Gauss Formula and Symmetric Extensions of the Laplacian on Configuration Spaces

D.L.Finkelshtein,* Yu.G.Kondratiev,† A.Yu.Konstantinov,‡ M.Röckner§

Abstract

We prove an analogue of the classical Gauss formula for the configuration space Γ_Λ over a domain Λ of $\mathbb{R}^d$ and study symmetric extensions of the corresponding Laplacian on Γ_Λ.

1 Introduction

The classical Gauss formula says that

$$\int_\Lambda \operatorname{div} v(x)\, m(dx) = \int_{\partial\Lambda} \langle v(s), n_s\rangle_{\mathbb{R}^d}\, \tilde{m}(ds). \tag{1.1}$$

Here Λ is a bounded domain with smooth boundary $\partial\Lambda$, $v \in C^1\left(\bar{\Lambda}\right)$, m is Lebesgue measure on $\mathbb{R}^d$, $\tilde{m}$ is the corresponding surface measure on $\partial\Lambda$ and n_s is the outer normal to $\partial\Lambda$ at the point s. Infinite-dimensional generalizations of (1.1) for the case of Gauss measures or, more generally, of a differentiable measure one can found in [4], [12], [14]. An interesting Poisson analogue of the Gauss formula was proved in [13]. In this paper we prove a different version of the Gauss formula for the configuration space which one can consider as a natural "lifting" (see [1], [11]) of the classical formula (1.1).

The article is arranged as follows. In Section 2 we give a brief review of the analysis on configurations spaces. In Section 3 we prove the corresponding Gauss formula. Section 4 is devoted to the study of symmetric realizations of the Laplace operator in $L^2\left(\Gamma_\Lambda, \pi_\sigma^\Lambda\right)$. Here Γ_Λ is the configuration space over a domain $\Lambda \subset \mathbb{R}^d$, π_σ denotes the Poisson measure with intensity σ and π_σ^Λ is the restriction of π_σ to Γ_Λ. In Section 5 we consider analogous questions for Gibbs measures.

*Institute for Mathematics of the Ukrainian National Science Academy, 3 Tereshchenkivs'ka St, MSP, Kyiv, 01601, Ukraine, e-mail: fdl@ln.ua.

†Department for Mathematics, Bielefeld University, D 33615, Bielefeld, Germany, e-mail: kondrat@mathematik.uni-bielefeld.de; BiBoS research center, Bielefeld University, D 33615, Bielefeld, Germany; Institute for Mathematics of the Ukrainian National Science Academy, 3 Tereshchenkivs'ka St, MSP, Kyiv, 01601, Ukraine.

‡Department for Mathematics, Kyiv University, 64 Volodymyrs'ka St, 252033, Kyiv, Ukraine, e-mail: konst@faust.kiev.ua.

§Department for Mathematics, Bielefeld University, D 33615, Bielefeld, Germany, e-mail: roeckner@mathematik.uni-bielefeld.de.

2 Poisson Analysis, Intrinsic and Extrinsic Differential Geometry of Poisson Spaces

In this section we provide a brief review of Poisson analysis and the intrinsic and extrinsic differential geometry of the corresponding Poisson spaces needed for this article. For a more detailed exposition of different aspects of Poisson analysis and differential geometry of Poisson spaces see [1], [11], [7], [8], [6] and the references therein.

Let $X=\mathbb{R}^d$. For each point $x\in X$ the tangent space to X at x will be denoted by $T_x(X)$ and the tangle bundle will be denoted $T(X)=\bigcup_{x\in X}T_x(X)$. Consider the inner product on $T_x(X)$, which we denote by $\langle\cdot,\cdot\rangle_{T_x(X)}$. Let m denote the Lebesgue measure on X.

The configuration space Γ over X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma:=\{\gamma\subset X\,||\gamma\cap K|<\infty \text{ for any compact } K\subset X\}, \tag{2.2}$$

where $|A|$ denotes the cardinality of a set A. Let $\Lambda\subset X$ and define

$$\Gamma_\Lambda=\{\gamma\in\Gamma\,|\gamma\cap(X\backslash\Lambda)=\emptyset\}. \tag{2.3}$$

We can identify any $\gamma\in\Gamma$ with the Radon measure $\sum_{x\in\gamma}\varepsilon_x$, where ε_x is the Dirac measure at the point x and $\sum_{x\in\emptyset}\varepsilon_x$:=zero measure. For any $f\in C_0(X)$ (the set of all continuous functions on X with compact support) we introduce the map $\Gamma\ni\gamma\longmapsto\langle f,\gamma\rangle:=\int_X f(x)\gamma(dx)=\sum_{x\in\gamma}f(x)$.

Let $\mathcal{O}_c(X)$ be the family of all open subsets of X which have compact closures. Define for any $\Lambda\in\mathcal{O}_c(X)$ and for any $n\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}$

$$\Gamma_\Lambda^{(n)}:=\{\gamma\in\Gamma_\Lambda\,||\gamma|=n\},\Gamma_\Lambda^{(0)}:=\{\emptyset\}.$$

Note that we have a bijection $\tilde{\Lambda}^n/S_n\to\Gamma_\Lambda^{(n)}$, where

$$\tilde{\Lambda}^n:=\{(x_1,\ldots,x_n)\in\Lambda^n|x_k\neq x_j,k\neq j\}$$

and S_n is the permutation group over $(1,\ldots,n)$. This bijection defines a locally compact metrizable Hausdorf topology on $\Gamma_\Lambda^{(n)}$. Let $s_\Lambda^n:\tilde{\Lambda}^n\to\Gamma_\Lambda^{(n)}$ be such that $s_\Lambda^n:(x_1,\ldots,x_n)\longmapsto\{x_1,\ldots,x_n\}\in\Gamma_\Lambda^{(n)}$. It is obvious that $\Gamma_\Lambda=\bigcup_{n=0}^\infty\Gamma_\Lambda^{(n)}$. This space is equipped with the usual topology of disjoint unions and corresponding Borel σ-algebra $\mathcal{B}(\Gamma_\Lambda)$. Let $\mathcal{B}(\Gamma)$ be the smallest σ-algebra on Γ such that all restriction mappings $\Gamma\ni\gamma\longmapsto p_\Lambda\gamma=\gamma\cap\Lambda=:\gamma_\Lambda\in\Gamma_\Lambda$ are $\mathcal{B}(\Gamma)/\mathcal{B}(\Gamma_\Lambda)$-measurable. So, $\mathcal{B}(\Gamma)\cap\Gamma_\Lambda=\mathcal{B}(\Gamma_\Lambda)$. Note that $\mathcal{B}(\Gamma)$ is the Borel σ-algebra corresponding to the smallest topology on Γ such that all maps $\Gamma\ni\gamma\longmapsto\langle f,\gamma\rangle$ are continuous.

Consider a C^1- density $\rho>0$ $m-$a.e. Set $\sigma(dx)=\rho(x)m(dx)$, then σ is a non-atomic Radon measure on X. For any $n\in\mathbb{N}$ we introduce the product-measure $\sigma^{\otimes n}$ on $(X^n,\mathcal{B}(X^n))$. Clearly, $\sigma^{\otimes n}\left(X^n\backslash\tilde{X}^n\right)=0$. The measure $\sigma^{\otimes n}$

can be considered as a finite measure on $\tilde{\Lambda}^n$, and by $\sigma_{\Lambda,n} := \sigma^{\otimes n} \circ (s^n_\Lambda)^{-1}$ we denote the corresponding image measure on $\Gamma^{(n)}_\Lambda$ under s^n_Λ. The Lebesgue-Poisson measure on $\mathcal{B}(\Gamma_\Lambda)$ with intensity measure σ is defined by

$$\lambda^\Lambda_\sigma := \sum_{n=0}^\infty \frac{1}{n!}\sigma_{\Lambda,n}, \tag{2.4}$$

where $\sigma_{\Lambda,0} := \varepsilon_\emptyset$ on $\Gamma^{(0)}_\Lambda = \{\emptyset\}$. It is obvious that the measure

$$\pi^\Lambda_\sigma := e^{-\sigma(\Lambda)}\lambda^\Lambda_\sigma \tag{2.5}$$

is a probability measure on $\mathcal{B}(\Gamma_\Lambda)$. The Poisson measure on $\mathcal{B}(\Gamma)$ with intensity measure σ is probability measure π_σ such that

$$\pi^\Lambda_\sigma = \pi_\sigma \circ (p_\Lambda)^{-1}, \ \Lambda \in \mathcal{O}_c(X). \tag{2.6}$$

Let $\mathbb{V}(X)$ be the set of all C^∞-vector fields on X (i.e., smooth sections of $T(X)$) and let $\mathbb{V}_0(X) \subset \mathbb{V}(X)$ be the set of all vector fields with compact support. Let $v \in \mathbb{V}_0(X)$ and for any $x \in X$ the curve $\mathbb{R} \ni t \longmapsto \phi^v_t(x) \in X$ be defined as the solution to the following Cauchy problem: $\frac{d}{dt}\phi^v_t(x) = v(\phi^v_t(x))$, $\phi^v_0(x) = x$. Then for any $t \in \mathbb{R}$, $\phi^v_t \in \mathrm{Diff}_0(X)$ (the set of all diffeomorphisms on X with compact support) and for any $t, s \in \mathbb{R}$ $\phi^v_t \circ \phi^v_s = \phi^v_{t+s}$.

For any $\phi \in \mathrm{Diff}_0(X)$ and for any $\gamma \in \Gamma$ we can define $\phi(\gamma) := \{\phi(x) \,|x \in \gamma\}$. For a function $F : \Gamma \to \mathbb{R}$ we define the directional derivative along the vector field $v \in \mathbb{V}_0(X)$ as

$$\left(\nabla^\Gamma_v F\right)(\gamma) := \left.\frac{d}{dt}F(\phi^v_t(\gamma))\right|_{t=0}, \tag{2.7}$$

provided the right hand side exists.

Let $\mathcal{D} = C^\infty_0(X)$ be the set of all C^∞ -functions on X with compact support and let $\mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$ be the set of all functions $F : \Gamma \to \mathbb{R}$ of the form

$$F(\gamma) = g_F(\langle\varphi_1, \gamma\rangle, \ldots, \langle\varphi_N, \gamma\rangle), \ \gamma \in \Gamma, \tag{2.8}$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ and $g_F \in C^\infty_b(\mathbb{R}^N)$. The set $\mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$ is a dense subset in the space $L^2(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma) =: L^2(\pi_\sigma)$. We have that for any $F \in \mathcal{F}C^\infty_b(\mathcal{D},\Gamma)$ of the form (2.8)

$$\left(\nabla^\Gamma_v F\right)(\gamma) = \sum_{j=1}^N \frac{\partial g_F}{\partial q_j}(\langle\varphi_1, \gamma\rangle, \ldots, \langle\varphi_N, \gamma\rangle) \cdot \langle\nabla_v\varphi_j, \gamma\rangle, \tag{2.9}$$

where $(\nabla_v\varphi)(x) := \langle\nabla\varphi(x), v(x)\rangle_{T_x(X)}$, ∇ denotes the gradient on X.

We introduce the tangent space $T_\gamma(\Gamma)$ to the configuration space Γ at the point $\gamma \in \Gamma$ as the Hilbert space of measurable γ-square-integrable sections (measurable vector fields) $V_\gamma : X \to T(X)$ with the scalar product

$$\left\langle V^1_\gamma, V^2_\gamma\right\rangle_{T_\gamma(\Gamma)} = \int_X \left\langle V^1_\gamma(x), V^2_\gamma(x)\right\rangle_{T_x(X)} \gamma(dx), \tag{2.10}$$

$V_\gamma^1, V_\gamma^2 \in T_\gamma(\Gamma)$. The corresponding tangent bundle is $T(\Gamma) = \bigcup_{\gamma\in\Gamma} T_\gamma(\Gamma)$. It is obvious that any $v \in \mathbb{V}_0(X)$ can be considered as "constant" vector field on Γ such that $\Gamma \ni \gamma \longmapsto V_\gamma(\cdot) = v(\cdot) \in T_\gamma(\Gamma)$. We define the intrinsic gradient of a function $F : \Gamma \to \mathbb{R}$ as the mapping $\Gamma \ni \gamma \longmapsto \left(\nabla^\Gamma F\right)(\gamma) \in T_\gamma(\Gamma)$ such that for any $v \in \mathbb{V}_0(X)$

$$\left(\nabla_v^\Gamma F\right)(\gamma) = \left\langle \nabla^\Gamma F(\gamma), v\right\rangle_{T_\gamma(\Gamma)}, \tag{2.11}$$

provided it exists. Then for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ of the form (2.8)

$$\left(\nabla^\Gamma F\right)(\gamma; x) = \sum_{j=1}^N \frac{\partial g_F}{\partial q_j}\left(\langle\varphi_1, \gamma\rangle, \ldots, \langle\varphi_N, \gamma\rangle\right) \cdot \nabla\varphi_j(x), \tag{2.12}$$

$$\gamma \in \Gamma, x \in X.$$

The logarithmic derivative of σ is defined as the vector field $X \ni x \longmapsto \beta^\sigma(x) := \frac{\nabla\rho(x)}{\rho(x)} \in T_x(X)$, where as usual $\beta^\sigma(x) := 0$ if $\rho(x) = 0$; and the logarithmic derivative of σ along $v \in \mathbb{V}_0(X)$ is defined as

$$\operatorname{div}_\sigma v(x) := \langle\beta^\sigma(x), v(x)\rangle_{T_x(X)} + \operatorname{div} v(x),$$

where $\operatorname{div} := \operatorname{div}_m$ is the divergence on X w.r.t. m. One has the following integration by parts formula:

$$\int_\Gamma \left(\nabla_v^\Gamma F\right)(\gamma)\, G(\gamma)\, \pi_\sigma(d\gamma) \tag{2.13}$$

$$= -\int_\Gamma F(\gamma)\left(\nabla_v^\Gamma G\right)(\gamma)\, \pi_\sigma(d\gamma) - \int_\Gamma F(\gamma)\, G(\gamma)\, \langle\operatorname{div}_\sigma v, \gamma\rangle\, \pi_\sigma(d\gamma).$$

for any $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$.

Let $V \in \mathbb{V}(\Gamma)$, i.e., a section $V : \Gamma \to T(\Gamma)$. The divergence $\operatorname{div}_{\pi_\sigma}^\Gamma V$ is defined via the duality relation

$$\int_\Gamma \left\langle V_\gamma, \nabla^\Gamma F(\gamma)\right\rangle_{T_\gamma(\Gamma)} \pi_\sigma(d\gamma) = -\int_\Gamma F(\gamma)\left(\operatorname{div}_{\pi_\sigma}^\Gamma V\right)(\gamma)\, \pi_\sigma(d\gamma) \tag{2.14}$$

for all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, provided it exists. For any vector field

$$V_\gamma(x) = \sum_{j=1}^N G_j(\gamma)\, v_j(x), \gamma \in \Gamma, x \in X \tag{2.15}$$

with $G_j \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma), v_j \in \mathbb{V}_0(X),\ j = 1, \ldots, N$ we have

$$\left(\operatorname{div}_{\pi_\sigma}^\Gamma V\right)(\gamma) = \sum_{j=1}^N \left(\nabla_{v_j}^\Gamma G_j\right)(\gamma) + \sum_{j=1}^N G_j(\gamma)\, \langle\operatorname{div}_\sigma v_j, \gamma\rangle. \tag{2.16}$$

Then obviously for all $G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$

$$\operatorname{div}_{\pi_\sigma}^\Gamma (G \cdot V) = G \cdot \operatorname{div}_{\pi_\sigma}^\Gamma V + \left\langle\nabla^\Gamma G, V\right\rangle_{T(\Gamma)}. \tag{2.17}$$

Let us define two classes of smooth functions on $\Gamma:\ \mathcal{F}C_p^{\infty}(\mathcal{D},\Gamma)$ is the set of functions of the form (2.8), where $g_F\in C_p^{\infty}\left(\mathbb{R}^N\right)$ (:= the set of all C^{∞}-functions f on $\mathbb{R}^N$ such that f and all its partial derivatives are polynomially bounded); and $\mathcal{FP}(\mathcal{D},\Gamma)$ is the set of functions of the form (2.8), where $g_F\in\mathcal{P}\left(\mathbb{R}^N\right)$ (:= the set of all polynomials on $\mathbb{R}^N$). For $F,G\in\mathcal{F}C_p^{\infty}(\mathcal{D},\Gamma)$ we introduce an intrinsic pre-Dirichlet form as

$$\mathcal{E}_{\pi_\sigma}^{\Gamma}(F,G):=\int_{\Gamma}\left\langle\nabla^{\Gamma}F(\gamma),\nabla^{\Gamma}G(\gamma)\right\rangle_{T_\gamma(\Gamma)}\pi_\sigma(d\gamma).\tag{2.18}$$

We introduce also a differential operator $H_{\pi_\sigma}^{\Gamma}$ on the domain $\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$ which is given on any $F\in\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$ of the form (2.8) by the formula

$$\begin{aligned}\left(H_{\pi_\sigma}^{\Gamma}F\right)(\gamma):=&\\ -\sum_{i,j=1}^{N}\frac{\partial^2 g_F}{\partial q_i\partial q_j}\left(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle\right)&\int_X\langle\nabla\varphi_i(x),\nabla\varphi_j(x)\rangle_{T_x(X)}\gamma(dx)\\ -\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}\left(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle\right)&\int_X\Delta\varphi_j(x)\gamma(dx)\\ -\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}\left(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle\right)&\int_X\langle\nabla\varphi_j(x),\beta^{\sigma}(x)\rangle_{T_x(X)}\gamma(dx),\end{aligned}\tag{2.19}$$

where Δ denotes the Laplace operator on X.

For all $F,G\in\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$ we can write

$$\mathcal{E}_{\pi_\sigma}^{\Gamma}(F,G)=\left(H_{\pi_\sigma}^{\Gamma}F,G\right)_{L^2(\pi_\sigma)},\tag{2.20}$$

$$H_{\pi_\sigma}^{\Gamma}=-\operatorname{div}_{\pi_\sigma}^{\Gamma}\nabla^{\Gamma}\text{ on }\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma).\tag{2.21}$$

Note that $\left(\mathcal{E}_{\pi_\sigma}^{\Gamma},\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)\right)$ (the form $\mathcal{E}_{\pi_\sigma}^{\Gamma}$ on the domain $\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$) is a closable bilinear form in $L^2(\pi_\sigma)$. Its closure $\left(\mathcal{E}_{\pi_\sigma}^{\Gamma},\mathcal{D}\left(\mathcal{E}_{\pi_\sigma}^{\Gamma}\right)\right)$ is associated with a positive definite self-adjoint operator (the Friedrichs' extension of $H_{\pi_\sigma}^{\Gamma}$) which we also denote by $H_{\pi_\sigma}^{\Gamma}$ (and its domain by $\mathcal{D}\left(H_{\pi_\sigma}^{\Gamma}\right)$).

For any $F\in\mathcal{F}C_p^{\infty}(\mathcal{D},\Gamma)$ we define the Poissonian gradient ∇^P as

$$\left(\nabla^P F\right)(\gamma,x)=F(\gamma+\varepsilon_x)-F(\gamma),\ \gamma\in\Gamma,\ x\in X.\tag{2.22}$$

Note that the operation $\Gamma\ni\gamma\longmapsto\gamma+\varepsilon_x\in\Gamma$ is a π_σ-a.e. well-defined map since $\pi_\sigma(\{\gamma\in\Gamma|x\in\gamma\})=0$.

Let B be a linear operator on $L^2(\sigma)$ and $\|B\|\le 1$. One can define the operator $\operatorname{Exp}B$ on

$$\operatorname{Exp}L^2(\sigma):=\bigoplus_{n=0}^{\infty}\operatorname{Exp}_n L^2(\sigma):=\bigoplus_{n=0}^{\infty}\left(L^2(\sigma)\right)^{\hat{\otimes}n}=\bigoplus_{n=0}^{\infty}\hat{L}^2\left(X^n,\sigma^{\otimes n}\right),\tag{2.23}$$

where $\mathrm{Exp}_0\, L^2(\sigma) := \mathbb{R}$, by $\mathrm{Exp}\, B \restriction_{\mathrm{Exp}_n L^2(\sigma)} := B^{\otimes n}, n \in \mathbb{N}, \mathrm{Exp}\, B \restriction_{\mathrm{Exp}_0 L^2(\sigma)} := 1$. Let A be a positive self-adjoint operator in $L^2(\sigma)$. Consider the contraction semi-group e^{-tA}, $t \geq 0$, and define a positive self-adjoint operator $d\,\mathrm{Exp}\, A$ as the generator of the semigroup $\mathrm{Exp}(e^{-tA}), t \geq 0$: $\mathrm{Exp}(e^{-tA}) = \exp(-t\, d\,\mathrm{Exp}\, A)$. The operator $d\,\mathrm{Exp}\, A$ is called the second quantization of the one-particle operator A. For any $\varphi \in L^2(\sigma)$ one can introduce the coherent state

$$\mathrm{Exp}\, \varphi := \left(\frac{1}{n!}\, \varphi^{\otimes n} \right)_{n=0}^{\infty} \in \mathrm{Exp}\, L^2(\sigma).$$

There is a canonical Wiener-Ito-Segal isomorphism between the spaces $\mathrm{Exp}\, L^2(\sigma)$ and $L^2(\pi_\sigma)$ such that

$$\begin{gathered}\mathrm{Exp}\, L^2(\sigma) \ni \mathrm{Exp}\, \varphi \longmapsto e_{\pi_\sigma}(\varphi, \cdot) := \exp\left(\langle \log(1+\varphi), \cdot \rangle - \langle \varphi \rangle_\sigma \right), \\ \varphi \in \mathcal{D}, \varphi > -1,\end{gathered}$$

where $\langle \varphi \rangle_\sigma = \int_X \varphi(x)\sigma(dx)$ (see, e.g., [1],[8]). We denote by H_A^P the image of the operator $d\,\mathrm{Exp}\, A$ under this isomorphism.

Suppose that $\mathcal{D} \subset \mathrm{Dom}\, A$. Then one can introduce the extrinsic pre-Dirichlet form with coefficient A on $\mathcal{FP}(\mathcal{D}, \Gamma)$ by

$$\mathcal{E}_{\pi_\sigma, A}^P(F, G) := \int_\Gamma \left(\nabla^P F, A \nabla^P G \right)_{L^2(\sigma)} \pi_\sigma(d\gamma). \tag{2.24}$$

Then the following equality holds

$$\mathcal{E}_{\pi_\sigma, A}^P(F, G) = \left(H_A^P F, G \right)_{L^2(\pi_\sigma)}. \tag{2.25}$$

In the case when A is the Dirichlet operator H_σ which is given on $\mathcal{D}$ by

$$\begin{aligned}(H_\sigma \varphi)(x) &:= -\operatorname{div}_\sigma \nabla \varphi(x) \qquad (2.26)\\ &= -\Delta \varphi(x) - \langle \beta^\sigma(x), \nabla \varphi(x) \rangle_{T_x(X)},\end{aligned}$$

we have

$$H_{H_\sigma}^P = H_{\pi_\sigma}^\Gamma \tag{2.27}$$

on the dense domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$.

3 Gauss Formula for the Space of Configurations

In this section we give a proof of a variant of the classical Gauss formula for the space of configurations Γ_Λ (cf. [13], [12]).

Let Λ be an open domain of $X = \mathbb{R}^d$ and let Γ_Λ be defined by (2.3). If $\Lambda \in O_c(X)$ then one can define π_σ^Λ by (2.5). In general case one can introduce this measure by the formula:

$$\pi_\sigma^\Lambda := \pi_\sigma \circ (p_\Lambda)^{-1}. \tag{3.1}$$

For any non-negative $\mathcal{B}(\Gamma_\Lambda)\times\mathcal{B}(\Lambda)$-measurable function U we have Mecke identity (see, e.g., [9], and note that $(\sigma\otimes\pi_\sigma)\{(\gamma,x)\,|x\in\gamma\}=0$)

$$\int_{\Gamma_\Lambda}\int_\Lambda U(\gamma+\varepsilon_x,x)\,\sigma(dx)\,\pi_\sigma^\Lambda(d\gamma)=\int_{\Gamma_\Lambda}\int_\Lambda U(\gamma,x)\,\gamma(dx)\,\pi_\sigma^\Lambda(d\gamma)\,. \qquad (3.2)$$

In particular, for any $G\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)\,,\ \varphi\in L^2(\sigma)$ the following formula holds

$$\int_{\Gamma_\Lambda}\int_\Lambda G(\gamma+\varepsilon_x)\,\varphi(x)\,\sigma(dx)\,\pi_\sigma^\Lambda(d\gamma)=\int_{\Gamma_\Lambda}G(\gamma)\,\langle\varphi,\gamma\rangle\,\pi_\sigma^\Lambda(d\gamma)\,. \qquad (3.3)$$

In the following we always suppose that the boundary $\partial\Lambda$ of Λ is piecewise C^1. By n_s we denote the outer normal to $\partial\Lambda$ (at the point $s\in\partial\Lambda$) . Let $\tilde m$ be the surface measure on $\partial\Lambda$ corresponding to Lebesgue measure m. Set

$$\tilde\sigma(ds):=\rho(s)\,\tilde m(ds)\,.$$

The following theorem gives an analog of the classical Gauss formula.

Theorem 3.1 (Gauss formula for Poisson measure)**.** *For any vector field V of the form (2.15) the following formula holds*

$$\int_{\Gamma_\Lambda}\left(\operatorname{div}_{\pi_\sigma}^\Gamma V\right)(\gamma)\,\pi_\sigma^\Lambda(d\gamma) \qquad (3.4)$$
$$=\int_{\Gamma_\Lambda}\int_{\partial\Lambda}\langle V(\gamma+\varepsilon_s,s)\,,n_s\rangle_{T_s(X)}\,\tilde\sigma(ds)\,\pi_\sigma^\Lambda(d\gamma)\,.$$

Proof. By linearity we see that it is sufficient to prove (3.4) for $N=1$. Consider $V(\gamma,x)=G(\gamma)\,v(x)$, where $v\in\mathbb{V}_0(X)$ and

$$G(\cdot)=g_G(\langle\psi_1,\cdot\rangle,\ldots,\langle\psi_M,\cdot\rangle)\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)\,. \qquad (3.5)$$

Then by (2.9) and (3.3)

$$\int_{\Gamma_\Lambda}\left(\operatorname{div}_{\pi_\sigma}^\Gamma V\right)(\gamma)\,\pi_\sigma^\Lambda(d\gamma)=\int_{\Gamma_\Lambda}\left(\nabla_v^\Gamma G(\gamma)+G(\gamma)\,\langle\operatorname{div}_\sigma v,\gamma\rangle\right)\pi_\sigma^\Lambda(d\gamma)$$
$$=\int_{\Gamma_\Lambda}\left(\left(\sum_{j=1}^N\frac{\partial g_G}{\partial q_j}\langle\psi_1,\gamma\rangle,\ldots,\langle\psi_M,\gamma\rangle\right)\langle\nabla_v\psi_j,\gamma\rangle+G(\gamma)\,\langle\operatorname{div}_\sigma v,\gamma\rangle\right)\pi_\sigma^\Lambda(d\gamma)$$
$$=\int_{\Gamma_\Lambda}\int_\Lambda\left(\sum_{j=1}^N\frac{\partial g_G}{\partial q_j}(\langle\psi_1,\gamma+\varepsilon_x\rangle,\ldots,\langle\psi_M,\gamma+\varepsilon_x\rangle)\,\nabla_v\psi_j(x)+\right.$$
$$\left.+\,G(\gamma+\varepsilon_x)\operatorname{div}_\sigma v(x)\right)\sigma(dx)\,\pi_\sigma^\Lambda(d\gamma)\,.$$

Denote for fixed $\gamma\in\Gamma_\Lambda$

$$a(x)=G(\gamma+\varepsilon_x)\,v(x)\,.$$

Then

$$\begin{aligned}&\left(\operatorname{div}_\sigma a\right)(x) \qquad (3.6)\\ &= \sum_{j=1}^{N} \frac{\partial g_G}{\partial q_j} \left(\langle \psi_1, \gamma + \varepsilon_x \rangle, \ldots, \langle \psi_M, \gamma + \varepsilon_x \rangle\right) \nabla_v \psi_j (x) + G(\gamma + \varepsilon_x) \operatorname{div}_\sigma v(x).\end{aligned}$$

By (3.6) and the classical Gauss formula,

$$\begin{aligned}\int_{\Gamma_\Lambda} \left(\operatorname{div}^\Gamma_{\pi_\sigma} V\right)(\gamma)\, \pi^\Lambda_\sigma(d\gamma) &= \int_{\Gamma_\Lambda} \int_\Lambda \operatorname{div}_\sigma \left(G(\gamma + \varepsilon_x) v(x)\right) \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma)\\ &= \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \langle G(\gamma + \varepsilon_s) v(s), n_s \rangle_{T_s(X)} \tilde{\sigma}(ds)\, \pi^\Lambda_\sigma(d\gamma)\\ &= \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \langle V(\gamma + \varepsilon_s, s), n_s \rangle_{T_s(X)} \tilde{\sigma}(ds)\, \pi^\Lambda_\sigma(d\gamma).\end{aligned}$$

□

Corollary 3.2. *For any vector field V of the form (2.15) and $G \in \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$ we have*

$$\begin{aligned}&\int_{\Gamma_\Lambda} \left\langle V(\gamma), \nabla^\Gamma G(\gamma) \right\rangle_{T_\gamma(\Gamma)} \pi^\Lambda_\sigma(d\gamma)\\ &= \int_{\Gamma_\Lambda} \int_{\partial\Lambda} G(\gamma + \varepsilon_s) \langle V(\gamma + \varepsilon_s, s), n_s \rangle_{T_s(X)} \tilde{\sigma}(ds)\, \pi^\Lambda_\sigma(d\gamma) \qquad (3.7)\\ &\quad - \int_{\Gamma_\Lambda} G(\gamma) \left(\operatorname{div}^\Gamma_{\pi_\sigma} V\right)(\gamma)\, \pi^\Lambda_\sigma(d\gamma)\end{aligned}$$

Proof. Formula (3.7) is a direct consequence of (2.17) and Theorem 3.1. □

Note that (see (2.12)), for any $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$,

$$\nabla F(\gamma + \varepsilon_x) = \nabla^\Gamma F(\gamma + \varepsilon_x, x), \qquad (3.8)$$

and let us set

$$\frac{\partial}{\partial n} F(\gamma + \varepsilon_s) := \langle \nabla F(\gamma + \varepsilon_s), n_s \rangle_{T_s(X)}. \qquad (3.9)$$

Proposition 3.3 (The first Green formula)**.** *Let $F, G \in \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$. Then*

$$\begin{aligned}&\int_{\Gamma_\Lambda} \left\langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \right\rangle_{T_\gamma(\Gamma)} \pi^\Lambda_\sigma(d\gamma) \qquad (3.10)\\ &= \int_{\Gamma_\Lambda} \left(H^\Gamma_{\pi_\sigma} F\right)(\gamma)\, G(\gamma)\, \pi^\Lambda_\sigma(d\gamma)\\ &\quad + \int_{\Gamma_\Lambda} \int_{\partial\Lambda} G(\gamma + \varepsilon_s) \frac{\partial}{\partial n} F(\gamma + \varepsilon_s) \tilde{\sigma}(ds)\, \pi^\Lambda_\sigma(d\gamma).\end{aligned}$$

Proof. Formula (3.10) directly follows from Corollary 3.2 and formulas (3.8) and (3.9). □

Proposition 3.4 (The second Green formula)**.** *Let* $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. *Then*

$$\int_{\Gamma_\Lambda} \left(\left(H_{\pi_\sigma}^\Gamma F \right)(\gamma) G(\gamma) - F(\gamma) \left(H_{\pi_\sigma}^\Gamma G \right)(\gamma) \right) \pi_\sigma^\Lambda(d\gamma) \tag{3.11}$$

$$= \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \left(F(\gamma+\varepsilon_s) \frac{\partial}{\partial n} G(\gamma+\varepsilon_s) - G(\gamma+\varepsilon_s) \frac{\partial}{\partial n} F(\gamma+\varepsilon_s) \right) \tilde{\sigma}(ds) \, \pi_\sigma^\Lambda(d\gamma)$$

Proof. Formula (3.11) is a direct consequence of Proposition 3.3. □

Define (cf. (2.24))

$$\mathcal{E}_{\pi_\sigma^\Lambda, H_\sigma}^P(F, G) := \int_{\Gamma_\Lambda} \left(\nabla^P F, H_\sigma \nabla^P G \right)_{L^2(\Lambda,\sigma)} \pi_\sigma^\Lambda(d\gamma), \tag{3.12}$$

$$F, G \in \mathcal{FP}(\mathcal{D}, \Gamma).$$

Note that the form (3.12) is not symmetric and does not coincide with the bilinear form of the operator $H_{\pi_\sigma}^\Gamma$ in $L^2\left(\Gamma_\Lambda, \pi_\sigma^\Lambda\right)$. Nevertheless, one can prove the following formula.

Proposition 3.5. *For any* $F, G \in \mathcal{FP}(\mathcal{D}, \Gamma)$

$$\mathcal{E}_{\pi_\sigma^\Lambda, H_\sigma}^P(F, G) \tag{3.13}$$

$$= \int_{\Gamma_\Lambda} F(\gamma) \left(H_{\pi_\sigma}^\Gamma G \right)(\gamma) \, \pi_\sigma^\Lambda(d\gamma) + \int_{\Gamma_\Lambda} F(\gamma) \int_{\partial\Lambda} \frac{\partial}{\partial n} G(\gamma+\varepsilon_s) \tilde{\sigma}(ds) \, \pi_\sigma^\Lambda(d\gamma).$$

Proof. By (3.3), (3.12), (2.22), (2.26), (3.8) and the classical Gauss formula

$$\begin{aligned}
\mathcal{E}_{\pi_\sigma^\Lambda, H_\sigma}^P(F, G) &= \int_{\Gamma_\Lambda} \left(\nabla^P F, H_\sigma \nabla^P G \right)_{L^2(\Lambda,\sigma)} \pi_\sigma^\Lambda(d\gamma) \\
&= \int_{\Gamma_\Lambda} \int_\Lambda \left(F(\gamma+\varepsilon_x) - F(\gamma) \right) \cdot H_\sigma \left(G(\gamma+\varepsilon_x) - G(\gamma) \right) \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma) \\
&= \int_{\Gamma_\Lambda} \int_\Lambda \left(F(\gamma+\varepsilon_x) - F(\gamma) \right) \cdot H_\sigma G(\gamma+\varepsilon_x) \, \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma) \\
&= -\int_{\Gamma_\Lambda} \int_\Lambda F(\gamma+\varepsilon_x) \cdot \operatorname{div}_\sigma \nabla G(\gamma+\varepsilon_x) \, \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma) \\
&\quad + \int_{\Gamma_\Lambda} \int_\Lambda F(\gamma) \cdot \operatorname{div}_\sigma \nabla G(\gamma+\varepsilon_x) \, \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma) \\
&= -\int_{\Gamma_\Lambda} \int_\Lambda \operatorname{div}_\sigma \left(F(\gamma+\varepsilon_x) \cdot \nabla G(\gamma+\varepsilon_x) \right) \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma) \\
&\quad + \int_{\Gamma_\Lambda} \int_\Lambda \left\langle \nabla^\Gamma F(\gamma+\varepsilon_x, x), \nabla^\Gamma G(\gamma+\varepsilon_x, x) \right\rangle_{T_x(X)} \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma) \\
&\quad + \int_{\Gamma_\Lambda} \int_{\partial\Lambda} F(\gamma) \frac{\partial}{\partial n} G(\gamma+\varepsilon_s) \, \tilde{\sigma}(ds) \, \pi_\sigma^\Lambda(d\gamma) \\
&= -\int_{\Gamma_\Lambda} \int_{\partial\Lambda} F(\gamma+\varepsilon_s) \cdot \frac{\partial}{\partial n} G(\gamma+\varepsilon_x) \, \tilde{\sigma}(ds) \, \pi_\sigma^\Lambda(d\gamma)
\end{aligned}$$

$$+\int_{\Gamma_\Lambda}\left\langle \nabla^\Gamma F(\gamma),\nabla^\Gamma G(\gamma)\right\rangle_{T_\gamma(\Gamma)}\pi_\sigma^\Lambda(d\gamma)$$
$$+\int_{\Gamma_\Lambda}\int_{\partial\Lambda}F(\gamma)\frac{\partial}{\partial n}G(\gamma+\varepsilon_s)\,\tilde\sigma(ds)\,\pi_\sigma^\Lambda(d\gamma).$$

Now the assertion follows by Proposition 3.3. □

4 Symmetric Extensions of $H^\Gamma_{\pi_\sigma}$ in $L^2\left(\Gamma_\Lambda,\pi_\sigma^\Lambda\right)$.

Let $\mathcal{D}(\Lambda):=C_0^\infty(\Lambda)$ be the set of all C^∞-functions on X with compact support in Λ. Define $\mathcal{F}C_b^\infty(\mathcal{D}(\Lambda),\Gamma_\Lambda)$ as the set of all functions $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_\Lambda)$ of the form (2.8) on Γ_Λ with $\varphi_j\in\mathcal{D}(\Lambda)$, $j=1,\ldots,N$. In this section we study symmetric extensions of the minimal operator $H_{\pi_\sigma,\min}:=\left(H^\Gamma_{\pi_\sigma},\mathcal{F}C_b^\infty(\mathcal{D}(\Lambda),\Gamma_\Lambda)\right)$ which are defined by the same differential expression (2.19). Note first that $H_{\pi_\sigma,\min}$ is a symmetric operator in $L^2(\Gamma_\Lambda,\pi_\sigma^\Lambda)$. This directly follows from Proposition 3.4 and the fact that for $F\in C_b^\infty(\mathcal{D}(\Lambda),\Gamma_\Lambda)$, $s\in\partial\Lambda$

$$\frac{\partial}{\partial n}F(\gamma+\varepsilon_s)=\left\langle\nabla^\Gamma F(\gamma+\varepsilon_s,s),n_s\right\rangle_{T_s(X)}= \tag{4.1}$$
$$\sum_{j=1}^N\frac{\partial g_F}{\partial q_j}\left(\langle\varphi_1,\gamma+\varepsilon_s\rangle,\ldots,\langle\varphi_N,\gamma+\varepsilon_s\rangle\right)\langle\nabla\varphi_j(s),n_s\rangle_{T_s(X)}=0.$$

Define the maximal operator $H_{\pi_\sigma,\max}$ by the standard relation

$$H_{\pi_\sigma,\max}:=(H_{\pi_\sigma,\min})^*,$$

where $(\ )^*$ denotes adjoint in $L^2\left(\Gamma_\Lambda,\pi_\sigma^\Lambda\right)$. Note that $H_{\pi_\sigma,\max}$ extends any symmetric extension of $H_{\pi_\sigma,\min}$.

Proposition 4.1. *We have* $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_\Lambda)\subset\mathrm{Dom}(H_{\pi_\sigma,\max})$ *and for any* $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$(H_{\pi_\sigma,\max}F)(\gamma)=(H^\Gamma_{\pi_\sigma}F)(\gamma)+\int_{\partial\Lambda}\frac{\partial}{\partial n}F(\gamma+\varepsilon_s)\,\tilde\sigma(ds). \tag{4.2}$$

Proof. It follows directly from Proposition 3.4 and (4.1) that for any $F\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_\Lambda)$ and $G\in\mathcal{F}C_b^\infty(\mathcal{D}(\Lambda),\Gamma_\Lambda)$

$$\int_{\Gamma_\Lambda}\left((H^\Gamma_{\pi_\sigma}F)(\gamma)G(\gamma)-F(\gamma)(H^\Gamma_{\pi_\sigma}G)(\gamma)\right)\pi_\sigma^\Lambda(d\gamma)$$
$$=-\int_{\Gamma_\Lambda}\int_{\partial\Lambda}G(\gamma)\frac{\partial}{\partial n}F(\gamma+\varepsilon_s)\,\tilde\sigma(ds)\,\pi_\sigma^\Lambda(d\gamma).$$

The latter relation implies (4.2). □

Corollary 4.2. *For any* $F,G\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_\Lambda)$

$$\mathcal{E}^P_{\pi_\sigma^\Lambda,H_\sigma}(F,G)=(F,H_{\pi_\sigma,\max}G)_{L^2(\Gamma_\Lambda)}. \tag{4.3}$$

Proof. Formula (4.3) is a direct consequence of (3.13) and (4.2). □

Remark 4.3. Suppose that

$$\mathcal{F}C_b^\infty\left(\mathcal{D}\left(\Lambda\right),\Gamma\right)\subset\mathcal{F}\subset\mathcal{F}C_b^\infty\left(\mathcal{D},\Gamma\right). \tag{4.4}$$

Then obviously by Corollary 4.2 $(H_{\pi_\sigma,\max},\mathcal{F})$ is a symmetric extension of $H_{\pi_\sigma,\min}$ if and only if $\left(\mathcal{E}^P_{\pi_\sigma^\Lambda,H_\sigma},\mathcal{F}\right)$ is a symmetric bilinear form.

In what follows, we describe a class of self-adjoint extensions of $H_{\pi_\sigma,\min}$ which are defined by the standard differential expression (2.19) (without any additional term). Besides, we give a differential expression corresponding to the Friedrichs extension of $H_{\pi_\sigma,\min}$. We start with the following simple proposition.

Proposition 4.4. *Suppose that the condition* (4.4) *is fulfilled and* $\left(H^\Gamma_{\pi_\sigma},\mathcal{F}\right)$ *is a symmetric extension of* $H_{\pi_\sigma,\min}$. *Then for any* $F\in\mathcal{F}$

$$\int_{\partial\Lambda}\frac{\partial}{\partial n}F\left(\gamma+\varepsilon_s\right)\tilde{\sigma}\left(ds\right)=0,\quad\text{for }\pi_\sigma^\Lambda\text{-a.e. }\gamma\in\Gamma_\Lambda$$

Proof. Since $H_{\pi_\sigma,\max}$ extends $\left(H^\Gamma_{\pi_\sigma},\mathcal{F}\right)$, the assertion follows from Proposition 4.1. □

In the following we will use the system of Charlier polynomials (see, e.g., [1], [8], [6]) which can be defined through the following generating functional

$$e^\Lambda_{\pi_\sigma}\left(\varphi,\cdot\right):=\exp\left(\left\langle\log\left(1+\varphi\right),\cdot\right\rangle-\left\langle\varphi\right\rangle_{\sigma,\Lambda}\right)=\sum_{n=0}^\infty\frac{1}{n!}Q_{n,\Lambda}\left(\varphi^{\otimes n},\cdot\right),$$

where $\varphi\in\mathcal{D}$, $\langle\varphi\rangle_{\sigma,\Lambda}=\int_\Lambda\varphi(x)\sigma(d\,x)$. Note that the Charlier polynomials of different order are mutually orthogonal in $L^2\left(\pi_\sigma^\Lambda\right)$. More precisely,

$$\int_{\Gamma_\Lambda}Q_{n,\Lambda}\left(\varphi^{(n)},\gamma\right)Q_{m,\Lambda}\left(\psi^{(m)},\gamma\right)\pi^\Lambda_\sigma\left(d\gamma\right)=n!\,\delta_{nm}\left(\varphi^{(n)},\psi^{(n)}\right)_{L^2(\Lambda^n,\sigma^{\otimes n})},$$
$$\varphi^{(n)}\in\mathcal{D}^{\hat{\otimes}n},\psi^{(m)}\in\mathcal{D}^{\hat{\otimes}m}$$

To prove the main result of this section we need the following simple lemma.

Lemma 4.5. *Let* $m\in\mathbb{N}$ *and* $\varphi\in\mathcal{D},y\in\Lambda$. *Then*

$$Q_{m,\Lambda}\left(\varphi^{\otimes m},\gamma+\varepsilon_y\right)=Q_{m,\Lambda}\left(\varphi^{\otimes m},\gamma\right)+m\varphi\left(y\right)Q_{m-1,\Lambda}\left(\varphi^{\otimes(m-1)},\gamma\right). \tag{4.5}$$

Proof. For any $z>0$ we have

$$e^\Lambda_{\pi_\sigma}\left(z\varphi,\gamma+\varepsilon_y\right)=\left(1+z\varphi\left(y\right)\right)e^\Lambda_{\pi_\sigma}\left(z\varphi,\gamma\right).$$

We can expand both sides of this equality into series

$$\sum_{n=0}^\infty z^nQ_{n,\Lambda}\left(\varphi^{\otimes n},\gamma+\varepsilon_y\right)=\left(1+z\varphi\left(y\right)\right)\sum_{n=0}^\infty z^nQ_{n,\Lambda}\left(\varphi^{\otimes n},\gamma\right)$$

and then a comparison of coefficients gives (4.5). □

Corollary 4.6. *Let $m \in \mathbb{N}, \varphi \in \mathcal{D}, y \in \Lambda$ and $s \in \partial\Lambda$. Then*

$$\nabla Q_{m,\Lambda}\left(\varphi^{\otimes m}, \gamma + \varepsilon_y\right) = m \nabla \varphi\left(y\right) Q_{m-1,\Lambda}\left(\varphi^{\otimes (m-1)}, \gamma\right) \tag{4.6}$$

$$\begin{aligned}\frac{\partial}{\partial n} Q_m(\varphi^{\otimes m}, \gamma + \varepsilon_s) &= \left\langle \nabla Q_{m,\Lambda}\left(\varphi^{\otimes n}, \gamma + \varepsilon_s\right), n_s \right\rangle_{T_s(X)} \\ &= m \frac{\partial}{\partial n} \varphi\left(s\right) Q_{m-1,\Lambda}\left(\varphi^{\otimes (m-1)}, \gamma\right) \end{aligned} \tag{4.7}$$

Let $\mathcal{A}$ be some subalgebra of $\mathcal{D}$ and $\mathcal{D}\left(\Lambda\right) \subset \mathcal{A}$. Define the class $\mathcal{FP}\left(\mathcal{A}, \Gamma_\Lambda\right)$ as the set of functions $F \in \mathcal{FP}\left(\mathcal{D}, \Gamma_\Lambda\right)$ of the form (2.8) on Γ_Λ, with $\varphi_j \in \mathcal{A}$, $j = 1, \ldots, N$. The following theorem describes all symmetric extensions of the operator $\left(H^\Gamma_{\pi_\sigma}, \mathcal{FP}\left(\mathcal{D}\left(\Lambda\right), \Gamma_\Lambda\right)\right)$ which are given by the differential expression (2.19) on the set $\mathcal{FP}\left(\mathcal{A}, \Gamma_\Lambda\right)$.

Theorem 4.7. $\left(H^\Gamma_{\pi_\sigma}, \mathcal{FP}\left(\mathcal{A}, \Gamma_\Lambda\right)\right)$ *is a symmetric operator in* $L^2\left(\pi^\Lambda_\sigma\right)$ *if and only if* $\left(H_\sigma, \mathcal{A}\right)$ *is a symmetric operator in* $L^2\left(\Lambda, \sigma\right)$ *and for any* $\varphi \in \mathcal{A}$

$$\int_{\partial\Lambda} \frac{\partial}{\partial n} \varphi\left(s\right) \tilde{\sigma}\left(ds\right) = 0. \tag{4.8}$$

Remark 4.8. Under the assumptions of Theorem 4.7 $\left(H^\Gamma_{\pi_\sigma}, \mathcal{FP}\left(\mathcal{A}, \Gamma\right)\right)$ is the image of the second quantization of the symmetric (in $L^2\left(\Lambda, \sigma\right)$) operator $\left(H_\sigma, \mathcal{A}\right)$.

Remark 4.9. It directly follows from the proof of Theorem 4.7 that the "only if" part of this theorem is valid without the assumption that $\mathcal{A}$ is an algebra. We need only the inclusion $\mathcal{D}(\Lambda) \subset \mathcal{A} \subset \mathcal{D}$.

Remark 4.10. It follows from the classical Gauss formula that the condition (4.8) is equivalent to the condition

$$\int_{\Lambda} \left(H_\sigma \varphi\right)\left(x\right) \sigma\left(dx\right) = 0. \tag{4.9}$$

Proof. First suppose that $\left(H_\sigma, \mathcal{A}\right)$ is a symmetric operator on $L^2\left(\Lambda, \sigma\right)$ and for any $\varphi \in \mathcal{A}$ the condition (4.8) is fulfilled. For $\varphi, \psi \in \mathcal{A}$ consider $F = Q_{k,\Lambda}\left(\varphi^{\otimes k}, \gamma\right)$, $G = Q_{m,\Lambda}\left(\psi^{\otimes m}, \gamma\right)$ (since $\mathcal{A}$ is an algebra, $F, G \in \mathcal{FP}\left(\mathcal{A}, \Gamma_\Lambda\right)$).

By (4.5), (4.7) and (4.8) we have

$$\begin{aligned} &\int_{\Gamma_\Lambda} \int_{\partial\Lambda} Q_{k,\Lambda}\left(\varphi^{\otimes k}, \gamma + \varepsilon_s\right) \frac{\partial}{\partial n} Q_{m,\Lambda}(\psi^{\otimes m}, \gamma + \varepsilon_s) \tilde{\sigma}\left(ds\right) \pi^\Lambda_\sigma\left(d\gamma\right) \\ &= m \int_{\Gamma_\Lambda} Q_{k,\Lambda}\left(\varphi^{\otimes k}, \gamma\right) Q_{m-1,\Lambda}\left(\psi^{\otimes (m-1)}, \gamma\right) \pi^\Lambda_\sigma\left(d\gamma\right) \int_{\partial\Lambda} \frac{\partial}{\partial n} \psi\left(s\right) \tilde{\sigma}\left(ds\right) \\ &\quad + km \int_{\Gamma_\Lambda} Q_{k-1,\Lambda}\left(\varphi^{\otimes (k-1)}, \gamma\right) Q_{m-1,\Lambda}\left(\psi^{\otimes (m-1)}, \gamma\right) \pi^\Lambda_\sigma\left(d\gamma\right) \\ &\times \int_{\partial\Lambda} \varphi\left(s\right) \frac{\partial}{\partial n} \psi\left(s\right) \tilde{\sigma}\left(ds\right) = k \cdot k!\, \delta_{km}\left(\varphi, \psi\right)^{k-1}_{L^2(\Lambda,\sigma)} \int_{\partial\Lambda} \varphi\left(s\right) \frac{\partial}{\partial n} \psi\left(s\right) \tilde{\sigma}\left(ds\right) \end{aligned}$$

and analogously

$$\int_{\Gamma_\Lambda}\int_{\partial\Lambda} Q_{m,\Lambda}\left(\psi^{\otimes m},\gamma+\varepsilon_s\right)\frac{\partial}{\partial n}Q_{k,\Lambda}(\varphi^{\otimes k},\gamma+\varepsilon_s)\tilde{\sigma}\left(ds\right)\pi_\sigma^\Lambda\left(d\gamma\right)$$
$$= k\cdot k!\,\delta_{km}\left(\varphi,\psi\right)^{k-1}_{L^2(\Lambda,\sigma)}\int_{\partial\Lambda}\psi\left(s\right)\frac{\partial}{\partial n}\varphi\left(s\right)\tilde{\sigma}\left(ds\right).$$

So, by standard Green formula

$$\begin{aligned}&\int_{\Gamma_\Lambda}\int_{\partial\Lambda} Q_{k,\Lambda}\left(\varphi^{\otimes k},\gamma+\varepsilon_s\right)\frac{\partial}{\partial n}Q_{m,\Lambda}(\psi^{\otimes m},\gamma+\varepsilon_s)\tilde{\sigma}\left(ds\right)\pi_\sigma^\Lambda\left(d\gamma\right)\\ &-\int_{\Gamma_\Lambda}\int_{\partial\Lambda} Q_{m,\Lambda}\left(\psi^{\otimes m},\gamma+\varepsilon_s\right)\frac{\partial}{\partial n}Q_{k,\Lambda}(\varphi^{\otimes k},\gamma+\varepsilon_s)\tilde{\sigma}\left(ds\right)\pi_\sigma^\Lambda\left(d\gamma\right)\\ &= k\cdot k!\,\delta_{km}\left(\varphi,\psi\right)^{k-1}_{L^2(\Lambda,\sigma)}\int_{\partial\Lambda}\left(\varphi\left(s\right)\frac{\partial}{\partial n}\psi\left(s\right)-\psi\left(s\right)\frac{\partial}{\partial n}\varphi\left(s\right)\right)\tilde{\sigma}\left(ds\right)\\ &= k\cdot k!\,\delta_{km}\left(\varphi,\psi\right)^{k-1}_{L^2(\Lambda,\sigma)}\int_{\Lambda}\left(\varphi\left(x\right)H_\sigma\psi\left(x\right)-\psi\left(x\right)H_\sigma\varphi\left(x\right)\right)\sigma\left(dx\right)=0.\end{aligned}$$

Then by (3.8) and (3.11) we have

$$\int_{\Gamma_\Lambda}\left(\left(H^\Gamma_{\pi_\sigma}F\right)\left(\gamma\right)G\left(\gamma\right)-F\left(\gamma\right)\left(H^\Gamma_{\pi_\sigma}G\right)\left(\gamma\right)\right)\pi_\sigma^\Lambda\left(d\gamma\right)=0 \tag{4.10}$$

By standard arguments it follows that $\left(H^\Gamma_{\pi_\sigma},\mathcal{FP}\left(\mathcal{A},\Gamma_\Lambda\right)\right)$ is a symmetric operator in $L^2\left(\pi_\sigma^\Lambda\right)$.

Conversely, suppose that $\left(H^\Gamma_{\pi_\sigma},\mathcal{FP}\left(\mathcal{A},\Gamma_\Lambda\right)\right)$ is a symmetric operator in $L^2\left(\pi_\sigma^\Lambda\right)$. Let $\varphi,\psi\in\mathcal{A}$. Then $Q_{1,\Lambda}\left(\varphi,\gamma\right)=\langle\varphi,\gamma\rangle-\langle\varphi\rangle_{\sigma,\Lambda}$, $Q_{1,\Lambda}\left(\psi,\gamma\right)=\langle\psi,\gamma\rangle-\langle\psi\rangle_{\sigma,\Lambda}$ and $Q_{0,\Lambda}=1$ (see, e.g., [8]). By the same arguments as in the first part of this proof

$$\begin{aligned}0=&\int_{\Gamma_\Lambda}\left(\left(H^\Gamma_{\pi_\sigma}Q_{1,\Lambda}\left(\varphi,\gamma\right)\right)Q_{1,\Lambda}\left(\psi,\gamma\right)-Q_{1,\Lambda}\left(\varphi,\gamma\right)\left(H^\Gamma_{\pi_\sigma}Q_{1,\Lambda}\left(\psi,\gamma\right)\right)\right)\pi_\sigma^\Lambda\left(d\gamma\right)\\ &=\int_{\Gamma_\Lambda}Q_{1,\Lambda}\left(\varphi,\gamma\right)Q_{0,\Lambda}\pi_\sigma^\Lambda\left(d\gamma\right)\int_{\partial\Lambda}\frac{\partial}{\partial n}\psi\left(s\right)\tilde{\sigma}\left(ds\right)\\ &+\int_{\Gamma_\Lambda}Q_{0,\Lambda}Q_{0,\Lambda}\pi_\sigma^\Lambda\left(d\gamma\right)\int_{\partial\Lambda}\varphi\left(s\right)\frac{\partial}{\partial n}\psi\left(s\right)\tilde{\sigma}\left(ds\right)\\ &-\int_{\Gamma_\Lambda}Q_{1,\Lambda}\left(\psi,\gamma\right)Q_{0,\Lambda}\pi_\sigma^\Lambda\left(d\gamma\right)\int_{\partial\Lambda}\frac{\partial}{\partial n}\varphi\left(s\right)\tilde{\sigma}\left(ds\right)\\ &-\int_{\Gamma_\Lambda}Q_{0,\Lambda}Q_{0,\Lambda}\pi_\sigma^\Lambda\left(d\gamma\right)\int_{\partial\Lambda}\psi\left(s\right)\frac{\partial}{\partial n}\varphi\left(s\right)\tilde{\sigma}\left(ds\right)\\ &=\int_{\partial\Lambda}\left(\varphi\left(s\right)\frac{\partial}{\partial n}\psi\left(s\right)-\psi\left(s\right)\frac{\partial}{\partial n}\varphi\left(s\right)\right)\tilde{\sigma}\left(ds\right).\end{aligned}$$

So, by the classical Green formula $\left(\varphi,H_\sigma\psi\right)_{L^2(\Lambda,\sigma)}=\left(H_\sigma\varphi,\psi\right)_{L^2(\Lambda,\sigma)}$

By (2.19) and (2.26) we see that $H_{\pi_\sigma}^{\Gamma} Q_{1,\Lambda}(\varphi,\gamma) = \langle H_\sigma \varphi, \gamma\rangle$, and $H_{\pi_\sigma}^{\Gamma} Q_{0,\Lambda} = 0$. So, by (3.3)

$$0 = \int_{\Gamma_\Lambda} \left(\left(H_{\pi_\sigma}^{\Gamma} Q_{1,\Lambda}(\varphi,\gamma)\right) Q_{0,\Lambda} - Q_{1,\Lambda}(\varphi,\gamma) \left(H_{\pi_\sigma}^{\Gamma} Q_{0,\Lambda}\right)\right) \pi_\sigma^\Lambda (d\gamma)$$

$$= \int_{\Gamma_\Lambda} \langle H_\sigma\varphi,\gamma\rangle \pi_\sigma^\Lambda(d\gamma) = \int_\Lambda H_\sigma \varphi(x)\,\sigma(dx) = \int_{\partial\Lambda} \frac{\partial}{\partial n}\varphi(s)\,\tilde{\sigma}(ds).$$

□

Theorem 4.11. *Suppose that $(H_\sigma, \mathcal{A})$ is an essentially self-adjoint operator in $L^2(\Lambda,\sigma)$. Then $\left(H_{\pi_\sigma}^{\Gamma}, \mathcal{FP}(\mathcal{A},\Gamma_\Lambda)\right)$ is an essentially self-adjoint operator in $L^2\left(\Gamma_\Lambda, \pi_\sigma^\Lambda\right)$ if and only if for any $\varphi\in\mathcal{A}$ condition (4.8) is fulfilled.*

Proof. The result follows immediately from Theorem 4.7, Remarks 4.8 and the fact that the second quantization of an essentially self-adjoint operator is an essentially self-adjoint operator in the corresponding Hilbert space. □

Denote by $\mathcal{D}_N$ the set of all functions from $\mathcal{D}$ satisfying the Neumann boundary condition $\frac{\partial}{\partial n}\varphi\restriction_{\partial\Lambda} = 0$. Clearly, $\mathcal{D}_N$ is a subalgebra of $\mathcal{D}$. The following result is an important special case of Theorem 4.11.

Theorem 4.12. *Suppose that $(H_\sigma, \mathcal{D}_N)$ is an essentially self-adjoint operator in $L^2(\Lambda,\sigma)$. Then $\left(H_{\pi_\sigma}^{\Gamma}, \mathcal{F}C_b^\infty(\mathcal{D}_N,\Gamma_\Lambda)\right)$ is an essentially self-adjoint operator in $L^2\left(\Gamma_\Lambda,\pi_\sigma^\Lambda\right)$. Moreover, the closure of this operator coincides with the second quantization of the closure of $(H_\sigma,\mathcal{D}_N)$.*

Let us give another simple example of an operator satisfying the conditions of Theorem 4.11.

Example 4.13. Put $X=\mathbb{R}$, $\Lambda=(0,1)$ and $\sigma(dx)=dx$ (the Lebesgue measure on $\mathbb{R}$). Then $(H_\sigma\varphi)(x) = -\varphi''(x)$. Then (4.8) is equivalent to the condition

$$\varphi'(1) = \varphi'(0).$$

Define $\mathcal{A} := \left\{\varphi\in C^2[0,1]\,|\,\varphi(0)=\varphi(1), \varphi'(0)=\varphi'(1)\right\}$. Then the operator $\left(-\frac{d^2}{dx^2},\mathcal{A}\right)$ is essentially self-adjoint in $L^2((0,1),dx)$ and the conditions of Corollary 4.11 are fulfilled. Note that for a general domain Λ with a bounded piecewise C^1 boundary H_σ is a symmetric operator on the algebra

$$\mathcal{A} = \{\varphi\in\mathcal{D}\,|\,\varphi \text{ satisfies (4.8) and } \varphi\restriction\partial\Lambda = c(\varphi) = \text{const}\}$$

in $L^2(\Lambda,\sigma)$. (This fact directly follows from the standard Green formula).

To end this section, we shall present an explicit formula for the action of the Friedrichs extension $H_{\pi_\sigma,D}^{\Gamma}$ of $H_{\pi_\sigma,\min}$ on smooth cylinder functions. By $\mathcal{D}_D$ denote the set of all functions from $\mathcal{D}$ satisfying the Dirichlet boundary condition on $\partial\Lambda$ and let $H_{\sigma,D}$ be the Friedrichs extension of $(H_\sigma,\mathcal{D}(\Lambda))$. The following proposition gives a formula for the action of $H_{\pi_\sigma,D}^{\Gamma}$ on the smooth cylinder functions.

Theorem 4.14. *Suppose that $(H_{\sigma,D},\mathcal{D}_D)$ is an essentially self-adjoint in $L^2(\Lambda,\sigma)$. Then the closure of the operator $\left(H^{\Gamma}_{\pi_\sigma,D},\mathcal{FP}(\mathcal{D}_D,\Gamma_\Lambda)\right)$ defined by the differential expression for $F=g_F(\langle\varphi_1,\cdot\rangle,\dots,\langle\varphi_N,\cdot\rangle)\in\mathcal{FP}(\mathcal{D}_D,\Gamma)$*

$$\left(H^{\Gamma}_{\pi_\sigma,D}F\right)(\gamma):= \tag{4.11}$$
$$\left(H^{\Gamma}_{\pi_\sigma}F\right)(\gamma)+\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}(\langle\varphi_1,\gamma\rangle,\dots,\langle\varphi_N,\gamma\rangle)\cdot\int_{\partial\Lambda}\frac{\partial\varphi_j}{\partial n}(s)\tilde{\sigma}(ds),$$

coincides with the Friedrichs extension of $H_{\pi_\sigma,\min}$ in $L^2\left(\pi^{\Lambda}_{\sigma}\right)$.

Proof. First, we recall that, for $F,G\in\mathcal{FP}(\mathcal{D}(\Lambda),\Gamma_\Lambda)$

$$(H^{\Gamma}_{\pi_\sigma}F,G)_{L^2(\pi^{\Lambda}_{\sigma})}=(H^{P}_{H_\sigma}F,G)_{L^2(\pi^{\Lambda}_{\sigma})}.$$

Here $H^{P}_{H_\sigma}$ is the image of the second quantization of the symmetric (in $L^2(\Lambda)$) operator $(H_\sigma,\mathcal{D}(\Lambda))$. Therefore, the Friedrichs extension $H^{\Gamma}_{\pi_\sigma,D}$ of the minimal operator $H_{\pi_\sigma,\min}$ is the image of the second quantization of $H_{\sigma,D}$. In particular, $\left(H^{\Gamma}_{\pi_\sigma,D},\mathcal{FP}(\mathcal{D}_D,\Gamma_\Lambda)\right)$ is essentially self-adjoint in $L^2\left(\pi^{\Lambda}_{\sigma}\right)$. Therefore, we only need to prove (4.11). This, however, directly follows from Proposition 4.1 and the operator inclusion $H_{\pi_\sigma,\min}\subset H^{\Gamma}_{\pi_\sigma,D}\subset H_{\pi_\sigma,\max}$. (Note that for $F\in\mathcal{FP}(\mathcal{D}_D,\Gamma_\Lambda)$ the differential expressions (4.2) and (4.11) coincide). □

5 Gibbsian case

Consider a function $\phi:\mathbb{R}^d\to\mathbb{R}\cup\{+\infty\}$, such that $\phi(-x)=\phi(x)$. For any $\Lambda\in\mathcal{O}_c\left(\mathbb{R}^d\right)$ the conditional energy $E^{\phi}_{\Lambda}:\Gamma\to\mathbb{R}\cup\{+\infty\}$ is defined by

$$E^{\phi}_{\Lambda}(\gamma)=E^{\phi}_{\Lambda}(\gamma_\Lambda)+W(\gamma_\Lambda|\gamma_{\Lambda^c}), \tag{5.1}$$

where

$$W(\gamma_\Lambda|\gamma_{\Lambda^c}):=\sum_{x\in\gamma_\Lambda,\ y\in\gamma_{\Lambda^c}}\phi(x-y) \tag{5.2}$$

describes the interaction energy between γ_Λ and γ_{Λ^c} ($\Lambda^c:=\mathbb{R}^d\setminus\Lambda$) and

$$E^{\phi}_{\Lambda}(\gamma_\Lambda):=\sum_{\{x,y\}\in\gamma_\Lambda}\phi(x-y) \tag{5.3}$$

is the conditional energy corresponding to Λ.

Consider for any $\gamma\in\Gamma,\Delta\in\mathcal{B}(\Gamma)$

$$\Pi^{\phi}_{\Lambda}(\gamma,\Delta):=1\!\!1_{\left\{Z^{\phi}_{\Lambda}<\infty\right\}}(\gamma)\left[Z^{\phi}_{\Lambda}(\gamma)\right]^{-1}\int_{\Gamma}1\!\!1_{\Delta}(\gamma_{\Lambda^c}\cup\gamma\prime_{\Lambda}) \tag{5.4}$$
$$\times\exp\left[-E^{\phi}_{\Lambda}(\gamma_{\Lambda^c}\cup\gamma\prime_{\Lambda})\right]\pi_m(d\gamma\prime),$$

where

$$Z_{\Lambda}^{\phi}(\gamma) := \int_{\Gamma} \exp\left[-E_{\Lambda}^{\phi}\left(\gamma_{\Lambda^c} \cup \gamma'_{\Lambda}\right)\right] \pi_m(d\gamma'). \tag{5.5}$$

A probability measure $\mu = \mu^{\phi}$ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a grand canonical Gibbs measure with potential ϕ, or Ruelle measure, if for all $\Lambda \in \mathcal{O}_c(X)$ and $\Delta \in \mathcal{B}(\Gamma)$ the following Dobrushin-Lenford-Ruelle equation is true

$$\int_{\Gamma} \Pi_{\Lambda}^{\phi}(\gamma, \Delta)\, \mu(d\gamma) = \mu(\Delta). \tag{5.6}$$

For any $r = (r_1, \ldots, r_d) \in \mathbb{Z}^d$ consider the cube

$$Q_r := \left\{ x \in \mathbb{R}^d \,\middle|\, r_i - \frac{1}{2} \leq x_i < r_i + \frac{1}{2} \right\}$$

and for any $\gamma \in \Gamma$ set $\gamma_r := \gamma_{Q_r}$. Let Λ_n be the cube with side length $2n-1$ centered at the origin in $\mathbb{R}^d$. In what follows we shall always assume the following conditions on the interaction ϕ.

(SS) (Superstability). There exists $A > 0, B \geq 0$ such that if $\gamma \in \Gamma_{\Lambda_n}$, then

$$E_{\Lambda_n}^{\phi}(\gamma) \geq \sum_{r \in \mathbb{Z}^d} \left(A\left|\gamma_r\right|^2 - B\left|\gamma\right| \right).$$

(LR) (Lower regularity). There exists a decreasing positive function $a : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^d} a(\|r\|) < \infty,$$

and for any Λ', Λ'' which are each finite unions cubes of the form Q_r and disjoint, and all $\gamma' \in \Lambda', \gamma'' \in \Lambda''$,

$$W(\gamma' | \gamma'') \geq - \sum_{r', r'' \in Z^d} a(\|r' - r''\|) \left|\gamma'_{r'}\right| \left|\gamma''_{r''}\right|.$$

Here $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^d$.

(D) (Differentiability). $e^{-\phi}$ is C^1 on $\mathbb{R}^d$, ϕ is C^1 on $\mathbb{R}^d \setminus \{0\}$ and the gradient $\nabla\phi$ satisfies the condition

$$\nabla\phi \in L^1\left(\mathbb{R}^d, e^{-\phi} dm\right) \cap L^2\left(\mathbb{R}^d, e^{-\phi} dm\right).$$

(C) ϕ has compact support.

Remark 5.1. The assumption that $e^{-\phi}$ is C^1 on $\mathbb{R}^d$, ϕ is C^1 on $\mathbb{R}^d \setminus \{0\}$ was made only to avoid purely technical complications below. Weak differentiability would have been enough.

For any $v \in V_0(X)$ consider the function:

$$L_v^{\phi}(\gamma) := -\sum_{\{x,y\}\subset\gamma} \langle \nabla\phi(x-y), v(x)-v(y)\rangle_{T_x(X)}. \tag{5.7}$$

It is well-known that under the assumptions above ϕ, $L_v^{\phi} \in L^2(\Gamma,\mu) =: L^2(\mu)$. Set

$$B_v^{\phi}(\gamma) := L_v^{\phi}(\gamma) + \langle \operatorname{div} v, \gamma\rangle.$$

Then the following integration by part formula is true:

$$\int_\Gamma \left(\nabla_v^\Gamma F\right)(\gamma)\, G(\gamma)\,\mu(d\gamma) \tag{5.8}$$

$$= -\int_\Gamma F(\gamma)\left(\nabla_v^\Gamma G\right)(\gamma)\,\mu(d\gamma) - \int_\Gamma F(\gamma)\, G(\gamma)\, B_v^{\phi}(\gamma)\,\mu(d\gamma).$$

for any $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$.

For any vector field of the form (2.15) set

$$\operatorname{div}_\mu^\Gamma V := \sum_{i=1}^N \left(\nabla_{v_i}^\Gamma F_i + B_{v_i}^{\phi} F_i\right). \tag{5.9}$$

Then obviously for $G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$\operatorname{div}_\mu^\Gamma(GV) = G\operatorname{div}_\mu^\Gamma(V) + \left\langle \nabla^\Gamma G, V\right\rangle_{T(\Gamma)}. \tag{5.10}$$

Then this divergence is dual to the gradient ∇^Γ w.r.t. μ:

$$\int_\Gamma \left\langle \nabla^\Gamma F, V\right\rangle_{T(\Gamma)} d\mu = -\int_\Gamma F \operatorname{div}_\mu^\Gamma V d\mu.$$

Set

$$H_\mu^\Gamma := -\operatorname{div}_\mu^\Gamma \nabla^\Gamma. \tag{5.11}$$

Then for any $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$\int_\Gamma \left\langle \nabla^\Gamma F, \nabla^\Gamma G\right\rangle_{T(\Gamma)} d\mu = \int_\Gamma F H_\mu^\Gamma G d\mu.$$

Let $\Lambda \in \mathcal{O}_c\left(\mathbb{R}^d\right)$. Set $\mu_\Lambda := \mu^\phi \circ p_\Lambda^{-1}$. First of all note that

$$L_v^{\phi}(\gamma) = -\sum_{x\in\gamma}\sum_{\substack{y\in\gamma\\ y\neq x}} \langle \nabla\phi(x-y), v(x)\rangle_{T_x(X)}. \tag{5.12}$$

Let $x \in \Lambda$. Then for μ_Λ-a.e. $\gamma \in \Gamma_\Lambda : x \notin \gamma$ and

$$E_{\{x\}}^{\phi}(\gamma+\varepsilon_x) = \sum_{\substack{\gamma'\subset\gamma\cup\{x\}\\ \gamma'(\{x\})>0}} \phi(\gamma') = \sum_{\substack{\gamma'\subset\gamma\cup\{x\}\\ |\gamma'|=2\\ \gamma'(\{x\})>0}} \phi(\gamma')$$

$$= \sum_{y\in\gamma} \phi(x-y) = \langle \phi(x-\cdot), \gamma\rangle.$$

Then by the Nguyen-Zessin identity (see [10])

$$\begin{aligned}&\int_{\Gamma_\Lambda}\int_\Lambda h(\gamma,x)\,\gamma(dx)\,\mu_\Lambda(d\gamma)\\ =&\int_{\Gamma_\Lambda}\int_\Lambda h(\gamma+\varepsilon_x,x)\,e^{-E^\phi_{\{x\}}(\gamma+\varepsilon_x)}m(dx)\,\mu_\Lambda(d\gamma)\\ =&\int_{\Gamma_\Lambda}\int_\Lambda h(\gamma+\varepsilon_x,x)\,e^{-\langle\phi(x-\cdot),\gamma\rangle}m(dx)\,\mu_\Lambda(d\gamma).\end{aligned}$$

Theorem 5.2 (Gauss formula for the Gibbsian case)**.** *For any vector field* V *of the form* (2.15) *we have*

$$\begin{aligned}&\int_{\Gamma_\Lambda}\left(\operatorname{div}^\Gamma_\mu V\right)(\gamma)\,\mu_\Lambda(d\gamma) \qquad (5.13)\\ =&\int_{\Gamma_\Lambda}\int_{\partial\Lambda}\langle V(\gamma+\varepsilon_s,s),n_s\rangle_{T_x(X)}\,e^{-\langle\phi(s-\cdot),\gamma\rangle}\tilde m(ds)\,\mu_\Lambda(d\gamma).\end{aligned}$$

Proof. Clearly, it is sufficient to prove (5.13) only for a field of the form

$$V(\gamma,x)=G(\gamma)\,v(x).$$

Then

$$\begin{aligned}&\int_{\Gamma_\Lambda}\left(\operatorname{div}^\Gamma_\mu V\right)(\gamma)\,\mu_\Lambda(d\gamma)\\ =&\int_{\Gamma_\Lambda}\int_\Lambda\Bigg[\sum_{j=1}^M\frac{\partial G}{\partial q_j}\left(\langle\psi_1,\gamma+\varepsilon_x\rangle,\ldots,\langle\psi_N,\gamma+\varepsilon_x\rangle\right)\nabla_v\psi_j(x)+G(\gamma+\varepsilon_x)\operatorname{div}v(x)\\ &-G(\gamma+\varepsilon_x)\sum_{\substack{y\in\gamma\cup\{x\}\\ y\neq x}}\langle\nabla\phi(x-y),v(x)\rangle_{T_x(X)}\Bigg]e^{-\langle\phi(x-\cdot),\gamma\rangle}m(dx)\,\mu_\Lambda(d\gamma)\end{aligned}$$

(see the beginning of the proof of Theorem 3.1).

Set

$$a(x):=G(\gamma+\varepsilon_x)\,v(x)\,e^{-\langle\phi(x-\cdot),\gamma\rangle}\in V_0(X)$$

(for μ_Λ-a.e. $\gamma\in\Gamma_\Lambda: x\notin\gamma$). Then

$$\begin{aligned}&(\operatorname{div}a)(x)=e^{-\langle\phi(x-\cdot),\gamma\rangle}\\ &\times\Big[\nabla_v G(\gamma+\varepsilon_x)+G(\gamma+\varepsilon_x)(\operatorname{div}v)(x)-G(\gamma+\varepsilon_x)\nabla_v\langle\phi(x-\cdot),\gamma\rangle\Big].\end{aligned}$$

Obviously,

$$\sum_{j=1}^M\frac{\partial G}{\partial q_j}\left(\langle\psi_1,\gamma+\varepsilon_x\rangle,\ldots,\langle\psi_N,\gamma+\varepsilon_x\rangle\right)\nabla_v\psi_j(x)=\nabla_v G(\gamma+\varepsilon_x),$$

$$\sum_{y\in\gamma}\langle\nabla\phi(x-y),v(x)\rangle_{T_x(X)}=\nabla_v\langle\phi(x-\cdot),\gamma\rangle,$$

and hence

$$\begin{aligned}\int_{\Gamma_\Lambda}\left(\mathrm{div}_\mu^\Gamma V\right)(\gamma)\,\mu_\Lambda(d\gamma) &= \int_{\Gamma_\Lambda}\int_\Lambda(\mathrm{div}\,a)(x)\,(dx)\,\mu_\Lambda(d\gamma)\\ &= \int_{\Gamma_\Lambda}\int_{\partial\Lambda}\langle a(s)\,,n_s\rangle_{T_x(X)}\,\tilde m(ds)\,\mu_\Lambda(d\gamma)\\ = \int_{\Gamma_\Lambda}\int_{\partial\Lambda}&\langle G(\gamma+\varepsilon_x)\,v(s)\,,n_s\rangle_{T_x(X)}\,e^{-\langle\phi(s-\cdot),\gamma\rangle}\tilde m(ds)\,\mu_\Lambda(d\gamma)\\ = \int_{\Gamma_\Lambda}\int_{\partial\Lambda}&\langle V(\gamma+\varepsilon_s,s)\,,n_s\rangle_{T_x(X)}\,e^{-\langle\phi(s-\cdot),\gamma\rangle}\tilde m(ds)\,\mu_\Lambda(d\gamma)\,.\end{aligned}$$

□

Proposition 5.3 (The first Green formula formula for Gibbsian case)**.** *For any* $F,G\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$\begin{aligned}\int_{\Gamma_\Lambda}\left\langle\nabla^\Gamma F(\gamma)\,,\nabla^\Gamma G(\gamma)\right\rangle_{T_\gamma(\Gamma)}\mu_\Lambda(d\gamma) &\qquad (5.14)\\ = \int_{\Gamma_\Lambda}\left(H_\mu^\Gamma F\right)(\gamma)\,G(\gamma)\,\mu_\Lambda(d\gamma)&\\ +\int_{\Gamma_\Lambda}\int_{\partial\Lambda}G(\gamma+\varepsilon_s)\,\frac{\partial}{\partial n}F(\gamma+\varepsilon_s)e^{-\langle\phi(s-\cdot),\gamma\rangle}\tilde m(ds)\,\mu_\Lambda(d\gamma)&\end{aligned}$$

Proof. Formula (5.14) directly follows from (5.10) and Theorem 5.2. □

Proposition 5.4 (The second Green formula formula for Gibbsian case)**.** *For any* $F,G\in\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$\begin{aligned}\int_{\Gamma_\Lambda}\left(\left(H_\mu^\Gamma F\right)(\gamma)\,G(\gamma)-F(\gamma)\left(H_\mu^\Gamma G\right)(\gamma)\right)\mu_\Lambda(d\gamma)&\qquad (5.15)\\ = \int_{\Gamma_\Lambda}\int_{\partial\Lambda}\Bigg(F(\gamma+\varepsilon_s)\,\frac{\partial}{\partial n}G(\gamma+\varepsilon_s)&\\ -G(\gamma+\varepsilon_s)\,\frac{\partial}{\partial n}F(\gamma+\varepsilon_s)\Bigg)\,e^{-\langle\phi(s-\cdot),\gamma\rangle}\tilde m(ds)\,\mu_\Lambda(d\gamma)&\end{aligned}$$

Proof. Formula (5.15) is a direct consequence of Proposition 5.3. □

As in Section 4, one can define the minimal operator $H_{\mu,\min}:=(H_\mu^\Gamma,\mathcal{F}C_b^\infty(\mathcal{D}(\Lambda),\Gamma_\Lambda))$. It directly follows from Proposition 5.4 that $H_{\mu,\min}$ is a symmetric operator in $L^2(\Gamma_\Lambda,\mu_\Lambda)$. Define the maximal operator $H_{\mu,\max}$ by

$$H_{\mu,\max}:=(H_{\mu,\min})^*,$$

where $(\quad)^*$ denotes adjoint in $L^2(\Gamma_\Lambda,\mu_\Lambda)$.

Proposition 5.5. *We have* $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma_\Lambda) \subset \mathrm{Dom}(H_{\mu,\,\max})$ *and for any* $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma_\Lambda)$

$$(H_{\mu,\,\max})(\gamma) = (H_\mu^\Gamma F)(\gamma) + \int_{\partial\Lambda} \frac{\partial}{\partial n} F(\gamma+\varepsilon_s) e^{-\langle \phi(s-\cdot),\gamma\rangle}\, \tilde{m}(ds).$$

Proof. The proof is analogous to that of Proposition 4.1. □

Let us give two examples of symmetric extensions of $H_{\mu,\,\min}$ corresponding to Neumann and Dirichlet boundary conditions, respectively.

Proposition 5.6. $(H_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}_N,\Gamma_\Lambda))$ *is a symmetric operator in* $L^2(\Gamma_\Lambda,\mu_\Lambda)$.

Proof. The proof directly follows from Proposition 5.4. □

Define the operator

$$(H_{\mu,D}^\Gamma F)(\gamma) = (H_\mu^\Gamma F)(\gamma) + \sum_{j=1}^N \frac{\partial g_F}{\partial q_j}(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_n,\gamma\rangle) \int_{\partial\Lambda} \frac{\partial \varphi_j}{\partial n}(s) e^{-\langle \phi(s-\cdot),\gamma\rangle}\, \tilde{m}(ds). \tag{5.16}$$

Clearly, formula (5.16) is a Gibbsian analogue of (4.11).

Proposition 5.7. $H_{\mu,D}^\Gamma$ *is a symmetric extension of* $H_{\mu,\min}$. *Moreover, for* $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}_D,\Gamma_\Lambda)$

$$(H_{\mu,D}^\Gamma F, G) = \int_{\Gamma_\Lambda} \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma)\rangle_{T_\gamma(\Gamma_X)}\, \mu_\Lambda(d\gamma). \tag{5.17}$$

Proof. It follows from Proposition (5.5) that $H_{\mu,\min} \subset H_{\mu,D}^\Gamma \subset H_{\mu,\max}$. Moreover, it is easy to see from (5.15) that $H_{\mu,D}^\Gamma$ is a symmetric operator. Equality (5.17) follows from (5.14). □

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