Second Order Differential Operators on the Configuration Spaces

Dmitri L. Finkelshtein †

† *Institute of Mathematics of NASU, Kiev, Ukraine*
E-mail: *fdl@ln.ua*

The procedure of construction of a big class of operators on configuration spaces is presented. Second order differential operators are studied on configuration space over domain with boundary. Symmetric extensions of the minimal operator are discovered.

1 Introduction

The aim of the present article is the construction of a big class of operators on configuration spaces over whole underlying space or only domain of them and discovery conditions under which this operators will be symmetric. Our underlying space is the usuall $\mathbb{R}^d$ with Borel σ-algebra $\mathcal{B}\left(\mathbb{R}^d\right)$ and the fixed measure σ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$, such that

$$d\sigma\left(x\right) = \rho\left(x\right) dx,$$

where $\rho > 0$ a.a., $\rho \in L^1_{loc}\left(\mathbb{R}^d, dx\right)$, $\rho^{1/2} \in W^{1,2}_{loc}\left(\mathbb{R}^d, dx\right)$.

Definition 1. Configuration space $\Gamma = \Gamma_{\mathbb{R}^d}$ is the space of all locally finite subsets (configurations) of $\mathbb{R}^d$:

$$\Gamma = \left\{ \gamma \subset \mathbb{R}^d \,\middle|\, |\gamma_\Lambda| < +\infty \text{ for any compact } \Lambda \subset \mathbb{R}^d \right\},$$

where $|\cdot|$ means the cardinality of a set and

$$\gamma_\Lambda = \gamma \cap \Lambda.$$

Let us define the σ-algebra $\mathcal{B}\left(\Gamma\right)$ as the minimal σ-algebra such that all mappings

$$\Gamma \ni \gamma \longmapsto |\gamma_\Lambda|$$

are $\mathcal{B}\left(\Gamma\right)$-measurable for any $\Lambda \in \mathcal{B}_c\left(\mathbb{R}^d\right)$, where $\mathcal{B}_c\left(\mathbb{R}^d\right)$ is the family of all Borel subsets of $\mathbb{R}^d$ with compact closure.

The space Γ can be naturally embedded into the space $\mathcal{M}\left(\mathbb{R}^d\right)$ of all measures on $\mathbb{R}^d$ in the following way

$$\gamma \in \Gamma \leftrightarrow \sum_{x\in\gamma} \varepsilon_x \in \mathcal{M}\left(\mathbb{R}^d\right)$$

where ε_x is a Dirac measure at the point x. Thus Γ can be endowed with topology generated by the weak topology on $\mathcal{M}\left(\mathbb{R}^d\right)$. Moreover, it can be shown, that the σ-algebra $\mathcal{B}\left(\Gamma\right)$ is really the Borel σ-algebra with respect to this topology.

For $f \in C_0\left(\mathbb{R}^d\right)$ we can define a pairing between function and configuration:

$$\langle f, \gamma\rangle := \sum_{x\in\gamma} f\left(x\right).$$

This definition is correct since the sum in the r.h.s. is finite due to definition of configurations.

Consider also the class of cylindrical smooth functions: $F \in \mathcal{F}C_b^\infty(\Gamma, \mathcal{D})$ iff

$$F(\gamma) := g_F(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle), \tag{1}$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D} := C_0^\infty(\mathbb{R}^d)$, $g_F \in C_b^\infty(\mathbb{R}^N)$ (the space of all infinitely differentiable functions on $\mathbb{R}^N$ which are bounded together with all its derivatives).

Note that, if $f \in C_0(\mathbb{R}^d)$ then for any $\gamma \in \Gamma$

$$\langle f, \gamma + \varepsilon.\rangle - \langle f, \gamma\rangle \in C_0\left(\mathbb{R}^d\right).$$

Analogously, if $F \in \mathcal{F}C_b^\infty(\Gamma, \mathcal{D})$ then for any $\gamma \in \Gamma$

$$F(\gamma + \varepsilon.) - F(\gamma) \in \mathcal{D}.$$

Consider also the space Γ_Λ of the all configurations of $\Lambda \in \mathcal{B}(\mathbb{R}^d)$:

$$\Gamma_\Lambda = \left\{\gamma \subset \mathbb{R}^d \,|\gamma \cap \Lambda^c = \emptyset\right\}, \tag{2}$$

where $\Lambda^c := \mathbb{R}^d \setminus \Lambda$.

2 Measures on the configuration spaces

Let us consider a class $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ of the probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ which have all finite locale moments, it means that

$$\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \Leftrightarrow \int_\Gamma |\gamma_\Lambda|^n \, d\mu(\gamma) < +\infty \tag{3}$$

for any $n \in \mathbb{N}$ and for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$.

We start from the non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function $r : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$. We suppose that $r(\gamma, x)$ is defined for μ-a.a. $\gamma \in \Gamma$ and a.a. $x \in \mathbb{R}^d$ (note that we always assume that $x \notin \gamma$).

Definition 2. The measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called the Gibbs measure corresponding to r if for any non-negative $\mathcal{B}(\Gamma) \times \mathcal{B}(\mathbb{R}^d)$-measurable function $h : \Gamma \times \mathbb{R}^d \to \mathbb{R}_+$ the following Campbell-Mecke identity holds:

$$\int_\Gamma \sum_{x\in\gamma} h(\gamma, x)\, d\mu(\gamma) = \int_\Gamma \int_{\mathbb{R}^d} h(\gamma + \varepsilon_x, x)\, r(\gamma, x)\, d\sigma(x)\, d\mu(\gamma). \tag{CM}$$

For examples of the such measures we start from the case when $r \equiv 1$. Mecke [3] proved that there exists only one such measure μ for given Radon measure σ on $\mathbb{R}^d$. This measure is called the Poisson measure with intensity σ and denotes by π_σ. There exists a direct construction of the Poisson measure. For explore it we start from the space Γ_Λ, where $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$. Clearly,

$$\Gamma_\Lambda := \bigsqcup_{n\in\mathbb{N}_0} \Gamma_\Lambda^{(n)},$$

where $\Gamma_\Lambda^{(n)}$ is the set of all n-particle configurations (subsets) of Λ, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. There is a bijection $\tilde{\Lambda}^n / S_n \to \Gamma_\Lambda^{(n)}$, where $\tilde{\Lambda}^n := \{(x_1, \ldots, x_n) \in \Lambda^n \,|x_k \neq x_j, k \neq j\}$, and S_n is the permutation group over $\{1, \ldots, n\}$. Therefore we can consider on $\Gamma_\Lambda^{(n)}$ the image $\sigma^{(n)}$ of the product measure σ^n under this bijection. Consider also a σ-algebra $\mathcal{B}(\Gamma_\Lambda)$ as the minimal σ-algebra such that all mappings

$$\Gamma_\Lambda \ni \gamma \longmapsto |\gamma_{\Lambda'}| \tag{4}$$

are $\mathcal{B}(\Gamma_\Lambda)$-measurable for any $\Lambda' \in \mathcal{B}_c(\Lambda)$. Then the Poisson measure π_σ^Λ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ is defined as

$$\pi_\sigma^\Lambda := e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{(n)}. \tag{5}$$

It can be shown that the measurable space $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ and that the family of measures $\{\pi_\sigma^\Lambda\}_{\Lambda\in\mathcal{B}_c(X)}$ is consistent. Therefore one has define the Poisson measure π_σ on $(\Gamma, \mathcal{B}(\Gamma))$ as the projective limit of this family.

Another examples were founded by Nguen and Zessin [4]. They shown that a big class of Gibbs measures constructed by the quite general potential are satisfied to the Campbell-Mecke identity. More precisely, let Φ be a potential, i.e., a measurable function $\Phi : \Gamma_0 \to \mathbb{R} \cup \{+\infty\}$, such that $\Phi(\emptyset) = 0$. Define for any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ the conditional energy $E_\Lambda^\Phi : \Gamma \to \mathbb{R} \cup \{+\infty\}$ such that $E_\Lambda^\Phi(\gamma) = \sum_{\eta\Subset\gamma, |\eta\cap\Lambda|>0} \Phi(\eta)$ if $\sum_{\eta\Subset\gamma,|\eta\cap\Lambda|>0} |\Phi(\eta)| < \infty$ and $E_\Lambda^\Phi(\gamma) = +\infty$ otherwise (the notation $\eta \Subset \gamma$ means that η is a finite subset of γ). Then for fixed $\beta > 0$ we define for $\gamma \in \Gamma, \Delta \in \mathcal{B}(\Gamma)$ a specification

$$\Pi_\Lambda^{\sigma,\beta,\Phi}(\gamma, \Delta) = \frac{1_{\left\{Z_\Lambda^{\sigma,\beta,\Phi}(\gamma)<+\infty\right\}}}{Z_\Lambda^{\sigma,\beta,\Phi}(\gamma)} \int_\Gamma 1_\Delta\left(\gamma_{\Lambda^c} \cup \gamma'_\Lambda\right) e^{-\beta E_\Lambda^\Phi\left(\gamma_{\Lambda^c}\cup\gamma'_\Lambda\right)} d\pi_\sigma\left(\gamma'\right), \tag{6}$$

where $Z_\Lambda^{\sigma,\beta,\Phi}(\gamma) = \int_\Gamma e^{-\beta E_\Lambda^\Phi\left(\gamma_{\Lambda^c}\cup\gamma'_\Lambda\right)} d\pi_\sigma(\gamma')$. A measure $\mu \in \mathcal{M}^1(\Gamma)$ is called the grand canonical Gibbs measure with interaction potential Φ iff for all $\Lambda \in \mathcal{F}_c(\mathbb{R}^d)$ and for all $\Delta \in \mathcal{B}(\Gamma)$ the following Dobrushin-Lanford-Ruelle identity holds

$$\mu(\Delta) = \int_\Gamma \Pi_\Lambda^{\sigma,\beta,\Phi}(\gamma, \Delta)\, d\mu(\gamma). \tag{7}$$

The set of all such probability measures μ will be denoted by $\mathcal{G}_{gc}(\sigma, \beta\Phi)$.

Therefore, let $\mu \in \mathcal{G}_{gc}(p\,dx, \beta\Phi)$ and let μ has the local first moment (i.e., (3) is true for $n = 1$). Then μ satisfies the Campbell-Mecke identity with

$$r(\gamma, x) = \exp\left(-\beta E_{\{x\}}^\Phi(\gamma + \varepsilon_x)\right). \tag{8}$$

Let us recall that

$$E_{\{x\}}^\Phi(\gamma + \varepsilon_x) = \begin{cases} \sum_{\{x\}\subset\eta\Subset\gamma\cup\{x\}} \Phi(\eta), & \text{if } \sum_{\{x\}\subset\eta\Subset\gamma\cup\{x\}} |\Phi(\eta)| < +\infty \\ +\infty, & \text{otherwise} \end{cases}. \tag{9}$$

Of course, all our considerations will be true in the general situation with function r, but with some additional conditions on it. Thus, if we will want to use our results for examples above, we have to check such conditions for corresponding r.

3 Construction of operators

First of all, for μ-a.a. $\gamma \in \Gamma$ let us consider a measure on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$:

$$d\sigma_\gamma(x) = r(\gamma, x)\, d\sigma(x).$$

Therefore, we have a family of spaces $\left\{L^2\left(\mathbb{R}^d,\sigma_\gamma\right)\right\}_{\gamma\in\Gamma}$ (without loss of generality we assume that this family is indexed by all space Γ).

Suppose that we have the family of operators $\{B(\gamma)\}_{\gamma\in\Gamma}$ in the corresponding spaces $\left\{L^2\left(\mathbb{R}^d,\sigma_\gamma\right)\right\}_{\gamma\in\Gamma}$ such that

$$\mathcal{D}\subset \mathrm{Dom}\left(B\left(\gamma\right)\right),\quad B\left(\gamma\right)1\in L^2\left(\mathbb{R}^d,\sigma_\gamma\right)\cap L^1\left(\mathbb{R}^d,\sigma_\gamma\right). \tag{10}$$

Suppose also that for $F\in\mathcal{F}C_b^\infty\left(\Gamma,\mathcal{D}\right)$ there are functions $F_k\in\mathcal{F}C_b^\infty\left(\Gamma,\mathcal{D}\right)$, $\psi_k:\Gamma\times\mathbb{R}^d\to\mathbb{R}, k=1,\ldots,n$ such that $\psi_k\left(\gamma',\cdot\right)\in L^2\left(\mathbb{R}^d,\sigma_\gamma\right)\cap L^1\left(\mathbb{R}^d,\sigma_\gamma\right)$ for μ-a.a. $\gamma,\gamma'\in\Gamma$ and for $x\in\gamma$ if we consider $F\left(\gamma\right)$ as function of x then

$$B\left(\gamma'\right)_x F\left(\gamma\right)=\sum_{k=1}^n F_k\left(\gamma\right)\psi_k\left(\gamma',x\right).$$

Then we can consider the bilinear form $\mathcal{E}$ on $\mathcal{F}C_b^\infty\left(\Gamma,\mathcal{D}\right)$ generated by the family $\{B(\gamma)\}_{\gamma\in\Gamma}$:

$$\mathcal{E}_\mu\left(F,G\right)=\int_\Gamma\int_{\mathbb{R}^d}B\left(\gamma'\right)_x\left(F\left(\gamma+\varepsilon_x\right)-F\left(\gamma\right)\right)\Big|_{\gamma'=\gamma}\cdot\left(G\left(\gamma+\varepsilon_x\right)-G\left(\gamma\right)\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)$$

for $F,G\in\mathcal{F}C_b^\infty\left(\Gamma,\mathcal{D}\right)$.

Note that for $x\notin\gamma$ one has

$$B\left(\gamma'\right)_x F\left(\gamma\right)=F\left(\gamma\right)\left(B\left(\gamma'\right)1\right)\left(x\right), \tag{11}$$

since then $F\left(\gamma\right)$ is a constant as a function of x.

Proposition 1. *For any $F,G\in\mathcal{F}C_b^\infty\left(\Gamma,\mathcal{D}\right)$ the following equality holds*

$$\mathcal{E}_\mu\left(F,G\right)=\int_\Gamma H_\mu F\left(\gamma\right)\cdot G\left(\gamma\right)d\mu\left(\gamma\right),$$

where

$$\begin{aligned}H_\mu F\left(\gamma\right) &= \sum_{x\in\gamma}B\left(\gamma'\right)_x F\left(\gamma\right)\Big|_{\gamma'=\gamma-\varepsilon_x}-\sum_{x\in\gamma}F\left(\gamma-\varepsilon_x\right)\left(B\left(\gamma-\varepsilon_x\right)1\right)\left(x\right)\\ &\quad -\int_{\mathbb{R}^d}B\left(\gamma\right)_x F\left(\gamma+\varepsilon_x\right)r\left(\gamma,x\right)d\sigma\left(x\right)+F\left(\gamma\right)\left(\int_{\mathbb{R}^d}\left(B\left(\gamma\right)1\right)\left(x\right)r\left(\gamma,x\right)d\sigma\left(x\right)\right).\end{aligned}$$

Proof.

$$\begin{aligned}&\int_\Gamma\int_{\mathbb{R}^d}B\left(\gamma\right)_x\left(F\left(\gamma+\varepsilon_x\right)-F\left(\gamma\right)\right)\cdot\left(G\left(\gamma+\varepsilon_x\right)-G\left(\gamma\right)\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)\\ =&\int_\Gamma\int_{\mathbb{R}^d}B\left(\gamma\right)_x F\left(\gamma+\varepsilon_x\right)\cdot G\left(\gamma+\varepsilon_x\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)\\ &-\int_\Gamma\int_{\mathbb{R}^d}B\left(\gamma\right)_x F\left(\gamma+\varepsilon_x\right)\cdot G\left(\gamma\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)\\ &-\int_\Gamma\int_{\mathbb{R}^d}F\left(\gamma\right)\cdot\left(B\left(\gamma\right)1\right)\left(x\right)\cdot G\left(\gamma+\varepsilon_x\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)\\ &+\int_\Gamma\int_{\mathbb{R}^d}F\left(\gamma\right)\cdot\left(B\left(\gamma\right)1\right)\left(x\right)\cdot G\left(\gamma\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)\\ =&\int_\Gamma\int_{\mathbb{R}^d}\sum_{k=1}^n F_k\left(\gamma+\varepsilon_x\right)\varphi_k\left(x,\gamma\right)\cdot G\left(\gamma+\varepsilon_x\right)r\left(\gamma,x\right)d\sigma\left(x\right)d\mu\left(\gamma\right)\end{aligned}$$

$$
\begin{aligned}
&-\int_{\Gamma}\int_{\mathbb{R}^d} B(\gamma)_x F(\gamma+\varepsilon_x)\cdot G(\gamma)\, r(\gamma,x)\, d\sigma(x)\, d\mu(\gamma)\\
&-\int_{\Gamma}\int_{\mathbb{R}^d} F(\gamma)\cdot (B(\gamma)1)(x)\cdot G(\gamma+\varepsilon_x)\, r(\gamma,x)\, d\sigma(x)\, d\mu(\gamma)\\
&+\int_{\Gamma}\int_{\mathbb{R}^d} F(\gamma)\cdot (B(\gamma)1)(x)\cdot G(\gamma)\, r(\gamma,x)\, d\sigma(x)\, d\mu(\gamma)\\
=\ &\int_{\Gamma}\sum_{x\in\gamma}\left(\sum_{k=1}^{n} F_k(\gamma)\,\varphi_k(x,\gamma-\varepsilon_x)\right) G(\gamma)\, d\mu(\gamma)\\
&-\int_{\Gamma}\left(\int_{\mathbb{R}^d} B(\gamma)_x F(\gamma+\varepsilon_x)\, r(\gamma,x)\, d\sigma(x)\right) G(\gamma)\, d\mu(\gamma)\\
&-\int_{\Gamma}\left(\sum_{x\in\gamma} F(\gamma-\varepsilon_x)\,(B(\gamma-\varepsilon_x)1)(x)\right) G(\gamma)\, d\mu(\gamma)\\
&+\int_{\Gamma} F(\gamma)\, G(\gamma)\left(\int_{\mathbb{R}^d} (B(\gamma)1)(x)\, r(\gamma,x)\, d\sigma(x)\right) d\mu(\gamma).
\end{aligned}
$$

■

Therefore if $B(\gamma)$ are symmetric operators on $C_0^\infty(\mathbb{R}^d)$ in $L^2\left(\mathbb{R}^d,\sigma_\gamma\right)$ for any $\gamma\in\Gamma$, then the corresponding operator H_μ is a symmetric operator on $\mathcal{F}C_b^\infty(\Gamma,\mathcal{D})$ in $L^2(\Gamma,\mu)$.

4 Second order differential operators

In this section we generalized results obtained in [1, 2] for the Poisson measure on our general case.

Suppose that $r(\gamma,\cdot)\in C^1\left(\mathbb{R}^d\right)$ for μ-a.a. $\gamma\in\Gamma$.

Consider now instead of $B(\gamma)$ the Dirichlet operator $A(\gamma)$ corresponding to the bilinear form

$$
\mathcal{E}_{\sigma_\gamma}(f,g)=\int_{\mathbb{R}^d}\langle\mathcal{A}(\gamma,x)\nabla f(x),\nabla g(x)\rangle\, d\sigma_\gamma(x),
$$

where $\mathcal{A}(\gamma,x)=(a_{ij}(\gamma,x))_{i,j=1,\dots,d}$ and $a_{ij}(\gamma,\cdot)=a_{ji}(\gamma,\cdot)$ is a smooth functions on $\mathbb{R}^d$. Since,

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\langle\mathcal{A}(\gamma,x)\nabla f(x),\nabla g(x)\rangle\, d\sigma_\gamma(x)\\
&=\int_{\mathbb{R}^d}\langle\mathcal{A}(\gamma,x)\nabla f(x),\nabla g(x)\rangle\, r(\gamma,x)\,\rho(x)\, dx\\
&=-\int_{\mathbb{R}^d}\operatorname{div}\left((\mathcal{A}(\gamma,x)\nabla f(x))\, r(\gamma,x)\,\rho(x)\right) g(x)\, dx\\
&=-\int_{\mathbb{R}^d}\left(\operatorname{div}(\mathcal{A}(\gamma,x)\nabla f(x))\right) r(\gamma,x)\,\rho(x)\, g(x)\, dx\\
&-\int_{\mathbb{R}^d}\langle\mathcal{A}(\gamma,x)\nabla f(x),\nabla(r(\gamma,x)\,\rho(x))\rangle\, g(x)\, dx\\
&=-\int_{\mathbb{R}^d}\operatorname{div}(\mathcal{A}(\gamma,x)\nabla f(x))\, g(x)\, d\sigma_\gamma(x)\\
&-\int_{\mathbb{R}^d}\langle\mathcal{A}(\gamma,x)\nabla f(x),\nabla\ln(r(\gamma,x)\,\rho(x))\rangle\, g(x)\, d\sigma_\gamma(x),
\end{aligned}
$$

then

$$
A(\gamma)f(x)=-\operatorname{div}(\mathcal{A}(\gamma,x)\nabla f(x))-\langle\mathcal{A}(\gamma,x)\nabla f(x),\beta(\gamma,x)\rangle, \tag{12}
$$

where $\beta(\gamma, x) = \nabla \ln(r(\gamma, x)\rho(x))$.

Since $A(\gamma)1 \equiv 0$ then the operator H_μ corresponding to the family $\{A(\gamma)\}_{\gamma\in\Gamma}$ has the simple form

$$H_\mu F(\gamma) = \sum_{x\in\gamma} A(\gamma')_x F(\gamma)\big|_{\gamma'=\gamma-\varepsilon_x}.$$

Let us now study the operator H_μ in the space $L^2(\Gamma_\Lambda, \mu_\Lambda)$, where μ_Λ is the projection of the measure μ onto Γ_Λ and Λ is the regular (bounded or not) domain of $\mathbb{R}^d$ with piecewise C^1 boundary $\partial\Lambda$.

Proposition 2. *The first Green formula for the operator H_μ holds:*

$$\begin{aligned}
&\int_{\Gamma_\Lambda} H_\mu F(\gamma) G(\gamma)\, d\mu_\Lambda(\gamma) \\
&= \int_{\Gamma_\Lambda}\int_\Lambda \langle \mathcal{A}(\gamma, x)\nabla_x F(\gamma+\varepsilon_x), \nabla_x G(\gamma+\varepsilon_x)\rangle\, d\sigma_\gamma(x)\, d\mu_\Lambda(\gamma) \\
&\quad - \int_{\Gamma_\Lambda}\int_{\partial\Lambda} \frac{\partial}{\partial n_s^\gamma} F(\gamma+\varepsilon_s) G(\gamma+\varepsilon_s)\, d\tilde{\sigma}_\gamma(s)\, d\mu_\Lambda(\gamma),
\end{aligned}$$

where

$$\frac{\partial}{\partial n_s^\gamma} f(s) = \langle \mathcal{A}(\gamma, s)\nabla f(s), n_s\rangle.$$

Proof. Due to Campbell-Mecke identity one has

$$\int_{\Gamma_\Lambda} H_\mu F(\gamma) G(\gamma)\, d\mu_\Lambda(\gamma) = \int_{\Gamma_\Lambda}\int_\Lambda (A_\gamma)_x F(\gamma+\varepsilon_x) G(\gamma+\varepsilon_x)\, d\sigma_\gamma(x)\, d\mu_\Lambda(\gamma).$$

Then, write for $A(\gamma)$ the first Green formula:

$$\begin{aligned}
&\int_\Lambda A(\gamma) f(x) g(x)\, d\sigma_\gamma(x) \\
&= -\int_\Lambda \operatorname{div}(\mathcal{A}(\gamma, x)\nabla f(x))\, g(x)\, d\sigma_\gamma(x) \\
&\quad - \int_\Lambda \langle \mathcal{A}(\gamma, x)\nabla f(x), \beta(\gamma, x)\rangle\, g(x)\, d\sigma_\gamma(x) \\
&= -\int_\Lambda \operatorname{div}(\mathcal{A}(\gamma, x)\nabla f(x))\, g(x)\, r(\gamma, x)\rho(x)\, d(x) \\
&\quad - \int_\Lambda \langle \mathcal{A}(\gamma, x)\nabla f(x), \beta(\gamma, x)\rangle\, g(x)\, d\sigma_\gamma(x) \\
&= \int_\Lambda \langle \mathcal{A}(\gamma, x)\nabla f(x), \beta(\gamma, x)\rangle\, g(x)\, d\sigma_\gamma(x) \\
&\quad + \int_\Lambda \langle \mathcal{A}(\gamma, x)\nabla f(x), \nabla g(x)\rangle\, d\sigma_\gamma(x) \\
&- \int_{\partial\Lambda} \langle \mathcal{A}(\gamma, s)\nabla f(s), n_s\rangle\, g(s)\, r(\gamma, s)\rho(s)\, dS(s) \\
&\quad - \int_\Lambda \langle \mathcal{A}(\gamma, x)\nabla f(x), \beta(\gamma, x)\rangle\, g(x)\, d\sigma_\gamma(x) \\
&= \int_\Lambda \langle \mathcal{A}(\gamma, x)\nabla f(x), \nabla g(x)\rangle\, d\sigma_\gamma(x) \\
&- \int_{\partial\Lambda} \langle \mathcal{A}(\gamma, s)\nabla f(s), n_s\rangle\, g(s)\, d\tilde{\sigma}_\gamma(s),
\end{aligned}$$

where $d\tilde{\sigma}_\gamma(s) = r(\gamma, s)\rho(s)\, dS(s)$ for $s \in \partial\Lambda$. As a result, we obtain the statement. ■

Corollary 1. *The second Green formula for the operator H_μ has the following form*

$$\int_{\Gamma_\Lambda} \left(H_\mu F(\gamma) G(\gamma) - F(\gamma) H_\mu G(\gamma)\right) d\mu_\Lambda(\gamma)$$
$$= \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \left(\frac{\partial}{\partial n_s^\gamma} G(\gamma+\varepsilon_s) \cdot F(\gamma+\varepsilon_s) - \frac{\partial}{\partial n_s^\gamma} F(\gamma+\varepsilon_s) \cdot G(\gamma+\varepsilon_s) \right) d\tilde{\sigma}_\gamma(s)\, d\mu_\Lambda(\gamma).$$

Remark 1. Note that for any function $F \in \mathcal{F}C_b^\infty(\Gamma, \mathcal{D})$ in the form (1)

$$\frac{\partial}{\partial n_s^\gamma} F(\gamma+\varepsilon_s) = \sum_{k=1}^{N} \hat{F}_k(\gamma+\varepsilon_s) \frac{\partial}{\partial n_s^\gamma} \varphi_k(s),$$

where

$$\hat{F}_k(\gamma) := \frac{\partial g_F}{\partial q_k} \left(\langle \varphi_1, \gamma\rangle, ..., \langle \varphi_N, \gamma\rangle\right), 1 \le k \le N.$$

Let us consider now the minimal operator

$$H_{\min} := \left(H_\mu, \mathcal{F}C_b^\infty(\Gamma_\Lambda, \mathcal{D}(\Lambda))\right),$$

which is symmetric in $L^2(\Gamma_\Lambda, \mu_\Lambda)$ (there $\mathcal{D}(\Lambda) = C_0^\infty(\Lambda)$). We define the maximal operator by the standard way:

$$H_{\max} = (H_{\min})^*$$

Proposition 3. $\mathcal{F}C_b^\infty(\Gamma_\Lambda, \mathcal{D}) \subset \mathrm{Dom}(H_{\max})$ *and for any* $G \in C_b^\infty(\Gamma_\Lambda, \mathcal{D})$

$$H_{\max} G(\gamma) = H_\mu G(\gamma) + \int_{\partial\Lambda} \frac{\partial}{\partial n_s^\gamma} G(\gamma+\varepsilon_s)\, d\tilde{\sigma}_\gamma(s)$$

Proof. It follows directly from the second Green formula that for any $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma_\Lambda)$ and $F \in \mathcal{F}C_b^\infty(\mathcal{D}(\Lambda), \Gamma_\Lambda)$

$$\int_{\Gamma_\Lambda} \big((H_\mu G)(\gamma) F(\gamma) - G(\gamma)(H_\mu F)(\gamma)\big)\, d\mu_\Lambda(\gamma) = -\int_{\Gamma_\Lambda} \int_{\partial\Lambda} F(\gamma) \frac{\partial}{\partial n_s^\gamma} G(\gamma+\varepsilon_s)\, d\tilde{\sigma}_\gamma(s)\, d\mu_\Lambda(\gamma).$$

■

In the case when $\mathcal{A}$ is not depend on γ: $\mathcal{A}(\gamma, x) \equiv \mathcal{A}(x)$ one has that

$$\frac{\partial}{\partial n_s^\gamma} f(s) \equiv \frac{\partial}{\partial \nu_s} f(s) = \langle \mathcal{A}(s) \nabla f(s), n_s\rangle$$

is the usual co-normal derivative.

Thus if we define the following set of functions, which satisfied Neumann-type boundary conditions on $\partial\Lambda$:

$$\mathcal{D}_{\mathcal{N}}(\Lambda) = \left\{ f \in \mathcal{D} \,\middle|\, \frac{\partial}{\partial \nu_s} f(s) = 0, s \in \partial\Lambda \right\},$$

then the operator $(H_\mu, \mathcal{F}C_b^\infty(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}}(\Lambda)))$ will be symmetric in $L^2(\Gamma_\Lambda, \mu_\Lambda)$.

Acknowledgements

I would like to thank the Prof. Yu. G. Kondratiev and Dr. A. Yu. Konstantinov for fruitful discussions.

[1] Finkelshtein D. L., Kondratiev Yu. G., Röckner M. and Konstantinov A. Yu. Gauss formula and symmetric extensions of the laplacian on configuration spaces. *Infinite Dimensional Analysis, Quantum Probabilities and Related Topics*, 2001, V.4, N 4, 489–509.

[2] Finkelshtein D. L., Kondratiev Yu. G., Röckner M. and Konstantinov A. Yu. Symmetric differential operators of the second order in poisson spaces. *Methods of Functional Analysis and Topology*, 2000, V.6, N 4, 14–25.

[3] Mecke J. Stationäre zufällige maße auf lokalkompakten abelschen gruppen. *Z. Wahrsch. verw. Gebiete*, 1967, N 9, 36–58.

[4] Nguyen X. X. and Zessin H. Integral and differential characterizations of the Gibbs process. *Math. Nachr.*, 1979, N 88, 105–115.