SPECTRAL GAP INEQUALITIES ON CONFIGURATION SPACES

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Abstract. In the first part we consider the Laplace operator with Neumann boundary conditions on a configuration space with Poisson measure over a bounded domain. The spectrum of this operator is considered and a structure of its vacuum space is studied. The corresponding spectral gap inequality is proved. The differences between Poincaré and spectral gap inequalities are shown, and absence of Poincaré inequality is presented. In the second part we study a second order differential operator with grown coefficients on a whole configuration space. The main properties of this operator are considered and Poncaré inequality is proved.

1. A spectral gap inequality on Γ_Λ

Let Λ be a bounded domain in $\mathbb{R}^d$ which satisfies the following conditions:

(1) For any smooth vector field w on $\bar{\Lambda}$ the Gauss formula holds:

$$\int_\Lambda (\operatorname{div} w)(x)\,dx = \int_{\partial\Lambda} (w(s), \nu_s)\,dS,$$

where ν is the outer normal to $\partial\Lambda$ at the point s;

(2) For any smooth function f on $\bar{\Lambda}$ the Poincaré inequality holds:

$$\int_\Lambda \left(f(x) - \frac{1}{m(\Lambda)}\int_\Lambda f(y)\,dy\right)^2 dx = \int_\Lambda f^2(x)\,dx - \frac{1}{m(\Lambda)}\left(\int_\Lambda f(x)\,dx\right)^2$$
$$\leq C\int_\Lambda |\nabla f|^2(x)\,dx,$$

where ∇ is a usual gradient on $\mathbb{R}^d$;

(3) Let $\mathcal{D}_{\mathcal{N}(\Lambda)}$ be a set of functions on $\bar{\Lambda}$ that satisfied Neumann boundary condition on boundary $\partial\Lambda$ of Λ, then an operator $H = \left(-\Delta, \mathcal{D}_{\mathcal{N}(\Lambda)}\right)$ is essentially self-adjoint in $L^2(\Lambda, dx)$ (Δ is a usual Laplace operator on $\mathbb{R}^d$).

The simple example of such domain is a ball or a cube.

Since $\operatorname{Ker} H = \{c \in \mathbb{R}\}$, then $\operatorname{Pr}_{\operatorname{Ker} H} f = \frac{1}{m(\Lambda)}\int_\Lambda f(x)dx$, and so, if $f \in \mathcal{D}_{\mathcal{N}(\Lambda)}$ then because of equality $\int_\Lambda |\nabla f|^2(x)\,dx = \int_\Lambda Hf(x)\cdot f(x)dx$ the Poincaré inequality may be written in a form of a "spectral gap inequality": for $f \in \mathcal{D}_{\mathcal{N}(\Lambda)}$

$$\int_\Lambda \left(f(x) - \operatorname{Pr}_{\operatorname{Ker} H} f(x)\right)^2 dx \leq C\int_\Lambda Hf(x)\cdot f(x)dx$$
$$= C\int_\Lambda H\left(f(x) - \operatorname{Pr}_{\operatorname{Ker} H} f(x)\right)(x)\cdot\left(f(x) - \operatorname{Pr}_{\operatorname{Ker} H} f(x)\right)dx,$$

it means that on the set $\mathcal{D}_{\mathcal{N}(\Lambda)} \bigcap (\operatorname{Ker} H)^\perp$ the operator H is positive: $H \geq \frac{1}{C} > 0$.

Let us consider a space of configurations (finite subsets) of Λ: $\Gamma_\Lambda = \{\gamma \subset \Lambda \,|\, |\gamma| < \infty\}$. Any configuration can be identified with a Radon measure on $\mathbb{R}^d$: $\gamma = \sum_{x\in\gamma}\varepsilon_x$, which gives a possibility

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to endow the configuration space with the relative topology as the subset of the space $\mathcal{D}'$ with vague topology, i.e., the weakest topology such that all maps $\gamma \mapsto \langle \varphi, \gamma \rangle = \sum_{x\in\gamma} \varphi(x)$, $\varphi \in \mathcal{D}$ are continuous $\left(\mathcal{D} = C_0^\infty\left(\mathbb{R}^d\right)\right)$.

Clearly, $\Gamma_\Lambda = \bigsqcup_{n=0}^\infty \Gamma_\Lambda^{(n)}$, where $\Gamma_\Lambda^{(n)} = \{\gamma \subset \Lambda \,|\, |\gamma| = n\} = \widetilde{\Lambda^n}/_{S_n}$ (there $\widetilde{\Lambda^n}$ is a set Λ^n without diagonals, S_n is a permutation group).

We define the Poisson measure on Γ_Λ via direct formula

$$\pi_\Lambda = e^{-m(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \widehat{m^n},$$

where $\widehat{m^n}$ is an image on $\Gamma_\Lambda^{(n)}$ of the Lebesgue measure m^n on Λ^n (so, m is a usual Lebesgue measure on $\mathbb{R}^d$).

In the Hilbert space $L^2(\Gamma_\Lambda)$ we consider a dense subset of cylindric functions:

$$\mathcal{F}C_b^\infty(\Gamma_\Lambda, \mathcal{D}) = \left\{ F(\cdot) = g_F(\langle \varphi_1, \cdot \rangle, \ldots, \langle \varphi_N, \cdot \rangle) \,\middle|\, \varphi_k \in \mathcal{D}; g_F \in C_b^\infty\left(\mathbb{R}^N\right) \right\}.$$

In [1] the differential geometry on configuration space was constructed. Note that the gradient of a function F is defined as the element of a tangent space (the space of vector fields indexed by points of a configuration), such that

$$\nabla^\Gamma F(\gamma) = \Big(\nabla_x F(\gamma)\Big)_{x\in\gamma} \in T_\gamma(\Gamma_\Lambda).$$

A Laplace operator is defined as following:

$$\Delta^\Gamma F(\gamma) = \sum_{x\in\gamma} \Delta_x F(\gamma).$$

It is not symmetric on all (smooth) cylindric functions. In [5] the necessary and sufficient conditions of its symmetry on smaller sets of functions were founded. In particular, if we consider a class of functions which satisfies "the Neumann boundary condition"

$$F \in \mathcal{F}C_b^\infty\left(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}(\Lambda)}\right) \Leftrightarrow \varphi_k \in \mathcal{D}_{\mathcal{N}(\Lambda)},$$

then an operator

$$H^{\Gamma_\Lambda} := -\Delta^\Gamma \upharpoonright_{\mathcal{F}C_b^\infty(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}(\Lambda)})}$$

will be the image of the second quantization of the one-particle operator $\left(-\Delta, \mathcal{D}_{\mathcal{N}(\Lambda)}\right)$ under the canonical isomorphism between the space $L^2(\Gamma_\Lambda, \pi_\Lambda)$ and the Fock space $\mathrm{Exp}\left(L^2(\Lambda, dx)\right)$; so, it is essentially self-adjoint in $L^2(\Gamma_\Lambda, \pi_\Lambda)$. Moreover, since the operator $\left(-\Delta, \mathcal{D}_{\mathcal{N}(\Lambda)}\right)$ has pure point spectrum only

$$0 = \mu_1 < \mu_2 \le \mu_3 \le \ldots,$$

then the operator H^{Γ_Λ} have this property too.

A new result is that the own subspace of the H^{Γ_Λ}, which correspond to zero eigenvalue, be infinite-dimensional. More precisely, the following statement is true.

Proposition 1.1. *Let* $\mathrm{Ker}\, H^{\Gamma_\Lambda}$ *be a kernel of the* H^{Γ_Λ} *in the* $L^2(\Gamma_\Lambda, \pi_\Lambda)$, $\chi_{\Gamma_\Lambda^{(n)}}$ *be a indicator of the space* $\Gamma_\Lambda^{(n)}$. *Then*

$$\mathrm{Ker}\, H^{\Gamma_\Lambda} = \left\{ F = \sum_{n=0}^\infty c_n \chi_{\Gamma_\Lambda^{(n)}} \,\middle|\, F \in \mathcal{F}C_b^\infty\left(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}(\Lambda)}\right) \right\}.$$

Sketch of the proof. Note that we consider a non-closed operator, its kernel is a not-closed set. For its closure we use the same notation. Clearly, the condition $\sum_{n=0}^\infty c_n \chi_{\Gamma_\Lambda^{(n)}} \in L^2(\Gamma_\Lambda, \pi_\Lambda)$ means the following

$$\sum_{n=0}^\infty \frac{c_n^2 (m(\Lambda))^n}{n!} < +\infty.$$

The main idea of the proof of Proposition 1.1 is the following. If $F(\cdot) = g_F(\langle\varphi_1, \cdot\rangle, \dots, \langle\varphi_N, \cdot\rangle) \in \mathcal{F}C_b^\infty\left(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}(\Lambda)}\right)$, then if we consider for any $n \geq 1$ a function

$$f^{(n)}(x_1, \dots, x_n) := g_F\left(\sum_{k=1}^n \varphi_1(x_k), \dots, \sum_{k=1}^n \varphi_N(x_k)\right) = F(\{x_1, \dots, x_n\}),$$

we obtain that it satisfies the usual Neumann boundary condition as a function of nd variables over domain $\Lambda^n \subset \mathbb{R}^{dn}$. □

Note that $\operatorname{Ker} H^{\Gamma_\Lambda}$ is non-empty. For example, the function $F(\gamma) = e^{-|\gamma|} = e^{\langle -1, \gamma\rangle} \in \mathcal{F}C_b^\infty\left(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}(\Lambda)}\right)$ is in this kernel.

Moreover, the following statement holds:

Corollary 1.2. $\operatorname{Ker} H^{\Gamma_\Lambda} = \{c(|\cdot|) \,|\, c \in C_b^\infty(\mathbb{R})\}$.

It is a direct consequence of the Proposition 1.1. Note that this corollary is true for a non-closed operator H^{Γ_Λ} only.

Clearly, for any $F \in \operatorname{Ker} H^{\Gamma_\Lambda}$ one has $\nabla^\Gamma F(\gamma) = 0$, so, in general, the Poincaré inequality

$$\int_{\Gamma_\Lambda}\left(F(\gamma) - \int_{\Gamma_\Lambda} F(\gamma)\, d\pi_\Lambda(\gamma)\right)^2 d\pi_\Lambda(\gamma) \leq \text{const.} \int_{\Gamma_\Lambda} \left|\nabla^\Gamma F(\gamma)\right|^2_{T_\gamma(\Gamma)} d\pi_\Lambda(\gamma),$$

can not be true on the space Γ_Λ, since if $F \in \operatorname{Ker} H^{\Gamma_\Lambda}$, then the right hand side is equal to 0, but F is not a constant on the whole space.

Using Proposition 1.1 it is easy to compute a projection of F on the $\operatorname{Ker} H^{\Gamma_\Lambda}$.

Proposition 1.3. *Let $F \in \mathcal{F}C_b^\infty(\Gamma_\Lambda, \mathcal{D})$, then*

$$\operatorname{Pr}_{\operatorname{Ker} H^{\Gamma_\Lambda}} F(\gamma) = \sum_{n=0}^\infty \chi_{\Gamma_\Lambda^{(n)}}(\gamma) \frac{1}{(m(\Lambda))^n} \int_{\Lambda^n} F(\{x_1, \dots, x_n\})\, dx_1 \dots dx_n.$$

It follows from the fact that if $H^{(n)}$ is the n-particle Laplace operator on functions over Λ^n, which satisfies Neumann boundary conditions, then

$$\operatorname{Pr}_{\operatorname{Ker} H^{(n)}} f^{(n)} = \frac{1}{(m(\Lambda))^n} \int_{\Lambda^n} f^{(n)}(\{x_1, \dots, x_n\})\, dx_1 \dots dx_n$$

It is well-known that the Poincaré inequality has a following multiplicative property (see, e.g., [7]):

Proposition 1.4. *Let $f^{(n)}$ be a symmetric smooth function on $\bar{\Lambda}^n$. Then*

$$\int_{\Lambda^n}\left(f^{(n)}(x_1, \dots, x_n) - \frac{1}{(m(\Lambda))^n}\int_{\Lambda^n} f^{(n)}(x_1, \dots, x_n)\, dx_1 \dots dx_n\right)^2 dx_1 \dots dx_n$$
$$\leq C \int_{\Lambda^n} \left|\nabla^{(n)} f^{(n)}\right|^2 (x_1, \dots, x_n)\, dx_1 \dots dx_n,$$

and a constant C doesn't depend on n.

Because of this fact we may obtain the spectral gap inequality on a configuration space Γ_Λ.

Theorem 1.5. *For any function $F \in \mathcal{F}C_b^\infty(\Gamma_\Lambda, \mathcal{D})$ the spectral gap inequality holds*

$$\int_{\Gamma_\Lambda} \left|F(\gamma) - \operatorname{Pr}_{\operatorname{Ker} H^{\Gamma_\Lambda}} F(\gamma)\right|^2 d\pi_\Lambda(\gamma) \leq C \int_{\Gamma_\Lambda} \left|\nabla^\Gamma F(\gamma)\right|^2_{T_\gamma(\Gamma)} d\pi_\Lambda(\gamma). \tag{1.1}$$

For a proof we need rewrite the left hand side of (1.1) using Proposition 1.3 and on each $\Gamma_\Lambda^{(n)}$ use Proposition 1.4.

Remark 1.6. The inequality (1.1) is really the spectral gap inequality, since (see [5])

$$\int_{\Gamma_\Lambda} \left|\nabla^\Gamma F(\gamma)\right|^2_{T_\gamma(\Gamma)} d\pi_\Lambda(\gamma) = \int_{\Gamma_\Lambda} H^{\Gamma_\Lambda} F(\gamma) \cdot F(\gamma)\, d\pi_\Lambda(\gamma)$$

for $F \in \mathcal{F}C_b^\infty\left(\Gamma_\Lambda, \mathcal{D}_{\mathcal{N}(\Lambda)}\right)$.

Remark 1.7. By a projection property one has

$$\begin{aligned}&\int_{\Gamma_\Lambda} \left|F(\gamma) - \mathrm{Pr}_{\mathrm{Ker}\, H^{\Gamma_\Lambda}} F(\gamma)\right|^2 d\pi_\Lambda(\gamma)\\ &= \int_{\Gamma_\Lambda} F^2(\gamma)\, d\pi_\Lambda(\gamma) - \int_{\Gamma_\Lambda} \left(\mathrm{Pr}_{\mathrm{Ker}\, H^{\Gamma_\Lambda}} F(\gamma)\right)^2 d\pi_\Lambda(\gamma).\end{aligned}$$

Since

$$\int_{\Gamma_\Lambda} \mathrm{Pr}_{\mathrm{Ker}\, H^{\Gamma_\Lambda}} F(\gamma)\, d\pi_\Lambda(\gamma) = \int_{\Gamma_\Lambda} F(\gamma)\, d\pi_\Lambda(\gamma),$$

then by Hölder inequality we have that

$$\int_{\Gamma_\Lambda} \left|F(\gamma) - \mathrm{Pr}_{\mathrm{Ker}\, H^{\Gamma_\Lambda}} F(\gamma)\right|^2 d\pi_\Lambda(\gamma) \le \int_{\Gamma_\Lambda} \left(F(\gamma) - \int_{\Gamma_\Lambda} F(\gamma)\, d\pi_\Lambda(\gamma)\right)^2 d\pi_\Lambda(\gamma).$$

So, we see that the Poincaré inequality is more strong than the spectral gap inequality.

2. A spectral gap inequality on Γ

In this section we consider a space of all configurations (locally finite subsets) of $\mathbb{R}^d$:

$$\Gamma := \left\{\gamma \subset \mathbb{R}^d \,\middle|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\right\}.$$

The Poisson measure π is defined as a measure on Γ such that its projection on Γ_Λ is π_Λ for any bounded measurable $\Lambda \subset \mathbb{R}^d$. For main properties of this space we again refer to [1].

Let us consider an operator A on $\mathcal{D}$ in $L^2\left(\mathbb{R}^d, dx\right)$ such that

$$Af(x) = -\operatorname{div}\left(a(x)\nabla f(x)\right),$$

where $a(x)$ is a positive matrix function which has enough growing. The simplest but useful example is the case

$$a(x) = \left(1 + \|x\|^2\right) 1\!\!1. \tag{2.1}$$

The corresponding Dirichlet form is

$$\mathcal{E}(f, g) = \int_{\mathbb{R}^d} a(x)\left|\nabla f(x)\right|^2 dx.$$

Let us collect useful for us properties of the operator A:

1. (See, e.g., [6], [2], [3]). $(A, \mathcal{D})$ is essential self-adjoint operator, if

$$\|a(x)\| = O\left(r^2 \log^2 r\right),\ r = \|x\| \to \infty.$$

2. (See [3]). The semigroup $T_t := e^{-tA}$ is conservative, that means that

$$T_t 1 = 1 \text{ for all } t \ge 0,$$

if

$$\|a(x)\| = O\left(r^2 \log r\right),\ r = \|x\| \to \infty.$$

3. (See [4]). If

$$a(x) \ge \left(1 + \|x\|^2\right) 1\!\!1,$$

then $(A, \mathcal{D})$ is a strongly positive operator in $L^2\left(\mathbb{R}^d, dx\right)$, more precisely:

$$A \geq \frac{d}{4}.$$

4. Under previous condition $(A, \mathcal{D})$ has discrete spectrum if $d \geq 3$.

As we see, the growth "between" r^2 and $r^2 \log r$ is satisfied for all conditions.

Let us consider the second quantization of A in the Fock space $\operatorname{Exp}\left(L^2\left(\mathbb{R}^d, dx\right)\right)$, and let A^Γ be an image of this second quantization under the canonical isomorphism between this Fock space and $L^2(\Gamma, \pi)$. It is known from a general result (see [1]) that

$$\begin{aligned}&\int_\Gamma A^\Gamma F(\gamma) \cdot G(\gamma)\, d\pi(\gamma)\\ &= \int_\Gamma \int_{\mathbb{R}^d} A_x \left(F(\gamma + \varepsilon_x) - F(\gamma)\right) \cdot \left(G(\gamma + \varepsilon_x) - G(\gamma)\right) d\pi(\gamma).\end{aligned}$$

From this equality and Mecke formula (see, e.g., [1], [5]) one has

$$\int_\Gamma A^\Gamma F(\gamma) \cdot G(\gamma)\, d\pi(\gamma) = \int_\Gamma \left\langle \mathcal{A}^\Gamma(\gamma) \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\right\rangle_{T_\gamma(\Gamma)} d\pi(\gamma),$$

where $\mathcal{A}^\Gamma(\gamma)$ is a diagonal matrix:

$$\mathcal{A}^\Gamma(\gamma) = \operatorname{diag}\{a(x)\}_{x \in \gamma}.$$

Then, if the conditions 1–4 on the growth of a hold, from the general theory of second quantization one has that the operator $\left(A^\Gamma, \mathcal{F}C_b^\infty(\Gamma, \mathcal{D})\right)$ is essentially self-adjoint in $L^2(\Gamma, \pi)$ and has a discrete spectrum if $d \geq 3$.

Moreover, under a conservative property (see [1]) we know that the corresponding stochastic process on the configuration space can be considered as a collection of independent processes (without interaction) on $\mathbb{R}^d$. (It means that the dynamics of a configuration is a collection of the dynamics of the points of this configuration).

Finally, the following theorem states that the corresponding spectral gap inequality (which really is a Poincaré inequality, since $\operatorname{Ker} A^\Gamma = \{c \in \mathbb{R}\}$) is true on a dense subset of cylindric polynomials:

$$\mathcal{FP}(\Gamma, \mathcal{D}) = \left\{F(\cdot) = g_F\left(\langle\varphi_1, \cdot\rangle, \ldots, \langle\varphi_N, \cdot\rangle\right) \middle| \varphi_k \in \mathcal{D}; g_F \in \mathcal{P}\left(\mathbb{R}^N\right)\right\},$$

where $\mathcal{P}\left(\mathbb{R}^N\right)$ is the set of all polynomials on $\mathbb{R}^N$ (see [1] for main properties of $\mathcal{FP}(\Gamma, \mathcal{D})$).

Theorem 2.1. *For any $F \in \mathcal{FP}(\Gamma, \mathcal{D})$ the following Poincaré inequality holds*

$$\begin{aligned}&\int_\Gamma \left(F(\gamma) - \int_\Gamma F(\gamma)\, d\pi(\gamma)\right)^2 d\pi(\gamma)\\ &\leq \frac{4}{d} \int_\Gamma \left\langle \mathcal{A}^\Gamma(\gamma) \nabla^\Gamma F(\gamma), \nabla^\Gamma F(\gamma)\right\rangle_{T_\gamma(\Gamma)} d\pi(\gamma).\end{aligned}$$

Proof. Let $F(\gamma) = F_N(\gamma) = f_N\left(\langle\psi_1, \gamma\rangle, \ldots, \langle\psi_M, \gamma\rangle\right) \in \mathcal{FP}(\Gamma, \mathcal{D}),\ f_N \in \mathcal{P}_N\left(\mathbb{R}^M\right)$. Then (see, e.g., [1])

$$F_N(\gamma) = \sum_{n=0}^N \sum_{k=1}^K b_{nk} Q_n\left(\varphi_k^{\otimes n}, \gamma\right),$$

where Q_n are the Charlier polynomials (see [1,5]), $\varphi_k \in \mathcal{D}$ and some b_{nk} may be equal to 0.

We want to prove that

$$\int_\Gamma (F_N(\gamma))^2 d\pi(\gamma) - \left(\int_\Gamma F_N(\gamma)\, d\pi(\gamma)\right)^2 \leq \frac{4}{d} \int_\Gamma \left\langle \mathcal{A}^\Gamma(\gamma) \nabla^\Gamma F_N(\gamma), \nabla^\Gamma F_N(\gamma)\right\rangle_{T_\gamma(\Gamma)} d\pi(\gamma).$$

One has

$$\begin{aligned}\left(\int_\Gamma F_N(\gamma)\,d\pi(\gamma)\right)^2 &= \left(\int_\Gamma \sum_{n=0}^{N}\sum_{k=1}^{K} b_{nk} Q_n\left(\varphi_k^{\otimes n},\gamma\right)d\pi(\gamma)\right)^2\\ &= \left(\sum_{k=1}^{K} b_{0k}\right)^2 = \sum_{k,j=1}^{K} b_{0k}b_{0j};\\ \left(F_N(\gamma)\right)^2 &= \sum_{n,m=0}^{N}\sum_{k,j=1}^{K} b_{nk}b_{mj}Q_n\left(\varphi_k^{\otimes n},\gamma\right)Q_m\left(\varphi_j^{\otimes m},\gamma\right);\\ \int_\Gamma \left(F_N(\gamma)\right)^2 d\pi(\gamma) &= \sum_{n,m=0}^{N}\sum_{k,j=1}^{K} b_{nk}b_{mj}\int_\Gamma Q_n\left(\varphi_k^{\otimes n},\gamma\right)Q_m\left(\varphi_j^{\otimes m},\gamma\right)d\pi(\gamma)\\ &= \sum_{n=0}^{N}\sum_{k,j=1}^{K} b_{nk}b_{nj}n!\left(\varphi_k,\varphi_j\right)^n.\end{aligned}$$

Combining these equalities we obtain that

$$\begin{aligned}&\int_\Gamma \left(F_N(\gamma)\right)^2 d\pi(\gamma) - \left(\int_\Gamma F_N(\gamma)\,d\pi(\gamma)\right)^2\\ &= \sum_{n=0}^{N}\sum_{k,j=1}^{K} b_{nk}b_{nj}n!\left(\varphi_k,\varphi_j\right)^n - \sum_{k,j=1}^{K} b_{0k}b_{0j} = \sum_{n=1}^{N}\sum_{k,j=1}^{K} b_{nk}b_{nj}n!\left(\varphi_k,\varphi_j\right)^n.\end{aligned}$$

Next, we have that

$$\begin{aligned}&\frac{4}{d}\int_\Gamma \left\langle \mathcal{A}^\Gamma(\gamma)\nabla^\Gamma F_N(\gamma),\nabla^\Gamma F_N(\gamma)\right\rangle_{T_\gamma(\Gamma)} d\pi(\gamma)\\ &= \frac{4}{d}\int_\Gamma\int_{\mathbb{R}^d} \left|a(x)\nabla_x F_N(\gamma+\varepsilon_x)\right|^2 dx d\pi(\gamma)\\ &= \frac{4}{d}\int_\Gamma\int_{\mathbb{R}^d} \left|a(x)\nabla_x\left(F_N(\gamma+\varepsilon_x)-F_N(\gamma)\right)\right|^2 dx d\pi(\gamma)\\ &\geq \frac{4}{d}\int_\Gamma \frac{d}{4}\int_{\mathbb{R}^d} \left(F_N(\gamma+\varepsilon_x)-F_N(\gamma)\right)^2 dx d\pi(\gamma)\\ &= \int_\Gamma\int_{\mathbb{R}^d} \left(F_N(\gamma+\varepsilon_x)-F_N(\gamma)\right)^2 dx d\pi(\gamma).\end{aligned}$$

Since

$$\begin{aligned}F_N(\gamma+\varepsilon_x)-F_N(\gamma) &= \sum_{n=0}^{N}\sum_{k=1}^{K} b_{nk}\left(Q_n\left(\varphi_k^{\otimes n},\gamma+\varepsilon_x\right)-Q_n\left(\varphi_k^{\otimes n},\gamma\right)\right)\\ &= \sum_{n=1}^{N}\sum_{k=1}^{K} b_{nk}n\varphi_k(x)\,Q_{n-1}\left(\varphi_k^{\otimes(n-1)},\gamma\right),\end{aligned}$$

then

$$\begin{aligned}&\frac{4}{d}\int_\Gamma \left\langle \mathcal{A}^\Gamma(\gamma)\nabla^\Gamma F_N(\gamma),\nabla^\Gamma F_N(\gamma)\right\rangle_{T_\gamma(\Gamma)} d\pi(\gamma)\\ &\geq \sum_{n,m=1}^{N}\sum_{k,j=1}^{K} b_{nk}b_{mj}nm\int_\Gamma\int_{\mathbb{R}^d}\varphi_k(x)\,Q_{n-1}\left(\varphi_k^{\otimes(n-1)},\gamma\right)\varphi_j(x)\,Q_{m-1}\left(\varphi_j^{\otimes(m-1)},\gamma\right)dx d\pi(\gamma)\\ &= \sum_{n=1}^{N}\sum_{k,j=1}^{K} b_{nk}b_{nj}n^2(n-1)!\left(\varphi_k,\varphi_j\right)^n = \sum_{n=1}^{N}\sum_{k,j=1}^{K} b_{nk}b_{nj}nn!\left(\varphi_k,\varphi_j\right)^n.\end{aligned}$$

So, it is enough to prove that

$$\sum_{n=1}^{N} n \sum_{k,j=1}^{K} b_{nk} b_{nj} n! \left(\varphi_k, \varphi_j\right)^n \geq \sum_{n=1}^{N} \sum_{k,j=1}^{K} b_{nk} b_{nj} n! \left(\varphi_k, \varphi_j\right)^n . \tag{2.2}$$

For this let us consider $\hat{F}_n(\gamma) = \sum_{k=1}^{K} b_{nk} Q_n\left(\varphi_k^{\otimes n}, \gamma\right)$, $n \geq 1$, then

$$0 \leq \int_\Gamma \left(\hat{F}_n(\gamma)\right)^2 d\pi(\gamma) = \sum_{k,j=1}^{K} b_{nk} b_{nj} \int_\Gamma Q_n\left(\varphi_k^{\otimes n}, \gamma\right) Q_n\left(\varphi_j^{\otimes m}, \gamma\right) d\pi(\gamma)$$
$$= \sum_{k,j=1}^{K} b_{nk} b_{nj} n! \left(\varphi_k, \varphi_j\right)^n ,$$

so (2.2) is true. □

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