=0}^{\infty}$ . This is an analog of the backward Kolmogorov equation (1.2) on sequences of functions. Note that  $\hat{L}^*$  is the dual operator of  $\hat{L}$  with respect to the duality (1.7). The resulting, so-called hierarchical equation (1.9) may be analyzed in a Fock-type space of sequences of functions which satisfy (1.8). The corresponding semigroup may be used for a construction of time evolution (1.5) for correlation functions using the duality (1.7).

This approach was successfully applied to the construction and analysis of state evolutions for different versions of the Glauber dynamics [23, 15, 10] and for some spatial ecology models [13]. Each of the considered models required its own specific version of the construction of a semigroup, which takes into account particular properties of corresponding birth and death rates.

In the present paper, we develop a general approach to the construction of the state evolution corresponding to the birth-and-death Markov generators. We present conditions on the birth and death intensities which are sufficient for the existence of corresponding evolutions as strongly continuous semigroups in proper Banach spaces of correlation functions satisfying the Ruelle bound (1.6).

Moreover, we apply this construction to study of the convergence of the considered stochastic dynamics in a Vlasov-type scaling. Originally, the notion of the Vlasov scaling was related to the Hamiltonian dynamics of interacting particle systems. This is a mean field scaling limit when the influence of weak long-range forces is taken into account. Rigorously, this limit was studied by W. Braun and K. Hepp in [5] for the Hamiltonian dynamics, and by R.L. Dobrushin [7] for more general deterministic dynamical systems. In [14], we proposed a general scheme for a Vlasov-type scaling of stochastic Markovian dynamics. Our approach is based on a proper scaling of the evolutions of correlation functions proposed by H. Spohn in [34] for the Hamiltonian dynamics. In

the present paper, we apply such an approach to the birth-and-death stochastic dynamics. This gives us a rigorous framework for the study of convergence of the scaled hierarchical equations to a solution of the limiting Vlasov hierarchy, and for the derivation of a resulting non-linear evolutionary equation for the density of the limiting system. We consider some special birth-and-death models to show how the general conditions proposed in the paper may be verified in applications.

The structure of the paper is as follows. In Section 2 we give a brief introduction to notions related to the configuration space. Subsection 3.1 is devoted to the evolution of quasi-observables in the Fock-type space which is the pre-dual of the space of correlation functions. We propose constructive conditions on the birth and death rates under which the corresponding dynamics exist. These conditions are verified for a number of particular examples. The evolution of correlation functions is considered in Subsection 3.2. The question concerning the existence and uniqueness of the solution to the corresponding stationary equation in the space of correlation functions is studied in Subsection 3.3. In Section 4 we discuss the Vlasov-type scaling for birth-and-death stochastic dynamics.

## 2 Basic facts and notation

Let  $\mathcal{B}(\mathbb{R}^d)$  be the family of all Borel sets in  $\mathbb{R}^d$ ,  $d \geq 1$ ;  $\mathcal{B}_b(\mathbb{R}^d)$  denotes the system of all bounded sets from  $\mathcal{B}(\mathbb{R}^d)$ .

The configuration space over space  $\mathbb{R}^d$  consists of all locally finite subsets (configurations) of  $\mathbb{R}^d$ . Namely,

$$\Gamma = \Gamma(\mathbb{R}^d) := \left\{ \gamma \subset \mathbb{R}^d \mid |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}. \quad (2.1)$$

Here  $|\cdot|$  means the cardinality of a set, and  $\gamma_\Lambda := \gamma \cap \Lambda$ . The space  $\Gamma$  is equipped with the vague topology, i.e., the weakest topology for which all mappings  $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x) \in \mathbb{R}$  are continuous for any continuous function  $f$  on  $\mathbb{R}^d$  with compact support. The corresponding Borel  $\sigma$ -algebra  $\mathcal{B}(\Gamma)$  is the smallest  $\sigma$ -algebra for which all mappings  $\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$  are measurable for any  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ , see e.g. [1]. It is worth noting that  $\Gamma$  is a Polish space (see e.g. [22] and references therein).

The space of  $n$ -point configurations in  $Y \in \mathcal{B}(\mathbb{R}^d)$  is defined by

$$\Gamma^{(n)}(Y) := \left\{ \eta \subset Y \mid |\eta| = n \right\}, \quad n \in \mathbb{N}.$$

We set  $\Gamma^{(0)}(Y) := \{\emptyset\}$ . As a set,  $\Gamma^{(n)}(Y)$  may be identified with the symmetrization of  $\tilde{Y}^n = \{(x_1, \dots, x_n) \in Y^n \mid x_k \neq x_l \text{ if } k \neq l\}$ . Hence one can introduce the corresponding Borel  $\sigma$ -algebra, which we denote by  $\mathcal{B}(\Gamma^{(n)}(Y))$ . The space of finite configurations in  $Y \in \mathcal{B}(\mathbb{R}^d)$  is defined as

$$\Gamma_0(Y) := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}(Y).$$

This space is equipped with the topology of the disjoint union. Let  $\mathcal{B}(\Gamma_0(Y))$  denote the corresponding Borel  $\sigma$ -algebra. In the case of  $Y = \mathbb{R}^d$  we will omit the index  $Y$  in the previously defined notations. Namely,  $\Gamma_0 := \Gamma_0(\mathbb{R}^d)$ ,  $\Gamma^{(n)} := \Gamma^{(n)}(\mathbb{R}^d)$ .

The restriction of the Lebesgue product measure  $(dx)^n$  to  $(\Gamma^{(n)}, \mathcal{B}(\Gamma^{(n)}))$  we denote by  $m^{(n)}$ . We set  $m^{(0)} := \delta_{\{\emptyset\}}$ . The Lebesgue–Poisson measure  $\lambda$  on  $\Gamma_0$  is defined by

$$\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}. \quad (2.2)$$

For any  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  the restriction of  $\lambda$  to  $\Gamma(\Lambda) := \Gamma_0(\Lambda)$  will be also denoted by  $\lambda$ . The space  $(\Gamma, \mathcal{B}(\Gamma))$  is the projective limit of the family of spaces  $\{(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda)))\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$ . The Poisson measure  $\pi$  on  $(\Gamma, \mathcal{B}(\Gamma))$  is given as the projective limit of the family of measures  $\{\pi^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$ , where  $\pi^\Lambda := e^{-m(\Lambda)} \lambda$  is the probability measure on  $(\Gamma(\Lambda), \mathcal{B}(\Gamma(\Lambda)))$  and  $m(\Lambda)$  is the Lebesgue measure of  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  (see e.g. [1] for details).

A set  $M \in \mathcal{B}(\Gamma_0)$  is called bounded if there exists  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  and  $N \in \mathbb{N}$  such that  $M \subset \bigsqcup_{n=0}^N \Gamma^{(n)}(\Lambda)$ . The set of bounded measurable functions with bounded support we denote by  $B_{bs}(\Gamma_0)$ , i.e.,  $G \in B_{bs}(\Gamma_0)$  if  $G \upharpoonright_{\Gamma_0 \setminus M} = 0$  for some bounded  $M \in \mathcal{B}(\Gamma_0)$ . Any  $\mathcal{B}(\Gamma_0)$ -measurable function  $G$  on  $\Gamma_0$ , in fact, is defined by a sequence of functions  $\{G^{(n)}\}_{n \in \mathbb{N}_0}$  where  $G^{(n)}$  is a  $\mathcal{B}(\Gamma^{(n)})$ -measurable function on  $\Gamma^{(n)}$ . The set of *cylinder functions* on  $\Gamma$  we denote by  $\mathcal{F}_{cyl}(\Gamma)$ . Each  $F \in \mathcal{F}_{cyl}(\Gamma)$  is characterized by the following relation:  $F(\gamma) = F(\gamma_\Lambda)$  for some  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ . Functions on  $\Gamma$  will be called *observables* whereas functions on  $\Gamma_0$  will be called *quasi-observables*.

There exists mapping from  $B_{bs}(\Gamma_0)$  into  $\mathcal{F}_{cyl}(\Gamma)$ , which plays the key role in our further considerations:

$$(KG)(\gamma) := \sum_{\eta \in \gamma} G(\eta), \quad \gamma \in \Gamma, \quad (2.3)$$

where  $G \in B_{bs}(\Gamma_0)$ , see e.g. [20, 28, 29]. The summation in (2.3) is taken over all finite subconfigurations  $\eta \in \Gamma_0$  of the (infinite) configuration  $\gamma \in \Gamma$ ; we denote this by the symbol,  $\eta \Subset \gamma$ . The mapping  $K$  is linear, positivity preserving, and invertible, with

$$(K^{-1}F)(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0. \quad (2.4)$$

Set  $(K_0G)(\eta) := (KG)(\eta)$ ,  $\eta \in \Gamma_0$ .

The so-called coherent state corresponding to a  $\mathcal{B}(\mathbb{R}^d)$ -measurable function  $f$  is defined by

$$e_\lambda(f, \eta) := \prod_{x \in \eta} f(x), \quad \eta \in \Gamma_0 \setminus \{\emptyset\}, \quad e_\lambda(f, \emptyset) := 1.$$

Then

$$(K_0e_\lambda(f))(\eta) = e_\lambda(f + 1, \eta), \quad \eta \in \Gamma_0 \quad (2.5)$$

and for any  $f \in L^1(\mathbb{R}^d, dx)$

$$\int_{\Gamma_0} e_\lambda(f, \eta) d\lambda(\eta) = \exp\left\{\int_{\mathbb{R}^d} f(x) dx\right\}. \quad (2.6)$$

A measure  $\mu \in \mathcal{M}_{\text{fin}}^1(\Gamma)$  is called locally absolutely continuous with respect to the Poisson measure  $\pi$  if for any  $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$  the projection of  $\mu$  onto  $\Gamma(\Lambda)$  is absolutely continuous with respect to the projection of  $\pi$  onto  $\Gamma(\Lambda)$ . In this case, according to [20], there exists a *correlation functional*  $k_\mu : \Gamma_0 \rightarrow \mathbb{R}_+$  such that for any  $G \in B_{\text{bs}}(\Gamma_0)$  the following equality holds

$$\int_{\Gamma} (KG)(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta). \quad (2.7)$$

The functions  $k_\mu^{(n)} : (\mathbb{R}^d)^n \rightarrow \mathbb{R}_+$  given by

$$k_\mu^{(n)}(x_1, \dots, x_n) := \begin{cases} k_\mu(\{x_1, \dots, x_n\}), & \text{if } (x_1, \dots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

are called *correlation functions* of the measure  $\mu$ . Note that  $k_\mu^{(0)} = 1$ .

Below we would like to mention without proof the partial case of the well-known technical lemma (see e.g. [26]) which plays very important role in our calculations.

**Lemma 2.1.** *For any measurable function  $H : \Gamma_0 \times \Gamma_0 \times \Gamma_0 \rightarrow \mathbb{R}$*

$$\int_{\Gamma_0} \sum_{\xi \subset \eta} H(\xi, \eta \setminus \xi, \eta) d\lambda(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\xi, \eta, \eta \cup \xi) d\lambda(\xi) d\lambda(\eta) \quad (2.8)$$

*if both sides of the equality make sense.*

### 3 Non-equilibrium evolutions

In a birth-and-death dynamics, particles appear and disappear randomly in  $\mathbb{R}^d$  according to birth and death rates which depend on the configuration of the whole system. Heuristically, the corresponding Markov generator is described by the following expression

$$\begin{aligned} (LF)(\gamma) &:= \sum_{x \in \gamma} d(x, \gamma \setminus x) [F(\gamma \setminus x) - F(\gamma)] \\ &\quad + \int_{\mathbb{R}^d} b(x, \gamma) [F(\gamma \cup x) - F(\gamma)] dx. \end{aligned} \quad (3.1)$$

Here the coefficient  $d(x, \gamma) \geq 0$  represents the rate at which particle of the configuration  $\gamma$  located at  $x$  dies (disappears), whereas, for a given configuration  $\gamma$ , the new particle appears at the site  $x$  with the rate  $b(x, \gamma) \geq 0$ .

We always suppose that, for all  $x \in \mathbb{R}^d$  and a.a.  $x' \in \mathbb{R}^d$ , the values  $d(x, \eta)$  and  $b(x', \eta)$  are finite at least for all configurations  $\eta \in \Gamma_0$  which do not contain the points  $x$  and  $x'$ . Here and below, we assume that, for a.a.  $x \in \mathbb{R}^d$ , the functions  $d(x, \cdot)$  and  $b(x, \cdot)$  are locally integrable, i.e., for all bounded  $M \in \mathcal{B}(\Gamma_0)$

$$\int_M (d(x, \eta) + b(x, \eta)) d\lambda(\eta) < \infty.$$

A natural way to study a Markov evolution with generator (3.1) is to construct a corresponding Markov semigroup with the generator  $L$ . This problem is related to the analysis of the initial value problem

$$\frac{\partial}{\partial t} F_t = L F_t, \quad t > 0, \quad F_t|_{t=0} = F_0 \quad (3.2)$$

in some space of functions on the configuration space  $\Gamma$ . However, a rigorous analysis of such evolutionary equations meets serious technical problems, and was realized for the case of birth and death generator in a finite volumes only, see [19]. On the other hand, there is a very important question concerning the state evolution associated with Markov dynamics. Namely, one can consider the initial value problem

$$\frac{d}{dt} \langle F, \mu_t \rangle = \langle L F, \mu_t \rangle, \quad t > 0, \quad \mu_t|_{t=0} = \mu_0, \quad F \in K(B_{\text{bs}}(\Gamma_0)) \quad (3.3)$$

in some space of probability measures on  $(\Gamma, \mathcal{B}(\Gamma))$ . Here the pairing between functions and measure on  $\Gamma$  is given by (1.4). In fact, the solution to (3.3) describes the time evolution of distributions instead of the evolution of initial points in the Markov process. Suppose now that a solution  $\mu_t \in \mathcal{M}_{\text{fm}}^1(\Gamma)$  to (3.3) exists and remains locally absolutely continuous with respect to the Poisson measure  $\pi$  for all  $t > 0$  provided  $\mu_0$  has such a property. Then one can consider the correlation functionals  $k_t := k_{\mu_t}$ ,  $t \geq 0$ . By (2.7), we may rewrite (3.3) in the following way

$$\frac{d}{dt} \langle\langle K^{-1} F, k_t \rangle\rangle = \langle\langle K^{-1} L F, k_t \rangle\rangle, \quad t > 0, \quad k_t|_{t=0} = k_0, \quad (3.4)$$

for all  $F \in K(B_{\text{bs}}(\Gamma_0))$ . Here the duality between functions on  $\Gamma_0$  is given by (3.41) below (cf. (1.7)). Next, if we substitute  $F = K G$ ,  $G \in B_{\text{bs}}(\Gamma_0)$  in (3.4), we derive

$$\frac{d}{dt} \langle\langle G, k_t \rangle\rangle = \langle\langle K^{-1} L K G, k_t \rangle\rangle, \quad t > 0, \quad k_t|_{t=0} = k_0 \quad (3.5)$$

for all  $G \in B_{\text{bs}}(\Gamma_0)$ . In applications, for concrete birth and death rates we may usually define  $(L F)(\eta)$  at least for all  $\eta \in \Gamma_0$ . In particular, this can be done under the conditions on birth and death rates described above. Therefore, the expression  $K^{-1} L F$  may be defined via (2.4) point-wisely. This fact allows us to consider the following operator

$$(\hat{L} G)(\eta) := (K^{-1} L K G)(\eta), \quad \eta \in \Gamma_0$$



for  $G \in B_{\text{bs}}(\Gamma_0)$ . As a result, we are interested in the weak solution to the equation

$$\frac{\partial}{\partial t} k_t = \hat{L}^* k_t, \quad t > 0, \quad k_t|_{t=0} = k_0, \quad (3.6)$$

where  $L^*$  is dual operator to  $\hat{L}$  with respect to the duality  $\langle\langle \cdot, \cdot \rangle\rangle$ . One of the main aims of the present paper is to study the classical solution to (3.6) in a proper functional space.

To solve (3.6), we will use the following strategy. We start with a pre-dual (with respect to the duality  $\langle\langle \cdot, \cdot \rangle\rangle$ ) initial value problem

$$\frac{\partial}{\partial t} G_t = \hat{L} G_t, \quad t > 0, \quad G_t|_{t=0} = G_0, \quad (3.7)$$

which will be solved in a Banach space (3.15) of so-called *quasi-observables*. Namely, we construct a holomorphic semigroup which gives a solution to (3.7). After this we consider the dual semigroup which produces a weak solution to (3.5). And, finally, we will find a Banach space in which a classical solution to (3.6) exists.

### 3.1 Evolutions in the space of quasi-observables

We start from the deriving of the expression for  $\hat{L}$ .

**Proposition 3.1.** *For any  $G \in B_{\text{bs}}(\Gamma_0)$  the following formula holds*

$$\begin{aligned} (\hat{L}G)(\eta) = & - \sum_{\xi \subset \eta} G(\xi) \sum_{x \in \xi} (K_0^{-1} d(x, \cdot \cup \xi \setminus x))(\eta \setminus \xi) \\ & + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) (K_0^{-1} b(x, \cdot \cup \xi))(\eta \setminus \xi) dx, \quad \eta \in \Gamma_0, \end{aligned} \quad (3.8)$$

provided all terms of the right hand side have sense.

*Proof.* Following [16], for

$$B_x = K_0^{-1} b(x, \cdot), \quad D_x = K_0^{-1} d(x, \cdot) \quad (3.9)$$

we have

$$(\hat{L}G)(\eta) = - \sum_{x \in \eta} (D_x \star G(\cdot \cup x))(\eta \setminus x) + \int_{\mathbb{R}^d} (B_x \star G(\cdot \cup x))(\eta) dx. \quad (3.10)$$

Here for the given  $\mathcal{B}(\Gamma_0)$ -measurable functions  $G_1$  and  $G_2$ , we define

$$(G_1 \star G_2)(\eta) = \sum_{(\eta_1, \eta_2, \eta_3) \in \mathcal{P}_3(\eta)} G_1(\eta_1 \cup \eta_2) G_2(\eta_2 \cup \eta_3), \quad \eta \in \Gamma_0, \quad (3.11)$$

where  $\mathcal{P}_3(\eta)$  denotes the set of all partitions of  $\eta$  in three parts which may be empty, see [20]. Rewriting (3.11) in the form

$$(G_1 \star G_2)(\eta) = \sum_{\xi \subset \eta} G_1(\xi) \sum_{\zeta \subset \xi} G_2((\eta \setminus \xi) \cup \zeta), \quad \eta \in \Gamma_0, \quad (3.12)$$

we get

$$\begin{aligned} (\hat{L}G)(\eta) &= - \sum_{x \in \eta} \sum_{\xi \subset \eta \setminus x} G(\xi \cup x) \sum_{\zeta \subset \xi} D_x((\eta \setminus x) \setminus \xi) \cup \zeta) \\ &\quad + \int_{\mathbb{R}^d} \sum_{\xi \subset \eta} G(\xi \cup x) \sum_{\beta \subset \xi} B_x((\eta \setminus \xi) \cup \beta) dx. \end{aligned}$$

Using the fact that for any  $\mathcal{B}(\Gamma_0)$ -measurable function  $G$

$$(K_0G)(\eta_1 \cup \eta_2) = \sum_{\xi \subset \eta_1 \cup \eta_2} G(\xi) = \sum_{\xi_1 \subset \eta_1} \sum_{\xi_2 \subset \eta_2} G(\xi_1 \cup \xi_2), \quad \eta_1 \cap \eta_2 = \emptyset,$$

for  $F = K_0G$  we get

$$(K_0^{-1}F(\cdot \cup \eta_2))(\xi_1) = (K_0G(\xi_1 \cup \cdot))(\eta_2), \quad \xi_1 \cap \eta_2 = \emptyset. \quad (3.13)$$

Now, the simple equality

$$\sum_{x \in \eta} \sum_{\xi \subset \eta \setminus x} h(x, \xi, \eta) = \sum_{\xi \subset \eta} \sum_{x \in \xi} h(x, \xi \setminus x, \eta), \quad (3.14)$$

which holds for any  $\mathcal{B}(\mathbb{R}^d) \times \mathcal{B}(\Gamma_0) \times \mathcal{B}(\Gamma_0)$ -measurable function  $h$  finishes the proposition.  $\square$

In general, the r.h.s. of (3.8) may be undefined. For arbitrary and fixed  $C > 1$  we consider the functional space

$$\mathcal{L}_C := L^1(\Gamma_0, C^{|\eta|} \lambda(d\eta)). \quad (3.15)$$

Throughout of the whole paper, symbol  $\|\cdot\|_C$  stands for the norm of the space (3.15). Now we proceed to study rigorous properties of the operator given by the expression (3.8) in the Banach space  $\mathcal{L}_C$ .

*Remark 3.1.*  $B_{\text{bs}}(\Gamma_0)$  is a dense set in  $\mathcal{L}_C$ .

*Remark 3.2.* The reason to consider the weight  $C^{|\cdot|}$  in the definition of  $\mathcal{L}_C$  is the following. As it was noted above we expect to find a solution to (3.6) in the space of functions on  $\Gamma_0$  which satisfy the Ruelle bound (1.6). Such space  $\mathcal{K}_C$  will be considered in Subsection 3.2 below. The space  $\mathcal{L}_C$  is pre-dual to  $\mathcal{K}_C$  with respect to duality (3.41).

Set,

$$D(\eta) := \sum_{x \in \eta} d(x, \eta \setminus x) \geq 0, \quad \eta \in \Gamma_0; \quad (3.16)$$

$$\mathcal{D} := \{G \in \mathcal{L}_C \mid D(\cdot)G \in \mathcal{L}_C\}. \quad (3.17)$$

Note that  $B_{\text{bs}}(\Gamma_0) \subset \mathcal{D}$ . In particular,  $\mathcal{D}$  is a dense set in  $\mathcal{L}_C$ .

We will show that  $(\hat{L}, \mathcal{D})$  given by (3.8), (3.17) generates  $C_0$ -semigroup on  $\mathcal{L}_C$ .

**Theorem 3.2.** *Suppose that there exists  $a_1 \geq 1$ ,  $a_2 > 0$  such that for all  $\xi \in \Gamma_0$  and a.a.  $x \in \mathbb{R}^d$*

$$\sum_{x \in \xi} \int_{\Gamma_0} |K_0^{-1}d(x, \cdot \cup \xi \setminus x)|(\eta) C^{|\eta|} d\lambda(\eta) \leq a_1 D(\xi), \quad (3.18)$$

$$\sum_{x \in \xi} \int_{\Gamma_0} |K_0^{-1}b(x, \cdot \cup \xi \setminus x)|(\eta) C^{|\eta|} d\lambda(\eta) \leq a_2 D(\xi). \quad (3.19)$$

and, moreover,

$$a_1 + \frac{a_2}{C} < \frac{3}{2}. \quad (3.20)$$

Then  $(\hat{L}, \mathcal{D})$  is the generator of a holomorphic semigroup  $\hat{T}(t)$  on  $\mathcal{L}_C$ .

*Remark 3.3.* Conditions (3.18)–(3.20) express an essential role of the death rate in our construction. They are crucial for the existence of the classical solution to the evolution equation (3.7) in the space  $\mathcal{L}_C$  of quasi-observables (cf. Remark 3.5 below). Note also, that alternatively to a semigroup approach one can study local in time solutions to (3.7) also. For a particular model it was realized in the recent paper [9].

*Proof.* Let us consider the multiplication operator  $(L_0, \mathcal{D})$  on  $\mathcal{L}_C$  given by

$$(L_0 G)(\eta) = -D(\eta) G(\eta), \quad G \in \mathcal{D}, \quad \eta \in \Gamma_0. \quad (3.21)$$

We recall that a densely defined closed operators  $A$  on  $\mathcal{L}_C$  is called sectorial of angle  $\omega \in (0, \frac{\pi}{2})$  if its resolvent set  $\rho(A)$  contains the sector

$$\text{Sect}\left(\frac{\pi}{2} + \omega\right) := \left\{z \in \mathbb{C} \mid |\arg z| < \frac{\pi}{2} + \omega\right\} \setminus \{0\}$$

and for each  $\varepsilon \in (0; \omega)$  there exists  $M_\varepsilon \geq 1$  such that

$$\|R(z, A)\| \leq \frac{M_\varepsilon}{|z|} \quad (3.22)$$

for all  $z \neq 0$  with  $|\arg z| \leq \frac{\pi}{2} + \omega - \varepsilon$ . Here and below we will use notation

$$R(z, A) := (z\mathbb{1} - A)^{-1}, \quad z \in \rho(A).$$

The set of all sectorial operators of angle  $\omega \in (0, \frac{\pi}{2})$  in  $\mathcal{L}_C$  we denote by  $\mathcal{H}_C(\omega)$ . Any  $A \in \mathcal{H}_C(\omega)$  is a generator of a bounded semigroup  $T(t)$  which is holomorphic in the sector  $|\arg t| < \omega$  (see e.g. [8, Theorem II.4.6]). One can prove the following lemma.

**Lemma 3.3.** *The operator  $(L_0, \mathcal{D})$  given by (3.21) is a generator of a contraction semigroup on  $\mathcal{L}_C$ . Moreover,  $L_0 \in \mathcal{H}_C(\omega)$  for all  $\omega \in (0, \frac{\pi}{2})$  and (3.22) holds with  $M_\varepsilon = \frac{1}{\cos \omega}$  for all  $\varepsilon \in (0; \omega)$ .*

*Proof of Lemma 3.3.* It is not difficult to show that the densely defined operator  $L_0$  is closed in  $\mathcal{L}_C$ . Let  $0 < \omega < \frac{\pi}{2}$  be arbitrary and fixed. Clear, that for all  $z \in \text{Sect}(\frac{\pi}{2} + \omega)$

$$|D(\eta) + z| > 0, \quad \eta \in \Gamma_0.$$

Therefore, for any  $z \in \text{Sect}(\frac{\pi}{2} + \omega)$  the inverse operator  $R(z, L_0) = (z\mathbb{1} - L_0)^{-1}$ , the action of which is given by

$$[R(z, L_0)G](\eta) = \frac{1}{D(\eta) + z} G(\eta), \quad (3.23)$$

is well defined on the whole space  $\mathcal{L}_C$ . Moreover,

$$|D(\eta) + z| = \sqrt{(D(\eta) + \text{Re } z)^2 + (\text{Im } z)^2} \geq \begin{cases} |z|, & \text{if } \text{Re } z \geq 0 \\ |\text{Im } z|, & \text{if } \text{Re } z < 0 \end{cases},$$

and for any  $z \in \text{Sect}(\frac{\pi}{2} + \omega)$

$$|\text{Im } z| = |z| |\sin \arg z| \geq |z| \left| \sin \left( \frac{\pi}{2} + \omega \right) \right| = |z| \cos \omega.$$

As a result, for any  $z \in \text{Sect}(\frac{\pi}{2} + \omega)$

$$\|R(z, L_0)\| \leq \frac{1}{|z| \cos \omega}, \quad (3.24)$$

that implies the second assertion. Note also that  $|D(\eta) + z| \geq \text{Re } z$  for  $\text{Re } z > 0$ , hence,

$$\|R(z, L_0)\| \leq \frac{1}{\text{Re } z}, \quad (3.25)$$

that proves the first statement by the classical Hille–Yosida theorem.  $\square$

For any  $G \in B_{bs}(\Gamma_0)$  we define

$$\begin{aligned} (L_1 G)(\eta) &:= (\hat{L}G)(\eta) - (L_0 G)(\eta) \\ &= - \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi} (K_0^{-1} d(x, \cdot \cup \xi \setminus x))(\eta \setminus \xi) \\ &\quad + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) (K_0^{-1} b(x, \cdot \cup \xi))(\eta \setminus \xi) dx. \end{aligned} \quad (3.26)$$

Next Lemma shows that, under conditions (3.18), (3.19) above, the operator  $L_1$  is relatively bounded by the operator  $L_0$ .

**Lemma 3.4.** *Let (3.18), (3.19) hold. Then  $(L_1, \mathcal{D})$  is a well-defined operator in  $\mathcal{L}_C$  such that*

$$\|L_1 R(z, L_0)\| \leq a_1 - 1 + \frac{a_2}{C}, \quad \text{Re } z > 0 \quad (3.27)$$

and

$$\|L_1 G\| \leq \left( a_1 - 1 + \frac{a_2}{C} \right) \|L_0 G\|, \quad G \in \mathcal{D}. \quad (3.28)$$

*Proof of Lemma 3.4.* By Lemma 2.1, we have for any  $G \in \mathcal{L}_C$ ,  $\operatorname{Re} z > 0$

$$\begin{aligned}
& \int_{\Gamma_0} \left| - \sum_{\xi \subsetneq \eta} \frac{1}{z + D(\xi)} G(\xi) \sum_{x \in \xi} (K_0^{-1} d(x, \cdot \cup \xi \setminus x))(\eta \setminus \xi) \right| C^{|\eta|} d\lambda(\eta) \\
& \leq \int_{\Gamma_0} \sum_{\xi \subsetneq \eta} \frac{1}{|z + D(\xi)|} |G(\xi)| \sum_{x \in \xi} |K_0^{-1} d(x, \cdot \cup \xi \setminus x)|(\eta \setminus \xi) C^{|\eta|} d\lambda(\eta) \\
& = \int_{\Gamma_0} \frac{1}{|z + D(\xi)|} |G(\xi)| \sum_{x \in \xi} \int_{\Gamma_0} |K_0^{-1} d(x, \cdot \cup \xi \setminus x)|(\eta) C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi) \\
& \quad - \int_{\Gamma_0} \frac{1}{|z + D(\eta)|} D(\eta) |G(\eta)| C^{|\eta|} d\lambda(\eta) \\
& \leq (a_1 - 1) \int_{\Gamma_0} \frac{1}{\operatorname{Re} z + D(\eta)} D(\eta) |G(\eta)| C^{|\eta|} d\lambda(\eta) \leq (a_1 - 1) \|G\|_C,
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Gamma_0} \left| \sum_{\xi \subsetneq \eta} \int_{\mathbb{R}^d} \frac{1}{z + D(\xi \cup x)} G(\xi \cup x) (K_0^{-1} b(x, \cdot \cup \xi))(\eta \setminus \xi) dx \right| C^{|\eta|} d\lambda(\eta) \\
& \leq \int_{\Gamma_0} \int_{\Gamma_0} \int_{\mathbb{R}^d} \frac{1}{|z + D(\xi \cup x)|} |G(\xi \cup x)| |K_0^{-1} b(x, \cdot \cup \xi)|(\eta) dx C^{|\eta|} C^{|\xi|} d\lambda(\xi) d\lambda(\eta) \\
& = \frac{1}{C} \int_{\Gamma_0} \frac{1}{|z + D(\xi)|} |G(\xi)| \sum_{x \in \xi} \int_{\Gamma_0} |K_0^{-1} b(x, \cdot \cup \xi \setminus x)|(\eta) C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi) \\
& \leq \frac{a_2}{C} \int_{\Gamma_0} \frac{1}{\operatorname{Re} z + D(\xi)} |G(\xi)| D(\xi) C^{|\xi|} d\lambda(\xi) \leq \frac{a_2}{C} \|G\|_C.
\end{aligned}$$

Combining these inequalities we obtain (3.27). The same considerations yield

$$\begin{aligned}
& \int_{\Gamma_0} \left| - \sum_{\xi \subsetneq \eta} G(\xi) \sum_{x \in \xi} (K_0^{-1} d(x, \cdot \cup \xi \setminus x))(\eta \setminus \xi) \right| C^{|\eta|} d\lambda(\eta) \\
& \quad + \int_{\Gamma_0} \left| \sum_{\xi \subsetneq \eta} \int_{\mathbb{R}^d} G(\xi \cup x) (K_0^{-1} b(x, \cdot \cup \xi))(\eta \setminus \xi) dx \right| C^{|\eta|} d\lambda(\eta) \\
& \leq \left( (a_1 - 1) + \frac{a_2}{C} \right) \int_{\Gamma_0} |G(\eta)| D(\eta) C^{|\eta|} d\lambda(\eta),
\end{aligned}$$

that proves (3.28) as well.  $\square$

And now we proceed to finish the proof of the Theorem 3.2. Let us set  $\theta := a_1 + \frac{a_2}{C} - 1 \in (0; \frac{1}{2})$ . Then  $\frac{\theta}{1-\theta} \in (0; 1)$ . Let  $\omega \in (0; \frac{\pi}{2})$  be such that  $\cos \omega < \frac{\theta}{1-\theta}$ . Then, by the proof of Lemma 3.3,  $L_0 \in \mathcal{H}_C(\omega)$  and  $\|R(z, L_0)\| \leq \frac{M}{|z|}$  for all  $z \neq 0$  with  $|\arg z| \leq \frac{\pi}{2} + \omega$ , where  $M := \frac{1}{\cos \omega}$ . Then

$$\theta = \frac{1}{1 + \frac{1-\theta}{\theta}} < \frac{1}{1 + \frac{1}{\cos \omega}} = \frac{1}{1 + M}.$$

Hence, by (3.28) and the proof of [8, Theorem III.2.10], we have that  $(\hat{L} = L_0 + L_1, \mathcal{D})$  is a generator of holomorphic semigroup on  $\mathcal{L}_C$ .  $\square$

*Remark 3.4.* By (3.16), the estimates (3.18), (3.19) are satisfied if

$$\int_{\Gamma_0} |K_0^{-1}d(x, \cdot \cup \xi)|(\eta) C^{|\eta|} d\lambda(\eta) \leq a_1 d(x, \xi), \quad (3.29)$$

$$\int_{\Gamma_0} |K_0^{-1}b(x, \cdot \cup \xi)|(\eta) C^{|\eta|} d\lambda(\eta) \leq a_2 d(x, \xi). \quad (3.30)$$

**Example 1.** (Glauber-type dynamics in continuum). Let  $L$  be given by (3.1) with

$$d(x, \gamma \setminus x) = \exp\left\{s \sum_{y \in \gamma \setminus x} \phi(x-y)\right\}, \quad x \in \gamma, \gamma \in \Gamma, \quad (3.31)$$

$$b(x, \gamma) = z \exp\left\{(s-1) \sum_{y \in \gamma} \phi(x-y)\right\}, \quad x \in \mathbb{R}^d \setminus \gamma, \gamma \in \Gamma, \quad (3.32)$$

where  $\phi : \mathbb{R}^d \rightarrow \mathbb{R}_+$  is a pair potential,  $\phi(-x) = \phi(x)$ ,  $z > 0$  is an activity parameter and  $s \in [0; 1]$ . For any  $s \in [0; 1]$  the operator  $L$  is well defined and, moreover, symmetric in the space  $L^2(\Gamma, \mu)$ , where  $\mu$  is a Gibbs measure, given by the pair potential  $\phi$  and activity parameter  $z$  (see e.g. [25] and references therein). This gives possibility to study the corresponding semigroup in  $L^2(\Gamma, \mu)$ . In the case  $s = 0$ , the corresponding dynamics was also studied in another Banach spaces, see e.g. [23, 15, 10]. Below we show that one of the main result of the paper stated in Theorem 3.2 can be applied to the case of arbitrary  $s \in [0; 1]$ . Set

$$\beta_\tau := \int_{\mathbb{R}^d} |e^{\tau\phi(x)} - 1| dx \in [0; \infty], \quad \tau \in [-1; 1]. \quad (3.33)$$

Let  $s$  be arbitrary and fixed. Suppose that  $\beta_s < \infty$ ,  $\beta_{s-1} < \infty$ . Then, by (3.31), (2.5), and (2.6)

$$\begin{aligned} K_0^{-1}d(x, \cdot \cup \xi)(\eta) &= d(x, \xi) e_\lambda(e^{s\phi(x-\cdot)} - 1, \eta), \\ \int_{\Gamma_0} |K_0^{-1}d(x, \cdot \cup \xi)|(\eta) C^{|\eta|} d\lambda(\eta) &= d(x, \xi) e^{C\beta_s}, \end{aligned}$$

and, analogously,

$$\int_{\Gamma_0} |K_0^{-1}b(x, \cdot \cup \xi)|(\eta) C^{|\eta|} d\lambda(\eta) = b(x, \xi) e^{C\beta_{s-1}} \leq z d(x, \xi) e^{C\beta_{s-1}},$$

since  $\phi \geq 0$ . Therefore, to apply Theorem 3.2 we should assume additionally that

$$e^{C\beta_s} + \frac{z}{C} e^{C\beta_{s-1}} < \frac{3}{2}. \quad (3.34)$$

In particular, for  $s = 0$  we obtain the condition (cf. [23])

$$\frac{z}{C} e^{C\beta_{-1}} < \frac{1}{2}. \quad (3.35)$$

**Example 2.** (Bolker–Dieckman–Law–Pacala (BDLP) model) This example describes the model of plant ecology, see [13] and references therein. Let  $L$  be given by (3.1) with

$$d(x, \gamma \setminus x) = m + \varkappa^- \sum_{y \in \gamma \setminus x} a^-(x - y), \quad x \in \gamma, \gamma \in \Gamma, \quad (3.36)$$

$$b(x, \gamma) = \varkappa^+ \sum_{y \in \gamma} a^+(x - y), \quad x \in \mathbb{R}^d \setminus \gamma, \gamma \in \Gamma, \quad (3.37)$$

where  $m > 0$ ,  $\varkappa^\pm \geq 0$ ,  $0 \leq a^\pm \in L^1(\mathbb{R}^d, dx) \cap L^\infty(\mathbb{R}^d, dx)$ ,  $\int_{\mathbb{R}^d} a^\pm(x) dx = 1$ . Then

$$\begin{aligned} K_0^{-1}d(x, \cdot \cup \xi)(\eta) &= d(x, \xi)0^{|\eta|} + \varkappa^- \mathbb{1}_{\Gamma(1)}(\eta) \sum_{y \in \eta} a^-(x - y), \\ \int_{\Gamma_0} |K_0^{-1}d(x, \cdot \cup \xi)|(\eta) C^{|\eta|} d\lambda(\eta) &= d(x, \xi) + C\varkappa^-, \end{aligned}$$

and, analogously,

$$\int_{\Gamma_0} |K_0^{-1}b(x, \cdot \cup \xi)|(\eta) C^{|\eta|} d\lambda(\eta) = b(x, \xi) + C\varkappa^+.$$

Therefore, if we suppose, for example, that (cf. [13])

$$4\varkappa^- C < m \quad (3.38)$$

$$4\varkappa^+ a^+(x) \leq C\varkappa^- a^-(x), \quad x \in \mathbb{R}^d, \quad (3.39)$$

then there exists  $\delta > 0$  such that

$$d(x, \xi) + C\varkappa^- \leq d(x, \xi) + \frac{m}{4 + \delta} \leq \left(1 + \frac{1}{4 + \delta}\right) d(x, \xi)$$

and

$$b(x, \xi) + C\varkappa^+ \leq \frac{C}{4} \varkappa^- \sum_{y \in \xi} a^-(x - y) + \frac{Cm}{16} < \frac{C}{4} d(x, \xi),$$

since  $4\varkappa^+ \leq C\varkappa^- < \frac{m}{4}$ . The last bound we get integrating both sides of (3.39) over  $\mathbb{R}^d$ .

Hence, (3.18), (3.19) hold and

$$a_1 + \frac{a_2}{C} = 1 + \frac{1}{4 + \delta} + \frac{1}{4} < \frac{3}{2}.$$

*Remark 3.5.* It was shown in [13] that the condition like (3.38) is essential. Namely, if  $m > 0$  is arbitrary small the operator  $\hat{L}$  will not be even accretive in  $\mathcal{L}_C$ .

### 3.2 Evolutions in the space of correlation functions

In this Subsection we will use the semigroup  $\hat{T}(t)$  acting on the space of quasi-observables for a construction of solution to the evolution equation (3.6) on space of correlation functions.

We denote  $d\lambda_C := C^{|\cdot|}d\lambda$ ; and the dual space  $(\mathcal{L}_C)' = (L^1(\Gamma_0, d\lambda_C))' = L^\infty(\Gamma_0, d\lambda_C)$ . The space  $(\mathcal{L}_C)'$  is isometrically isomorphic to the Banach space

$$\mathcal{K}_C := \left\{ k : \Gamma_0 \rightarrow \mathbb{R} \mid k \cdot C^{-|\cdot|} \in L^\infty(\Gamma_0, \lambda) \right\}$$

with the norm

$$\|k\|_{\mathcal{K}_C} := \|C^{-|\cdot|}k(\cdot)\|_{L^\infty(\Gamma_0, \lambda)},$$

where the isomorphism is given by the isometry  $R_C$

$$(\mathcal{L}_C)' \ni k \longmapsto R_C k := k \cdot C^{|\cdot|} \in \mathcal{K}_C. \quad (3.40)$$

In fact, one may consider the duality between the Banach spaces  $\mathcal{L}_C$  and  $\mathcal{K}_C$  given by the following expression

$$\langle\langle G, k \rangle\rangle := \int_{\Gamma_0} G \cdot k \, d\lambda, \quad G \in \mathcal{L}_C, \, k \in \mathcal{K}_C \quad (3.41)$$

with  $|\langle\langle G, k \rangle\rangle| \leq \|G\|_C \cdot \|k\|_{\mathcal{K}_C}$ . It is clear that  $k \in \mathcal{K}_C$  implies

$$|k(\eta)| \leq \|k\|_{\mathcal{K}_C} C^{|\eta|} \quad \text{for } \lambda\text{-a.a. } \eta \in \Gamma_0.$$

Let  $(\hat{L}', \text{Dom}(\hat{L}'))$  be an operator in  $(\mathcal{L}_C)'$  which is dual to the closed operator  $(\hat{L}, \mathcal{D})$ . We consider also its image on  $\mathcal{K}_C$  under the isometry  $R_C$ . Namely, let  $\hat{L}^* = R_C \hat{L}' R_C^{-1}$  with the domain  $\text{Dom}(\hat{L}^*) = R_C \text{Dom}(\hat{L}')$ .

Similarly, one can consider the adjoint semigroup  $\hat{T}'(t)$  in  $(\mathcal{L}_C)'$  and its image  $\hat{T}^*(t)$  in  $\mathcal{K}_C$ . The space  $\mathcal{L}_C$  is not reflexive, hence,  $\hat{T}^*(t)$  is not  $C_0$ -semigroup in  $\mathcal{K}_C$ . However, from the general theory (see e.g. [8]) the last semigroup will be weak\*-continuous, weak\*-differentiable at 0 and  $\hat{L}^*$  will be weak\*-generator of  $\hat{T}^*(t)$ . Therefore, one has an evolution in the space of correlation functions. In fact, we have a solution to the evolution equation (3.6), in a weak\*-sense. This subsection is devoted to the study of a strong solution to this equation.

**Proposition 3.5.** *Let (3.18), (3.19) be satisfied. Suppose that there exists  $A > 0$ ,  $N \in \mathbb{N}_0$ ,  $\nu \geq 1$  such that for  $\xi \in \Gamma_0$  and  $x \notin \xi$*

$$d(x, \xi) \leq A(1 + |\xi|)^N \nu^{|\xi|}, \quad (3.42)$$

Then for any  $\alpha \in (0; \frac{1}{\nu})$

$$\mathcal{K}_{\alpha C} \subset \text{Dom}(\hat{L}^*). \quad (3.43)$$

*Proof.* In order to show (3.43) it is enough to verify that for any  $k \in \mathcal{K}_{\alpha C}$  there exists  $k^* \in \mathcal{K}_C$  such that for any  $G \in \text{Dom}(\hat{L})$

$$\langle\langle \hat{L}G, k \rangle\rangle = \langle\langle G, k^* \rangle\rangle. \quad (3.44)$$



According to [16], (3.44) is valid for any  $k \in \mathcal{K}_{\alpha C}$  with  $k^* = \hat{L}^*k$ , where

$$\begin{aligned} (\hat{L}^*k)(\eta) &= - \int_{\Gamma_0} k(\zeta \cup \eta) \sum_{x \in \eta} \sum_{\xi \subset \eta \setminus x} D_x(\zeta \cup \xi) d\lambda(\zeta) \\ &\quad + \int_{\Gamma_0} \sum_{x \in \eta} k(\zeta \cup (\eta \setminus x)) \sum_{\xi \subset \eta \setminus x} B_x(\zeta \cup \xi) d\lambda(\zeta), \end{aligned}$$

provided  $k^* \in \mathcal{K}_C$ . Using (3.13), one can rewrite the last expression

$$\begin{aligned} (\hat{L}^*k)(\eta) &= - \sum_{x \in \eta} \int_{\Gamma_0} k(\zeta \cup \eta) (K_0^{-1}d(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta) \\ &\quad + \sum_{x \in \eta} \int_{\Gamma_0} k(\zeta \cup (\eta \setminus x)) (K_0^{-1}b(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta). \end{aligned}$$

Then, by (3.18), (3.19), and (3.42),

$$\begin{aligned} &C^{-|\eta|} \left| (\hat{L}^*k)(\eta) \right| \\ &\leq C^{-|\eta|} \sum_{x \in \eta} \int_{\Gamma_0} |k(\zeta \cup \eta)| |K_0^{-1}d(x, \cdot \cup \eta \setminus x)|(\zeta) d\lambda(\zeta) \\ &\quad + C^{-|\eta|} \sum_{x \in \eta} \int_{\Gamma_0} |k(\zeta \cup (\eta \setminus x))| |K_0^{-1}b(x, \cdot \cup \eta \setminus x)|(\zeta) d\lambda(\zeta) \\ &\leq \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} \sum_{x \in \eta} \int_{\Gamma_0} (\alpha C)^{|\zeta|} |K_0^{-1}d(x, \cdot \cup \eta \setminus x)|(\zeta) d\lambda(\zeta) \\ &\quad + \frac{1}{\alpha C} \|k\|_{\mathcal{K}_{\alpha C}} \alpha^{|\eta|} \sum_{x \in \eta} \int_{\Gamma_0} (\alpha C)^{|\zeta|} |K_0^{-1}b(x, \cdot \cup \eta \setminus x)|(\zeta) d\lambda(\zeta) \\ &\leq \|k\|_{\mathcal{K}_{\alpha C}} \left( a_1 + \frac{a_2}{\alpha C} \right) \alpha^{|\eta|} \sum_{x \in \eta} d(x, \eta \setminus x) \\ &\leq A \|k\|_{\mathcal{K}_{\alpha C}} \left( a_1 + \frac{a_2}{\alpha C} \right) \alpha^{|\eta|} (1 + |\eta|)^{N+1} \nu^{|\eta|-1}. \end{aligned}$$

Using elementary inequality

$$(1+t)^b a^t \leq \frac{1}{a} \left( \frac{b}{-e \ln a} \right)^b, \quad b \geq 1, \quad a \in (0, 1), \quad t \geq 0, \quad (3.45)$$

we have for  $\alpha \nu < 1$

$$\operatorname{ess\,sup}_{\eta \in \Gamma_0} C^{-|\eta|} \left| (\hat{L}^*k)(\eta) \right| \leq \|k\|_{\mathcal{K}_{\alpha C}} \left( a_1 + \frac{a_2}{\alpha C} \right) \frac{A}{\alpha \nu^2} \left( \frac{N+1}{-e \ln(\alpha \nu)} \right)^{N+1} < \infty. \quad \square$$

**Lemma 3.6.** *Let (3.42) holds. We define for any  $\alpha \in (0; 1)$*

$$\mathcal{D}_\alpha := \{G \in \mathcal{L}_{\alpha C} \mid D(\cdot)G \in \mathcal{L}_{\alpha C}\}.$$

*Then for any  $\alpha \in (0; \frac{1}{\nu})$*

$$\mathcal{D} \subset \mathcal{L}_C \subset \mathcal{D}_\alpha \subset \mathcal{L}_{\alpha C} \quad (3.46)$$

*Proof.* The first and last inclusions are obvious. To prove the second one, we use (3.42), (3.45) and obtain for any  $G \in \mathcal{L}_C$

$$\begin{aligned} \int_{\Gamma_0} D(\eta) |G(\eta)| (\alpha C)^{|\eta|} d\lambda(\eta) &\leq \int_{\Gamma_0} \alpha^{|\eta|} \sum_{x \in \eta} A(1 + |\eta|)^N \nu^{|\eta|-1} |G(\eta)| C^{|\eta|} d\lambda(\eta) \\ &\leq \text{const} \int_{\Gamma_0} |G(\eta)| C^{|\eta|} d\lambda(\eta) < \infty. \quad \square \end{aligned}$$

**Proposition 3.7.** *Let (3.18), (3.19), and (3.42) hold with*

$$a_1 + \frac{a_2}{\alpha C} < \frac{3}{2} \quad (3.47)$$

for some  $\alpha \in (0; 1)$ . Then  $(\hat{L}, \mathcal{D}_\alpha)$  is a generator of a holomorphic semigroup  $\hat{T}_\alpha(t)$  on  $\mathcal{L}_{\alpha C}$ .

*Proof.* The proof is similar to the proof of Theorem 3.2, taking into account that bounds (3.19), (3.18) imply the same bounds for  $\alpha C$  instead of  $C$ . Note also that (3.47) is stronger than (3.20).  $\square$

Under conditions (3.18), (3.19), (3.47), and (3.42), we consider the adjoint semigroup  $\hat{T}'(t)$  in  $(\mathcal{L}_C)'$  and its image  $\hat{T}^*(t)$  in  $\mathcal{K}_C$ . By e.g. [8, Subsection II.2.6], the restriction  $\hat{T}^\circ(t)$  of the semigroup  $\hat{T}^*(t)$  onto its invariant Banach subspace  $\overline{\text{Dom}(\hat{L}^*)}$  (here and below all closures are in the norm of the space  $\mathcal{K}_C$ ) is a strongly continuous semigroup. Moreover, its generator  $\hat{L}^\circ$  will be a part of  $\hat{L}^*$ , namely,

$$\text{Dom}(\hat{L}^\circ) = \left\{ k \in \text{Dom}(\hat{L}^*) \mid \hat{L}^* k \in \overline{\text{Dom}(\hat{L}^*)} \right\}$$

and  $\hat{L}^* k = \hat{L}^\circ k$  for any  $k \in \text{Dom}(\hat{L}^\circ)$ .

**Theorem 3.8.** *Let (3.18), (3.19), and (3.42) hold with*

$$1 \leq \nu < \frac{C}{a_2} \left( \frac{3}{2} - a_1 \right). \quad (3.48)$$

Then for any  $\alpha \in \left( \frac{a_2}{C(\frac{3}{2} - a_1)}; \frac{1}{\nu} \right)$  the set  $\overline{\mathcal{K}_{\alpha C}}$  is a  $\hat{T}^\circ(t)$ -invariant Banach subspace of  $\mathcal{K}_C$ .

*Proof.* First of all note that the condition on  $\alpha$  implies (3.47). Next, we prove that  $\hat{T}'_\alpha(t)G = \hat{T}'(t)G$  for any  $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$ . Let  $\hat{L}_\alpha = (\hat{L}, \mathcal{D}_\alpha)$  is the operator in  $\mathcal{L}_{\alpha C}$ . There exists  $\omega > 0$  such that  $(\omega; +\infty) \subset \rho(\hat{L}) \cap \rho(\hat{L}_\alpha)$ , see e.g. [8, Section III.2]. For some fixed  $z \in (\omega; +\infty)$  we denote by  $R(z, \hat{L}) = (z\mathbb{1} - \hat{L})^{-1}$  the resolvent of  $(\hat{L}, \mathcal{D})$  in  $\mathcal{L}_C$  and by  $R(z, \hat{L}_\alpha) = (z\mathbb{1} - \hat{L}_\alpha)^{-1}$  the resolvent of  $\hat{L}_\alpha$  in  $\mathcal{L}_{\alpha C}$ . Then for any  $G \in \mathcal{L}_C$  we have  $R(z, \hat{L})G \in \mathcal{D} \subset \mathcal{D}_\alpha$  and

$$R(z, \hat{L})G - R(z, \hat{L}_\alpha)G = R(z, \hat{L}_\alpha)((z\mathbb{1} - \hat{L}_\alpha) - (z\mathbb{1} - \hat{L}))R(z, \hat{L})G = 0,$$

since  $\hat{L}_\alpha = \hat{L}$  on  $\mathcal{D}$ . As a result,  $\hat{T}_\alpha(t)G = \hat{T}(t)G$  on  $\mathcal{L}_C$ .

Note that for any  $G \in \mathcal{L}_C \subset \mathcal{L}_{\alpha C}$  and for any  $k \in \mathcal{K}_{\alpha C} \subset \mathcal{K}_C$  we have  $\hat{T}_\alpha(t)G \in \mathcal{L}_{\alpha C}$  and

$$\left\langle\left\langle \hat{T}_\alpha(t)G, k \right\rangle\right\rangle = \left\langle\left\langle G, \hat{T}_\alpha^*(t)k \right\rangle\right\rangle,$$

where, by the same construction as before,  $\hat{T}_\alpha^*(t)k \in \mathcal{K}_{\alpha C}$ . But  $G \in \mathcal{L}_C, k \in \mathcal{K}_C$  implies

$$\left\langle\left\langle \hat{T}_\alpha(t)G, k \right\rangle\right\rangle = \left\langle\left\langle \hat{T}(t)G, k \right\rangle\right\rangle = \left\langle\left\langle G, \hat{T}^*(t)k \right\rangle\right\rangle.$$

Hence,  $\hat{T}^*(t)k = \hat{T}_\alpha^*(t)k \in \mathcal{K}_{\alpha C}$  that proves the statement due to continuity of the family  $\hat{T}^*(t)$ .  $\square$

Therefore, one can consider the restriction  $\hat{T}^{\circ\alpha}$  of the semigroup  $\hat{T}^\circ$  onto  $\overline{\mathcal{K}_{\alpha C}}$ . It will be strongly continuous semigroup with the generator  $\hat{L}^{\circ\alpha}$  which is a restriction of  $\hat{L}^\circ$  onto  $\overline{\mathcal{K}_{\alpha C}}$  (see e.g. [8, Subsection II.2.3]). Hence, we have the strong solution (in the sense of the norm in  $\mathcal{K}_C$ ) to the evolution equation (3.6) on the linear subspace  $\mathcal{K}_{\alpha C}$ .

*Remark 3.6.* Let us clarify the reasons we avoid a construction of this evolution in  $\mathcal{K}_C$  directly, via e.g. perturbation techniques. First of all  $(L_0, \mathcal{K}_{\alpha C})$  is not closed operator neither in  $\mathcal{K}_C$  nor in  $\overline{\mathcal{K}_{\alpha C}}$ . To make it closed, one can consider the operator  $L_0$  in  $\mathcal{K}_C$  on its maximal domain  $\mathcal{D}^* := \{G \in \mathcal{K}_C \mid DG \in \mathcal{K}_C\}$ . However, this domain is not dense in  $\mathcal{K}_C$ . Under condition of Proposition 3.5 one can show that  $\mathcal{K}_{\alpha C} \subset \mathcal{D}^*$ , but it is not clear whether  $\mathcal{D}^* \subset \overline{\mathcal{K}_{\alpha C}}$ . Therefore, we are not able to work in the space  $\overline{\mathcal{K}_{\alpha C}}$ , staying on the operator-dependent space  $\mathcal{D}^*$ . Suppose one can prove estimate like (3.27). Then one can show that  $(\hat{L}^*, \mathcal{D}^*)$  will be a generator of a  $C_0$ -semigroup  $W(t)$  on  $\mathcal{D}^*$ . Even in this case it seems to be very difficult to show that this semigroup will be  $\mathcal{K}_{\alpha C}$ -invariant.

**Example 1** (revisited). To apply Theorem 3.8 to Example 1 it is enough to check (3.42) and (3.48). One has

$$d(x, \xi) = \exp\left\{s \sum_{y \in \xi} \phi(x - y)\right\} \leq \nu^{|\xi|},$$

where  $\nu = 1$  for  $s = 0$  and  $\nu = e^{s\bar{\phi}} \geq 1$ ,  $\bar{\phi} = \max_{x \in \mathbb{R}^d} \phi(x)$  for  $s \in (0; 1]$  provided  $\phi$  is bounded on  $\mathbb{R}^d$ . If  $s = 0$  then (3.48) is true (whenever condition (3.35) is satisfied). For the bounded  $\phi$  and  $s \in (0; 1]$  one may rewrite (3.48) in the following form:

$$e^{C\beta_s} + \frac{z}{C} e^{s\bar{\phi} + C\beta_{s-1}} < \frac{3}{2}. \quad (3.49)$$

Note, that (3.49) is the stronger version of condition (3.34).

**Example 2** (revisited). According to (3.38)–(3.39),

$$\begin{aligned} d(x, \xi) &= m + \varkappa^- \sum_{y \in \xi} a^-(x - y) \leq m + A^- \varkappa^- |\xi| \\ &< m + A^- \frac{m}{4C} |\xi| < m \left(1 + \frac{A^-}{4C}\right) (1 + |\xi|), \end{aligned}$$

where  $A^- = \|a^-\|_{L^\infty(\mathbb{R}^d)}$ . Therefore, (3.42) holds with  $\nu = 1$ , which makes (3.48) obvious.

### 3.3 Stationary equation

In this subsection we study the question about stationary solutions to (3.6). For any  $s \geq 0$ , we consider the following subset of  $\mathcal{K}_C$

$$\mathcal{K}_{\alpha C}^{(s)} := \{k \in \mathcal{K}_{\alpha C} \mid k(\emptyset) = s\}.$$

We define  $\tilde{\mathcal{K}}$  to be the closure of  $\mathcal{K}_{\alpha C}^{(0)}$  in the norm of  $\mathcal{K}_C$ . It is clear that  $\tilde{\mathcal{K}}$  with the norm of  $\mathcal{K}_C$  is a Banach space.

**Proposition 3.9.** *Let (3.18), (3.19), and (3.42) be satisfied with*

$$a_1 + \frac{a_2}{C} < 2. \quad (3.50)$$

Assume, additionally, that

$$d(x, \emptyset) > 0, \quad x \in \mathbb{R}^d. \quad (3.51)$$

Then for any  $\alpha \in (0; \frac{1}{\nu})$  the stationary equation

$$\hat{L}^* k = 0 \quad (3.52)$$

has a unique solution  $k_{\text{inv}}$  from  $\mathcal{K}_{\alpha C}^{(1)}$  which is given by the expression

$$k_{\text{inv}} = 1^* + (\mathbb{1} - S)^{-1} E. \quad (3.53)$$

Here  $1^*$  denotes the function defined by  $1^*(\eta) = 0^{|\eta|}$ ,  $\eta \in \Gamma_0$ , the function  $E \in \mathcal{K}_{\alpha C}^{(0)}$  is such that

$$E(\eta) = \mathbb{1}_{\Gamma^{(1)}}(\eta) \sum_{x \in \eta} \frac{b(x, \emptyset)}{d(x, \emptyset)}, \quad \eta \in \Gamma_0,$$

and  $S$  is a generalized Kirkwood–Salzburg operator on  $\tilde{\mathcal{K}}$ , given by

$$\begin{aligned} (Sk)(\eta) &= -\frac{1}{D(\eta)} \sum_{x \in \eta} \int_{\Gamma_0 \setminus \{\emptyset\}} k(\zeta \cup \eta) (K_0^{-1} d(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta) \\ &\quad + \frac{1}{D(\eta)} \sum_{x \in \eta} \int_{\Gamma_0} k(\zeta \cup (\eta \setminus x)) (K_0^{-1} b(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta), \end{aligned} \quad (3.54)$$

for  $\eta \neq \emptyset$  and  $(Sk)(\emptyset) = 0$ . In particular, if  $b(x, \emptyset) = 0$  for a.a.  $x \in \mathbb{R}^d$  then this solution is such that

$$k_{\text{inv}}^{(n)} = 0, \quad n \geq 1. \quad (3.55)$$

*Remark 3.7.* It is worth noting that (3.29), (3.30) imply (3.51).

*Proof.* Suppose that (3.52) holds for some  $k \in \mathcal{K}_{\alpha C}^{(1)}$ . Then

$$\begin{aligned} D(\eta)k(\eta) &= - \sum_{x \in \eta} \int_{\Gamma_0 \setminus \{\emptyset\}} k(\zeta \cup \eta) (K_0^{-1}d(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta) \\ &\quad + \sum_{x \in \eta} \int_{\Gamma_0} k(\zeta \cup (\eta \setminus x)) (K_0^{-1}b(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta). \end{aligned} \quad (3.56)$$

The equality (3.56) is satisfied for any  $k \in \mathcal{K}_{\alpha C}^{(1)}$  at the point  $\eta = \emptyset$ . Using the fact that  $D(\emptyset) = 0$  one may rewrite (3.56) in terms of the function  $\tilde{k} = k - 1^* \in \mathcal{K}_{\alpha C}^{(0)}$ . Namely,

$$\begin{aligned} D(\eta)\tilde{k}(\eta) &= - \sum_{x \in \eta} \int_{\Gamma_0 \setminus \{\emptyset\}} \tilde{k}(\zeta \cup \eta) (K_0^{-1}d(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta) \\ &\quad + \sum_{x \in \eta} \int_{\Gamma_0} \tilde{k}(\zeta \cup (\eta \setminus x)) (K_0^{-1}b(x, \cdot \cup \eta \setminus x))(\zeta) d\lambda(\zeta). \\ &\quad + \sum_{x \in \eta} 0^{|\eta \setminus x|} b(x, \eta \setminus x). \end{aligned} \quad (3.57)$$

As a result,

$$\tilde{k}(\eta) = (S\tilde{k})(\eta) + E(\eta), \quad \eta \in \Gamma_0.$$

Next, for  $\eta \neq \emptyset$

$$\begin{aligned} &C^{-|\eta|} |(Sk)(\eta)| \\ &\leq \frac{C^{-|\eta|}}{D(\eta)} \sum_{x \in \eta} \int_{\Gamma_0 \setminus \{\emptyset\}} |k(\zeta \cup \eta)| |(K_0^{-1}d(x, \cdot \cup \eta \setminus x))(\zeta)| d\lambda(\zeta) \\ &\quad + \frac{C^{-|\eta|}}{D(\eta)} \sum_{x \in \eta} \int_{\Gamma_0} |k(\zeta \cup (\eta \setminus x))| |(K_0^{-1}b(x, \cdot \cup \eta \setminus x))(\zeta)| d\lambda(\zeta) \\ &\leq \frac{\|k\|_{\mathcal{K}_C}}{D(\eta)} \sum_{x \in \eta} \int_{\Gamma_0 \setminus \{\emptyset\}} C^{|\zeta|} |(K_0^{-1}d(x, \cdot \cup \eta \setminus x))(\zeta)| d\lambda(\zeta) \\ &\quad + \frac{\|k\|_{\mathcal{K}_C}}{D(\eta)} \frac{1}{C} \sum_{x \in \eta} \int_{\Gamma_0} C^{|\zeta|} |(K_0^{-1}b(x, \cdot \cup \eta \setminus x))(\zeta)| d\lambda(\zeta) \\ &\leq \frac{\|k\|_{\mathcal{K}_C}}{D(\eta)} D(\eta) \left( a_1 - 1 + \frac{a_2}{C} \right) = \left( a_1 - 1 + \frac{a_2}{C} \right) \|k\|_{\mathcal{K}_C}. \end{aligned}$$

Hence,

$$\|S\| = a_1 + \frac{a_2}{C} - 1 < 1$$

in  $\tilde{K}$ . This finishes the proof.  $\square$

*Remark 3.8.* The name of the operator (3.54) is motivated by Example 1. Namely, if  $s = 0$  then the operator (3.54) has form

$$(Sk)(\eta) = \frac{1}{m|\eta|} \sum_{x \in \eta} e_\lambda(e^{-\phi(x^\cdot)}, \eta \setminus x) \int_{\Gamma_0} k(\zeta \cup (\eta \setminus x)) e_\lambda(e^{-\phi(x^\cdot)} - 1, \zeta) d\lambda(\zeta),$$

that is quite similar of the so-called Kirkwood–Salsburg operator known in mathematical physics (see e.g. [32, 21]). For  $s = 0$  condition (3.50) has form  $\frac{z}{C} e^{C\beta-1} < 1$  (cf. (3.35)). Under this condition, the stationary solution to (3.52) is a unique and coincides with the correlation function of the Gibbs measure, corresponding to potential  $\phi$  and activity  $z$ .

*Remark 3.9.* It is worth pointing out that  $b(x, \emptyset) = 0$  in the case of Example 2. Therefore, if we suppose (cf. (3.38), (3.39)) that  $2\kappa^- C < m$  and  $2\kappa^+ a^+(x) \leq C\kappa^- a^-(x)$ , for  $x \in \mathbb{R}^d$ , condition (3.50) will be satisfied. However, the unique solution to (3.52) will be given by (3.55). In the next example we improve this statement.

**Example 3.** Let us consider the following natural modification of BDLP-model coming from Example 2: let  $d$  be given by (3.36) and

$$b(x, \gamma) = \kappa + \kappa^+ \sum_{y \in \gamma} a^+(x - y), \quad x \in \mathbb{R}^d \setminus \gamma, \quad \gamma \in \Gamma, \quad (3.58)$$

where  $\kappa^+, a^+$  are as before and  $\kappa > 0$ . Then, under assumptions

$$2 \max\left\{\kappa^- C; \frac{2\kappa}{C}\right\} < m \quad (3.59)$$

and

$$2\kappa^+ a^+(x) \leq C\kappa^- a^-(x), \quad x \in \mathbb{R}^d, \quad (3.60)$$

we obtain for some  $\delta > 0$

$$\begin{aligned} \int_{\Gamma_0} |K_0^{-1} d(x, \cdot \cup \xi)| (\eta) C^{|\eta|} d\lambda(\eta) &= d(x, \xi) + C\kappa^- \leq \left(1 + \frac{1}{2 + \delta}\right) d(x, \xi) \\ \int_{\Gamma_0} |K_0^{-1} b(x, \cdot \cup \xi)| (\eta) C^{|\eta|} d\lambda(\eta) &= b(x, \xi) + C\kappa^+ \\ &\leq \kappa + \frac{1}{2} C\kappa^- \sum_{y \in \xi} a^-(x - y) + \frac{m}{4} C < \frac{C}{2} d(x, \xi). \end{aligned}$$

The latter inequalities imply (3.50). In this case,  $E(\eta) = \mathbb{1}_{\Gamma^{(1)}}(\eta) \frac{\kappa}{m}$ .

*Remark 3.10.* If  $a^+(x) = a^-(x)$ ,  $x \in \mathbb{R}^d$  and  $\kappa^+ = z\kappa^-$ ,  $\kappa = zm$  for some  $z > 0$  then  $b(x, \gamma) = zd(x, \gamma)$  and the Poisson measure  $\pi_z$  with the intensity  $z$  will be symmetrizing measure for the operator  $L$ . In particular, it will be invariant measure. This fact means that its correlation function  $k_z(\eta) = z^{|\eta|}$  is a solution to (3.52). Conditions (3.59) and (3.60) in this case are equivalent to  $4z < C$  and  $2\kappa^- C < m$ . As a result, due to uniqueness of such solution,

$$1^*(\eta) + z(\mathbb{1} - S)^{-1} \mathbb{1}_{\Gamma^{(1)}}(\eta) = z^{|\eta|}, \quad \eta \in \Gamma_0.$$

## 4 Scalings

For the reader convenience, we start from the idea of the Vlasov-type scaling. The general scheme for the birth-and-death dynamics as well as for the conservative ones may be found in [14]. The realizations of this approach for the Glauber dynamics (Example 1 with  $s = 0$ ) and for the BDLP dynamics (Example 2) were considered in [12, 11], correspondingly. The idea of the Vlasov-type scaling consists in the following.

We would like to construct some scaling  $L_\varepsilon$ ,  $\varepsilon > 0$ , of the generator  $L$ , such that the following scheme holds. Suppose that we have a semigroup  $\hat{U}_\varepsilon(t)$  with the generator  $\hat{L}_\varepsilon$  in some  $\mathcal{L}_{C_\varepsilon}$ ,  $\varepsilon > 0$ . Consider the dual semigroup  $\hat{U}_\varepsilon^*(t)$ . Let us choose an initial function of the corresponding Cauchy problem with a singularity in  $\varepsilon$ . Namely,  $\varepsilon^{|\eta|} k_0^{(\varepsilon)}(\eta) \sim r_0(\eta)$ ,  $\varepsilon \rightarrow 0$ ,  $\eta \in \Gamma_0$  for some function  $r_0$ , which is independent of  $\varepsilon$ . The scaling  $L \mapsto L_\varepsilon$  should be chosen in such a way that first of all the corresponding semigroup  $\hat{U}_\varepsilon^*(t)$  preserves the order of the singularity:

$$\varepsilon^{|\eta|} (\hat{U}_\varepsilon^*(t) k_0^{(\varepsilon)})(\eta) \sim r_t(\eta), \quad \varepsilon \rightarrow 0, \quad \eta \in \Gamma_0, \quad (4.1)$$

and, secondly, the dynamics  $r_0 \mapsto r_t$  preserves the Lebesgue–Poisson exponents. Namely, if  $r_0(\eta) = e_\lambda(\rho_0, \eta)$  then  $r_t(\eta) = e_\lambda(\rho_t, \eta)$ . There exists explicit (non-linear, in general) differential equation for  $\rho_t$ :

$$\frac{\partial}{\partial t} \rho_t(x) = v(\rho_t)(x) \quad (4.2)$$

which will be called the Vlasov-type equation.

Now we explain an informal way to realize such a scheme. Let us consider for any  $\varepsilon > 0$  the following mapping (cf. (3.40)) defined for functions on  $\Gamma_0$

$$(R_\varepsilon r)(\eta) := \varepsilon^{|\eta|} r(\eta). \quad (4.3)$$

This mapping is “self-dual” with respect to the duality (3.41), moreover,  $R_\varepsilon^{-1} = R_{\varepsilon^{-1}}$ . Having  $R_\varepsilon k_0^{(\varepsilon)} \sim r_0$ ,  $\varepsilon \rightarrow 0$ , we need  $r_t \sim R_\varepsilon \hat{U}_\varepsilon^*(t) k_0^{(\varepsilon)} \sim R_\varepsilon \hat{U}_\varepsilon^*(t) R_{\varepsilon^{-1}} r_0$ ,  $\varepsilon \rightarrow 0$ . Therefore, we have to show that for any  $t \geq 0$  the operator family  $R_\varepsilon \hat{U}_\varepsilon^*(t) R_{\varepsilon^{-1}}$ ,  $\varepsilon > 0$  has limiting (in a proper sense) operator  $U(t)$  and

$$U(t) e_\lambda(\rho_0) = e_\lambda(\rho_t). \quad (4.4)$$

But, heuristically,  $\hat{U}_\varepsilon^*(t) = \exp\{t \hat{L}_\varepsilon^*\}$  and  $R_\varepsilon \hat{U}_\varepsilon^*(t) R_{\varepsilon^{-1}} = \exp\{t R_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}\}$ . Let us consider the “renormalized” operator

$$\hat{L}_{\varepsilon, \text{ren}}^* := R_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}. \quad (4.5)$$

In fact, we need that there exists an operator  $\hat{L}_V^*$  such that  $\exp\{t R_\varepsilon \hat{L}_\varepsilon^* R_{\varepsilon^{-1}}\} \rightarrow \exp\{t \hat{L}_V^*\} =: U(t)$  satisfying (4.4). Therefore, an heuristic way to produce scaling  $L \mapsto L_\varepsilon$  is to demand that

$$\lim_{\varepsilon \rightarrow 0} \left( \frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) - \hat{L}_{\varepsilon, \text{ren}}^* e_\lambda(\rho_t, \eta) \right) = 0, \quad \eta \in \Gamma_0$$

provided  $\rho_t$  satisfies (4.2). The point-wise limit of  $\hat{L}_{\varepsilon, \text{ren}}^*$  will be natural candidate for  $\hat{L}_V^*$ .

Note that (4.5) implies informally that  $\hat{L}_{\varepsilon, \text{ren}} = R_{\varepsilon^{-1}} \hat{L}_{\varepsilon} R_{\varepsilon}$ . We propose below the scheme to give rigorous meaning to the idea introduced above. We consider, for a proper scaling  $L_{\varepsilon}$ , the ‘‘renormalized’’ operator  $\hat{L}_{\varepsilon, \text{ren}}$  and prove that it is a generator of a strongly continuous contraction semigroup  $\hat{U}_{\varepsilon, \text{ren}}(t)$  in  $\mathcal{L}_C$ . Next, we show that the formal limit  $\hat{L}_V$  of  $\hat{L}_{\varepsilon, \text{ren}}$  is a generator of a strongly continuous contraction semigroup  $\hat{U}_V(t)$  in  $\mathcal{L}_C$ . Finally, we prove that  $\hat{U}_{\varepsilon, \text{ren}}(t) \rightarrow \hat{U}_V(t)$  strongly in  $\mathcal{L}_C$ . This implies weak\*-convergence of the dual semigroups  $\hat{U}_{\varepsilon, \text{ren}}^*(t)$  to  $\hat{U}_V^*(t)$ . We explain also in which sense  $\hat{U}_V^*(t)$  satisfies the properties above.

Let us consider for any  $\varepsilon \in (0; 1]$  the following scaling of (3.1)

$$\begin{aligned} (L_{\varepsilon}F)(\gamma) &:= \sum_{x \in \gamma} d_{\varepsilon}(x, \gamma \setminus x) [F(\gamma \setminus x) - F(\gamma)] \\ &\quad + \varepsilon^{-1} \int_{\mathbb{R}^d} b_{\varepsilon}(x, \gamma) [F(\gamma \cup x) - F(\gamma)] dx, \end{aligned} \quad (4.6)$$

and define the renormalized operator  $\hat{L}_{\varepsilon, \text{ren}} := R_{\varepsilon^{-1}} K^{-1} L_{\varepsilon} K R_{\varepsilon}$ . Using the same arguments as in the proof of Proposition 3.1, we get

$$\begin{aligned} (\hat{L}_{\varepsilon, \text{ren}}G)(\eta) &= - \sum_{\xi \subset \eta} G(\xi) \varepsilon^{-|\eta \setminus \xi|} \sum_{x \in \xi} (K_0^{-1} d_{\varepsilon}(x, \cdot \cup \xi \setminus x))(\eta \setminus \xi) \\ &\quad + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) \varepsilon^{-|\eta \setminus \xi|} (K_0^{-1} b_{\varepsilon}(x, \cdot \cup \xi))(\eta \setminus \xi) dx. \end{aligned} \quad (4.7)$$

Below we generalize slightly the previous introduced notations: for  $\varepsilon \in (0; 1]$ ,  $\alpha \in (0; 1)$

$$\begin{aligned} D_{\varepsilon}(\eta) &:= \sum_{x \in \eta} d_{\varepsilon}(x, \eta \setminus x); \\ \mathcal{D}^{(\varepsilon)} &:= \{G \in \mathcal{L}_C \mid D^{(\varepsilon)}(\cdot)G \in \mathcal{L}_C\}; \\ (L_0^{(\varepsilon)}G)(\eta) &:= -D_{\varepsilon}(\eta)G(\eta), \quad G \in \mathcal{D}^{(\varepsilon)}; \\ (L_1^{(\varepsilon)}G)(\eta) &:= (\hat{L}_{\varepsilon, \text{ren}}G)(\eta) - (L_0^{(\varepsilon)}G)(\eta), \quad G \in \mathcal{D}^{(\varepsilon)}. \end{aligned}$$

Suppose that there exists  $a_1 \geq 1$ ,  $a_2 > 0$ ,  $A > 0$ ,  $N \in \mathbb{N}_0$ ,  $\nu \geq 1$  such that for all  $\xi \in \Gamma_0$ , for a.a.  $x \in \mathbb{R}^d$ , and for any  $\varepsilon \in (0; 1]$

$$\sum_{x \in \xi} \int_{\Gamma_0} |K_0^{-1} d_{\varepsilon}(x, \cdot \cup \xi \setminus x)|(\eta) \varepsilon^{-|\eta|} C^{|\eta|} d\lambda(\eta) \leq a_1 D_{\varepsilon}(\xi), \quad (4.8)$$

$$\sum_{x \in \xi} \int_{\Gamma_0} |K_0^{-1} b_{\varepsilon}(x, \cdot \cup \xi \setminus x)|(\eta) \varepsilon^{-|\eta|} C^{|\eta|} d\lambda(\eta) \leq a_2 D_{\varepsilon}(\xi), \quad (4.9)$$

$$d_{\varepsilon}(x, \xi) \leq A(1 + |\xi|)^N \nu^{|\xi|}. \quad (4.10)$$



Without loss of generality we will assume that all constant in (4.8)–(4.10) are the same as before.

**Proposition 4.1.** 1. Let conditions (4.8) and (4.9) hold with

$$a_1 + \frac{a_2}{C} < \frac{3}{2}. \quad (4.11)$$

Then, for any  $\varepsilon \in (0; 1]$ ,  $(\hat{L}_{\varepsilon, \text{ren}}, \mathcal{D}^{(\varepsilon)})$  is a generator of the holomorphic semigroup  $\hat{U}_\varepsilon(t)$  on  $\mathcal{L}_C$ .

2. Assume, additionally, that (4.10) is satisfied with

$$1 \leq \nu < \frac{C}{a_2} \left( \frac{3}{2} - a_1 \right). \quad (4.12)$$

Then there exists  $\alpha_0 \in (0; \frac{1}{\nu})$  such that for any  $\alpha \in (\alpha_0; \frac{1}{\nu})$  and for any  $\varepsilon \in (0; 1]$  there exists a strongly continuous semigroup  $\hat{U}_\varepsilon^{\odot \alpha}(t)$  on the space  $\overline{\mathcal{K}_{\alpha C}}$  with the generator  $\hat{L}_\varepsilon^{\odot \alpha} = \hat{L}_{\varepsilon, \text{ren}}^*$  on the domain

$$\text{Dom}(L_\varepsilon^{\odot \alpha}) = \{k \in \overline{\mathcal{K}_{\alpha C}} \mid \hat{L}_{\varepsilon, \text{ren}}^* k \in \overline{\mathcal{K}_{\alpha C}}\}.$$

Note that, for  $k \in \mathcal{K}_{\alpha C}$

$$\begin{aligned} (\hat{L}_{\varepsilon, \text{ren}}^* k)(\eta) &= - \sum_{x \in \eta} \int_{\Gamma_0} k(\xi \cup \eta) \varepsilon^{-|\xi|} (K_0^{-1} d_\varepsilon(x, \cdot \cup \eta \setminus x))(\xi) d\lambda(\xi) \\ &\quad + \sum_{x \in \eta} \int_{\Gamma_0} k(\xi \cup (\eta \setminus x)) \varepsilon^{-|\xi|} (K_0^{-1} b_\varepsilon(x, \cdot \cup \eta \setminus x))(\xi) d\lambda(\xi). \end{aligned} \quad (4.13)$$

*Proof.* 1. Identically to the proof of Lemma 3.3 we show that  $(L_0^{(\varepsilon)}, \mathcal{D}^{(\varepsilon)}) \in \mathcal{H}_C(\omega)$  for any  $\omega \in (0; \frac{\pi}{2})$ . Next, in the same way as in the proof of Lemma 3.4 we prove that, for any  $\text{Re } z > 0$ ,

$$\|L_1^{(\varepsilon)} R(z, L_0^{(\varepsilon)})\| \leq a_1 - 1 + \frac{a_2}{C} < \frac{1}{2}, \quad (4.14)$$

since (3.50) is satisfied. Note also that we may show also another bound (cf. (3.28)):

$$\|L_1^{(\varepsilon)} G\| < \frac{1}{2} \|L_0^{(\varepsilon)} G\|, \quad G \in \mathcal{L}_C. \quad (4.15)$$

Hence, one can prove the statement in the same way as Theorem 3.2.

2. Similarly to Proposition 3.5, we obtain that, under condition (4.10),  $\mathcal{K}_{\alpha C} \subset \text{Dom}(\hat{L}_{\varepsilon, \text{ren}}^*)$  for any  $\alpha \in (0; \frac{1}{\nu})$ . Using (4.12), we are able to choose  $\theta \in (a_1 + \frac{a_2 \nu}{C}; \frac{3}{2})$ . Then (3.48) is satisfied, and  $\alpha_0 := \frac{a_2}{C(\theta - a_1)} \in (0; \frac{1}{\nu})$ . The same considerations as in Theorem 3.8 finish the proof.  $\square$

**Assumption 4.1.** For all  $\eta, \xi \in \Gamma_0$  and a.a.  $x \in \mathbb{R}^d$  the following limits exist and coincide:

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K_0^{-1} d_\varepsilon(x, \cdot \cup \xi))(\eta) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K_0^{-1} d_\varepsilon(x, \cdot))(\eta) =: D_x^V(\eta); \quad (4.16)$$

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K_0^{-1} b_\varepsilon(x, \cdot \cup \xi))(\eta) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K_0^{-1} b_\varepsilon(x, \cdot))(\eta) =: B_x^V(\eta). \quad (4.17)$$

We would like to emphasize, that above limits should not depend on  $\xi$ . The collection of examples for such  $d_\varepsilon, b_\varepsilon$  can be found in [14]. Note that (4.16), (4.17) imply, in particular,

$$\lim_{\varepsilon \rightarrow 0} d_\varepsilon(x, \xi) = D_x^V(\emptyset), \quad \lim_{\varepsilon \rightarrow 0} b_\varepsilon(x, \xi) = B_x^V(\emptyset), \quad (4.18)$$

for all  $\xi \in \Gamma_0$  and a.a.  $x \in \mathbb{R}^d$ .

Combining condition (4.16) with (4.17), we have point-wise limit for  $\hat{L}_{\varepsilon, \text{ren}}$ :

$$(\hat{L}_V G)(\eta) := - \sum_{\xi \subset \eta} G(\xi) \sum_{x \in \xi} D_x^V(\eta \setminus \xi) + \sum_{\xi \subset \eta} \int_{\mathbb{R}^d} G(\xi \cup x) B_x^V(\eta \setminus \xi) dx. \quad (4.19)$$

Set

$$\begin{aligned} D_V(\eta) &:= \sum_{x \in \eta} D_x^V(\emptyset); \\ \mathcal{D}^V &:= \{G \in \mathcal{L}_C \mid D_V(\cdot)G \in \mathcal{L}_C\}; \\ (L_0^V G)(\eta) &:= -D_V(\eta)G(\eta), \quad G \in \mathcal{D}^V; \\ (L_1^V G)(\eta) &:= (\hat{L}_V G)(\eta) - (L_0^V G)(\eta), \quad G \in \mathcal{D}^V. \end{aligned}$$

Suppose that for a.a.  $x \in \mathbb{R}^d$

$$\int_{\Gamma_0} |D_x^V(\eta)| C^{|\eta|} d\lambda(\eta) \leq a_1 D_x^V(\emptyset), \quad (4.20)$$

$$\int_{\Gamma_0} |B_x^V(\eta)| C^{|\eta|} d\lambda(\eta) \leq a_2 D_x^V(\emptyset), \quad (4.21)$$

$$D_x^V(\emptyset) \leq A, \quad (4.22)$$

where the constants are the same as before.

*Remark 4.1.* It is worth pointing out that conditions (4.20)–(4.22), in general, are weaker than (4.8)–(4.10). Indeed, if  $b_\varepsilon(x, \gamma) = b'(x, \gamma) + \varepsilon \cdot b''(x, \gamma)$  then (4.21) is an assumption on function  $b'$  only, whereas (4.9) requires additional conditions on  $b''$ .

Let  $c > 0$ . We define  $\bar{B}_c^\infty$  to be the closed ball of radius  $c$  in the Banach space  $L^\infty(\mathbb{R}^d)$ .

**Proposition 4.2.** *1. Let conditions (4.20), (4.21), and (4.11) hold. Then  $(\hat{L}_V, \mathcal{D}^V)$  is a generator of the holomorphic semigroup  $\hat{U}_V(t)$  on  $\mathcal{L}_C$ .*

2. Suppose, additionally, (4.22) is satisfied. Then, there exists  $\alpha_0 \in (0; 1)$  such that for any  $\alpha \in (\alpha_0; 1)$  there exists a strongly continuous semigroup  $\tilde{U}_V^{\odot\alpha}(t)$  on the space  $\overline{\mathcal{K}_{\alpha C}}$  with the generator  $\hat{L}_V^{\odot\alpha} = \hat{L}_V^*$ ,

$$\text{Dom}(L_V^{\odot\alpha}) = \{k \in \overline{\mathcal{K}_{\alpha C}} \mid \hat{L}_V^* k \in \overline{\mathcal{K}_{\alpha C}}\}.$$

Moreover, for  $k \in \mathcal{K}_{\alpha C}$

$$\begin{aligned} (\hat{L}_V^* k)(\eta) &= - \sum_{x \in \eta} \int_{\Gamma_0} k(\xi \cup \eta) D_x^V(\xi) d\lambda(\xi) \\ &\quad + \sum_{x \in \eta} \int_{\Gamma_0} k(\xi \cup (\eta \setminus x)) B_x^V(\xi) d\lambda(\xi). \end{aligned} \quad (4.23)$$

3. Let  $\alpha \in (\alpha_0; 1)$ ,  $\rho_0 \in \bar{B}_{\alpha C}^\infty$ . Then the evolution equation

$$\begin{cases} \frac{\partial}{\partial t} k_t = \hat{L}_V^* k_t \\ k_t|_{t=0} = e_\lambda(\rho_0, \eta) \end{cases} \quad (4.24)$$

has a unique solution  $k_t = e_\lambda(\rho_t)$  in  $\overline{\mathcal{K}_{\alpha C}}$  provided  $\rho_t$  belongs to  $\bar{B}_{\alpha C}^\infty$  and satisfies the Vlasov-type equation

$$\begin{aligned} \frac{\partial}{\partial t} \rho_t(x) &= - \rho_t(x) \int_{\Gamma_0} e_\lambda(\rho_t, \xi) D_x^V(\xi) d\lambda(\xi) \\ &\quad + \int_{\Gamma_0} e_\lambda(\rho_t, \xi) B_x^V(\xi) d\lambda(\xi). \end{aligned} \quad (4.25)$$

*Proof.* 1. The proof for the first statement is similar to the analogous one in Proposition 4.1.

2. The same arguments as for the proof of Proposition 3.5 show that, for any  $\alpha \in (0; 1)$ ,  $\mathcal{K}_{\alpha C} \subset \text{Dom}(\hat{L}_V^*)$ . Next, by (4.11), let us now take  $\theta \in (a_1 + \frac{a_2}{C}; \frac{3}{2})$ . Then we can set  $\alpha_0 := \frac{a_2}{C(\theta - a_1)} \in (0; 1)$ . The second statement can be handled now in much the same way as in Theorem 3.8.

3. Since  $\rho_0 \in \bar{B}_{\alpha C}^\infty$  implies  $k_0 \in \overline{\mathcal{K}_{\alpha C}}$  then the Cauchy problem (4.24) has a unique solution in  $\overline{\mathcal{K}_{\alpha C}}$ . On the other hand, according to (4.13), for any  $\rho_t \in \bar{B}_{\alpha C}^\infty$

$$\begin{aligned} (\hat{L}_V^* e_\lambda(\rho_t))(\eta) &= - \sum_{x \in \eta} e_\lambda(\rho_t, \eta) \int_{\Gamma_0} e_\lambda(\rho_t, \xi) D_x^V(\xi) d\lambda(\xi) \\ &\quad + \sum_{x \in \eta} e_\lambda(\rho_t, \eta \setminus x) \int_{\Gamma_0} e_\lambda(\rho_t, \xi) B_x^V(\xi) d\lambda(\xi). \end{aligned} \quad (4.26)$$

Combining (4.26) with the equality

$$\frac{\partial}{\partial t} e_\lambda(\rho_t, \eta) = \sum_{x \in \eta} \rho_t(x) e_\lambda(\rho_t, \eta \setminus x),$$

we can assert that  $k_t = e_\lambda(\rho_t)$  is a solution to (4.24), with  $\rho_t$  given by (4.25).  $\square$

*Remark 4.2.* The question about existence and uniqueness of solutions to the Vlasov-type equation (4.25) in some ball  $\bar{B}_{\alpha C}^\infty$  of  $L^\infty(\mathbb{R}^d)$  shall be solved separately in each concrete model, see e.g. [12, 11].

Our next goal is to study the question about convergence of the semigroups  $\hat{U}_\varepsilon(t)$  to  $\hat{U}_V(t)$  in  $\mathcal{L}_C$ .

We begin by proving the following abstract statement.

**Lemma 4.3.** *Let  $X$  be a Banach space, and let  $(A_\varepsilon, \mathfrak{D}_\varepsilon)$ ,  $(B_\varepsilon, \mathfrak{D}_\varepsilon)$ ,  $\varepsilon \geq 0$  be closed, densely defined operators on  $X$ . Suppose that there exists  $\beta > 0$  and  $z \in \mathbb{C}$  with  $\operatorname{Re} z > \beta$  such that  $z \in \rho(A_\varepsilon)$  for all  $\varepsilon \geq 0$  and*

$$\kappa := \sup_{\varepsilon > 0} \|(A_\varepsilon - z\mathbb{1})^{-1}\| < \infty, \quad (4.27)$$

$$\sigma := \sup_{\varepsilon \geq 0} \|B_\varepsilon (A_\varepsilon - z\mathbb{1})^{-1}\| < 1, \quad (4.28)$$

$$(A_\varepsilon - z\mathbb{1})^{-1} \xrightarrow{s} (A_0 - z\mathbb{1})^{-1}, \quad \varepsilon \rightarrow 0, \quad (4.29)$$

$$B_\varepsilon (A_\varepsilon - z\mathbb{1})^{-1} \xrightarrow{s} B_0 (A_0 - z\mathbb{1})^{-1}, \quad \varepsilon \rightarrow 0. \quad (4.30)$$

Then  $z$  belongs to the resolvent set of  $L_\varepsilon := A_\varepsilon + B_\varepsilon$ ,  $\varepsilon \geq 0$  and

$$(L_\varepsilon - z\mathbb{1})^{-1} \xrightarrow{s} (L_0 - z\mathbb{1})^{-1}, \quad \varepsilon \rightarrow 0.$$

*Proof.* For any  $\varepsilon \geq 0$  we set  $C_\varepsilon := (A_\varepsilon - z\mathbb{1})^{-1}$ , then we have  $\operatorname{Ran}(C_\varepsilon) = \operatorname{Dom}(A_\varepsilon) = \operatorname{Dom}(B_\varepsilon) = \operatorname{Dom}(L_\varepsilon) = \mathfrak{D}_\varepsilon$ . Therefore, for any  $z \in \rho(A_\varepsilon)$  one can write

$$L_\varepsilon - z\mathbb{1} = A_\varepsilon + B_\varepsilon - z\mathbb{1} = (B_\varepsilon (A_\varepsilon - z\mathbb{1})^{-1} + \mathbb{1}) (A_\varepsilon - z\mathbb{1}).$$

By (4.28), the operator  $B_\varepsilon (A_\varepsilon - z\mathbb{1})^{-1} + \mathbb{1} = B_\varepsilon C_\varepsilon + \mathbb{1}$  is invertible with bounded inverse  $D_\varepsilon$ . Moreover,

$$\|D_\varepsilon\| \leq \frac{1}{1 - \|B_\varepsilon C_\varepsilon\|} \leq \frac{1}{1 - \sigma}. \quad (4.31)$$

Therefore, we have that  $z \in \rho(L_\varepsilon)$  and

$$(L_\varepsilon - z\mathbb{1})^{-1} = (A_\varepsilon - z\mathbb{1})^{-1} (B_\varepsilon C_\varepsilon + \mathbb{1})^{-1} = C_\varepsilon D_\varepsilon. \quad (4.32)$$

Next,

$$\begin{aligned} D_\varepsilon - D_0 &= (B_\varepsilon C_\varepsilon + \mathbb{1})^{-1} - (B_0 C_0 + \mathbb{1})^{-1} \\ &= (B_\varepsilon C_\varepsilon + \mathbb{1})^{-1} ((B_0 C_0 + \mathbb{1}) - (B_\varepsilon C_\varepsilon + \mathbb{1})) (B_0 C_0 + \mathbb{1})^{-1} \\ &= D_\varepsilon (B_0 C_0 - B_\varepsilon C_\varepsilon) D_0, \end{aligned}$$

thus, according to (4.31) and (4.30), for any  $x \in X$

$$\begin{aligned} \|D_\varepsilon x - D_0 x\| &\leq \|D_\varepsilon\| \cdot \|(B_0 C_0 - B_\varepsilon C_\varepsilon) D_0 x\| \\ &\leq \frac{1}{1 - \sigma} \|(B_0 C_0 - B_\varepsilon C_\varepsilon) D_0 x\| \rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

Hence,  $D_\varepsilon \xrightarrow{s} D_0$ . Then, using (4.32) and (4.29), we have for any  $x \in X$

$$\begin{aligned} & \|(L_\varepsilon - z\mathbb{1})^{-1}x - (L_0 - z\mathbb{1})^{-1}x\| \\ &= \|C_\varepsilon D_\varepsilon x - C_0 D_0 x\| = \|C_\varepsilon(D_\varepsilon - D_0)x + (C_\varepsilon - C_0)D_0 x\| \\ &\leq \|C_\varepsilon\| \cdot \|(D_\varepsilon - D_0)x\| + \|(C_\varepsilon - C_0)D_0 x\| \\ &\leq \kappa \cdot \|(D_\varepsilon - D_0)x\| + \|(C_\varepsilon - C_0)D_0 x\| \rightarrow 0, \quad \varepsilon \rightarrow 0. \end{aligned}$$

The statement is proven.  $\square$

Now we are able to prove result about convergence in  $\mathcal{L}_C$ .

**Theorem 4.4.** *Let conditions (4.8), (4.9), and (4.11) are satisfied. Suppose that convergences (4.16), (4.17) take place for all  $\eta \in \Gamma_0$  as well as in the sense of  $\mathcal{L}_C$ . Assume also that there exists  $\sigma > 0$  such that (cf. (4.18)) either*

$$d_\varepsilon(x, \xi) \leq \sigma D_x^V(\emptyset) \quad \text{or} \quad d_\varepsilon(x, \xi) \geq \sigma D_x^V(\emptyset) \quad (4.33)$$

*is satisfied for all  $\xi \in \Gamma_0$  and for a.a.  $x \in \mathbb{R}^d$ . Then  $\hat{U}_\varepsilon(t) \xrightarrow{s} \hat{U}_V(t)$  in  $\mathcal{L}_C$  uniformly on finite time intervals.*

*Proof.* First of all note that  $\mathcal{L}_C$ -convergence in (4.16), (4.17) together with (4.18) yields (4.20), (4.21) provided (4.8), (4.9) hold. Then, by Propositions 4.1, 4.2, the semigroups  $\hat{U}_\varepsilon(t)$ ,  $\hat{U}_V(t)$  exist in  $\mathcal{L}_C$ . To prove their convergence it is enough to show the strong convergence of the resolvent corresponding to the generators of this semigroup, see e.g. [8, Theorem III.4.8]. To verify this, we apply Lemma 4.3 taking  $A_\varepsilon = L_0^{(\varepsilon)}$ ,  $B_\varepsilon = L_1^{(\varepsilon)}$ ,  $L_\varepsilon = \hat{L}_{\varepsilon, \text{ren}}$ ,  $\mathfrak{D}_0 = \mathcal{D}^V$ ,  $\mathfrak{D}_\varepsilon = \mathcal{D}^{(\varepsilon)}$ ,  $\varepsilon > 0$ . Below we check the conditions of this lemma.

Let us fix any  $z > 0$ . It is easily seen that (4.27) is satisfied since

$$\|(L_0^{(\varepsilon)} - z\mathbb{1})^{-1}\| \leq \frac{1}{z}$$

for all  $\varepsilon \in (0; 1]$ . Clearly, (4.14) implies (4.28). Let  $G \in \mathcal{L}_C$ . Then

$$\begin{aligned} & \|(L_0^{(\varepsilon)} - z\mathbb{1})^{-1}G - (L_0^V - z\mathbb{1})^{-1}G\|_C \\ &\leq \int_{\Gamma_0} \frac{|D^{(\varepsilon)}(\eta) - D^V(\eta)|}{(z + D^V(\eta))(z + D^{(\varepsilon)}(\eta))} |G(\eta)| d\lambda(\eta). \end{aligned}$$

By (4.18), for all  $\eta \in \Gamma_0$

$$D^{(\varepsilon)}(\eta) \rightarrow D^V(\eta), \quad \varepsilon \rightarrow 0. \quad (4.34)$$

Then the inequality

$$\frac{|D^{(\varepsilon)}(\eta) - D^V(\eta)|}{(z + D^V(\eta))(z + D^{(\varepsilon)}(\eta))} \leq \frac{1}{z + D^V(\eta)} + \frac{1}{z + D^{(\varepsilon)}(\eta)} \leq \frac{2}{z}$$

implies (4.29) by the dominated convergence theorem.

Let inequality  $d_\varepsilon(x, \xi) \leq \sigma D_x^V(\emptyset)$  hold for all  $\xi \in \Gamma_0$  and a.a.  $x \in \mathbb{R}^d$ . Then, by Lemma 2.1,

$$\begin{aligned}
& \left\| L_1^{(\varepsilon)} (L_0^{(\varepsilon)} - z\mathbb{1})^{-1} G - L_1^V (L_0^V - z\mathbb{1})^{-1} G \right\|_C \\
& \leq \left\| (L_1^{(\varepsilon)} - L_1^V) (L_0^V - z\mathbb{1})^{-1} G \right\|_C + \left\| L_1^{(\varepsilon)} \left( (L_0^{(\varepsilon)} - z\mathbb{1})^{-1} - (L_0^V - z\mathbb{1})^{-1} \right) G \right\|_C \\
& \leq \int_{\Gamma_0} \frac{|G(\xi)|}{z + D^V(\xi)} \sum_{x \in \xi} \int_{\Gamma_0} \left| \varepsilon^{-|\eta|} K_0^{-1} d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - D_x^V(\eta) \right| C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi) \\
& \quad + \frac{1}{C} \int_{\Gamma_0} \frac{|G(\xi)|}{z + D^V(\xi)} \sum_{x \in \xi} \int_{\Gamma_0} \left| \varepsilon^{-|\eta|} K_0^{-1} b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - B_x^V(\eta) \right| C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi) \\
& \quad + \int_{\Gamma_0} |G(\xi)| \frac{|D^{(\varepsilon)}(\xi) - D^V(\xi)|}{(z + D^V(\xi))(z + D^{(\varepsilon)}(\xi))} \sum_{x \in \xi} \int_{\Gamma_0} \varepsilon^{-|\eta|} \left( |K_0^{-1} d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta)| \right. \\
& \quad \left. + \frac{1}{C} |K_0^{-1} b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta)| \right) C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi). \tag{4.35}
\end{aligned}$$

Convergence in  $\mathcal{L}_C$  for (4.16), (4.17) together with (4.34) implies that all three integrand functions of  $\xi$  appearing in (4.35) converge to 0  $\lambda$ -a.s., as  $\varepsilon \rightarrow 0$ . To use dominated convergence theorem we will show that the following functions are uniformly bounded. Using (4.8), (4.20), and (4.33), we get

$$\begin{aligned}
& \frac{1}{z + D^V(\xi)} \sum_{x \in \xi} \int_{\Gamma_0} \left| \varepsilon^{-|\eta|} K_0^{-1} d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - D_x^V(\eta) \right| C^{|\eta|} d\lambda(\eta) \\
& \leq \frac{a_1}{z + D^V(\xi)} \sum_{x \in \xi} (d_\varepsilon(x, \xi) + D_x^V(\emptyset)) \leq \frac{a_1(1 + \sigma)}{z + D^V(\xi)} \sum_{x \in \xi} D_x^V(\emptyset) \leq a_1(1 + \sigma).
\end{aligned}$$

Analogously, by (4.9), (4.21), and (4.33),

$$\frac{1}{z + D^V(\xi)} \sum_{x \in \xi} \int_{\Gamma_0} \left| \varepsilon^{-|\eta|} K_0^{-1} b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - B_x^V(\eta) \right| C^{|\eta|} d\lambda(\eta) \leq a_1(1 + \sigma).$$

According to (4.8), (4.9), and (4.33),

$$\begin{aligned}
& \frac{|D^{(\varepsilon)}(\xi) - D^V(\xi)|}{(z + D^V(\xi))(z + D^{(\varepsilon)}(\xi))} \sum_{x \in \xi} \int_{\Gamma_0} \varepsilon^{-|\eta|} \left( |K_0^{-1} d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta)| \right. \\
& \quad \left. + \frac{1}{C} |K_0^{-1} b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta)| \right) C^{|\eta|} d\lambda(\eta) \\
& \leq \frac{D^{(\varepsilon)}(\xi) + D^V(\xi)}{(z + D^V(\xi))(z + D^{(\varepsilon)}(\xi))} \sum_{x \in \xi} \left( a_1 d_\varepsilon(x, \xi) + \frac{a_2}{C} d_\varepsilon(x, \xi) \right) \\
& \leq \frac{D^{(\varepsilon)}(\xi)}{(z + D^V(\xi))(z + D^{(\varepsilon)}(\xi))} \left( a_1 + \frac{a_2}{C} \right) \sigma D^V(\xi) \\
& \quad + \frac{D^V(\xi)}{(z + D^V(\xi))(z + D^{(\varepsilon)}(\xi))} \left( a_1 + \frac{a_2}{C} \right) D^{(\varepsilon)}(\xi) \leq \left( a_1 + \frac{a_2}{C} \right) (1 + \sigma).
\end{aligned}$$

Hence, (4.30) is proved.

In the case  $d_\varepsilon(x, \xi) \geq \sigma D_x^V(\emptyset)$ ,  $\xi \in \Gamma_0$ , we rewrite l.h.s. of (4.30) in a another manner. Namely,

$$\begin{aligned}
& \left\| L_1^{(\varepsilon)} (L_0^{(\varepsilon)} - z\mathbb{1})^{-1} G - L_1^V (L_0^V - z\mathbb{1})^{-1} G \right\|_C \\
& \leq \left\| (L_1^{(\varepsilon)} - L_1^V) (L_0^{(\varepsilon)} - z\mathbb{1})^{-1} G \right\|_C + \left\| L_1^V \left( (L_0^{(\varepsilon)} - z\mathbb{1})^{-1} - (L_0^V - z\mathbb{1})^{-1} \right) G \right\|_C \\
& \leq \int_{\Gamma_0} \frac{|G(\xi)|}{z + D^{(\varepsilon)}(\xi)} \sum_{x \in \xi} \int_{\Gamma_0} \left| \varepsilon^{-|\eta|} K_0^{-1} d_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - D_x^V(\eta) \right| C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi) \\
& \quad + \frac{1}{C} \int_{\Gamma_0} \frac{|G(\xi)|}{z + D^{(\varepsilon)}(\xi)} \sum_{x \in \xi} \int_{\Gamma_0} \left| \varepsilon^{-|\eta|} K_0^{-1} b_\varepsilon(x, \cdot \cup \xi \setminus x)(\eta) - B_x^V(\eta) \right| C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi) \\
& \quad + \int_{\Gamma_0} |G(\xi)| \frac{|D^{(\varepsilon)}(\xi) - D^V(\xi)|}{(z + D^V(\xi))(z + D^{(\varepsilon)}(\xi))} \sum_{x \in \xi} \int_{\Gamma_0} \left( D_x^V(\eta) + \frac{1}{C} B_x^V(\eta) \right) C^{|\eta|} d\lambda(\eta) C^{|\xi|} d\lambda(\xi).
\end{aligned}$$

Repeating all estimates done for the first alternative of (4.33) we get the desired result.  $\square$

*Remark 4.3.* Note that in all examples considered in [14] the function  $d_\varepsilon(x, \xi)$  is monotone in  $\varepsilon$ . Taking into account (4.18), condition (4.33) becomes natural.

**Example 1** (revisited). Let us consider for  $\varepsilon \in [0; 1]$ ,  $s \in [0; 1]$

$$d_\varepsilon(x, \gamma) = \exp \left\{ \varepsilon s \sum_{y \in \gamma \setminus x} \phi(x - y) \right\}, \quad b_\varepsilon(x, \gamma) = z \exp \left\{ \varepsilon(s - 1) \sum_{y \in \gamma} \phi(x - y) \right\},$$

Analogously to the previous computations,

$$\begin{aligned}
& \int_{\Gamma_0} |K_0^{-1} d_\varepsilon(x, \cdot \cup \xi)|(\eta) \varepsilon^{-|\eta|} C^{|\eta|} d\lambda(\eta) = d_\varepsilon(x, \xi) e^{C\varepsilon^{-1}\beta_\varepsilon s} \\
& \int_{\Gamma_0} |K_0^{-1} b_\varepsilon(x, \cdot \cup \xi)|(\eta) \varepsilon^{-|\eta|} C^{|\eta|} d\lambda(\eta) = b_\varepsilon(x, \xi) e^{C\varepsilon^{-1}\beta_\varepsilon(s-1)} \\
& \leq z d_\varepsilon(x, \xi) e^{C\varepsilon^{-1}\beta_\varepsilon(s-1)},
\end{aligned}$$

since  $\phi \geq 0$ . Let  $s \in (0; 1]$ . Suppose that  $\tilde{\beta} := \int_{\mathbb{R}^d} \phi(x) e^{\phi(x)} dx < \infty$ . Then for  $\tau \in [-1; 1]$ ,  $\varepsilon \in [0, 1]$

$$\varepsilon^{-1} \beta_{\varepsilon\tau} \leq \varepsilon^{-1} \int_{\mathbb{R}^d} \varepsilon |\tau| \phi(x) \sup_{\tau \in [-1, 1]} e^{\varepsilon\tau\phi(x)} dx \leq \tilde{\beta}.$$

The bound (4.11) will be proved once we show  $e^{C\tilde{\beta}} \left(1 + \frac{z}{C}\right) < \frac{3}{2}$ . If  $s = 0$  then, similarly, we need  $\beta := \int_{\mathbb{R}^d} \phi(x) dx < \infty$  and  $\frac{z}{C} e^{C\beta} < \frac{1}{2}$ . Note also that the conditions  $\beta < \infty$  and  $\bar{\phi} = \sup_{\mathbb{R}^d} \phi(x) < \infty$  yield  $\tilde{\beta} \leq e^{\bar{\phi}} \beta < \infty$ . For the case  $s = 0$  condition (4.10) holds automatically. If  $s \in (0; 1]$  one should assume  $\bar{\phi} < \infty$  then  $\nu = e^{s\bar{\phi}}$  (uniformly by  $\varepsilon \in (0; 1]$ ). Then to guarantee (4.12) we

need  $e^{C\bar{\beta}}(1 + \frac{z}{C}e^{s\bar{\phi}}) < \frac{3}{2}$ . Therefore, under such conditions we obtain statement of Proposition 4.1. Next,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K_0^{-1} d_\varepsilon(x, \cdot \cup \xi))(\eta) &= \lim_{\varepsilon \rightarrow 0} \exp\left\{\varepsilon s \sum_{y \in \xi} \phi(x-y)\right\} e_\lambda\left(\frac{e^{\varepsilon s \phi(x-\cdot)} - 1}{\varepsilon}, \eta\right) \\ &= e_\lambda(s\phi(x-\cdot), \eta) =: D_x^V(\eta); \end{aligned} \quad (4.36)$$

and, analogously,

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{-|\eta|} (K_0^{-1} b_\varepsilon(x, \cdot \cup \xi))(\eta) = z e_\lambda((s-1)\phi(x-\cdot), \eta) =: B_x^V(\eta). \quad (4.37)$$

Since  $D_x^V(\emptyset) = 1 \leq d_\varepsilon(x, \eta)$ , the second alternative of (4.33) is satisfied. In order to use Proposition 4.2 and Theorem 4.4 we need to verify the convergences (4.36) and (4.37) in  $\mathcal{L}_C$  (recall that this implies (4.20) and (4.21), see the proof of Theorem 4.4). To do this let us note that for any  $\tau \in [-1; 1]$

$$\begin{aligned} &\left| \exp\left\{\varepsilon \tau \sum_{y \in \xi} \phi(x-y)\right\} e_\lambda\left(\frac{e^{\varepsilon \tau \phi(x-\cdot)} - 1}{\varepsilon}, \eta\right) - e_\lambda(\tau\phi(x-\cdot), \eta) \right| \\ &\leq \max\left\{\exp\left\{\tau \sum_{y \in \xi} \phi(x-y)\right\}, 1\right\} e_\lambda\left(\frac{|e^{\varepsilon \tau \phi(x-\cdot)} - 1|}{\varepsilon}, \eta\right) + e_\lambda(|\tau|\phi(x-\cdot), \eta) \\ &\leq \left(\max\left\{\exp\left\{\tau \sum_{y \in \xi} \phi(x-y)\right\}, 1\right\} + 1\right) e_\lambda(\phi(x-\cdot), \eta), \end{aligned}$$

and the last function of  $\eta$  belongs to  $\mathcal{L}_C$  for all  $\xi \in \Gamma_0$  and a.a.  $x \in \mathbb{R}^d$  provided  $\phi \in L^1(\mathbb{R}^d)$ . By (2.6), the Vlasov equation (4.25) now has the following form

$$\frac{\partial}{\partial t} \rho_t(x) = -\rho_t(x) \exp\{s(\rho_t * \phi)(x)\} + z \exp\{(s-1)(\rho_t * \phi)(x)\}.$$

Here and below  $*$  means usual convolution of functions in  $\mathbb{R}^d$ .

**Example 2** (revisited). Let  $d_\varepsilon(x, \gamma \setminus x) = m + \varepsilon \varkappa^- \sum_{y \in \gamma \setminus x} a^-(x-y)$ ,  $b_\varepsilon(x, \gamma) = \varepsilon \varkappa^+ \sum_{y \in \gamma} a^+(x-y)$ . Comparing with the previous notations we have changed  $\varkappa^\pm$  onto  $\varepsilon \varkappa^\pm$ . Clearly, conditions (3.38), (3.39) implies the same inequalities for  $\varepsilon \varkappa^\pm$ . Note also that  $d_\varepsilon$  is decreasing in  $\varepsilon \rightarrow 0$ . Therefore, to apply all results of this section to BDLP-model we should prove the convergence (4.16), (4.17) in  $\mathcal{L}_C$ . Note, that

$$\begin{aligned} \varepsilon^{-|\eta|} K_0^{-1} d_\varepsilon(x, \cdot \cup \xi)(\eta) &= d_\varepsilon(x, \xi) \varepsilon^{-|\eta|} 0^{|\eta|} + \varepsilon \varepsilon^{-|\eta|} \varkappa^- \mathbf{1}_{\Gamma(1)}(\eta) \sum_{y \in \eta} a^-(x-y) \\ &= d_\varepsilon(x, \xi) 0^{|\eta|} + \mathbf{1}_{\Gamma(1)}(\eta) \sum_{y \in \eta} a^-(x-y) \\ &\rightarrow m 0^{|\eta|} + \mathbf{1}_{\Gamma(1)}(\eta) \sum_{y \in \eta} a^-(x-y) =: D_x^V(\eta) \end{aligned}$$



and, analogously,

$$\begin{aligned} \varepsilon^{-|\eta|} K_0^{-1} b_\varepsilon(x, \cdot \cup \xi)(\eta) &= b_\varepsilon(x, \xi) 0^{|\eta|} + \mathbb{1}_{\Gamma(1)}(\eta) \sum_{y \in \eta} a^+(x - y) \\ &\rightarrow \mathbb{1}_{\Gamma(1)}(\eta) \sum_{y \in \eta} a^+(x - y) =: B_x^V(\eta). \end{aligned}$$

The convergence in  $\mathcal{L}_C$  is obvious now. The Vlasov equation has the following form

$$\frac{\partial}{\partial t} \rho_t(x) = \varkappa^+(a^+ * \rho_t)(x) - \varkappa^-(a^- * \rho_t)(x) - m \rho_t(x).$$

The existence and uniqueness of the solution to this equation was studied in [11].

*Remark 4.4.* By duality (3.41), Theorem 4.4 yields weak\*-convergence of the semigroups  $\hat{U}_\varepsilon^{\odot\alpha}(t)$  to  $\hat{U}_V^{\odot\alpha}(t)$  in  $\overline{\mathcal{K}_{\alpha C}}$ . To prove such convergence in the strong sense we need additional analysis of their generators. The problem concerns the fact that we have explicit expression for the generator  $\hat{L}_V^{\odot\alpha} = \hat{L}_V^*$  only on the core  $\{k \in \mathcal{K}_{\alpha C} \mid \hat{L}_V^* k \in \overline{\mathcal{K}_{\alpha C}}\}$ . However, we are able to show such convergence for the Glauber dynamics described in Example 1 for  $s = 0$  using modified technique (see [12]).

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