MEASURES ON CONFIGURATION SPACES DEFINED BY RELATIVE ENERGIES

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Dedicated to our admired teacher Yu. M. Berezansky on occasion of his 80th birthday.

Abstract. A construction of measures on configuration spaces defined by relative energies is presented. Integral equations for corresponding correlation functionals are studied. Conditions for the existence and uniqueness of measures in terms of the relative energies are found.

Introduction

The configuration space Γ_X over some manifold X is the space of locally finite subsets (configurations) from X.

The specific role of configuration spaces in the general structure of infinite dimensional analysis is related with several aspects. First of all, these spaces present a class of infinite dimensional manifolds, which may be equipped with a natural differentiable structure, but are neither Banach nor Fréchet manifolds, see [25], [26].

Another source of interests for configuration spaces analysis is the theory of point processes. From the analytical point of view this theory deals with measures on configuration spaces and related structures, see e.g. [6]–[11], [16]–[19].

And the big influence configuration spaces analysis comes from the side of mathematical physics. For a mathematical description of gases or fluids (as systems with indistinguishable particles) we need to use the notion of configuration spaces and corresponding analysis.

Let us describe the content of the work in more detail.

Main objects and preliminary constructions are presented in Section 1. A construction of Gibbs measures via relative energies is discussed in Section 2. There we show the characterization properties of such measures. Main properties of correlation functionals corresponding to these measures are considered in Section 3. In Section 4 we discuss stability and superstability conditions in terms of the relative energy density. A proof of the Kirkwood—Salsburg identity is the main result of Section 5. Sections 6 and 7 are devoted to the existence and uniqueness problem for Gibbs measures.

1. Preliminaries

Our underlying space is the Euclidean space $\mathbb{R}^d$ with the Borel σ-algebra $\mathcal{B}\left(\mathbb{R}^d\right)$ and a non-atomic Radon measure σ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ such that $\sigma\left(\mathbb{R}^d\right) = \infty$. We define the

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configuration space $\Gamma = \Gamma_{\mathbb{R}^d}$ as the space of all locally finite subsets (configurations) of $\mathbb{R}^d$:

$$\Gamma = \left\{ \gamma \subset \mathbb{R}^d \,\middle|\, |\gamma_\Lambda| < +\infty \text{ for any compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ means the cardinality of a set and $\gamma_\Lambda = \gamma \cap \Lambda$.

Consider a σ-algebra $\mathcal{B}(\Gamma)$ as the minimal σ-algebra such that all mappings $\Gamma \ni \gamma \longmapsto |\gamma_\Lambda|$ are $\mathcal{B}(\Gamma)$-measurable for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ (the family of all Borel subsets of $\mathbb{R}^d$ with compact closure). For $A \in \mathcal{B}(\mathbb{R}^d)$ we define also a σ-algebra $\mathcal{B}_A(\Gamma)$ as the minimal σ-algebra such that all mappings $\gamma \mapsto |\gamma_\Lambda|$ are $\mathcal{B}_A(\Gamma)$-measurable for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d), \Lambda \subset A$.

Any configuration $\gamma \in \Gamma$ can be identified with the Radon measure $\sum_{x\in\gamma} \varepsilon_x$ on $\mathbb{R}^d$, where ε_x is the Dirac measure at the point x. In this sense the configuration space can be naturally embedded into the space $\mathcal{M}(\mathbb{R}^d)$ of all Radon measures on $\mathbb{R}^d$. Then the configuration space can be endowed with topology generating by the weak topology on $\mathcal{M}(\mathbb{R}^d)$. Moreover, the σ-algebra $\mathcal{B}(\Gamma)$ is in fact the Borel σ-algebra with respect to this topology.

Let us denote by $\mathcal{M}^1(\Gamma)$ the class of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$. We consider a subclass $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ of all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ with finite locale moments: it means that $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ iff

$$\int_\Gamma |\gamma_\Lambda|^n \, d\mu(\gamma) < +\infty \tag{1.1}$$

for any $n \in \mathbb{N}$ and for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ (the family of all open subsets of $\mathbb{R}^d$ with compact closures).

For a measure $\mu \in \mathcal{M}^1(\Gamma)$ we can consider a (reduced) Campbell measure $\mathcal{C}^!_\mu$ on the space $\Gamma \times \mathbb{R}^d$ with the σ-algebra $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$ as the measure such that for any non-negative measurable function $h : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$ holds

$$\int_\Gamma \sum_{x\in\gamma} h(\gamma - \varepsilon_x, x)\, d\mu(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} h(\gamma, x)\, d\mathcal{C}^!_\mu(\gamma, x). \tag{1.2}$$

This relation which uniquely defines $\mathcal{C}^!_\mu$ is called Campbell identity, see e. g. [11], [9], [19], [16], [10], [24].

We will consider a special case of the Campbell measure such that

$$d\mathcal{C}^!_\mu(\gamma, x) = r(\gamma, x)\, d\sigma(x)\, d\mu(\gamma), \tag{1.3}$$

where r is a non-negative measurable function. Of course, the first question is about examples of such μ.

First of all, Mecke [17] proved that there exists only one measure μ such that

$$d\mathcal{C}^!_\mu(\gamma, x) = d\sigma(x)\, d\mu(\gamma)$$

for given Radon measure σ on $\mathbb{R}^d$. This measure is called the Poisson measure with intensity σ and is denoted by π_σ. There exists a direct construction of the Poisson measure. For explore it we start from the space Γ_Λ of all finite configurations in $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$:

$$\Gamma_\Lambda = \{\gamma \in \Gamma \,|\, \gamma \cap \Lambda^c = \emptyset\},$$

where $\Lambda^c := \mathbb{R}^d \setminus \Lambda$. Clearly,

$$\Gamma_\Lambda := \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}_\Lambda,$$

where $\Gamma^{(n)}_\Lambda$ is the set of all n-particle configurations (subsets) of Λ, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. There is a bijection $\tilde{\Lambda}^n / S_n \to \Gamma^{(n)}_\Lambda$, where $\tilde{\Lambda}^n := \{(x_1, \ldots, x_n) \in \Lambda^n \,|\, x_k \neq x_j, k \neq j\}$ and S_n

is the permutation group over $\{1, \ldots, n\}$. Therefore, we can consider the image $\sigma^{(n)}$ on $\Gamma_\Lambda^{(n)}$ of the product measure

$$\sigma^n = \sigma \times \cdots \times \sigma$$

under this bijection. Consider also a σ-algebra $\mathcal{B}(\Gamma_\Lambda)$ as the minimal σ-algebra such that all mappings $\Gamma_\Lambda \ni \gamma \longmapsto |\gamma_{\Lambda'}|$ are $\mathcal{B}(\Gamma_\Lambda)$-measurable for any $\Lambda' \in \mathcal{B}_c(\Lambda)$. Projection mappings p_Λ from Γ into Γ_Λ for any Λ can be define as $\Gamma \ni \gamma \mapsto p_\Lambda \gamma = \gamma_\Lambda \in \Gamma_\Lambda$. Then the Poisson measure π_σ^Λ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is defined as

$$\pi_\sigma^\Lambda := e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}.$$

It can be shown that the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ and that the family of measures $\{\pi_\sigma^\Lambda\}_{\Lambda \in \mathcal{B}_c(X)}$ is consistent. Therefore, one has define the Poisson measure π_σ on $(\Gamma, \mathcal{B}(\Gamma))$ as the projective limit of this family due to Kolmogorov theorem.

Consider also the space Γ_0 of all finite configurations (subsets) in $\mathbb{R}^d$. Clearly,

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)},$$

where $\Gamma_0^{(n)}$ is the set of all n-particle configurations of $\mathbb{R}^d$. We can define a σ-algebra $\mathcal{B}(\Gamma_0)$ as the minimal σ-algebra such that all mappings $\Gamma_0 \ni \gamma \longmapsto |\gamma_\Lambda|$ are $\mathcal{B}(\Gamma_0)$-measurable for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. The Lebesgue-Poisson measure λ_σ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined as

$$\lambda_\sigma := \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}.$$

Note that we have a big class of examples of the measures μ satisfying (1.3) coming from applications in statistical physics. Actually, it was shown by X.Nguen and H.Zessin [18] that the wide class of tempered Gibbs measures satisfies Campbell identity, where $d\mathcal{C}_\mu^!(\gamma, x) = r(\gamma, x)\, d\sigma(x)\, d\mu(\gamma)$ with some r. Moreover, they prove that this properties is characteristic for this class of measures if we will use the function $r(\gamma, x)$ of a special type.

In the present work we realize this approach for construction of a useful and wide class of measures on Γ and study sufficient conditions for existence and uniqueness of these measures.

Let us recall previously a classical approach to the construction of grand canonical Gibbs measures. Let Φ be a potential, i.e., a measurable function $\Phi : \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$ such that $\Phi(\emptyset) = 0$. Define for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ the conditional energy $E_\Lambda^\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ such that

$$E_\Lambda^\Phi(\gamma) = \begin{cases} \displaystyle\sum_{\eta \Subset \gamma, |\eta \cap \Lambda| > 0} \Phi(\eta) & \text{if } \displaystyle\sum_{\eta \Subset \gamma, |\eta \cap \Lambda| > 0} |\Phi(\eta)| < \infty \\ +\infty & \text{otherwise} \end{cases}$$

(the notation $\eta \Subset \gamma$ means that η is a finite subset of γ). Then for fixed $\beta > 0$ we define for $\gamma \in \Gamma, \Delta \in \mathcal{B}(\Gamma)$ a specification

$$\Pi_\Lambda^{\sigma,\beta,\Phi}(\gamma, \Delta) = \frac{1_{\{Z_\Lambda^{\sigma,\beta,\Phi}(\gamma) < +\infty\}}}{Z_\Lambda^{\sigma,\beta,\Phi}(\gamma)} \int_\Gamma 1_\Delta(\gamma_{\Lambda^c} \cup \gamma'_\Lambda)\, e^{-\beta E_\Lambda^\Phi(\gamma_{\Lambda^c} \cup \gamma'_\Lambda)} d\pi_\sigma(\gamma'),$$

where

$$Z_\Lambda^{\sigma,\beta,\Phi}(\gamma) = \int_\Gamma e^{-\beta E_\Lambda^\Phi(\gamma_{\Lambda^c} \cup \gamma'_\Lambda)} d\pi_\sigma(\gamma').$$

A measure $\mu \in \mathcal{M}^1(\Gamma)$ is called the grand canonical Gibbs measure with interaction potential Φ iff for all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and for all $\Delta \in \mathcal{B}(\Gamma)$

$$\mu(\Delta) = \int_\Gamma \Pi_\Lambda^{\sigma,\beta,\Phi}(\gamma, \Delta)\, d\mu(\gamma).$$

The set of all such measures μ will be denoted by $\mathcal{G}_{gc}(\sigma, \beta\Phi)$.

Actually we want to reconstruct a probability measure by the family of its conditional probabilities. The sufficient conditions for this were discovered by R.L.Dobrushin [3]. This approach was realized for pair potentials Φ in detail and some sufficient conditions on non-pair potentials Φ were formulated too (see [2], [4], [20], [1], [13]).

Another approach was developed by D.Ruelle [23]. This approach uses integral equations for the correlation functions generating by μ. In some sense existence and uniqueness of the solutions of these equations are equivalent to existence and uniqueness of μ. But all considerations were developed only for pair potentials Φ.

In the present work we generalize this approach to the case of a general measure μ satisfying Campbell identity (1.2) under condition (1.3).

2. Characterization properties

We start with a probability measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and consider a non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function $r : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$. We suppose that $r(\gamma, x)$ is defined for μ-a.a. $\gamma \in \Gamma$ and for σ-a.a. $x \in \mathbb{R}^d$ (note that we always assume that $x \notin \gamma$). The function r is called the *relative energy density.*

Definition 2.1. The measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is said to be the Gibbs measure corresponding to the relative energy density r if for any non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function $h : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$ the following Campbell-Mecke identity holds

$$\int_\Gamma \sum_{x\in\gamma} h(\gamma, x)\, d\mu(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} h(\gamma + \varepsilon_x, x)\, r(\gamma, x)\, d\sigma(x)\, d\mu(\gamma). \tag{2.1}$$

The following example shows that the class of such measures μ includes a big subclass of grand canonical Gibbs measures.

Example 1 (General grand canonical Gibbs measure). Let $\mu \in \mathcal{G}_{gc}(p\, dx, \beta\Phi)$ with $p > 0$ a.s. and $p \in L^1_{loc}(\mathbb{R}^d, dx)$ and suppose μ has the first local moment (i.e., (1.1) is true for $n = 1$). Then (see Lemma 6.7 in [22]) μ satisfies the Campbell–Mecke identity with

$$r(\gamma, x) = \exp\left(-\beta E^\Phi_{\{x\}}(\gamma + \varepsilon_x)\right).$$

Recall that

$$E^\Phi_{\{x\}}(\gamma + \varepsilon_x) = \begin{cases} \displaystyle\sum_{\{x\}\subset\eta\Subset\gamma\cup\{x\}} \Phi(\eta), & \text{if } \displaystyle\sum_{\{x\}\subset\eta\Subset\gamma\cup\{x\}} |\Phi(\eta)| < +\infty \\ +\infty, & \text{otherwise} \end{cases}.$$

Example 2 (Pair potential case). The second example is a particular case of the first one

$$\Phi(\eta) = \begin{cases} \phi(x, y), & \text{if } \eta = \{x, y\} \\ 0, & \text{if } |\eta| \neq 2 \end{cases},$$

where $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ is a symmetric measurable function. Let $\mu \in \mathcal{G}_{gc}(pdx, \beta\phi)$; then

$$r(\gamma, x) = \begin{cases} \exp\left(-\beta \sum_{y\in\gamma} \phi(x,y)\right), & \text{if } \sum_{y\in\gamma} |\phi(x,y)| < +\infty \\ 0, & \text{otherwise} \end{cases}.$$

The following fact is a direct corollary of the definition, actually it was shown in [6].

Lemma 2.2. *For μ-a.a. $\gamma \in \Gamma$ and for σ-a.a. $x, y \in \mathbb{R}^d$ so-called "cocycle identity" holds*

$$r(\gamma + \varepsilon_x, y)\, r(\gamma, x) = r(\gamma + \varepsilon_y, x)\, r(\gamma, y). \tag{2.2}$$

Proof. For any non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable functions h_1, $h_2 : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$ one has

$$\begin{aligned}
&\int_\Gamma \left(\sum_{x\in\gamma} h_1(\gamma,x)\right)\left(\sum_{y\in\gamma} h_2(\gamma,y)\right) d\mu(\gamma) \\
&= \int_\Gamma \sum_{x\in\gamma} \left(h_1(\gamma,x) \sum_{y\in\gamma} h_2(\gamma,y)\right) d\mu(\gamma) \\
&= \int_\Gamma \int_{\mathbb{R}^d} h_1(\gamma+\varepsilon_x, x) \sum_{y\in\gamma\cup\{x\}} h_2(\gamma+\varepsilon_x, y)\, r(\gamma,x)\, d\sigma(x)\, d\mu(\gamma) \\
&= \int_\Gamma \int_{\mathbb{R}^d} h_1(\gamma+\varepsilon_x, x) \sum_{y\in\gamma} h_2(\gamma+\varepsilon_x, y)\, r(\gamma,x)\, d\sigma(x)\, d\mu(\gamma) \\
&\quad + \int_\Gamma \int_{\mathbb{R}^d} h_1(\gamma+\varepsilon_x, x)\, h_2(\gamma+\varepsilon_x, x)\, r(\gamma,x)\, d\sigma(x)\, d\mu(\gamma) \\
&= \int_\Gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_1(\gamma+\varepsilon_x+\varepsilon_y, x)\, h_2(\gamma+\varepsilon_x+\varepsilon_y, y) \\
&\qquad\qquad \times r(\gamma+\varepsilon_y, x)\, r(\gamma,y)\, d\sigma(y)\, d\sigma(x)\, d\mu(\gamma) \\
&\quad + \int_\Gamma \int_{\mathbb{R}^d} h_1(\gamma+\varepsilon_x, x)\, h_2(\gamma+\varepsilon_x, x)\, r(\gamma,x)\, d\sigma(x)\, d\mu(\gamma).
\end{aligned}$$

These considerations are correct, since $h_1(\gamma+\varepsilon_x, x)\, h_2(\gamma+\varepsilon_x, y)\, r(\gamma,x)$ is a non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function of the variables γ and y for σ-a.a. $x \in \mathbb{R}^d$ and $h_1(\gamma,x) \sum_{y\in\gamma} h_2(\gamma,y)$ is a non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function of the variables γ and x. Analogously, we obtain that

$$\begin{aligned}
&\int_\Gamma \left(\sum_{x\in\gamma} h_1(\gamma,x)\right)\left(\sum_{y\in\gamma} h_2(\gamma,y)\right) d\mu(\gamma) \\
&= \int_\Gamma \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} h_2(\gamma+\varepsilon_y+\varepsilon_x, y)\, h_1(\gamma+\varepsilon_y+\varepsilon_x, x) \\
&\qquad\qquad \times r(\gamma+\varepsilon_x, y)\, r(\gamma,x)\, d\sigma(x)\, d\sigma(y)\, d\mu(\gamma) \\
&\quad + \int_\Gamma \int_{\mathbb{R}^d} h_2(\gamma+\varepsilon_y, y)\, h_1(\gamma+\varepsilon_y, y)\, r(\gamma,y)\, d\sigma(y)\, d\mu(\gamma).
\end{aligned}$$

Comparing the left hand sides of these equalities we have (2.2). □

Let us now formulate our basic condition

Condition 1. There exist sets $\tilde{\Gamma} \in \mathcal{B}(\Gamma)$ and $\tilde{\Gamma}_0 \in \mathcal{B}(\Gamma_0)$ with $\mu\left(\tilde{\Gamma}\right) = 1$ and $\lambda_\sigma\left(\Gamma_0 \setminus \tilde{\Gamma}_0\right) = 0$ such that for any $\gamma \in \tilde{\Gamma}$, for any $\eta \in \tilde{\Gamma}_0$, and for σ-a.a. $x, y \in \mathbb{R}^d$ the value $r(\gamma, x)$ is defined, the identity (2.2) holds and $\gamma \cup \eta \in \tilde{\Gamma}$.

Under this condition we can construct a new function $\mathcal{R}(\gamma,\eta)$ ($\gamma \in \tilde{\Gamma}, \eta \in \tilde{\Gamma}_0, \gamma \cap \eta = \emptyset$) in the following way. Fix some order of the finite configuration $\eta \in \tilde{\Gamma}_0$: $\eta = \{x_1, \ldots, x_n\}$ and set

$$\begin{aligned}\mathcal{R}(\gamma,\eta) &= \mathcal{R}(\gamma, \{x_1,\ldots,x_n\}) \\ &:= r(\gamma, x_1)\cdot r(\gamma\cup\{x_1\}, x_2)\cdot r(\gamma\cup\{x_1,x_2\},x_3)\cdot\ldots \\ &\quad \times r(\gamma\cup\{x_1,x_2,x_{3,}\ldots,x_{n-2}\},x_{n-1})\, r(\gamma\cup\{x_1,x_2,x_{3,}\ldots,x_{n-1}\},x_n).\end{aligned}$$

Lemma 2.3. *(See also* [11]*). The definition of the function $\mathcal{R}(\gamma,\eta)$ does not depend on the order of points in η and*

$$\mathcal{R}(\gamma, \eta_1\cup\eta_2) = \mathcal{R}(\gamma,\eta_1)\,\mathcal{R}(\gamma\cup\eta_1,\eta_2), \tag{2.3}$$

where $\gamma \in \tilde{\Gamma}, \eta_1, \eta_2 \in \tilde{\Gamma}_0, \gamma\cap\eta_1 = \gamma\cap\eta_2 = \eta_1\cap\eta_2 = \emptyset$.

Proof. The second statement is a direct consequence of the definition of $\mathcal{R}$, therefore, we need only to check the correctness of this definition. The case then $|\eta| = 2$ is identical to (2.2). Suppose we prove this for any $\eta \in \tilde{\Gamma}_0, |\eta| \le n$. Consider now $\eta = \{x_1,\ldots,x_{n+1}\}$ and any permutation $\tau \in S_{n+1}$. One has

$$\begin{aligned}&\mathcal{R}\left(\gamma, \left\{x_{\tau(1)},\ldots,x_{\tau(n+1)}\right\}\right) \\ &\quad := r\left(\gamma, x_{\tau(1)}\right)\cdot r\left(\gamma\cup\left\{x_{\tau(1)}\right\}, x_{\tau(2)}\right) r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)}\right\},x_{\tau(3)}\right)\cdot\ldots \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},x_{\tau(3),}\ldots,x_{\tau(n-2)}\right\},x_{\tau(n-1)}\right) \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},x_{\tau(3),}\ldots,x_{\tau(n-1)}\right\},x_{\tau(n)}\right) \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},x_{\tau(3),}\ldots,x_{\tau(n)}\right\},x_{\tau(n+1)}\right).\end{aligned}$$

Let $i = \tau^{-1}(1)$; then $\tau(i) = 1$. Therefore, by our assumption (if the apply transposition $\tau = (\tau(1), \tau(i))$ to order $\{\tau(1),\ldots,\tau(i)\}$),

$$\begin{aligned}&r\left(\gamma, x_{\tau(1)}\right)\cdot r\left(\gamma\cup\left\{x_{\tau(1)}\right\},x_{\tau(2)}\right)\cdot\ldots\cdot r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},x_{\tau(3),}\ldots,x_{\tau(i-1)}\right\},x_{\tau(i)}\right) \\ &\quad = r\left(\gamma, x_{\tau(i)}\right)\cdot r\left(\gamma\cup\left\{x_{\tau(i)}\right\},x_{\tau(2)}\right)\cdot\ldots\cdot r\left(\gamma\cup\left\{x_{\tau(i)},x_{\tau(2)},x_{\tau(3),}\ldots,x_{\tau(i-1)}\right\},x_{\tau(1)}\right) \\ &\quad = r(\gamma,x_1)\cdot r\left(\gamma\cup\{x_1\},x_{\tau(2)}\right)\cdot r\left(\gamma\cup\left\{x_1,x_{\tau(2)}\right\},x_{\tau(3)}\right)\cdot\ldots \\ &\quad\quad \times r\left(\gamma\cup\left\{x_1,x_{\tau(2)},x_{\tau(3),}\ldots,x_{\tau(i-1)}\right\},x_{\tau(1)}\right)\end{aligned}$$

Let $\gamma' = \gamma \cup x_1$; then

$$\begin{aligned}&\mathcal{R}\left(\gamma,\left\{x_{\tau(1)},\ldots,x_{\tau(n+1)}\right\}\right) \\ &\quad = r\left(\gamma,x_{\tau(1)}\right)\cdot r\left(\gamma\cup\left\{x_{\tau(1)}\right\},x_{\tau(2)}\right) \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)}\right\},x_{\tau(3)}\right)\cdot\ldots\cdot r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},\ldots,x_{\tau(i-1)}\right\},x_{\tau(i)}\right) \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},\ldots,x_{\tau(i)}\right\},x_{\tau(i+1)}\right)\cdot\ldots \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},\ldots,x_{\tau(n)}\right\},x_{\tau(n+1)}\right) \\ &\quad = r(\gamma,x_1)\cdot r\left(\gamma\cup\{x_1\},x_{\tau(2)}\right)\cdot r\left(\gamma\cup\left\{x_1,x_{\tau(2)}\right\},x_{\tau(3)}\right)\cdot\ldots \\ &\quad\quad \times r\left(\gamma\cup\left\{x_1,x_{\tau(2)},\ldots,x_{\tau(i-1)}\right\},x_{\tau(1)}\right) \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},\ldots,x_{\tau(i-1)},x_1\right\},x_{\tau(i+1)}\right)\cdot\ldots \\ &\quad\quad \times r\left(\gamma\cup\left\{x_{\tau(1)},x_{\tau(2)},\ldots,x_{\tau(i-1)},x_1,x_{\tau(i+1)},\ldots,x_{\tau(n)}\right\},x_{\tau(n+1)}\right) \\ &\quad = r(\gamma,x_1)\cdot r\left(\gamma',x_{\tau(2)}\right)\cdot r\left(\gamma'\cup\left\{x_{\tau(2)}\right\},x_{\tau(3)}\right)\cdot\ldots\end{aligned}$$

$$
\begin{aligned}
&\times r\left(\gamma' \cup \left\{x_{\tau(2)}, x_{\tau(3),} \ldots, x_{\tau(i-1)}\right\}, x_{\tau(1)}\right) \\
&\times r\left(\gamma' \cup \left\{x_{\tau(2)}, x_{\tau(3),} \ldots, x_{\tau(i-1)}, x_{\tau(1)}\right\}, x_{\tau(i+1)}\right) \cdot \ldots \\
&\times r\left(\gamma' \cup \left\{x_{\tau(2)}, \ldots, x_{\tau(i-1)}, x_{\tau(1)}, x_{\tau(i+1)}, \ldots, x_{\tau(n)}\right\}, x_{\tau(n+1)}\right) \\
&= r(\gamma, x_1)\, \mathcal{R}\left(\gamma', \left\{x_{\tau(2)}, \ldots, x_{\tau(i-1)}, x_{\tau(1)}, x_{\tau(i+1)}, \ldots, x_{\tau(n+1)}\right\}\right) \\
&= r(\gamma, x_1)\, \mathcal{R}\left(\gamma \cup x_1, \{x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_{n+1}\}\right),
\end{aligned}
$$

by induction assumption, under permutation

$$
\begin{pmatrix} \tau(2) & \ldots & \tau(i-1) & \tau(1) & \tau(i+1) & \ldots & \tau(n+1) \\ 2 & \ldots & i-1 & i & i+1 & \ldots & n+1 \end{pmatrix} \in S_n.
$$

Thus

$$
\begin{aligned}
&\mathcal{R}\left(\gamma, \left\{x_{\tau(1)}, \ldots, x_{\tau(n+1)}\right\}\right) \\
&\qquad = r(\gamma, x_1)\, \mathcal{R}\left(\gamma \cup x_1, \{x_2, \ldots, x_{n+1}\}\right) = \mathcal{R}\left(\gamma, \{x_1, \ldots, x_{n+1}\}\right),
\end{aligned}
$$

that finishes the proof. □

Remark 2.4. This result has combinatorial sense and follows directly from condition (2.2).

Remark 2.5. In the case of Example 1 for $\xi = \{x_1, x_2, x_{3,} \ldots, x_{m-1}\}$ one has

$$
\begin{aligned}
&r\left(\gamma \cup \{x_1, x_2, x_{3,} \ldots, x_{m-1}\}, x_m\right) \\
&\quad = \exp\left(-\beta E^{\Phi}_{\{x_m\}}\left((\gamma \cup \xi) + \varepsilon_{x_m}\right)\right) = \exp\left(-\beta \sum_{\gamma' \subset \gamma \cup \xi} \Phi\left(\gamma' + \varepsilon_{x_m}\right)\right) \\
&\quad = \exp\left(-\beta \sum_{\gamma' \subset \gamma} \Phi\left(\gamma' + \varepsilon_{x_m}\right) - \beta \sum_{\gamma' \subset \gamma} \sum_{\substack{\xi' \subset \xi: \\ \xi' \neq \emptyset}} \Phi\left((\gamma' \cup \xi') + \varepsilon_{x_m}\right)\right).
\end{aligned}
$$

Then

$$
\mathcal{R}(\gamma, \eta) = \exp\left(-\beta \sum_{\gamma' \subset \gamma} \sum_{\substack{\eta' \subset \eta: \\ \eta' \neq \emptyset}} \Phi\left(\gamma' \cup \eta'\right)\right).
$$

In the following we will use the next notations

$$
E^{\Phi}(\eta) = \sum_{\eta' \subset \eta} \Phi(\eta'); \quad W^{\Phi}(\eta, \gamma) = \sum_{\substack{\gamma' \subset \gamma: \\ \gamma' \neq \emptyset}} \sum_{\substack{\eta' \subset \eta: \\ \eta' \neq \emptyset}} \Phi\left(\gamma' \cup \eta'\right).
$$

Therefore, we get

$$
\mathcal{R}(\gamma, \eta) = \exp\left(-\beta E^{\Phi}(\eta) - \beta W^{\Phi}(\eta, \gamma) + \beta \Phi(0)\right)
$$

Note also that for any $\eta_1, \eta_2 \in \Gamma_0$

$$
W^{\Phi}(\eta_1, \eta_2) = E^{\Phi}(\eta_1 \cup \eta_2) - E^{\Phi}(\eta_1) - E^{\Phi}(\eta_2). \tag{2.4}
$$

Remark 2.6. In the case of Example 2 one has

$$
\mathcal{R}(\gamma, \eta) = \exp\left(-\beta E^{\phi}(\eta) - \beta W^{\phi}(\eta, \gamma)\right),
$$

where

$$
E^{\phi}(\eta) = \sum_{\{x,y\} \subset \eta} \phi(x, y); \quad W^{\phi}(\eta, \gamma) = \sum_{\substack{x \in \eta \\ y \in \gamma}} \phi(x, y).
$$

Theorem 2.7. *For any non-negative $\mathcal{B}(\Gamma)$-measurable functions $F:\Gamma\to\mathbb{R}_+$ and for all $\Lambda\in\mathcal{B}_c\left(\mathbb{R}^d\right)$ the following generalization of the Ruelle identity holds*

$$\int_\Gamma F(\gamma)\,d\mu(\gamma)=\int_{\Gamma_\Lambda}\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}}F(\gamma\cup\eta)\,\mathcal{R}(\gamma,\eta)\,d\mu(\gamma)\,d\lambda_\sigma(\eta).\tag{2.5}$$

Proof. Set in (2.1) $h(\gamma,x)=1\!\!1_{\{N_\Lambda=n\}}(\gamma)\,1\!\!1_\Lambda(x)\,F(\gamma)$. Then

$$\int_\Gamma\sum_{x\in\gamma}h(\gamma,x)\,d\mu(\gamma)=n\int_\Gamma 1\!\!1_{\{N_\Lambda=n\}}(\gamma)\,F(\gamma)\,d\mu(\gamma);$$

$$\begin{aligned}\int_\Gamma\int_{\mathbb{R}^d}&h(\gamma+\varepsilon_x,x)\,r(\gamma,x)\,d\sigma(x)\,d\mu(\gamma)\\&=\int_\Gamma\int_\Lambda 1\!\!1_{\{N_\Lambda=n\}}(\gamma+\varepsilon_x)\,F(\gamma+\varepsilon_x)\,r(\gamma,x)\,d\sigma(x)\,d\mu(\gamma).\end{aligned}$$

Hence,

$$\int_\Gamma 1\!\!1_{\{N_\Lambda=n\}}(\gamma)\,F(\gamma)\,d\mu(\gamma)=\frac{1}{n}\int_\Lambda\int_\Gamma 1\!\!1_{\{N_\Lambda=n-1\}}(\gamma)\,F(\gamma+\varepsilon_x)\,r(\gamma,x)\,d\mu(\gamma)\,d\sigma(x).$$

Applying this formula for $\widetilde{F}(\gamma):=F(\gamma+\varepsilon_x)\,r(\gamma,x)$ one has

$$\begin{aligned}\int_\Gamma &1\!\!1_{\{N_\Lambda=n\}}(\gamma)\,F(\gamma)\,d\mu(\gamma)\\&=\frac{1}{n(n-1)}\int_{\Lambda^2}\int_\Gamma 1\!\!1_{\{N_\Lambda=n-2\}}(\gamma)\,F(\gamma+\varepsilon_{x_1}+\varepsilon_{x_2})\\&\qquad\times r(\gamma+\varepsilon_{x_2},x_1)\,r(\gamma,x_2)\,d\mu(\gamma)\,d\sigma(x_1)\,d\sigma(x_2).\end{aligned}$$

Further,

$$\begin{aligned}\int_\Gamma &1\!\!1_{\{N_\Lambda=n\}}(\gamma)F(\gamma)\,d\mu(\gamma)\\&=\frac{1}{n!}\int_{\Lambda^n}\int_\Gamma 1\!\!1_{\{N_\Lambda=0\}}(\gamma)\,F(\gamma\cup\{x_1,\dots,x_n\})\\&\qquad\times\mathcal{R}(\gamma,\{x_1,\dots,x_n\})\,d\mu(\gamma)\,d\sigma(x_1)\,d\sigma(x_2)\dots d\sigma(x_n)\\&=\frac{1}{n!}\int_{\Lambda^n}\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}}F(\gamma\cup\{x_1,\dots,x_n\})\\&\qquad\times\mathcal{R}(\gamma,\{x_1,\dots,x_n\})\,d\mu(\gamma)\,d\sigma(x_1)\,d\sigma(x_2)\dots d\sigma(x_n).\end{aligned}$$

After summarizing on n we obtain (2.5). □

We will say that a measure $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is locally absolutely continuous w.r.t. the Poisson measure π_σ if for all $\Lambda\in\mathcal{B}_c(X)$ the projection μ^Λ of μ on Γ_Λ is absolutely continuous w.r.t. π^Λ_σ.

Proposition 2.8. *Let the measure μ is as above; then it is locally absolutely continuous w.r.t. the Poisson measure π_σ and for π^Λ_σ-a.a. $\eta\in\Gamma_\Lambda$*

$$\frac{d\mu^\Lambda}{d\pi^\Lambda_\sigma}(\eta)=e^{\sigma(\Lambda)}\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}}\mathcal{R}(\gamma,\eta)\,d\mu(\gamma).$$

Proof. Let f be a bounded non-negative $\mathcal{B}(\Gamma_\Lambda)$-measurable function and set $F=f\circ p_\Lambda$.

Then, by (2.5), we obtain

$$\int_{\Gamma_\Lambda} f(\gamma)\, d\mu^\Lambda(\gamma) = \int_\Gamma F(\gamma)\, d\mu(\gamma) = \int_{\Gamma_\Lambda}\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}} F(\gamma\cup\eta)\,\mathcal{R}(\gamma,\eta)\, d\mu(\gamma)\, d\lambda_\sigma(\eta)$$
$$= \int_{\Gamma_\Lambda} f(\eta)\left(\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}} \mathcal{R}(\gamma,\eta)\, d\mu(\gamma)\right) d\lambda_\sigma(\eta)$$
$$= e^{\sigma(\Lambda)}\int_{\Gamma_\Lambda} f(\eta)\left(\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}} \mathcal{R}(\gamma,\eta)\, d\mu(\gamma)\right) d\pi_\sigma^\Lambda(\eta).$$

Thus we obtain just the required result. □

Corollary 2.9. *From the general results (see, e.g., [21]), one has*

1. *For all $\gamma\in\Gamma$ the set $\{\gamma'\in\Gamma|\,\gamma\cap\gamma'=\emptyset\}$ has μ-measure zero.*
2. *The set $\{(\gamma,\gamma')\in\Gamma\times\Gamma|\,\gamma\cap\gamma'=\emptyset\}$ has $\mu\otimes\mu$-measure zero.*
3. *The set $\{\gamma\in\Gamma|\,\gamma\cap A=\emptyset\}$ has full μ-measure for any $A\in\mathcal{B}(\mathbb{R}^d)$ such that $\sigma(A)=0$.*

Let us now construct on the configuration spaces specifications corresponding to r. For any $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ we consider the partition function

$$Z_\Lambda(\gamma) := \int_{\Gamma_\Lambda} \mathcal{R}\left(\gamma_{\mathbb{R}^d\setminus\Lambda},\eta\right) d\lambda_\sigma(\eta), \quad \gamma\in\Gamma.$$

Then a specification Π_Λ is defined for any $\gamma\in\tilde{\Gamma}$, $\Delta\in\mathcal{B}(\Gamma)$ by

$$\Pi_\Lambda(\Delta,\gamma) := \frac{1\!\!1_{\{Z_\Lambda<\infty\}}(\gamma)}{Z_\Lambda(\gamma)}\int_{\Gamma_\Lambda} 1\!\!1_\Delta\left(\gamma_{\mathbb{R}^d\setminus\Lambda}\cup\eta\right)\mathcal{R}\left(\gamma_{\mathbb{R}^d\setminus\Lambda},\eta\right) d\lambda_\sigma(\eta).$$

Note that $Z_\Lambda(\cdot)$ and $\Pi_\Lambda(\Delta,\cdot)$ are $\mathcal{B}_{\mathbb{R}^d\setminus\Lambda}(\Gamma)$-measurable for any $\Delta\in\mathcal{B}(\Gamma),\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$.

Theorem 2.10. *For all $\Delta\in\mathcal{B}(\Gamma)$ the following Dobrushin-Lanford-Ruelle equation holds*

$$\int_\Gamma \Pi_\Lambda(\Delta,\gamma)\, d\mu(\gamma) = \mu(\Delta) \tag{2.6}$$

Let us now formulate an inverse result.

Theorem 2.11. *Let $\mu\in\mathcal{M}^1_{fm}(\Gamma),\tilde{\Gamma}\in\mathcal{B}(\Gamma),\mu(\tilde{\Gamma})=1, r:\Gamma\times\mathbb{R}^d\to\mathbb{R}_+$ is $\mathcal{B}(\Gamma)\times\mathcal{B}(\mathbb{R}^d)$-measurable and defined for any $\gamma\in\tilde{\Gamma}$ and for σ-a.a. $x\in\mathbb{R}^d$. Let also for any $\gamma\in\tilde{\Gamma}$ and for σ-a.a. $x,y\in\mathbb{R}^d$ the identity (2.2) holds. Then we can define $\mathcal{R}$ and $\{\Pi_\Lambda\}_{\Lambda\in\mathcal{B}_c(\mathbb{R}^d)}$ and*

1. *If μ satisfies the equation (2.6) then μ satisfies the equation (2.1).*
2. *If μ satisfies (2.5) then μ satisfies the equation (2.1).*

The proofs of these theorems are analogously to the proofs in [18].

3. Correlation functionals

Recall that the measure μ is locally absolutely continuous w.r.t. π_σ (see Proposition 2.8). Therefore, the set of finite configuration has zero measure. Due to this fact there is no canonical way for restriction of the function r onto finite configurations. Let us now fix the most useful restriction.

We describe some classes of functions. On Γ_0 one consider $L^0(\Gamma_0)$ the set of all measurable functions on Γ_0. Let $L^0_{ls}(\Gamma_0)$ be the set of all measurable functions with local support, i.e., $G\in L^0_{ls}(\Gamma_0)$ if there exists $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ such that $G\restriction_{\Gamma_0\setminus\Gamma_\Lambda}=0$. $L^0_{bs}(\Gamma_0)$ denotes the set of all measurable functions with bounded support, the latter means that there exist $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $G \upharpoonright_{\Gamma_0 \setminus \left(\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}\right)} = 0$. $B(\Gamma_0)$ the set of bounded measurable functions. Analogously, $B_{ls}(\Gamma_0)$ and $B_{bs}(\Gamma_0)$ are defined.

We consider also sets of cylinder measurable functions on Γ and Γ_0: one say that $F \in \mathcal{F}_{cyl}(\Gamma)$ or $G \in \mathcal{F}_{cyl}(\Gamma_0)$ if F and G are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ and $\mathcal{B}(\Gamma_\Lambda)$ correspondingly for some $\Lambda \in \mathcal{B}_c(X)$. These functions can be characterized by the following relations: $F(\gamma) = F \upharpoonright_{\Gamma_\Lambda} (\gamma_\Lambda)$, $G(\eta) = G \upharpoonright_{\Gamma_\Lambda} (\eta_\Lambda)$

One may introduce the following "key-mapping" between functions on Γ_0 and Γ (see [12], [15] for more details). Let $G \in L^0_{ls}(\Gamma_0)$, then we put

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi), \quad \gamma \in \Gamma. \tag{3.1}$$

The summation in the latter expression is extends over all finite sub configurations of γ (in symbols $\xi \Subset \gamma$). If $G \in L^0_{ls}(\Gamma_0)$, then this sum is finite and moreover, $KG \in \mathcal{F}_{cyl}(\Gamma)$. The mapping $K : L^0_{ls}(\Gamma_0) \to \mathcal{F}_{cyl}(\Gamma)$ is linear, positivity preserving and invertible with inverse

$$K^{-1}F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \quad \eta \in \Gamma_0.$$

Note also that the K-transform maps a function $G \in B_{bs}(\Gamma_0)$ into the cylinder polynomial bounded function. In particular, if $G \upharpoonright_{\Gamma_0 \setminus \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}} = 0$ for some $\Lambda \in \mathcal{B}_c(X)$ and $N \in \mathbb{N}$ then there exists $C > 0$ such that $|KG(\gamma)| \le C\,(1 + |\gamma_\Lambda|)^N$.

We denote the restriction of K onto Γ_0 by K_0. That means for any $G \in L^0(\Gamma_0)$

$$K_0 G(\eta) = \sum_{\xi \subset \eta} G(\xi), \quad K_0^{-1} G(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} G(\xi), \quad \eta \in \Gamma_0.$$

Note that K_0 and K_0^{-1} are well defined.

Note again that $K^{-1} : \mathcal{F}_{cyl}(\Gamma) \to L^0_{ls}(\Gamma_0)$ and $K_0^{-1} : \mathcal{F}_{cyl}(\Gamma_0) \to L^0_{ls}(\Gamma_0)$.

A set $B \in \mathcal{B}(\Gamma_0)$ is called bounded iff there exists $N \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(X)$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$. The family $\mathcal{B}_b(\Gamma_0)$ of all bounded sets $A \in \mathcal{B}(\Gamma_0)$ forms a ring of sets. This ring generates $\mathcal{B}(\Gamma_0)$.

Define a measure $K^*\mu$ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ due to the relation

$$(K^*\mu)(A) = \int_\Gamma K 1\!\!1_A(\gamma)\, \mu(d\gamma), \quad \text{for all } A \in \mathcal{B}_b(\Gamma_0).$$

Due to $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, one has $(K^*\mu)(A) < +\infty$ for all $A \in \mathcal{B}_b(\Gamma_0)$ (see [12]). Then we say that $K^*\mu$ is locally finite and denote this by $K^*\mu \in \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$. Note that, in particular, $K^*\pi_\sigma = \lambda_\sigma$.

Moreover, if $G \in L^1(\Gamma_0, K^*\mu)$ then the series in the right hand side of (3.1) μ-a.s. converges and $KG \in L^1(\Gamma, \mu)$ (see, e.g., [12]).

One can introduce also a convolution

$$(G_1, G_2) \mapsto (G_1 \star G_2)(\eta) := \sum_{(\xi_1, \xi_2, \xi_3) \in \mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1 \cup \xi_2)\; G_2(\xi_2 \cup \xi_3),$$

where $\mathcal{P}^3_\emptyset(\eta)$ denotes the set of all partitions (ξ_1, ξ_2, ξ_3) of η into 3 parts, i.e., all triples (ξ_1, ξ_2, ξ_3) with $\xi_i \subset \eta$, $\xi_i \cap \xi_j = \emptyset$ if $i \ne j$ and $\xi_1 \cup \xi_2 \cup \xi_3 = \eta$. For $G_1, G_2 \in L^0_{ls}(\Gamma_0)$

$$K(G_1 \star G_2) = KG_1 \cdot KG_2,$$

that holds if $G_1, G_2 \ge 0$ or if $|G_1| \star |G_2| \in L^1(\Gamma_0, K^*\mu)$ or if $G_1, G_2 \in L^1(\Gamma_0, K^*\mu)$ (then $K(G_1 \star G_2) \in L^1(\Gamma, \mu)$ too, see [12]).

Note that if $F \in \mathcal{F}_{cyl}(\Gamma)$ then we can consider the function $F_\Lambda := F \upharpoonright_{\Gamma_\Lambda}$ as measurable cylinder function on Γ_0. Then $K_0^{-1} F_\Lambda \in L^0_{ls}(\Gamma_0)$ [12] and, therefore, the K-transform is well defined on $K_0^{-1}F_\Lambda$:

$$\left(KK_0^{-1}F_\Lambda\right)(\gamma) = F(\gamma) = F(\gamma_\Lambda), \quad \gamma \in \Gamma.$$

And, vise versa: any cylinder function $G \in \mathcal{F}_{cyl}(\Gamma_0)$ such that there exists $\Lambda \in \mathcal{B}_c(X)$ with $G(\eta) = G(\eta_\Lambda)$, $\eta \in \Gamma_0$ may be considered as measurable cylinder function on Γ, putting $G(\gamma) := G(\gamma_\Lambda)$. Then also

$$\left(KK_0^{-1}G\right)(\gamma) = G(\gamma), \quad \gamma \in \Gamma.$$

In the other words, the mapping KK_0^{-1} is a kind of a continuation mapping on cylinder functions.

Clearly, we can not extend in this way general measurable functions. For example for so-called Lebesgue-Poisson exponent

$$e_\lambda(\varphi, \eta) := \prod_{x\in\eta} \varphi(x), \quad \eta \in \Gamma_0,$$

where $\varphi \in L^1\left(\mathbb{R}^d, d\sigma(x)\right)$, one has e_λ belongs to $L^1(\Gamma_0, \lambda_\sigma)$ but there is not sensible definition of $KK_0^{-1}e_\lambda(\varphi, \cdot)$.

Let us present now our main condition for fixing a useful restriction of r.

Condition 2. Let $\tilde{r} : \Gamma_0 \times \mathbb{R}^d \to \mathbb{R}_+$ be a non-negative function such that $\tilde{r}(\eta, x)$ is defined for **any** $\eta \in \Gamma_0$ and for **any** $x \in \mathbb{R}^d \setminus \eta$. Suppose that

(1) For any $\gamma \in \tilde{\Gamma}$, for σ-a.a. $x \in \mathbb{R}^d$, and for any $\Lambda \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ one has

$$\tilde{r}(\gamma_\Lambda, x) = r(\gamma_\Lambda, x);$$

and for any $F \in \mathcal{F}_{cyl}(\Gamma), \varphi \in C_0\left(\mathbb{R}^d\right)$

$$\int_\Gamma \int_{\mathbb{R}^d} \varphi(x) F(\gamma) \tilde{r}(\gamma_\Lambda, x) \, d\sigma(x) \, d\mu(\gamma) \to \int_\Gamma \int_{\mathbb{R}^d} \varphi(x) F(\gamma) r(\gamma, x) \, d\sigma(x) \, d\mu(\gamma), \quad \Lambda \uparrow \mathbb{R}^d.$$

(2) For any $\eta \in \Gamma_0$ and for any $x, y \in \mathbb{R}^d \setminus \eta$ the "cocycled identity" holds

$$\tilde{r}(\eta + \varepsilon_x, y) \tilde{r}(\eta, x) = \tilde{r}(\eta + \varepsilon_y, x) \tilde{r}(\eta, y).$$

(3) For any $\eta \in \Gamma_0$ and for any $x \in \mathbb{R}^d$

$$\tilde{r}(\eta, x) > 0, \quad x \notin \eta;$$
$$\tilde{r}(\emptyset, x) = 1.$$

Sometimes (see Theorem 4.5 below and its consequences) we need translation invariance condition:

(4) For any $\{a, x, x_1, \ldots, x_n\} \subset \mathbb{R}^d$

$$\tilde{r}(\{x_1 + a, \ldots, x_n + a\}, x + a) = \tilde{r}(\{x_1, \ldots, x_n\}, x).$$

Note that in Examples 1–2 these conditions are fulfilled (see, e.g., [12], [18], [7], [8]). In that follows we will use for $\tilde{r}$ the same notation r.

In the following we need important lemma (about proof see, e.g., [12]).

Lemma 3.1 (Ruelle)**.** *Let H, G_1, G_2 be $\mathcal{B}(\Gamma_0)$-measurable functions. Then the following equality holds*

$$\int_{\Gamma_0} H(\eta) (G_1 * G_2)(\eta) \, \lambda_\sigma(d\eta) = \int_{\Gamma_0} \int_{\Gamma_0} H(\eta_1 \cup \eta_2) G_1(\eta_1) G_2(\eta_2) \lambda_\sigma(d\eta_1) \lambda_\sigma(d\eta_2), \tag{3.2}$$

where

$$(G_1 * G_2)(\eta) := \sum_{\xi\subset\eta} G_1(\eta\setminus\xi) G_2(\xi),$$

if either all functions are positive or one side of (3.2) exists for $|G_1|$, $|G_2|$, $|H|$.

Definition 3.2. An energy $E:\Gamma_0\to\mathbb{R}$ is defined via relation

$$\mathcal{R}(\emptyset,\eta)=e^{-E(\eta)}.$$

The following simple but very important statement holds

Proposition 3.3. *For any* $\eta\in\Gamma_0,\ x\in\mathbb{R}^d\setminus\eta$ *one has*

$$r(\eta,x)=e^{-(E(\eta+\varepsilon_x)-E(\eta))}. \tag{3.3}$$

Proof. Let us apply (2.3) for the case $\gamma=\emptyset,\ \eta_1=\eta,\ \eta_2=\{x\}$. Then

$$\mathcal{R}(\emptyset,\eta\cup\{x\})=\mathcal{R}(\emptyset,\eta)\,\mathcal{R}(\eta,\{x\}),$$

hence,

$$e^{-E(\eta+\varepsilon_x)}=e^{-E(\eta)}r(\eta,x),$$

that fulfilled the proof. □

In fact, we prove that if μ and r are such that (2.1) is true and r has the restriction, satisfying Condition 2, then r has to have the special form (3.3). Of course, we can not define the energy on infinite configurations but formally (3.3) will be true if we change η onto $\gamma\in\Gamma$.

Since μ is locally absolutely continuous w.r.t. the Poisson measure π_σ one has $K^*\mu$ is absolutely continuous w.r.t. the Lebesgue-Poisson measure λ_σ (see [12]). Therefore, we may consider the correlation functional $\rho_\mu:\Gamma_0\to\mathbb{R}_+$ corresponding to μ:

$$\rho_\mu(\eta):=\frac{d(K^*\mu)}{d\lambda_\sigma}(\eta),$$

and we know (see [12]) that for all $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$ and for λ_σ-a.a. $\eta\in\Gamma_\Lambda$

$$\rho_\mu(\eta)=\int_{\Gamma_\Lambda}\frac{d\mu^\Lambda}{d\lambda_\sigma}(\eta\cup\xi)\,d\lambda_\sigma(\xi). \tag{3.4}$$

As a result, the following statement is true.

Proposition 3.4. *For* λ_σ*-a.e.* $\eta\in\Gamma_0$

$$\rho_\mu(\eta)=\int_\Gamma\mathcal{R}(\gamma,\eta)\,\mu(d\gamma).$$

Proof. From Proposition 2.8 one has

$$\frac{d\mu^\Lambda}{d\lambda_\sigma}(\eta)=\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}}\mathcal{R}(\gamma,\eta)\,d\mu(\gamma).$$

Using (2.5) and (2.3), we get

$$\int_{\Gamma_\Lambda}\frac{d\mu^\Lambda}{d\lambda_\sigma}(\eta\cup\xi)\,d\lambda_\sigma(\xi)=\int_{\Gamma_\Lambda}\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}}\mathcal{R}(\gamma,\eta\cup\xi)\,d\mu(\gamma)\,d\lambda_\sigma(\xi)$$
$$=\int_{\Gamma_\Lambda}\int_{\Gamma_{\mathbb{R}^d\setminus\Lambda}}\mathcal{R}(\gamma,\xi)\,\mathcal{R}(\gamma\cup\xi,\eta)\,d\mu(\gamma)\,d\lambda_\sigma(\xi)=\int_\Gamma\mathcal{R}(\gamma,\eta)\,\mu(d\gamma).$$

Thus, according to (3.4), we obtain the assertion. □

Definition 3.5. Let $\Lambda\in\mathcal{B}_c(\mathbb{R}^d)$. A finite volume correlation functional $\rho_\Lambda:\Gamma_\Lambda\to(0;+\infty)$ (in the volume Λ) is defined as

$$\rho_\Lambda(\eta)=\frac{1}{Z_\Lambda(\emptyset)}\int_{\Gamma_\Lambda}\mathcal{R}(\emptyset,\eta\cup\xi)\,d\lambda_\sigma(\xi),$$

where

$$Z_\Lambda(\emptyset)=\int_{\Gamma_\Lambda}\mathcal{R}(\emptyset,\eta)\,d\lambda_\sigma(\eta).$$

Remark 3.6. Note that if arbitrary non-negative measurable function r satisfies (2.2) then we can construct the function $\mathcal{R}$ and, therefore, E, as well as define the functional ρ_Λ.

Proposition 3.7. *The finite volume correlation functional is normalized, i.e.,* $\rho_\Lambda(\emptyset)=1$*, and Lenard-positive, i.e.,*

$$\int_{\Gamma_\Lambda}(-1)^{|\xi|}\rho_\Lambda(\eta\cup\xi)\,d\lambda_\sigma(\xi)\geq 0. \tag{3.5}$$

Proof. The proof of the first statement is clear. For the second one we use (3.2). One has

$$\begin{aligned}\int_{\Gamma_\Lambda}(-1)^{|\xi|}\rho_\Lambda(\eta\cup\xi)\,d\lambda_\sigma(\xi)&=\int_{\Gamma_\Lambda}e_\lambda(-1,\xi)\frac{1}{Z_\Lambda(\emptyset)}\int_{\Gamma_\Lambda}\mathcal{R}(\emptyset,\eta\cup\xi\cup\zeta)\,d\lambda_\sigma(\zeta)\,d\lambda_\sigma(\xi)\\&=\frac{1}{Z_\Lambda(\emptyset)}\int_{\Gamma_\Lambda}(e_\lambda(-1,\xi)*e_\lambda(1,\xi))\,\mathcal{R}(\emptyset,\eta\cup\xi)\,d\lambda_\sigma(\xi)=\frac{\mathcal{R}(\emptyset,\eta)}{Z_\Lambda(\emptyset)}\geq 0,\end{aligned}$$

where we use the identity

$$(e_\lambda(f_1,\cdot)*e_\lambda(f_2,\cdot))(\eta)=e_\lambda(f_1+f_2,\eta)$$

that may be obtained via direct calculation. □

4. Stability condition and Ruelle bounds

Stability condition is a classical condition in statistical physics. In our case it has the following form.

(S) There exists a constant $B_{st}\geq 1$ such that

$$\mathcal{R}(\emptyset,\eta)\leq(B_{st})^{|\eta|},\quad \eta\in\Gamma_0.$$

Remark 4.1. In the case of Example 2

$$\mathcal{R}(\emptyset,\eta)=\exp\left(-\beta E^\phi(\eta)-\beta W^\phi(\eta,\emptyset)\right)=\exp\left(-\beta E^\phi(\eta)\right),$$

Recall that the stability condition for a pair potential Gibbs measure is that there exists $B\geq 0$ such that

$$E^\phi(\eta)\geq -B|\eta|,\quad \eta\in\Gamma_0. \tag{4.1}$$

Then our condition is equivalent to this with $B_{st}=e^{\beta B}$.

At present, there are not good sufficient conditions on general many-body potential for stability. Such conditions are known just for pair-potential (Dobrushin-Fisher-Ruelle criterion, see, e.g., [23]) and for some special many-body potentials [5].

But by definition for $\eta=\{x_1,\ldots,x_n\}$

$$\mathcal{R}(\emptyset,\eta)=r(\{x_1\},x_2)\,r(\{x_1,x_2\},x_3)\ldots r(\{x_1,\ldots,x_{n-1}\},x_n), \tag{4.2}$$

whence, if ϕ is pair stable potential and for any $\eta\in\Gamma_0, x\notin\eta$

$$r(\eta,x)\leq r^\phi(\eta,x):=\exp\left(-\beta\sum_{y\in\eta}\phi(x,y)\right),$$

then (S) holds.

Using stability condition, we can prove some useful facts.

Proposition 4.2. *Let (S) holds; then for all* $\Lambda \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ *and for* λ_σ*-a.a.* $\eta \in \Gamma_0$

$$1 \leq Z_\Lambda(\emptyset) \leq \exp\left(B_{st}\sigma(\Lambda)\right);$$
$$0 \leq \rho_\Lambda(\eta) \leq \left(B_{st}\right)^{|\eta|} \exp\left(B_{st}\sigma(\Lambda)\right).$$

Proof. Using (S), we obtain

$$1 = \int_{\{\emptyset\}} \mathcal{R}(\emptyset,\eta)\, d\lambda_\sigma(\eta) \leq \int_{\Gamma_\Lambda} \mathcal{R}(\emptyset,\eta)\, d\lambda_\sigma(\eta) \leq \int_{\Gamma_\Lambda} \left(B_{st}\right)^{|\eta|} d\lambda_\sigma(\eta) = \exp\left(B_{st}\sigma(\Lambda)\right)$$

and

$$0 \leq \rho_\Lambda(\eta) = \frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \mathcal{R}(\emptyset, \eta\cup\xi)\, d\lambda_\sigma(\xi)$$
$$\leq \int_{\Gamma_\Lambda} \left(B_{st}\right)^{|\eta|+|\xi|} d\lambda_\sigma(\xi) = \left(B_{st}\right)^{|\eta|} \exp\left(B_{st}\sigma(\Lambda)\right),$$

that finishes the proof. □

Remark 4.3. The inequality

$$\rho_\Lambda(\eta) \leq \left(B_{st}\right)^{|\eta|} \exp\left(B_{st}\sigma(\Lambda)\right)$$

is called *the local Ruelle bounds*.

Definition 4.4. The correlation functional ρ_μ is said to be satisfied the Ruelle bounds if there exists $C_R > 0$ such that for λ_σ-a.a. $\eta \in \Gamma_0$

$$\rho_\mu(\eta) \leq \left(C_R\right)^{|\eta|}. \tag{4.3}$$

The family $\{\rho_\Lambda\}_{\Lambda\in\mathcal{B}_c(\mathbb{R}^d)}$ is said to be satisfied the uniform Ruelle bounds if there exists $C_R > 0$ such that for all $\Lambda \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ and for λ_σ-a.a. $\eta \in \Gamma_0$

$$\rho_\Lambda(\eta) \leq \left(C_R\right)^{|\eta|}. \tag{4.4}$$

Note that, by (4.3),

$$\int_{\Gamma_\Lambda} 2^{|\eta|}\rho_\mu(\eta)\, d\lambda_\sigma(\eta) \leq \int_{\Gamma_\Lambda} \left(2C_R\right)^{|\eta|} d\lambda_\sigma(\eta) = \exp\left(2C_R|\Lambda|\right) < +\infty,$$

then (see [12]) for λ_σ-a.e. $\eta \in \Gamma_\Lambda$

$$\frac{d\mu^\Lambda}{d\lambda_\sigma}(\eta) = \int_{\Gamma_\Lambda} (-1)^{|\xi|} \rho_\mu(\eta\cup\xi)\, d\lambda_\sigma(\xi).$$

The sufficient conditions for (4.4) was found in [23]. For demonstrate them we need some additional objects. Let us fix a positive number $t > 0$. For every $a \in \mathbb{Z}^d$ we define a cube

$$\mathcal{L}(a) = \left\{ x \in \mathbb{R}^d \,\middle|\, \left(a_i - \frac{1}{2}\right)t \leq x_i \leq \left(a_i + \frac{1}{2}\right)t \right\}$$

and set $n(\eta, a) = |\eta \cap \mathcal{L}(a)|$. Now we formulate the superstability condition.

(SS) There exist $A > 0, B \geq 0$ such that if $\mathcal{A}$ is a finite subset of $\mathbb{Z}^d$ and $\eta \in \cup_{a\in\mathcal{A}}\mathcal{L}(a)$ then

$$\mathcal{R}(\emptyset,\eta) \leq \exp\left(\sum_{a\in\mathcal{A}} \left[Bn(\eta,a) - An^2(\eta,a)\right]\right). \tag{4.5}$$

Since $\sum_{a\in\mathcal{A}} Bn(\eta,a) = B|\eta|$, we see that superstability condition is stronger than stability.

Note also that usually (4.5) is wrote in the equivalent form

$$E(\eta) \geq \sum_{a\in\mathcal{A}} \left[An^2(\eta,a) - Bn(\eta,a)\right].$$

Let us consider (cf. (2.4)) for any $\eta_1, \eta_2 \in \Gamma_0, \eta_1 \cap \eta_2 = \emptyset$

$$W(\eta_1, \eta_2) := E(\eta_1 \cup \eta_2) - E(\eta_1) - E(\eta_2).$$

And now we can formulate the lower regularity condition.

(LR) There exists a decreasing positive function Ψ on the positive integers such that

$$\sum_{a \in \mathbb{Z}^d} \Psi(|a|) < +\infty$$

and if $\mathcal{A}_1, \mathcal{A}_2$ are finite subsets of $\mathbb{Z}^d$ and $\eta_1 \in \cup_{a_1 \in \mathcal{A}_1} \mathcal{L}(a_1)$, $\eta_2 \in \cup_{a_2 \in \mathcal{A}_2} \mathcal{L}(a_2)$, then

$$W(\eta_1, \eta_2) \geq -\frac{1}{2} \sum_{a_1 \in \mathcal{A}_1} \sum_{a_2 \in \mathcal{A}_2} \Psi(|a_1 - a_2|) \left[n^2(\eta_1, a_1) + n^2(\eta_2, a_2) \right].$$

Note that conditions (SS) and (LR) are translation invariant and independent of the choice of t.

And now we formulate the Ruelle theorem. We assume that forth part of the Condition 2 holds.

Theorem 4.5 (Ruelle, [23])**.** *Let conditions (SS) and (LR) hold; then the family* $\{\rho_\Lambda : \Lambda \in \mathcal{B}_c(\mathbb{R}^d)\}$ *satisfies the uniform Ruelle bounds with proper* C_R*.*

Again, by (4.2) and (4.5), we see that if ϕ is pair superstable potential and for any $\eta \in \Gamma_0, x \notin \eta$

$$r(\eta, x) \leq r^\phi(\eta, x), \tag{4.6}$$

then (SS) holds.

Due to this considerations, let us now to find some sufficient conditions for (SS) (and, therefore, (S)). We define two functions:

$$\delta_*(\eta, x) = \min_{y \in \eta} |x - y|, \quad \delta^*(\eta, x) = \max_{y \in \eta} |x - y|, \quad \eta \in \Gamma_0, x \notin \eta$$

and let $B(x, s)$ be the ball with center at x and radius $s > 0$.

Proposition 4.6. *Let there exist* $0 < a_1 < a_2 < +\infty$*,* $C_1, C_2 > 0$*,* $\varepsilon_1, \varepsilon_2 > 0$*,* $b > \dfrac{C_2}{a_2^{d+\varepsilon_2}}$ *such that*

$$r(\eta, x) \leq \exp\left(\sum_{x \in \eta_3} \frac{C_2}{(\delta^*(\eta, x))^{d+\varepsilon_2}} + b|\eta_2| - \sum_{x \in \eta_1} \frac{C_1}{(\delta_*(\eta, x))^{d+\varepsilon_1}} \right), \tag{4.7}$$

for $\eta \in \Gamma_0$*,* $x \notin \eta$*, where* $\eta_1 = \eta \cap B(x, a_1)$*,* $\eta_2 = \eta \cap (B(x, a_2) \setminus B(x, a_1))$*,* $\eta_3 = \eta \setminus (\eta_1 \cup \eta_2)$*. Then (SS) holds.*

Proof. By the Dobrushin-Fisher-Ruelle criterion (see, e.g., [5], [23]) one has that if $\phi(x, y) = V(x - y)$, where

$$V(x) = \begin{cases} \dfrac{C_1}{|x|^{d+\varepsilon_1}}, & |x| < a_1, \\ -\dfrac{C_2}{|x|^{d+\varepsilon_2}}, & |x| > a_2, \\ -b, & a_1 \leq |x| \leq a_2, \end{cases} \tag{4.8}$$

thus ϕ is superstable.

It now follows that

$$\sum_{y\in\eta} V(x-y) = \sum_{y\in\eta_1} \frac{C_1}{|x-y|^{d+\varepsilon_1}} - \sum_{y\in\eta_2} b - \sum_{y\in\eta_3} \frac{C_2}{|x-y|^{d+\varepsilon_2}}$$
$$\leq \sum_{x\in\eta_1} \frac{C_1}{(\delta_*(\eta,x))^{d+\varepsilon_1}} + b|\eta_2| - \sum_{x\in\eta_3} \frac{C_2}{(\delta^*(\eta,x))^{d+\varepsilon_2}}.$$

Further, using (4.7), we obtain

$$r(\eta,x) \leq \exp\left(-\sum_{y\in\eta} V(x-y)\right) = r^{\phi}(\eta,x),$$

thus, by (4.6), condition (SS) holds. □

Remark 4.7. The inequality (4.7) is natural in the sense that $r(\eta,x)\to 0$ if $dist(\eta;x)\to 0$ and $r(\eta,x)\to 1$ if $dist(\eta;x)\to\infty$.

Remark 4.8. Since the potential ϕ generating by (4.8) is also the lower-regular, we can formulate some sufficient condition on r for (LR). Really, for $\eta_2=\{y_1,\dots y_n\}$ one has

$$e^{-W(\eta_1,\eta_2)} = \frac{e^{-E(\eta_1\cup\eta_2)}}{e^{-E(\eta_1)}e^{-E(\eta_2)}} = \frac{\mathcal{R}(\emptyset,\eta_1\cup\eta_2)}{\mathcal{R}(\emptyset,\eta_1)\,\mathcal{R}(\emptyset,\eta_2)}$$
$$= \frac{\mathcal{R}(\eta_1,\eta_2)}{\mathcal{R}(\emptyset,\eta_2)} = \prod_{k=1}^{n} \frac{r(\eta_1\cup\{y_1,\dots,y_{k-1}\},y_k)}{r(\{y_1,\dots,y_k-1\},y_k)}.$$

As a result, if for any $\eta,\xi\in\Gamma_0, x\notin\eta\cup\xi$

$$\frac{r(\eta\cup\xi,x)}{r(\xi,x)} \leq r^{\phi}(\eta,x),$$

then (LS) holds. Therefore, if we change the left hand side of (4.7) onto $\dfrac{r(\eta\cup\xi,x)}{r(\xi,x)}$, then we obtain the sufficient condition on r for (LR). But this condition will be very specific and non-natural.

5. The Kirkwood—Salsburg identities

We start from the so-called "integrability-stability condition" (we interpret this name below).

(IS) (Integrability-stability condition) There exists a increasing function $b_{is}:[0;+\infty)\times(0;+\infty)\to[1;+\infty)$ such that for any $\eta\in\Gamma_0$, for σ-a.e. $x\in\mathbb{R}^d\setminus\eta$, and for any $c\in(0;+\infty)$

$$\int_{\Gamma_0} \left|\left(K_0^{-1}r(\cdot\cup\eta,x)\right)(\xi)\right| c^{|\xi|}d\lambda_\sigma(\xi) \leq b_{is}(|\eta|,c).$$

Remark 5.1. In the case of the Example 2 under stability condition (4.1)

$$r(\xi\cup\eta,x) = \prod_{y\in\xi\cup\eta} e^{-\beta\phi(x,y)} = \prod_{y\in\eta} e^{-\beta\phi(x,y)} \prod_{y\in\xi} e^{-\beta\phi(x,y)}$$
$$= \prod_{y\in\eta} e^{-\beta\phi(x,y)} \prod_{y\in\xi}\left(1+\left(e^{-\beta\phi(x,y)}-1\right)\right),$$

Hence,

$$\left|\left(K_0^{-1}r(\cdot\cup\eta,x)\right)(\xi)\right| = \left|\prod_{y\in\eta} e^{-\beta\phi(x,y)} e_\lambda\left(e^{-\beta\phi(x,\cdot)}-1,\xi\right)\right|$$
$$\leq e^{2\beta B|\eta|} e_\lambda\left(\left|e^{-\beta\phi(x,\cdot)}-1\right|,\xi\right),$$

and

$$\int_{\Gamma_0} \left|\left(K_0^{-1} r\left(\cdot \cup \eta, x\right)\right)(\xi)\right| c^{|\xi|} d\lambda_\sigma(\xi) \leq e^{2\beta B|\eta|} \int_{\Gamma_0} e_\lambda\left(c\left|e^{-\beta\phi(x,\cdot)}-1\right|, \xi\right) d\lambda_\sigma(\xi)$$
$$= e^{2\beta B|\eta|} \exp\left(\int_{\mathbb{R}^d} c\left|e^{-\beta\phi(x,y)}-1\right| d\sigma(y)\right).$$

In the case of the pair potential the following integrability condition plays an essential role:

$$C(\beta) := \operatorname*{ess\,sup}_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left|e^{-\beta\phi(x,y)}-1\right| d\sigma(y) < +\infty. \tag{5.1}$$

Then, by (4.1) and (5.1), condition (IS) is true in this case with

$$b_{is}\left(|\eta|, c\right) = e^{2\beta B|\eta| + cC(\beta)}$$

These are arguments for name of this condition: we use both integrability and stability conditions.

Let us consider also a weaker condition.

(IS-loc) (Local integrability-stability condition) For any $\eta \in \Gamma_0$, for σ-a.e. $x \in \mathbb{R}^d \setminus \eta$, for any $c \in (0; +\infty)$, and for any $\Lambda \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ one has

$$\int_{\Gamma_\Lambda} \left|\left(K_0^{-1} r\left(\cdot \cup \eta, x\right)\right)(\xi)\right| c^{|\xi|} d\lambda_\sigma(\xi) < +\infty.$$

Proposition 5.2. *Under (S) and (IS-loc) the finite-volume correlation functional ρ_Λ satisfies the following finite-volume Kirkwood—Salsburg equation*

$$\rho_\Lambda\left(\eta \cup \{x\}\right) = e_\lambda\left(1\!\!1_\Lambda, \eta \cup \{x\}\right) \int_{\Gamma_\Lambda} K_0^{-1}\left(r\left(\eta \cup \cdot, x\right)\right)(\xi)\, \rho_\Lambda\left(\eta \cup \xi\right) d\lambda_\sigma(\xi) \tag{5.2}$$

for $\lambda_\sigma \otimes \sigma$-a.a. $(\eta, x) \in \Gamma_\Lambda \times \Lambda$.

Proof. We have for $\eta \in \Gamma_\Lambda, x \in \Lambda$

$$\begin{aligned}
\rho_\Lambda\left(\eta \cup \{x\}\right) &= \frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \mathcal{R}\left(\emptyset, \eta \cup \xi \cup \{x\}\right) d\lambda_\sigma(\xi) \\
&= \frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \mathcal{R}\left(\emptyset, \eta \cup \xi\right) \mathcal{R}\left(\eta \cup \xi, \{x\}\right) d\lambda_\sigma(\xi) \\
&= \frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \mathcal{R}\left(\emptyset, \eta \cup \xi\right) r\left(\eta \cup \xi, x\right) d\lambda_\sigma(\xi) \\
&= \frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \mathcal{R}\left(\emptyset, \eta \cup \xi\right) K_0\left(K_0^{-1} r\left(\eta \cup \cdot, x\right)\right)(\xi)\, d\lambda_\sigma(\xi) \\
&= \frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \mathcal{R}\left(\emptyset, \eta \cup \xi\right) \left(K_0^{-1} r\left(\eta \cup \cdot, x\right) * e_\lambda\left(1, \cdot\right)\right)(\xi)\, d\lambda_\sigma(\xi).
\end{aligned}$$

And now we may use identity (3.2), since

$$\begin{aligned}
&\frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \mathcal{R}\left(\emptyset, \eta \cup \xi\right) \left|\left(K_0^{-1} r\left(\eta \cup \cdot, x\right) * e_\lambda\left(1, \cdot\right)\right)(\xi)\right| d\lambda_\sigma(\xi) \\
&\leq \frac{1}{Z_\Lambda(\emptyset)} \int_{\Gamma_\Lambda} \int_{\Gamma_\Lambda} \mathcal{R}\left(\emptyset, \eta \cup \xi \cup \zeta\right) \left|K_0^{-1}\left(r\left(\eta \cup \cdot, x\right)\right)(\xi)\right| e_\lambda\left(1, \zeta\right) d\lambda_\sigma(\zeta)\, d\lambda_\sigma(\xi) \\
&= \int_{\Gamma_\Lambda} \left|K_0^{-1}\left(r\left(\eta \cup \cdot, x\right)\right)(\xi)\right| \rho_\Lambda\left(\eta \cup \xi\right) d\lambda_\sigma(\xi) \\
&\leq \int_{\Gamma_\Lambda} \left|K_0^{-1}\left(r\left(\eta \cup \cdot, x\right)\right)(\xi)\right| \left(B_{st}\right)^{|\eta|+|\xi|} \exp\left(B_{st}\sigma(\Lambda)\right) d\lambda_\sigma(\xi) < +\infty.
\end{aligned}$$

Thus,

$$\begin{aligned}&\rho_\Lambda(\eta\cup\{x\})\\&=\frac{1}{Z_\Lambda(\emptyset)}\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}\mathcal{R}(\emptyset,\eta\cup\xi\cup\zeta)\,K_0^{-1}(r(\eta\cup\cdot,x))(\xi)\,e_\lambda(1,\zeta)\,d\lambda_\sigma(\zeta)\,d\lambda_\sigma(\xi)\\&=\int_{\Gamma_\Lambda}K_0^{-1}(r(\eta\cup\cdot,x))(\xi)\,\rho_\Lambda(\eta\cup\xi)\,d\lambda_\sigma(\xi).\end{aligned}$$

□

Remark 5.3. We may set $\rho_\Lambda\equiv 0$ on $\Gamma_0\setminus\Gamma_\Lambda$ and extend the equation (5.2) to $\lambda_\sigma\otimes\sigma$-a.a. $(\eta,x)\in\Gamma_0\times\mathbb{R}^d$.

We need some lemmas for following considerations.

Lemma 5.4. *Suppose that (IS) and (4.3) are true; then for any $\eta\in\Gamma_0$ and for σ-a.a. x*

$$K_0^{-1}r(\cdot\cup\eta,x)\in L^1(\Gamma_0,\rho_\mu d\lambda_\sigma).$$

Proof. Really,

$$\begin{aligned}&\int_{\Gamma_0}\left|\left(K_0^{-1}r(\cdot\cup\eta,x)\right)(\xi)\right|\rho_\mu(\xi)\,d\lambda_\sigma(\xi)\\&\qquad\le\int_{\Gamma_0}\left|\left(K_0^{-1}r(\cdot\cup\eta,x)\right)(\xi)\right|(C_R)^{\xi}\,d\lambda_\sigma(\xi)\le b_{is}(|\eta|,C_R)<+\infty,\end{aligned}$$

by (IS) and (4.3). □

Lemma 5.5. *Suppose that (IS) and (4.3) are true; then for any $G\in B_{bs}(\Gamma_0),\varphi\in C_0(\mathbb{R}^d)$ the following equality holds*

$$\begin{aligned}&\int_{\Gamma_0}\sum_{x\in\eta}\varphi(x)\,G(\eta\setminus\{x\})\,\rho_\mu(\eta)\,d\lambda_\sigma(\eta)\\&=\int_{\mathbb{R}^d}\varphi(x)\left(\int_{\Gamma_0}\left(G\star K_0^{-1}r(\cdot,x)\right)(\eta)\,\rho_\mu(\eta)\,d\lambda_\sigma(\eta)\right)d\sigma(x).\end{aligned}$$

Proof. Let $F=KG$ and apply Campbell-Mecke identity (2.1) for

$$h(\gamma,x)=\varphi(x)\,F(\gamma\setminus\{x\}),$$

we obtain

$$\int_\Gamma\sum_{x\in\gamma}\varphi(x)\,F(\gamma\setminus\{x\})\,d\mu(\gamma)=\int_\Gamma\int_{\mathbb{R}^d}F(\gamma)\,\varphi(x)\,r(\gamma,x)\,d\sigma(x)\,d\mu(\gamma).$$

Note that

$$\sum_{x\in\gamma}\varphi(x)\,F(\gamma\setminus\{x\})=K\left(\sum_{x\in\cdot}\varphi(x)\left(K^{-1}F\right)(\cdot\setminus\{x\})\right)(\gamma),$$

since

$$\begin{aligned}K\left(\sum_{x\in\cdot}\varphi(x)\left(K^{-1}F\right)(\cdot\setminus\{x\})\right)(\gamma)&=\sum_{\eta\Subset\gamma}\sum_{x\in\eta}\varphi(x)\left(K^{-1}F\right)(\eta\setminus\{x\})\\=\sum_{x\in\gamma}\varphi(x)\sum_{\eta\Subset\gamma:x\in\eta}\left(K^{-1}F\right)(\eta\setminus\{x\})&=\sum_{x\in\gamma}\varphi(x)\sum_{\eta\Subset\gamma\setminus\{x\}}\left(K^{-1}F\right)(\eta)\\&=\sum_{x\in\gamma}\varphi(x)\,F(\gamma\setminus\{x\}).\end{aligned}$$

Therefore,

$$\int_{\Gamma}\sum_{x\in\gamma}\varphi(x)\,F(\gamma\setminus\{x\})\,d\mu(\gamma)=\int_{\Gamma_0}\sum_{x\in\eta}\varphi(x)\left(K^{-1}F\right)(\eta\setminus\{x\})\,\rho_\mu(\eta)\,d\lambda_\sigma(\eta).$$

Since $K_0^{-1}r(\cdot,x)\in L^1(\Gamma_0,\rho_\mu d\lambda_\sigma)$ and $G\in L^1(\Gamma_0,\rho_\mu d\lambda_\sigma)$, we, by Condition 2, obtain

$$\begin{aligned}
&\int_{\Gamma}\int_{\mathbb{R}^d}F(\gamma)\,\varphi(x)\,r(\gamma,x)\,d\sigma(x)\,d\mu(\gamma)\\
&=\lim_{\Lambda\uparrow\mathbb{R}^d}\int_{\Gamma}\int_{\mathbb{R}^d}F(\gamma)\,\varphi(x)\,r(\gamma_\Lambda,x)\,d\sigma(x)\,d\mu(\gamma)\\
&=\lim_{\Lambda\uparrow\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\left(\int_{\Gamma}\left(KK^{-1}F\right)(\gamma)\left(KK_0^{-1}r(\cdot_\Lambda,x)\right)(\gamma)\,d\mu(\gamma)\right)d\sigma(x)\\
&=\lim_{\Lambda\uparrow\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\left(\int_{\Gamma}K\left(\left(K^{-1}F\right)(\cdot)\star\left(K_0^{-1}r(\cdot_\Lambda,x)\right)\right)(\gamma)\,d\mu(\gamma)\right)d\sigma(x)\\
&=\lim_{\Lambda\uparrow\mathbb{R}^d}\int_{\mathbb{R}^d}\varphi(x)\left(\int_{\Gamma_0}\left(K^{-1}F\right)(\eta)\star\left(K_0^{-1}r(\eta_\Lambda,x)\right)\rho_\mu(\eta)\,d\lambda_\sigma(\eta)\right)d\sigma(x)\\
&=\int_{\mathbb{R}^d}\varphi(x)\left(\int_{\Gamma_0}G(\eta)\star\left(K_0^{-1}r(\eta,x)\right)\rho_\mu(\eta)\,d\lambda_\sigma(\eta)\right)d\sigma(x),
\end{aligned}$$

and finally, one get

$$\begin{aligned}
&\int_{\Gamma_0}\sum_{x\in\eta}\varphi(x)\,G(\eta\setminus\{x\})\,\rho_\mu(\eta)\,d\lambda_\sigma(\eta)\\
&=\int_{\mathbb{R}^d}\varphi(x)\left(\int_{\Gamma_0}\left(G\star K_0^{-1}r(\cdot,x)\right)(\eta)\,\rho_\mu(\eta)\,d\lambda_\sigma(\eta)\right)d\sigma(x).
\end{aligned}$$

The statement is proved. □

The following lemma is needed for the sequel as well as it is interesting itself.

Lemma 5.6. *For any $G_1,G_2,H\in L^1_{loc}(\Gamma_0,d\lambda_\sigma)$ the following identity holds provided one of its sides exists*

$$\begin{aligned}
&\int_{\Gamma_0}\left(G_1\star K_0^{-1}G_2\right)(\eta)\,H(\eta)\,d\lambda_\sigma(\eta)\\
&=\int_{\Gamma_0}\int_{\Gamma_0}G_1(\eta_1)\left(K_0^{-1}G_2(\cdot\cup\eta_1)\right)(\eta_2)\,H(\eta_1\cup\eta_2)\,d\lambda_\sigma(\eta_1)\,d\lambda_\sigma(\eta_2).
\end{aligned}$$

Proof. We prove it at first for $H_\Lambda:=1\!\!1_{\Gamma_\Lambda}H$. Using (3.2), one has

$$\begin{aligned}
&\int_{\Gamma_0}\left(G_1\star K_0^{-1}G_2\right)(\eta)\,H_\Lambda(\eta)\,d\lambda_\sigma(\eta)\\
&=\int_{\Gamma_\Lambda}\left(K_0^{-1}K_0\left(G_1\star K_0^{-1}G_2\right)\right)(\eta)\,H(\eta)\,d\lambda_\sigma(\eta)\\
&=\int_{\Gamma_\Lambda}\left(K_0^{-1}\left(K_0G_1\cdot G_2\right)\right)(\eta)\,H(\eta)\,d\lambda_\sigma(\eta)\\
&=\int_{\Gamma_\Lambda}\left(\left(K_0G_1\cdot G_2\right)*e_\lambda(-1,\cdot)\right)(\eta)\,H(\eta)\,d\lambda_\sigma(\eta)\\
&=\int_{\Gamma_\Lambda}\int_{\Gamma_\Lambda}K_0G_1(\eta)\,G_2(\eta)\,e_\lambda(-1,\xi)\,H(\eta\cup\xi)\,d\lambda_\sigma(\xi)\,d\lambda_\sigma(\eta)\\
&=\int_{\Gamma_\Lambda}\left(\int_{\Gamma_\Lambda}\left(G_1*e_\lambda(1,\cdot)\right)(\eta)\,G_2(\eta)\,H(\eta\cup\xi)\,d\lambda_\sigma(\eta)\right)e_\lambda(-1,\xi)\,d\lambda_\sigma(\xi)
\end{aligned}$$

$$= \int_{\Gamma_\Lambda} \left(\int_{\Gamma_\Lambda} \int_{\Gamma_\Lambda} G_1(\eta)\, e_\lambda(1,\zeta)\, G_2(\eta\cup\zeta)\, H(\eta\cup\xi\cup\zeta)\, d\lambda_\sigma(\zeta)\, d\lambda_\sigma(\eta) \right) \times e_\lambda(-1,\xi)\, d\lambda_\sigma(\xi)$$

$$= \int_{\Gamma_\Lambda} G_1(\eta) \left(\int_{\Gamma_\Lambda} \int_{\Gamma_\Lambda} G_2(\eta\cup\zeta)\, e_\lambda(-1,\xi)\, H(\eta\cup\xi\cup\zeta)\, d\lambda_\sigma(\zeta)\, d\lambda_\sigma(\xi) \right) d\lambda_\sigma(\eta)$$

$$= \int_{\Gamma_\Lambda} G_1(\eta) \left(\int_{\Gamma_\Lambda} (G_2(\eta\cup\cdot) * e_\lambda(-1,\cdot))(\xi)\, H(\eta\cup\xi)\, d\lambda_\sigma(\xi) \right) d\lambda_\sigma(\eta)$$

$$= \int_{\Gamma_\Lambda} \int_{\Gamma_\Lambda} G_1(\eta) \left(K_0^{-1} G_2(\eta\cup\cdot)\right)(\xi)\, H(\eta\cup\xi)\, d\lambda_\sigma(\xi)\, d\lambda_\sigma(\eta).$$

Since the limit of the one side exists as $\Lambda \uparrow \mathbb{R}^d$, the assertion is true. □

Theorem 5.7 (Kirkwood—Salsburg equation)**.** *Let (IS) and (4.3) are true; then for* λ_σ*-a.e.* $\eta \in \Gamma_0$ *and for* σ*-a.e.* $x \notin \eta$

$$\rho_\mu(\eta\cup\{x\}) = \int_{\Gamma_0} \left(K_0^{-1} r(\cdot\cup\eta, x)\right)(\xi)\, \rho_\mu(\eta\cup\xi)\, d\lambda_\sigma(\xi). \tag{5.3}$$

Proof. Using (3.2) with

$$\hat\varphi(\xi) := \begin{cases} \varphi(x), & \xi = \{x\} \\ 0, & \text{otherwise,} \end{cases}$$

one has

$$\int_{\Gamma_0} \sum_{x\in\eta} \varphi(x)\, G(\eta\setminus\{x\})\, \rho_\mu(\eta)\, d\lambda_\sigma(\eta) = \int_{\Gamma_0} \sum_{\xi\subset\eta} \hat\varphi(\xi)\, G(\eta\setminus\xi)\, \rho_\mu(\eta)\, d\lambda_\sigma(\eta)$$

$$= \int_{\Gamma_0} \int_{\Gamma_0} \hat\varphi(\xi)\, G(\eta)\, \rho_\mu(\eta\cup\xi)\, d\lambda_\sigma(\xi)\, d\lambda_\sigma(\eta)$$

$$= \int_{\mathbb{R}^d} \varphi(x) \int_{\Gamma_0} G(\eta)\, \rho_\mu(\eta\cup\{x\})\, d\lambda_\sigma(\eta)\, d\sigma(x).$$

Further, by Lemma 5.5,

$$\int_{\mathbb{R}^d} \varphi(x) \int_{\Gamma_0} G(\eta)\, \rho_\mu(\eta\cup\{x\})\, d\lambda_\sigma(\eta)\, d\sigma(x) = \int_{\mathbb{R}^d} \varphi(x) \left(\int_{\Gamma_0} \left(G \star K^{-1} r(\cdot, x)\right)(\eta)\, \rho_\mu(\eta)\, d\lambda_\sigma(\eta) \right) d\sigma(x)$$

for any $\varphi \in C_0\left(\mathbb{R}^d\right)$, that follows

$$\int_{\Gamma_0} G(\eta)\, \rho_\mu(\eta\cup\{x\})\, d\lambda_\sigma(\eta) = \int_{\Gamma_0} \left(G \star K^{-1} r(\cdot, x)\right)(\eta)\, \rho_\mu(\eta)\, d\lambda_\sigma(\eta).$$

And by Lemma 5.6, we get

$$\int_{\Gamma_0} \left(G \star K^{-1} r(\cdot, x)\right)(\eta)\, \rho_\mu(\eta)\, d\lambda_\sigma(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\eta) \left(K_0^{-1} r(\cdot\cup\eta, x)\right)(\xi)\, \rho_\mu(\eta\cup\xi)\, d\lambda_\sigma(\eta)\, d\lambda_\sigma(\xi),$$

hence, for any $G \in L^1(\Gamma_0, \rho_\mu d\lambda_\sigma)$

$$\int_{\Gamma_0} G(\eta)\, \rho_\mu(\eta\cup\{x\})\, d\lambda_\sigma(\eta) = \int_{\Gamma_0} \int_{\Gamma_0} G(\eta) \left(K_0^{-1} r(\cdot\cup\eta, x)\right)(\xi)\, \rho_\mu(\eta\cup\xi)\, d\lambda_\sigma(\eta)\, d\lambda_\sigma(\xi),$$

and, as a result,

$$\rho_\mu(\eta\cup\{x\}) = \int_{\Gamma_0} \left(K_0^{-1} r(\cdot\cup\eta, x)\right)(\xi)\,\rho_\mu(\eta\cup\xi)\,d\lambda_\sigma(\xi),$$

that is (5.3). □

6. Existence results

Theorem 6.1. *Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an order generating sequence in $\mathcal{B}_c(\mathbb{R}^d)$ and $(\rho_{\Lambda_n})_{n\in\mathbb{N}}$ be the sequence of the corresponding finite volume correlation functionals that fulfills the equation (5.2) and the uniform Ruelle bound, i.e., there exists a constant $C_R>0$ such that for λ_σ-a.a. $\eta\in\Gamma_0$ and for all $n\in\mathbb{N}$*

$$\rho_{\Lambda_n}(\eta) \le (C_R)^{|\eta|}.$$

Let also (IS) holds for $c=C_R$ and (S) holds too; then there exists a measure $\mu\in\mathcal{M}^1_{fm}(\Gamma)$ satisfying the equation (2.1).

Proof. Let us consider the space $L^\infty(\Gamma_0,\lambda_\sigma)$ as the dual space of $L^1(\Gamma_0,\lambda_\sigma)$ equipped with the weak*-topology. According to the Banach-Alaoglu theorem, the set

$$B=\left\{ f\in L^\infty(\Gamma_0,\lambda_\sigma) \,\middle|\, \operatorname*{ess\,sup}_{\gamma\in\Gamma_0} |f(\gamma)|\le 1\right\}$$

is compact. Put

$$\tilde\rho_{\Lambda_n}(\xi)=\frac{\rho_{\Lambda_n}(\xi)}{(C_R)^{|\xi|}}.$$

Under the uniform Ruelle bound, we have $\tilde\rho_{\Lambda_n}\in B$. Then there exists a subsequence $\left(\tilde\rho_{\Lambda_{n_i}}(\xi)\right)_{i\in\mathbb{N}}$ that weak*-converges. Without loss of generality we assume that $(\tilde\rho_{\Lambda_n})_{n\in\mathbb{N}}$ weak*-converge to $\tilde\rho\in B$ itself. Let $\rho(\eta)=\tilde\rho(\eta)(C_R)^{|\eta|}$ then for λ_σ-a.a. $\eta\in\Gamma_0$ $\rho(\eta)\le (C_R)^{|\eta|}$. Clearly, $\rho d\lambda_\sigma\in\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ and for all $G\in B_{bs}(\Gamma_0)$ one has $\left(G\star\bar G\right)(\cdot)(C_R)^{|\cdot|}\in L^1(\Gamma_0,\lambda_\sigma)$; further,

$$\int_{\Gamma_0}\left(G\star\bar G\right)(\eta)\,\rho(\eta)\,d\lambda_\sigma(\eta)=\int_{\Gamma_0}\left(G\star\bar G\right)(\eta)\,(C_R)^{|\eta|}\,\tilde\rho(\eta)\,d\lambda_\sigma(\eta)$$
$$=\lim_{n\to\infty}\int_{\Gamma_0}\left(G\star\bar G\right)(\eta)\,(C_R)^{|\eta|}\,\tilde\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta)=\lim_{n\to\infty}\int_{\Gamma_0}\left(G\star\bar G\right)(\eta)\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta)\ge 0,$$

since

$$\int_{\Gamma_0}\left(G\star\bar G\right)(\eta)\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta)$$
$$=\int_{\Gamma_{\Lambda_n}}\left(\left(K_0\left(G\star\bar G\right)\right)*e_\lambda(-1,\cdot)\right)(\eta)\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta)$$
$$=\int_{\Gamma_{\Lambda_n}}|K_0G(\eta)|^2\int_{\Gamma_{\Lambda_n}}(-1)^{|\xi|}\rho_{\Lambda_n}(\xi\cup\eta)\,d\lambda_\sigma(\xi)\,d\lambda_\sigma(\eta)\ge 0,$$

due to (3.5). Therefore, $\rho d\lambda_\sigma$ is a positive semi-defined locally finite measure. Since

$$\int_{\{\emptyset\}}\rho(\eta)\,d\lambda_\sigma(\eta)=\int_{\Gamma_0}1_{\{\emptyset\}}(\eta)\,\rho(\eta)\,d\lambda_\sigma(\eta)=\lim_{n\to\infty}\int_{\Gamma_0}1_{\{\emptyset\}}(\eta)\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta)=1,$$

this measure is normalized. Then there exist a unique measure $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ such that $\rho d\lambda_\sigma=K^*\mu$ (see Theorem 6.2 in [12]). Next, note that if for any $c>0$ $G\in$

$L^1\left(\Gamma_0, c^{|\cdot|}d\lambda_\sigma\right)$ then

$$\lim_{n\to\infty}\int_{\Gamma_0} G(\eta)\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta) = \lim_{n\to\infty}\int_{\Gamma_0} G(\eta)\,(C_R)^{|\eta|}\,\tilde{\rho}_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta)$$
$$= \int_{\Gamma_0} G(\eta)\,(C_R)^{|\eta|}\,\tilde{\rho}(\eta)\,d\lambda_\sigma(\eta) = \int_{\Gamma_0} G(\eta)\,\rho(\eta)\,d\lambda_\sigma(\eta).$$

Let us consider for fixed $G \in B_{bs}(\Gamma_0)$ functions

$$f_n(x) = 1\!\!1_{\Lambda_n}(x)\int_{\Gamma_0} G(\eta)\,\rho_{\Lambda_n}(\eta\cup\{x\})\,d\lambda_\sigma(\eta).$$

Using (5.2) and Lemma 5.6, we get

$$f_n(x) = 1\!\!1_{\Lambda_n}(x)\int_{\Gamma_0}\int_{\Gamma_0} G(\eta)\,K_0^{-1}\left(r(\eta\cup\cdot,x)\right)(\xi)\,\rho_{\Lambda_n}(\eta\cup\xi)\,d\lambda_\sigma(\xi)\,d\lambda_\sigma(\eta)$$
$$= 1\!\!1_{\Lambda_n}(x)\int_{\Gamma_0}\left(G\star K_0^{-1}r(\cdot,x)\right)(\eta)\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta).$$

Next,

$$\int_{\Gamma_0}\left(G\star K_0^{-1}r(\cdot,x)\right)(\eta)\,c^{|\eta|}d\lambda_\sigma(\eta)$$
$$= \int_{\Gamma_0}\int_{\Gamma_0} G(\eta)\,K_0^{-1}\left(r(\eta\cup\cdot,x)\right)(\xi)\,c^{|\eta|+|\xi|}d\lambda_\sigma(\xi)\,d\lambda_\sigma(\eta)$$
$$\le \int_{\Gamma_0}|G(\eta)|\,c^{|\eta|}b_{is}(|\eta|,c)\,d\lambda_\sigma(\eta) < \infty,$$

since $G \in B_{bs}(\Gamma_0)$. Then for σ-a.a. x

$$f_n(x) \to \int_{\Gamma_0}\left(G\star K^{-1}r(\cdot,x)\right)(\eta)\,\rho(\eta)\,d\lambda_\sigma(\eta) =: f(x), \quad n\to\infty.$$

Using the Lebesgue dominate theorem, for any $\varphi \in C_0\left(\mathbb{R}^d\right)$

$$\int_{\mathbb{R}^d}\varphi(x)\,f_n(x)\,d\sigma(x) \to \int_{\mathbb{R}^d}\varphi(x)\,f(x)\,d\sigma(x),$$

since

$$|f_n(x)| \le \int_{\Gamma_0}|G(\eta)|\,c^{|\eta|}b_{is}(|\eta|,C_R)\,d\lambda_\sigma(\eta) < \infty.$$

But by Mecke identity,

$$\int_{\mathbb{R}^d}\varphi(x)\,f_n(x)\,d\sigma(x) = \int_{\mathbb{R}^d}\varphi(x)\,1\!\!1_{\Lambda_n}(x)\int_{\Gamma_0} G(\eta)\,\rho_{\Lambda_n}(\eta\cup\{x\})\,d\lambda_\sigma(\eta)\,d\sigma(x)$$
$$= \int_{\Gamma_0}\sum_{x\in\eta}\varphi(x)\,1\!\!1_{\Lambda_n}(x)\,G(\eta\setminus\{x\})\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta)$$
$$= \int_{\Gamma_0}\sum_{x\in\eta}\varphi(x)\,G(\eta\setminus\{x\})\,\rho_{\Lambda_n}(\eta)\,d\lambda_\sigma(\eta),$$

starting from some big n. Since for every $c > 0$

$$\begin{aligned}
&\int_{\Gamma_0} \left| \sum_{x\in\eta} \varphi(x) G(\eta \setminus \{x\}) \right| c^{|\eta|} d\lambda_\sigma(\eta) \\
&\leq \int_{\Gamma_0} \sum_{x\in\eta} |\varphi(x)| \, |G(\eta \setminus \{x\})| \, c^{|\eta|} d\lambda_\sigma(\eta) \\
&= \int_{\Gamma_0} \int_{\mathbb{R}^d} |\varphi(x)| \, |G(\eta)| \, c^{|\eta|+1} d\sigma(x) \, d\lambda_\sigma(\eta) < \infty,
\end{aligned}$$

one has

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi(x) f_n(x) \, d\sigma(x) &\to \int_{\Gamma_0} \sum_{x\in\eta} \varphi(x) G(\eta \setminus \{x\}) \rho(\eta) \, d\lambda_\sigma(\eta) \\
&= \int_{\mathbb{R}^d} \varphi(x) \int_{\Gamma_0} G(\eta) \rho(\eta \cup \{x\}) \, d\lambda_\sigma(\eta) \, d\sigma(x).
\end{aligned}$$

Thus we have

$$f(x) = \int_{\Gamma_0} G(\eta) \rho(\eta \cup \{x\}) \, d\lambda_\sigma(\eta).$$

As a result, for every $G \in B_{bs}(\Gamma_0)$

$$\begin{aligned}
\int_{\Gamma_0} G(\eta) \rho(\eta \cup \{x\}) \, d\lambda_\sigma(\eta) &= \int_{\Gamma_0} \left(G \star K^{-1} r(\cdot, x)\right)(\eta) \rho(\eta) \, d\lambda_\sigma(\eta) \\
&= \int_{\Gamma_0} \int_{\Gamma_0} G(\eta) K_0^{-1} (r(\eta \cup \cdot, x))(\xi) \rho(\eta \cup \xi) \, d\lambda_\sigma(\xi) \, d\lambda_\sigma(\eta),
\end{aligned}$$

Thus, ρ satisfies (5.3).

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi(x) f(x) \, d\sigma(x) &= \int_{\mathbb{R}^d} \varphi(x) \left(\int_{\Gamma_0} \left(G \star K_0^{-1} r(\cdot, x)\right)(\eta) \rho(\eta) \, d\lambda_\sigma(\eta) \right) d\sigma(x) \\
&= \int_{\Gamma_0} \sum_{x\in\eta} \varphi(x) G(\eta \setminus \{x\}) \rho(\eta) \, d\lambda_\sigma(\eta).
\end{aligned}$$

Then (see the proof of Lemma 5.5) for any $G \in B_{bs}(\Gamma_0)$ and $F = KG$ we have

$$\int_\Gamma \sum_{x\in\gamma} \varphi(x) F(\gamma \setminus \{x\}) \, d\mu(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} F(\gamma) \varphi(x) r(\gamma, x) \, d\sigma(x) \, d\mu(\gamma). \tag{6.1}$$

Since ρ satisfies (5.3) and due to Lemma 5.6 and the Mecke identity, we may write

$$\begin{aligned}
&\int_\Gamma K(G(\cdot) \langle \varphi, \cdot \rangle)(\gamma) \, d\mu(\gamma) = \int_{\Gamma_0} G(\eta) \langle \varphi, \eta \rangle \rho(\eta) \, d\lambda_\sigma(\eta) \\
&= \int_{\Gamma_0} \int_{\mathbb{R}^d} \varphi(x) G(\eta \cup \{x\}) \rho(\eta \cup \{x\}) \, d\sigma(x) \, d\lambda_\sigma(\eta) \\
&= \int_{\mathbb{R}^d} \int_{\Gamma_0} \varphi(x) G(\eta \cup \{x\}) \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup \eta, x)\right)(\xi) \rho(\eta \cup \xi) \, d\lambda_\sigma(\xi) \, d\lambda_\sigma(\eta) \, d\sigma(x) \\
&= \int_{\mathbb{R}^d} \int_{\Gamma_0} \varphi(x) \left(G(\cdot \cup \{x\}) \star K_0^{-1} r(\cdot, x)\right)(\eta) \rho(\eta) \, d\lambda_\sigma(\eta) \, d\sigma(x) \\
&= \int_{\mathbb{R}^d} \int_\Gamma \varphi(x) K \left(G(\cdot \cup \{x\}) \star K_0^{-1} r(\cdot, x)\right)(\gamma) \, d\mu(\gamma) \, d\sigma(x) \\
&= \int_{\mathbb{R}^d} \int_\Gamma \varphi(x) (KG(\cdot \cup \{x\}))(\gamma) r(\gamma, x)(\eta) \, d\mu(\gamma) \, d\sigma(x).
\end{aligned}$$

But by direct computation, one has

$$K\left(G\left(\cdot\right)\langle\varphi,\cdot\rangle\right)\left(\gamma\right)=\left(KG\right)\left(\gamma\right)\langle\varphi,\gamma\rangle-\sum_{x\in\gamma}\left(KG\right)\left(\gamma\setminus\{x\}\right)\varphi\left(x\right),$$

further, using (6.1), we obtain

$$\int_{\Gamma}\left(KG\right)\left(\gamma\right)\langle\varphi,\gamma\rangle\,d\mu\left(\gamma\right)=\int_{\mathbb{R}^d}\int_{\Gamma}\varphi\left(x\right)\left(KG\left(\cdot\cup\{x\}\right)\right)\left(\gamma\right)r\left(\gamma,x\right)\left(\eta\right)d\mu\left(\gamma\right)d\sigma\left(x\right)$$
$$+\int_{\mathbb{R}^d}\int_{\Gamma}\varphi\left(x\right)\left(KG\right)\left(\gamma\right)r\left(\gamma,x\right)\left(\eta\right)d\mu\left(\gamma\right)d\sigma\left(x\right),$$

but, by definition of K, we see that

$$\left(KG\right)\left(\gamma\cup\{x\}\right)=\left(KG\left(\cdot\cup\{x\}\right)\right)\left(\gamma\right)+\left(KG\right)\left(\gamma\right),$$

and, as a result,

$$\int_{\Gamma}F\left(\gamma\right)\langle\varphi,\gamma\rangle\,d\mu\left(\gamma\right)=\int_{\Gamma}\int_{\mathbb{R}^d}F\left(\gamma\cup\{x\}\right)\varphi\left(x\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)$$

for any $\varphi\in C_0\left(\mathbb{R}^d\right)$ and $F=KG, G\in B_{bs}\left(\Gamma_0\right)$. □

Remark 6.2. In the pair-potential case the proof is more simpler since from the $*$-weak convergence we can obtain convergence a.s. due to the Mayer—Montroll equation (see, e.g., [23]). In this case it was true under the same conditions as Kirkwood—Salsburg equation. In the our general case we can write Mayer—Montroll equation too:

$$\rho_\mu\left(\eta\cup\xi\right)=\int_{\Gamma_0}\left(K_0^{-1}\mathcal{R}\left(\cdot\cup\eta,\xi\right)\right)\left(\zeta\right)\rho_\mu\left(\eta\cup\zeta\right)d\lambda_\sigma\left(\zeta\right)\tag{6.2}$$

for $\lambda_\sigma\otimes\lambda_\sigma$-a.e. $(\eta,\xi)\in\Gamma_0\times\Gamma_0$. But it is true under just more strong condition than condition (IS):

(EIS) (Exponential integrability-stability condition) There exist a increasing function $b_{eis}:[0;+\infty)\times\mathbb{N}\to[1;+\infty)$ and constant $B_{eis}\geq 0$ such that for any $\eta,\zeta\in\Gamma_0,\eta\cap\zeta=\emptyset$ and for any $c\in(0;+\infty)$

$$\int_{\Gamma_0}\left|\left(K_0^{-1}\mathcal{R}\left(\cdot\cup\eta,\zeta\right)\right)\left(\xi\right)\right|c^{|\xi|}d\lambda_\sigma\left(\xi\right)\leq\left(B_{eis}\right)^{|\eta||\zeta|}b_{eis}\left(|\zeta|,c\right).$$

Due to exponential growth on $|\eta|$ this condition is useful just for positive many-body potential.

7. Uniqueness conditions

Let us start from the condition with another bound for the same integral as in (IS).

(MIS) (Modified integrability-stability condition) There exist a increasing function $b_{mis}:[0;+\infty)\to[1;+\infty)$ and constant $B_{mis}\geq 0$ such that for any $\eta\in\Gamma_0$, for any $x\in\mathbb{R}^d\setminus\eta$, and for any $c\in(0;+\infty)$

$$\int_{\Gamma_0}\left|\left(K_0^{-1}r\left(\cdot\cup\eta,x\right)\right)\left(\xi\right)\right|c^{|\xi|}d\lambda_\sigma\left(\xi\right)\leq p\left(\eta,x\right)b_{mis}\left(c\right),$$

where $p:\Gamma_0\times\mathbb{R}^d\to[0;+\infty)$ such that for any $\eta\in\Gamma_0$ there exists $x_0\in\eta$ such that

$$p\left(\eta\setminus\{x_0\},x_0\right)\leq B_{mis}.$$

Remark 7.1. In the case of Example 2 under condition (4.1) one has

$$\int_{\Gamma_0}\left|\left(K_0^{-1}r\left(\cdot\cup\eta,x\right)\right)\left(\xi\right)\right|c^{|\xi|}d\lambda_\sigma\left(\xi\right)=\prod_{y\in\eta}e^{-\beta\phi(x,y)}\int_{\Gamma_0}\left|e_\lambda\left(e^{-\beta\phi(x,\cdot)}-1,\xi\right)\right|c^{|\xi|}d\lambda_\sigma\left(\xi\right)$$
$$=e^{-\beta W(\{x\},\eta)}\int_{\Gamma_0}\left|e_\lambda\left(c\left(e^{-\beta\phi(x,\cdot)}-1\right),\xi\right)\right|d\lambda_\sigma\left(\xi\right)\leq e^{-\beta W(\{x\},\eta)}e^{cC(\beta)},$$

and under (4.1) for any $\eta \in \Gamma_0$ there exists $x_0 \in \eta$ such that

$$W(\{x_0\}, \eta \setminus \{x_0\}) \geq -2B.$$

Thus, by (4.1) and (5.1), condition (MIS) is true in this case with

$$b_{mis}(c) = e^{cC(\beta)};\ p(\eta, x) = e^{-\beta W(\{x\}, \eta)};\ B_{mis} = e^{2\beta B}.$$

And now we prove so-called modified Kirkwood—Salsburg equation. We will use notation of condition (MIS). First of all, let us define for $\eta \in \Gamma_0, x \in \eta$ a function

$$\tilde{\varkappa}(\eta, x) := \begin{cases} 1, & p(\eta \setminus \{x\}, x) \leq B_{mis} \\ 0, & \text{otherwise} \end{cases},$$

now, by (MIS), we has

$$\sum_{x \in \eta} \tilde{\varkappa}(\eta, x) \geq 1.$$

Also we define a function

$$\varkappa(\eta, x) := \frac{\tilde{\varkappa}(\eta, x)}{\sum_{x \in \eta} \tilde{\varkappa}(\eta, x)} \in [0; 1].$$

Theorem 7.2 (Modified Kirkwood—Salsburg equation)**.** *Let (IS), (4.3) and (MIS) are true; then for any* $\eta \neq \emptyset$

$$\rho_\mu(\eta) = \sum_{x \in \eta} \varkappa(\eta, x) \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup (\eta \setminus \{x\}), x)\right)(\xi)\, \rho_\mu(\xi \cup (\eta \setminus \{x\}))\, d\lambda_\sigma(\xi). \tag{7.1}$$

Proof. Using (5.3), one has

$$\sum_{x \in \eta} \varkappa(\eta, x) \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup (\eta \setminus \{x\}), x)\right)(\xi)\, \rho_\mu(\xi \cup (\eta \setminus \{x\}))\, d\lambda_\sigma(\xi)$$
$$= \sum_{x \in \eta} \varkappa(\eta, x)\, \rho_\mu((\eta \setminus \{x\}) \cup \{x\}) = \rho_\mu(\eta) \sum_{x \in \eta} \varkappa(\eta, x) = \rho_\mu(\eta).$$

□

Remark 7.3. Actually, we prove that if $\rho : \Gamma_0 \to [0; +\infty)$ such that
1) $\rho(\emptyset) = 1$;
2) there exists $C_R > 0 :\ \rho(\eta) \leq (C_R)^{|\eta|}$ for λ_σ-a.a. $\eta \in \Gamma_0$;
3) for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ one has that ρ is Lenard-positive, i.e.,

$$\int_{\Gamma_\Lambda} (-1)^{|\xi|} \rho(\eta \cup \xi)\, d\lambda_\sigma(\xi) \geq 0 \text{ for } \lambda_\sigma\text{-a.a. } \eta \in \Gamma_0,$$

thus under (IS) and (MIS) (5.3) ⇒ (7.1) and under (EIS) (6.2) ⇔ (5.3) (clearly, we have to change ρ_μ onto ρ in the equations).

Remark 7.4. We can prove also that under condition (MIS) (and moreover, under the local condition (MIS)) the following finite-volume modified Kirkwood—Salsburg equation holds

$$\rho_\Lambda(\eta) = e_\lambda(1\!\!1_\Lambda, \eta) \times \sum_{x \in \eta} \varkappa(\eta, x) \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup (\eta \setminus \{x\}), x)\right)(\xi)\, \rho_\Lambda(\xi \cup (\eta \setminus \{x\}))\, d\lambda_\sigma(\xi). \tag{7.2}$$

In the following we always assume that (S) and (MIS) hold.

Definition 7.5. For arbitrary constant $C_R > 0$ define a space E_{C_R} in the following way

$$E_{C_R} = \left\{ F \in L^0(\Gamma_0, \lambda_\sigma) \,\middle|\, \|F\|_{C_R} := \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \left(\frac{|F(\eta)|}{C_R^{|\eta|}} \right) < \infty \right\}.$$

Lemma 7.6. *E_{C_R} is a Banach space.*

About proof see, e.g., Lemma 3.4.2 in [14].

Remark 7.7. Clearly, if ρ_μ is correlation functional and fulfills (4.3) with constant C_R, then $\rho_\mu \in E_{C_R}$.

Definition 7.8. Consider a "modified" Kirkwood—Salsburg operator $S : B_{bs}(\Gamma_0) \to L^\infty(\Gamma_0, \lambda_\sigma)$ given for $\eta \neq \emptyset$ by

$$SF(\eta) = \sum_{x\in\eta} \varkappa(\eta, x) \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x)\right)(\xi)\, F(\xi \cup \eta \setminus \{x\})\, d\lambda_\sigma(\xi)$$

and $SF(\emptyset) = 0$.

Note that really $SF \in L^\infty(\Gamma_0, \lambda_\sigma)$, since

$$\begin{aligned}
&\left| \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x)\right)(\xi)\, F(\xi \cup \eta \setminus \{x\})\, d\lambda_\sigma(\xi) \right| \\
&\leq \|F\|_{L^\infty(\Gamma_0,\lambda_\sigma)} \int_{\Gamma_0} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x)\right)(\xi) \right| d\lambda_\sigma(\xi) \\
&\leq \|F\|_{L^\infty(\Gamma_0,\lambda_\sigma)}\, p(\eta \setminus \{x\}, x)\, b_{mis}(1),
\end{aligned}$$

it follows that, for λ_σ-a.a. $\eta \in \Gamma_0$

$$|SF(\eta)| \leq \|F\|_{L^\infty(\Gamma_0,\lambda_\sigma)}\, b_{mis}(1) \sum_{x\in\eta} \varkappa(\eta, x)\, p(\eta \setminus \{x\}, x) \leq \|F\|_{L^\infty(\Gamma_0,\lambda_\sigma)}\, b_{mis}(1)\, B_{mis}.$$

Theorem 7.9. *S is bounded operator in E_{C_R} with*

$$\|S\| \leq \frac{B_{mis} b_{mis}(C_R)}{C_R}.$$

Proof. One has

$$\begin{aligned}
\|SF\|_{C_R} &\leq \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \sum_{x\in\eta} \varkappa(\eta, x) \frac{1}{C_R^{|\eta|}} \\
&\quad\times \left| \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x)\right)(\xi)\, F(\xi \cup \eta \setminus \{x\})\, d\lambda_\sigma(\xi) \right| \\
&= \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \sum_{x\in\eta} \varkappa(\eta, x) \frac{1}{C_R^{|\eta|}} \\
&\quad\times \int_{\Gamma_0} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x)\right)(\xi) \frac{F(\xi \cup \eta \setminus \{x\})}{C_R^{|\xi\cup\eta\setminus\{x\}|}} \right| C_R^{|\xi\cup\eta\setminus\{x\}|} d\lambda_\sigma(\xi) \\
&\leq \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \sum_{x\in\eta} \varkappa(\eta, x) \frac{\|F\|_{C_R}}{C_R} \int_{\Gamma_0} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x)\right)(\xi) \right| C_R^{|\xi|} d\lambda_\sigma(\xi) \\
&\leq \frac{\|F\|_{C_R}}{C_R} \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \sum_{x\in\eta} \varkappa(\eta, x)\, p(\eta \setminus \{x\}, x) \cdot b_{mis}(C_R) \leq \frac{\|F\|_{C_R}}{C_R} B_{mis} b_{mis}(C_R),
\end{aligned}$$

that fulfilled the proof. □

Proposition 7.10. *If function r be such that*

$$\frac{B_{mis}b_{mis}\left(C_R\right)}{C_R}<1,$$

then there exists a unique solution in E_{C_R} of the equation

$$F-SF=e_\lambda\left(0,\cdot\right). \tag{7.3}$$

Proof. Really, if

$$\|S\|_{C_R}\leq\frac{B_{mis}b_{mis}\left(C_R\right)}{C_R}<1,$$

then S is a contraction and $(1-S)^{-1}:=\sum_{j=0}^{\infty}S^j$ exists as a linear bounded operator. Then the unique solution of (7.3) has the form $(1-S)^{-1}e_\lambda\left(0,\cdot\right).$ □

And now we can prove the uniqueness result.

Theorem 7.11. *Let (S) and (MIS) hold. Let $C_R>0$ be given. Then for all r such that*

$$\frac{B_{mis}b_{mis}\left(C_R\right)}{C_R}<1$$

there exists at most one correlation functional ρ that fulfills equations (5.3) and (4.3) with constant C_R.

Proof. Let ρ be a correlation functional such that equalities (5.3) and (4.3) with constant C_R hold. Then $\rho\in E_{C_R}$ and it fulfills the equation (7.1). Therefore, it is a solution to (7.3) and hence unique in E_{C_R}. □

Definition 7.12. Consider for the volume $\Lambda\in\mathcal{B}_c\left(\mathbb{R}^d\right)$ an operator $S_\Lambda:E_{C_R}\to E_{C_R}$ in the following way

$$S_\Lambda=1\!\!1_{\Gamma_\Lambda}S1\!\!1_{\Gamma_\Lambda}.$$

Since $0\leq\rho_\Lambda\left(\eta\right)\leq\left(B_{st}\right)^{|\eta|}\exp\left(B_{st}\sigma\left(\Lambda\right)\right)$, one has $\rho_\Lambda\in E_{B_{st}}$ and

$$\|\rho_\Lambda\|_{B_{st}}=\exp\left(B_{st}\sigma\left(\Lambda\right)\right).$$

By (7.2), it follows that

$$\left(1-S_\Lambda\right)\rho_\Lambda=e_\lambda\left(0,\cdot\right).$$

Note that $\|S_\Lambda\|_{C_R}\leq\|S\|_{C_R}$ and if $C_R\geq C_R'$ then $E_{C_R'}\subset E_{C_R}$. As a result, if $C_R\geq B_{st}$, then $\rho_\Lambda\in E_{C_R}$ and S_Λ is a contraction in E_{C_R}. We get that

$$\rho_\Lambda=\left(1-S_\Lambda\right)^{-1}e_\lambda\left(0,\cdot\right)=\sum_{j=0}^{\infty}S_\Lambda^je_\lambda\left(0,\cdot\right).$$

For every generating sequence $\left(\Lambda_n\right)_{n\in\mathbb{N}}$

$$\|\rho_{\Lambda_n}\|_{C_R}\leq\sum_{j=0}^{\infty}\left(\|S_{\Lambda_n}\|_{C_R}\right)^j\leq\sum_{j=0}^{\infty}\left(\|S\|_{C_R}\right)^j=\frac{1}{1-\|S\|_{C_R}},$$

then $\left(\rho_{\Lambda_n}\right)_{n\in\mathbb{N}}$ fulfills the uniform Ruelle bound. Then due to the proof of the existence theorem, one has that there exists $\rho\in E$ satisfying the equation (5.3) and then under condition (MIS) satisfying the equation (7.1), thus, it is unique. Moreover, the corresponding measure μ is unique too. As a result, we have the following theorem.

Theorem 7.13. *Let r satisfies (S), (IS) and (MIS) and let $C_R>B_{st}$ be such that*

$$\frac{B_{mis}b_{mis}\left(C_R\right)}{C_R}<1.$$

Then there exists a unique measure $\mu\in\mathcal{M}_{fm}^1\left(\Gamma\right)$ such that ρ_μ satisfies (4.3) with the Ruelle constant C_R.

The following theorem gives more strong converges result for correlation functions. But previously we need additional condition on the relative energy density.

(UMIS) (Uniform modified integrability-stability condition) There exists a increasing function $b_{umis} : [0;+\infty) \to [1;+\infty)$ such that for any $\Lambda \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ and any $\varepsilon > 0$ there exists $\Lambda_1 \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ such that for σ-a.a. $x \in \Lambda$ and for λ_σ-a.a. $\eta \in \Gamma_0$

$$\int_{\Gamma_0\setminus\Gamma_{\Lambda_1}} \left|\left(K_0^{-1} r\left(\cdot \cup \eta, x\right)\right)(\xi)\right| c^{|\xi|} d\lambda_\sigma(\xi) \le \varepsilon p(\eta, x)\, b_{umis}(c).$$

Remark 7.14. In the case of the Example 2 under condition (I') from [21] one has

$$\begin{aligned}
&\int_{\Gamma_0\setminus\Gamma_{\Lambda_1}} \left|\left(K_0^{-1} r\left(\cdot \cup \eta, x\right)\right)(\xi)\right| c^{|\xi|} d\lambda_\sigma(\xi) \\
&= e^{-\beta W(\{x\},\eta)} \int_{\Gamma_0\setminus\Gamma_{\Lambda_1}} \left| e_\lambda\left(c\left(e^{-\beta\phi(x,\cdot)} - 1\right), \xi\right)\right| d\lambda_\sigma(\xi) \\
&= p(\eta, x)\left(\int_{\Gamma_0} \left| e_\lambda\left(c\left(e^{-\beta\phi(x,\cdot)} - 1\right), \xi\right)\right| d\lambda_\sigma(\xi)\right. \\
&\qquad\qquad \left. - \int_{\Gamma_{\Lambda_1}} \left| e_\lambda\left(c\left(e^{-\beta\phi(x,\cdot)} - 1\right), \xi\right)\right| d\lambda_\sigma(\xi)\right) \\
&= p(\eta, x)\left(\exp\left(\int_{\mathbb{R}^d} c\left|e^{-\beta\phi(x,y)} - 1\right| d\sigma(y)\right) - \exp\left(\int_{\Lambda_1} c\left|e^{-\beta\phi(x,y)} - 1\right| d\sigma(y)\right)\right) \\
&\le p(\eta, x) \exp\left(\int_{\mathbb{R}^d} c\left|e^{-\beta\phi(x,y)} - 1\right| d\sigma(y)\right)\left(\int_{\mathbb{R}^d\setminus\Lambda_1} c\left|e^{-\beta\phi(x,y)} - 1\right| d\sigma(y)\right) \\
&< \varepsilon p(\eta, x)\, e^{cC(\beta)},
\end{aligned}$$

then in this case $b_{umis}(c) = b_{mis}(c) = e^{cC(\beta)}$.

Theorem 7.15. *Let (S), (MIS) and (UMIS) hold and suppose r be such that $C_R \ge B_{st}$ and $\dfrac{B_{mis} b_{mis}(C_R)}{C_R} < 1$. Then for every generating sequence $(\Lambda_n)_{n\in\mathbb{N}}$ and every $\Lambda_0 \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ the sequence of finite volume correlation functionals $(\rho_{\Lambda_n})_{n\in\mathbb{N}}$ converge to $\rho = (1 - S)^{-1} e_\lambda(0, \cdot)$, the unique solution of (7.3), in the following sense:*

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{\eta\in\Gamma_0} \left|\rho(\eta) - \rho_{\Lambda_n}(\eta)\right| \left(\frac{1}{C_R}\right)^{|\eta|} = 0.$$

Proof. Note that for every $\Lambda \in \mathcal{B}_c\left(\mathbb{R}^d\right)$ with $\Lambda_0 \subset \Lambda$ and for $\eta \in \Gamma_{\Lambda_0}$ we have (see [14]) that for any $l \in \mathbb{N}$

$$\begin{aligned}
(7.4)\quad \left|\rho(\eta) - \rho_\Lambda(\eta)\right| (C_R)^{|\eta|} &\left\| 1\!\!1_{\Gamma_{\Lambda_0}} (1 - S)^{-1} - (1 - S_\Lambda)^{-1}\right\|_{C_R} \\
&\le (C_R)^{|\eta|}\left(\frac{2\|S\|_{C_R}^{l+1}}{1 - \|S\|_{C_R}} + \sum_{j=0}^{l} \left\| 1\!\!1_{\Gamma_{\Lambda_0}}\left(S_\Lambda^j - S^j\right)\right\|_{C_R}\right).
\end{aligned}$$

Next, for any order generating sequence $(\Lambda_s)_{s\in\mathbb{N}_0}$ and for any fixed $j \in \mathbb{N}$ we have (see Lemma 3.4.11 from [14]) that for all $s \ge j$

$$\begin{aligned}
(7.5)\quad &\left\| 1\!\!1_{\Gamma_{\Lambda_0}}\left(S_{\Lambda_s}^j - S^j\right)\right\|_{C_R} \\
&\le \|S\|_{C_R}^{j-1} \sum_{r=1}^{j} \left(\left\| 1\!\!1_{\Gamma_{\Lambda_{r-1}}}\left(S_{\Lambda_r} - S\right)\right\|_{C_R} + \left\| 1\!\!1_{\Gamma_{\Lambda_{r-1}}}\left(S_{\Lambda_n} - S_{\Lambda_r}\right)\right\|_{C_R}\right).
\end{aligned}$$

Next, for any $Y_1, Y_2 \in \mathcal{B}(\mathbb{R}^d)$ such that $\Lambda_0 \subset Y_1 \subset Y_2$ and for any $F \in E_{C_R}$

$$\begin{aligned}
&\left\| 1\!\!1_{\Gamma_{\Lambda_0}} (S_{Y_1} - S_{Y_2}) F \right\|_{C_R} \\
&= \operatorname*{ess\,sup}_{\eta \in \Gamma_0} \left| 1\!\!1_{\Gamma_{\Lambda_0}} (\eta) \left((S_{Y_1} - S_{Y_2}) F \right) (\eta) \left(\frac{1}{C_R} \right)^{|\eta|} \right| \\
&= \operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}} \left| \left((S_{Y_1} - S_{Y_2}) F \right) (\eta) \left(\frac{1}{C_R} \right)^{|\eta|} \right| \\
&= \operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}} \left| \sum_{x \in \eta} \varkappa(\eta, x) \int_{\Gamma_0} \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x) \right)(\xi) F(\xi \cup \eta \setminus \{x\}) \right. \\
&\quad \left. \times \left(1\!\!1_{\Gamma_{Y_1}} (\xi \cup \eta \setminus \{x\}) - 1\!\!1_{\Gamma_{Y_2}} (\xi \cup \eta \setminus \{x\}) \right) d\lambda_\sigma(\xi) \left(\frac{1}{C_R} \right)^{|\eta|} \right| \\
&\le \operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}} \sum_{x \in \eta} \varkappa(\eta, x) \int_{\Gamma_0} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x) \right)(\xi) \right| \left| F(\xi \cup \eta \setminus \{x\}) \right| \left(\frac{1}{C_R} \right)^{|\eta|+|\xi|-1} \\
&\quad \times (C_R)^{|\eta|+|\xi|-1} \left| 1\!\!1_{\Gamma_{Y_1}} (\xi \cup \eta \setminus \{x\}) - 1\!\!1_{\Gamma_{Y_2}} (\xi \cup \eta \setminus \{x\}) \right| d\lambda_\sigma(\xi) \left(\frac{1}{C_R} \right)^{|\eta|} \\
&\le \frac{\|F\|_{C_R}}{C_R} \operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}} \sum_{x \in \eta} \varkappa(\eta, x) \\
&\quad \times \int_{\Gamma_0} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x) \right)(\xi) \right| \left| 1\!\!1_{\Gamma_{Y_1}} (\xi) - 1\!\!1_{\Gamma_{Y_2}} (\xi) \right| (C_R)^{|\xi|} d\lambda_\sigma(\xi) \\
&= \frac{\|F\|_{C_R}}{C_R} \operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}} \sum_{x \in \eta} \varkappa(\eta, x) \\
&\quad \times \int_{\Gamma_0} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x) \right)(\xi) \right| 1\!\!1_{\Gamma_{Y_2} \setminus \Gamma_{Y_1}} (\xi) (C_R)^{|\xi|} d\lambda_\sigma(\xi) \\
&= \frac{\|F\|_{C_R}}{C_R} \operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}} \sum_{x \in \eta} \varkappa(\eta, x) \int_{\Gamma_{0,Y_2} \setminus \Gamma_{0,Y_1}} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x) \right)(\xi) \right| (C_R)^{|\xi|} d\lambda_\sigma(\xi),
\end{aligned}$$

since for any $\xi \in \Gamma_0$

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}, x \in \eta} \left| F(\xi \cup \eta \setminus \{x\}) \right| \left(\frac{1}{C_R} \right)^{|\eta|+|\xi|-1} \left| 1\!\!1_{\Gamma_{Y_1}} (\xi \cup \eta \setminus \{x\}) - 1\!\!1_{\Gamma_{Y_2}} (\xi \cup \eta \setminus \{x\}) \right| \\
&\quad \le \operatorname*{ess\,sup}_{\beta \in \Gamma_0} |F(\beta)| \left(\frac{1}{C_R} \right)^{|\beta|} \left| 1\!\!1_{\Gamma_{Y_1}} (\xi) - 1\!\!1_{\Gamma_{Y_2}} (\xi) \right| = \|F\|_{C_R} \left| 1\!\!1_{\Gamma_{Y_1}} (\xi) - 1\!\!1_{\Gamma_{Y_2}} (\xi) \right|.
\end{aligned}$$

As a result,

$$\begin{aligned}
(7.6) \quad &\left\| 1\!\!1_{\Gamma_{\Lambda_0}} (S_{Y_1} - S_{Y_2}) \right\|_{C_R} \\
&\le \frac{1}{C_R} \operatorname*{ess\,sup}_{\eta \in \Gamma_{\Lambda_0}} \sum_{x \in \eta} \varkappa(\eta, x) \int_{\Gamma_{0,Y_2} \setminus \Gamma_{0,Y_1}} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x) \right)(\xi) \right| (C_R)^{|\xi|} d\lambda_\sigma(\xi).
\end{aligned}$$

Let $\varepsilon > 0$ be given. Let us choose l such that

$$\frac{2 \|S\|_{C_R}^{l+1}}{1 - \|S\|_{C_R}} < \frac{\varepsilon}{2}.$$

Then, due to (7.4),

$$|\rho(\eta) - \rho_{\Lambda_n}(\eta)| \frac{1}{(C_R)^{|\eta|}} < \frac{\varepsilon}{2} + \sum_{j=0}^{l} \left\| 1\!\!1_{\Gamma_{\Lambda_0}} \left(S^j_{\Lambda_n} - S^j \right) \right\|_{C_R}. \tag{7.7}$$

And now we construct a sequence $(n_r)_{r\in\mathbb{N}_0}$ in the following way. Let

$$\varepsilon_1 = \varepsilon \frac{C_R}{2l(l+1) B_{mis} b_{umis}(C_R) \|S\|_{C_R}^{j-1}}$$

Take $n_0 = 0$. Assume n_{r-1} is already chosen, then according to (UMIS) we can choose n_r such that for σ-a.a. $x \in \Gamma_{\Lambda_{n_{r-1}}}$ and for λ_σ-a.a. $\eta \in \Gamma_0$

$$\int_{\Gamma_0 \setminus \Gamma_{\Lambda_{n_r}}} \left| \left(K_0^{-1} r(\cdot \cup \eta, x) \right)(\xi) \right| c^{|\xi|} d\lambda_\sigma(\xi) \le \varepsilon_1 p(\eta, x) b_{umis}(c).$$

Let $n \ge n_l \ge n_j$ then due to (7.5) for the sequence $(\Lambda_n, \Lambda_{n_r})_{r\in\mathbb{N}}$ and (7.6)

$$\begin{aligned}
&\left\| 1\!\!1_{\Gamma_{\Lambda_0}} \left(S^j_{\Lambda_n} - S^j \right) \right\|_{C_R} \\
&\le \|S\|_{C_R}^{j-1} \sum_{r=1}^{j} \left(\left\| 1\!\!1_{\Gamma_{\Lambda_{n_{r-1}}}} \left(S_{\Lambda_{n_r}} - S \right) \right\|_{C_R} + \left\| 1\!\!1_{\Gamma_{\Lambda_{n_{r-1}}}} \left(S_{\Lambda_n} - S_{\Lambda_{n_r}} \right) \right\|_{C_R} \right) \\
&\le \|S\|_{C_R}^{j-1} \frac{2}{C_R} \\
&\quad \times \sum_{r=1}^{j} \left(\operatorname*{ess\,sup}_{\eta\in\Gamma_{\Lambda_{n_{r-1}}}} \sum_{x\in\eta} \varkappa(\eta, x) \int_{\Gamma_0 \setminus \Gamma_{0,\Lambda_{n_r}}} \left| \left(K_0^{-1} r(\cdot \cup \eta \setminus \{x\}, x) \right)(\xi) \right| (C_R)^{|\xi|} d\lambda_\sigma(\xi) \right) \\
&\le \|S\|_{C_R}^{j-1} \frac{2}{C_R} \sum_{r=1}^{j} \left(\operatorname*{ess\,sup}_{\eta\in\Gamma_{\Lambda_{n_{r-1}}}} \sum_{x\in\eta} \varkappa(\eta, x) p(\eta \setminus \{x\}, x) b_{umis}(C_R) \varepsilon_1 \right) \\
&< \|S\|_{C_R}^{j-1} \frac{2}{C_R} B_{mis} b_{umis}(C_R) j \varepsilon_1 \\
&< \frac{2}{C_R} \|S\|_{C_R}^{j-1} B_{mis} b_{umis}(C_R) j \varepsilon_1.
\end{aligned}$$

Therefore,

$$\sum_{j=0}^{l} \left\| 1\!\!1_{\Gamma_{\Lambda_0}} \left(S^j_{\Lambda_n} - S^j \right) \right\|_{C_R} < \frac{2}{C_R} B_{mis} b_{umis}(C_R) \frac{l(l+1)}{2} \varepsilon_1.$$

And using (7.7), one has

$$|\rho(\eta) - \rho_{\Lambda_n}(\eta)| \frac{1}{(C_R)^{|\eta|}} < \frac{\varepsilon}{2} + \frac{l(l+1) B_{mis} b_{umis}(C_R) \|S\|_{C_R}^{j-1}}{C_R} \varepsilon_1 < \varepsilon.$$

This yields the required result. □

Bibliography

1. V. Belitsky and E. A. Pechersky, *Uniqueness of Gibbs state for non-ideal gas in $\mathbb{R}^d$: the case of multibody interaction*, J. Statist. Phys. **106** (2002), no. 5-6, 931–955.
2. R. L. Dobrushin, *Gibbsian random fields for particles without hard core*, Theor. I Math. Fyz. **4** (1970), 101–118.
3. R. L. Dobrushin, *Prescribing a system of random variables by conditional distributions*, Theor. Veroat. Primen. **15** (1970), 458–486 .
4. R. L. Dobrushin and E. A. Pechersky, *A criterion of the uniqueness of Gibbs field in the non-compact case*, Lecture Notes in Mathematics **70** (1982), no. 1021, 97–110.

5. M. E. Fisher and D. Ruelle, *The stability of many-particle systems* , J. of Math. Phys. **7** (1966), no. 2, 260–270.
6. H.-O. Georgii and T. Kuneth, *Stochastic comparison of point random fields*, J. Appl. Probab. **34** (1997), no. 4, 868–881.
7. E. Glötz, *Bemerkungen zu einer Arbeit von O. K. Kozlov* , Math. Nachr. **94** (1980), 277–289.
8. E. Glötz, *Lokale Energien und Potentiale für Punktprozesse*, Math. Nachr. **96** (1980), 195–206.
9. O. Kallenberg, *Random Measures*, second edition, Academic Press, London, New York, 1976.
10. O. Kallenberg, *On conditional inensities of point processes*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete **41** (1978), 205–220.
11. J. Kerstan, K. Matthes, and J. Mecke, *Infinite Divisible Point Processes*, Akademie-Verlag, Berlin, 1978.
12. Yu. G. Kondratiev and T. Kuna, *Harmonic analysis on configuration space I. General theory*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **5** (2002), no. 2, 201–233.
13. Yu. G. Kondratiev, O. V. Kutoviy, and E. A. Pechersky, *Existence of Gibbs state for non-ideal gas in $\mathbb{R}^d$: the case of pair, long-range interaction*, In preparation (2003).
14. T. Kuna, *Studies in configuration space analysis and applications*, Bonner Mathematische Schriften, Uni. Bonn. PhD-Thesis **324** (1999).
15. A. Lenard, *Correlation functions and the uniqueness of the state in classical statistical mechanics*, Comm. Math. Phys. **30** (1973), 35–44.
16. K. Matthes, J. Mecke, and W. Warmuth, *Bemerkungen zu einer Arbeit von Nguyen Xuan Xanh und Hans Zessin*, Math. Nachr. **88** (1979), 117–127.
17. J. Mecke, *Stationäre zufällige Maße auf lokalkompakten Abelschen Gruppen*, Z. Wahrsch. verw. Gebiete **9** (1967), 36–58.
18. X. X. Nguyen and H. Zessin, *Integral and differential characterizations of the Gibbs process*, Math. Nachr. **88** (1979), 105–115.
19. F. Papangelou, *The conditional intensity of general point processes and an application to line processes*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete **28** (1974), 207–226.
20. E. A. Pechersky and Yu. Zhukov, *Uniqueness of Gibbs state for non-ideal gas in $\mathbb{R}^d$: The case of pair potentials*, J. Statistical Physics **97** (1999), 145–172.
21. Yu. G. Kondratiev and T. Kuna, *Correlation functionals for Gibbs measures and Ruelle bounds*, Meth. of Funct. Anal. and Topol. **9** (1999), no. 1, 9–59.
22. M. Röckner, *Stochastic analysis on configuration spaces: Basic ideas and recent results*, In J. Jost, W. Kendall, U. Mosco, M. Röckner, and K.-T. Sturm, editors, New Directions in Dirichlet Forms, volume 8 of Studies in Advanced Mathematics, American Mathematical Society, International Press (1998), 157–232.
23. D. Ruelle, *Superstable interactions in classical statistical mechanics*, Comm. Math. Phys. **18** (1970), 127–159.
24. A. Wakolbinger and G. Eder, *A condition Σ_λ^c for point processes*, Math. Nachr. **116** (1984), 209–232.
25. S. Albeverio, Yu. G. Kondratiev, M. Röckner, *Analysis and geometry on configuration spaces*, J. Func. Anal. **154** (1998), 444–500.
26. S. Albeverio, Yu. G. Kondratiev, M. Röckner, *Analysis and geometry on configuration spaces: The Gibbsian case*, J. Func. Anal. **157** (1998), 242–291.

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