Symmetric differential operators of the second order in Poisson spaces

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Abstract

We study symmetric differential operators of the second order in Poisson spaces.

1 Introduction

Recently [4] we have proved an analogue of the classical Gauss formula for the configuration space Γ_Λ over a domain Λ of $\mathbb{R}^d$ and have studied symmetric extensions of the corresponding Laplacian on Γ_Λ. (We refer to [1], [2], [6], [8] for the basis notions of analysis and geometry on configuration spaces.) The purpose of this paper is to extend some of the results of [4] to the case of general second order differential operators.

The article is arranged as follows. In Section 2 we present some well-known facts on second order differential operators in $L^2\left(\mathbb{R}^d\right)$. In Section 3 we construct the second quantization of a closable operator and define symmetric second order differential operators $H^P_{A_\sigma}$ and $H^{P,\Lambda}_{A_\sigma}$ in $L^2\left(\Gamma, \pi_\sigma\right)$ and $L^2\left(\Gamma_\Lambda, \pi^\Lambda_\sigma\right)$ respectively. Here π_σ is the Poisson measure with intensity σ and π^Λ_σ is the restriction of π_σ to Γ_Λ. In Section 4 we prove an analogue of the classical Green formula for $H^P_{A_\sigma}$. It gives the possibility to describe symmetric extensions of the corresponding minimal operator in $L^2\left(\Gamma_\Lambda, \pi^\Lambda_\sigma\right)$ (see Section 5).

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2 Differential operators of the second order in $L^2\left(\mathbb{R}^d, \sigma\right)$.

Let A be the second order symmetric differential expression of the form

$$(Af)(x) := \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial f}{\partial x_k}(x)\right) + c(x) f(x), \tag{2.1}$$

where $a_{jk} \in C^2\left(\mathbb{R}^d\right), c \in C\left(\mathbb{R}^d\right)$ and $a_{jk} = a_{kj}$ (all the functions a_{jk}, c are assumed to be real).

Let Λ be an open domain of $\mathbb{R}^d$. In the following we always suppose that the boundary $\partial\Lambda$ of Λ is piecewise C^1. Let $\tilde{m}$ be the surface measure on $\partial\Lambda$ corresponding to Lebesgue measure m on $\mathbb{R}^d$. By n_s we denote the outer normal to $\partial\Lambda$ (at the point $s \in \partial\Lambda$) and for any $f \in C^1\left(\bar{\Lambda}\right)$ we consider the co-normal derivative

$$\frac{\partial f}{\partial n^a}(s) := \sum_{j,k=1}^{d} a_{jk}(s)\frac{\partial f}{\partial s_j}(s)\cos\left(n_s, s_k\right), \quad s \in \partial\Lambda. \tag{2.2}$$

Let us introduce also the bilinear form

$$a(u,v)(x) := \sum_{j,k=1}^{d} a_{jk}(x)\frac{\partial u}{\partial x_j}(x)\frac{\partial v}{\partial x_k}(x), \quad u,v \in C^2\left(\mathbb{R}^d\right). \tag{2.3}$$

Consider a C^1-density $\rho > 0$ on $\mathbb{R}^d$. Set $\sigma(dx) = \rho(x)\, m(dx)$, then σ is a non-atomic Radon measure on $\mathbb{R}^d$. Let $\tilde{\sigma}$ be the surface measure on $\partial\Lambda$ corresponding to σ

$$\tilde{\sigma}(ds) = \rho(s)\,\tilde{m}(ds).$$

It's easy to see, that for any $f, g \in C_0^2\left(\mathbb{R}^d\right)$

$$\int_{\mathbb{R}^d}\Big((Af)(x)\, g(x) - f(x)\,(Ag)(x)\Big)\,\sigma(dx)$$
$$= \int_{\mathbb{R}^d}\Big(f(x)\, a(g, \ln\rho)(x) - g(x)\, a(f, \ln\rho)(x)\Big)\,\sigma(dx).$$

Then let us construct a new differential expression A_σ which acts on the smooth function f on $\mathbb{R}^d$ as follows

$$(A_\sigma f)(x) := (Af)(x) + a(f, \ln\rho)(x). \tag{2.4}$$

It defines the second order symmetric differential operator:

$$C_0^2\left(\mathbb{R}^d\right) \ni f \mapsto (A_\sigma f) \in L^2(\sigma),$$

where $L^p(\sigma) := L^p\left(\mathbb{R}^d, \sigma\right)$.

In the following we always suppose that

$$c \in L^2(\sigma) \cap L^1(\sigma). \tag{2.5}$$

Definition 2.1. Introduce a class of smooth functions, which are equal to constant outside some compact set:

$$\mathcal{K}_2 := \textit{linear hull of } \left\{1, C_0^2\left(\mathbb{R}^d\right)\right\}. \tag{2.6}$$

The following proposition is a variant of well-known integration by part formula.

Proposition 2.2. *For any $f, g \in \mathcal{K}_2$ the following statements are true:*

1. $\{A_\sigma f, A_\sigma f \cdot g\} \in L^2(\sigma) \cap L^1(\sigma)$.
2. $(A_\sigma f, g)_{L^2(\sigma)} = (f, A_\sigma g)_{L^2(\sigma)}$.
3. (The first Green formula)

$$\begin{aligned}\int_\Lambda (A_\sigma f)(x)\, g(x)\, \sigma(dx) &= \int_\Lambda c(x)\, f(x)\, g(x)\, \sigma(dx) \\ &+ \int_\Lambda a(f, g)(x)\, \sigma(dx) + \int_{\partial\Lambda} \frac{\partial}{\partial n^a} f(s)\, g(s)\, \tilde{\sigma}(ds).\end{aligned} \tag{2.7}$$

4. (The second Green formula)

$$\begin{aligned}&\int_\Lambda ((A_\sigma f)(x)\, g(x) - f(x)\, (A_\sigma g)(x))\, \sigma(dx) \\ &= \int_{\partial\Lambda} \left(\frac{\partial}{\partial n^a} f(s) \cdot g(s) - f(s) \cdot \frac{\partial}{\partial n^a} g(s)\right) \tilde{\sigma}(ds).\end{aligned} \tag{2.8}$$

3 The image of the second quantization operator in Poisson spaces

Let $(B, D(B))$ be a closable operator in a real separable Hilbert space $\mathcal{H}$ with a dense domain $D(B)$. Define the Fock space as follows

$$\operatorname{Exp} \mathcal{H} := \bigoplus_{n=0}^{\infty} \operatorname{Exp}_n \mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\hat{\otimes} n}, \tag{3.1}$$

where $\operatorname{Exp}_0 \mathcal{H} := \mathbb{R}$. Consider an operator $\left(B^{(n)}, (D(B))^{\hat{\otimes} n}\right)$ in $\operatorname{Exp}_n \mathcal{H}$, $n \in \mathbb{N}$, such as

$$B^{(n)} := B \otimes 1 \ldots \otimes 1 + 1 \otimes B \otimes 1 \ldots \otimes 1 + \ldots + 1 \otimes \ldots \otimes 1 \otimes B, \tag{3.2}$$

and set $B^{(0)} := 0$ in $\operatorname{Exp}_0 \mathcal{H}$. Let

$$\operatorname{Exp}_{fin}(D(B)) := \left\{(f_0, \ldots, f_k, 0, \ldots) \,\middle|\, f_j \in (D(B))^{\hat{\otimes} j}, j = 0, \ldots, k, k \in \mathbb{N}_0\right\}, \tag{3.3}$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Define the second quantization $d\operatorname{Exp} B$ of the operator B as an operator on the dense domain $\operatorname{Exp}_{fin}(D(B))$ in $\operatorname{Exp}\mathcal{H}$ such that

$$d\operatorname{Exp} B \upharpoonright \operatorname{Exp}_n \mathcal{H} = B^{(n)}, n \in \mathbb{N}_0. \tag{3.4}$$

Proposition 3.1. *The operator* $\left(d\operatorname{Exp} B, \operatorname{Exp}_{fin}(D(B))\right)$ *is closable in* $\operatorname{Exp}\mathcal{H}$.

Proof. It's easy to see that for any $n \in \mathbb{N}_0$ $D\left(\left(B^{(n)}\right)^*\right)$ is a dense domain in $\operatorname{Exp}_n \mathcal{H}$. □

In the following we retain the notation $d\operatorname{Exp} B$ for the closure of this operator.

Set $\mathcal{H} := L^2(\sigma)$ and suppose that $\mathcal{D} := C_0^\infty\left(\mathbb{R}^d\right) \subset D(B) \subset C^\infty\left(\mathbb{R}^d\right)$. There is a canonical isomorphism between $\operatorname{Exp} L^2(\sigma)$ and $L^2(\Gamma, \pi_\sigma) =: L^2(\pi_\sigma)$ (where $\Gamma := \Gamma_{\mathbb{R}^d}, \pi_\sigma$ is a Poisson measure with intensity measure σ) such that

$$\operatorname{Exp}_n L^2(\sigma) \ni \varphi^{\otimes n} \mapsto Q_n\left(\varphi^{\otimes n}, \cdot\right) \in L^2(\pi_\sigma), \varphi \in \mathcal{D} \subset D(B), \tag{3.5}$$

where $Q_n(\varphi^{\otimes n}, \cdot)$ is a Charlier polynomial on Γ (see [1] , [4] for details). Let H_B^P be the image of the operator $d\operatorname{Exp} B$ under this isomorphism. The proof of the following proposition is analogous to that of Theorem 5.1 of [1].

Proposition 3.2. *For all* $F, G \in \mathcal{FP}(\mathcal{D}, \Gamma)$ *the following formula holds*

$$\int_\Gamma \left(H_B^P F\right)(\gamma) \cdot G(\gamma)\, \pi_\sigma(d\gamma) = \int_\Gamma \int_{\mathbb{R}^d} B\nabla^P F(\gamma, x) \cdot \nabla^P G(\gamma, x)\, \sigma(dx)\, \pi_\sigma(d\gamma). \tag{3.6}$$

Note that $\mathcal{F}C_p^\infty(\mathcal{D}, \Gamma) \subset D\left(H_B^P\right)$ and by standard approximation arguments one can extend (3.6) on $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma)$ (see, e.g., [1]).

Set now $\mathcal{H} := L^2(\Lambda, \sigma)$ and suppose that $\mathcal{D}(\Lambda) \subset D(B) \subset \mathcal{D}$, where $\mathcal{D}(\Lambda) = C_0^\infty(\Lambda)$. We consider the canonical isomorphism between $\operatorname{Exp} L^2(\Lambda, \sigma)$ and $L^2\left(\Gamma_\Lambda, \pi_\sigma^\Lambda\right)$ such that

$$\operatorname{Exp}_n L^2(\Lambda, \sigma) \ni \varphi^{\otimes n} \mapsto Q_{n,\Lambda}\left(\varphi^{\otimes n}, \cdot\right) \in L^2\left(\pi_\sigma^\Lambda\right), \tag{3.7}$$

where $Q_{\Lambda,n}(\varphi^{\otimes n}, \cdot)$ is a Charlier polynomial on Γ_Λ. Denote by $H_B^{P,\Lambda}$ the image of $d\operatorname{Exp} B$ under this isomorphism. Clearly (cf. (3.6)) for any $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_\Lambda)$.

$$\int_{\Gamma_\Lambda} \left(H_B^{P,\Lambda} F\right)(\gamma) \cdot G(\gamma)\, \pi_\sigma^\Lambda(d\gamma) = \int_{\Gamma_\Lambda} \int_\Lambda B\nabla^P F(\gamma, x) \cdot \nabla^P G(\gamma, x)\, \sigma(dx)\, \pi_\sigma^\Lambda(d\gamma). \tag{3.8}$$

Suppose that the operator B is associated with the differential expression A_σ :

$$Bf = A_\sigma f,\ f \in D(B).$$

In this situation we'll write $H^P_{A_\sigma}$ and $H^{P,\Lambda}_{A_\sigma}$ instead of H^P_B and $H^{P,\Lambda}_B$, respectively.

Let $F \in \mathcal{F}C^\infty_p(\mathcal{D},\Gamma)$, i.e.,

$$F(\gamma) := g_F(\langle\varphi_1,\gamma\rangle, ..., \langle\varphi_N,\gamma\rangle), \tag{3.9}$$
$$\varphi_j \in \mathcal{D}, j = 1,\dots,N, g_F \in C^\infty_p(\mathbb{R}^N),$$

where $C^\infty_p(\mathbb{R}^N)$ is the set of all C^∞-functions g on $\mathbb{R}^N$ such that g and all its partial derivatives are polynomially bounded. Set

$$\hat{F}_r(\gamma) := \frac{\partial g_F}{\partial q_r}(\langle\varphi_1,\gamma\rangle, ..., \langle\varphi_N,\gamma\rangle), 1 \le r \le N, \tag{3.10}$$

$$\hat{F}_{r,s}(\gamma) := \frac{\partial^2 g_F}{\partial q_r \partial q_s}(\langle\varphi_1,\gamma\rangle, ..., \langle\varphi_N,\gamma\rangle), 1 \le r,s \le N. \tag{3.11}$$

The direct calculation gives the next result:

Proposition 3.3. *For any $F \in \mathcal{F}C^\infty_p(\mathcal{D},\Gamma)$ the following formula holds*

$$\begin{aligned} A_\sigma F(\gamma+\varepsilon_x) = &\sum_{r,s=1}^N \hat{F}_{r,s}(\gamma+\varepsilon_x)\, a(\varphi_r,\varphi_s)(x) \\ &+\sum_{r=1}^N \hat{F}_r(\gamma+\varepsilon_x)\left((A_\sigma\varphi_r)(x) + a(\varphi_r,\ln\rho)(x) - c(x)\varphi_r(x)\right) \\ &+ F(\gamma+\varepsilon_x)\, c(x). \end{aligned}$$

Remark 3.4. 1. By (3.9) for any function $F \in \mathcal{F}C^\infty_p(\mathcal{D},\Gamma)$ and for any $\gamma \in \Gamma$ we have

$$F(\gamma+\varepsilon_\cdot) \in \mathcal{K}_2. \tag{3.12}$$

2. Note that when we write, for example, $A_\sigma F(\gamma)$ we mean that the expression A_σ acts on **constant** $F(\gamma)$ as on the constant function of $x \in \mathbb{R}^d$. But we can also consider the action of the expression A_σ on function $F(\gamma)$ as function of some $x \in \gamma$ when others point of γ are fixed. In this situation we will write $(A_\sigma)_x F(\gamma), x \in \gamma$. It's easy to see that for any $x \in \gamma$

$$(A_\sigma)_x F(\gamma) = \sum_{r,s=1}^N \hat{F}_{r,s}(\gamma)\, a(\varphi_r,\varphi_s)(x) \tag{3.13}$$

$$+\sum_{r=1}^N \hat{F}_r(\gamma)\left((A_\sigma\varphi_r)(x) + a(\varphi_r,\ln\rho)(x) - c(x)\varphi_r(x)\right) + F(\gamma)\, c(x).$$

Proposition 3.5. *For any $F \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma)$ the following formula holds*

$$\begin{aligned}\left(H^P_{A_\sigma} F\right)(\gamma) &= \langle (A_\sigma)_{.} F(\gamma), \gamma\rangle - \langle F(\gamma - \varepsilon_{.})\, c(\cdot), \gamma\rangle \\ &\quad - \int_{\mathbb{R}^d} F(\gamma + \varepsilon_x)\, c(x)\, \sigma(dx) + F(\gamma) \int_{\mathbb{R}^d} c(x)\, \sigma(dx).\end{aligned} \tag{3.14}$$

Proof. By (3.6), Proposition 2.2 and the Mecke identity we have

$$\begin{aligned}
&\int_\Gamma \left(H^P_{A_\sigma} F\right)(\gamma) \cdot G(\gamma)\, \pi_\sigma(d\gamma) \\
&= \int_\Gamma \int_{\mathbb{R}^d} A_\sigma F(\gamma + \varepsilon_x) \cdot G(\gamma + \varepsilon_x)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&- \int_\Gamma \int_{\mathbb{R}^d} A_\sigma F(\gamma) \cdot G(\gamma + \varepsilon_x)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&- \int_\Gamma \int_{\mathbb{R}^d} A_\sigma F(\gamma + \varepsilon_x) \cdot G(\gamma)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&+ \int_\Gamma \int_{\mathbb{R}^d} A_\sigma F(\gamma) \cdot G(\gamma)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&= \int_\Gamma \int_{\mathbb{R}^d} \left(\sum_{r,s=1}^N \hat{F}_{r,s}(\gamma + \varepsilon_x)\, a(\varphi_r, \varphi_s)(x) \right. \\
&\qquad + \sum_{r=1}^N \hat{F}_r(\gamma + \varepsilon_x) \left((A_\sigma \varphi_r)(x) + a(\varphi_r, \ln \rho)(x) - c(x)\, \varphi_r(x) \right) \\
&\qquad \left. + F(\gamma + \varepsilon_x)\, c(x) \right) G(\gamma + \varepsilon_x)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&- \int_\Gamma \int_{\mathbb{R}^d} F(\gamma)\, c(x)\, G(\gamma + \varepsilon_x)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&- \int_\Gamma \int_{\mathbb{R}^d} F(\gamma + \varepsilon_x)\, c(x)\, G(\gamma)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&+ \int_\Gamma \int_{\mathbb{R}^d} F(\gamma)\, c(x)\, G(\gamma)\, \sigma(dx)\, \pi_\sigma(d\gamma) \\
&= \int_\Gamma \left(\sum_{r,s=1}^N \hat{F}_{r,s}(\gamma) \langle a(\varphi_r, \varphi_s), \gamma\rangle \right. \\
&\qquad + \sum_{r=1}^N \hat{F}_r(\gamma) \langle A_\sigma \varphi_r + a(\varphi_r, \ln \rho) - \varphi_r c, \gamma\rangle \\
&\qquad \left. + F(\gamma) \langle c, \gamma\rangle \right) G(\gamma)\, \pi_\sigma(d\gamma) \\
&- \int_\Gamma \langle F(\gamma - \varepsilon_{.})\, c(\cdot), \gamma\rangle\, G(\gamma)\, \pi_\sigma(d\gamma)
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\Gamma} \left(\int_{\mathbb{R}^d} F(\gamma + \varepsilon_x) \, c(x) \, \sigma(dx) \right) G(\gamma) \, \pi_\sigma(d\gamma) \\
&+ \int_{\Gamma} \left(F(\gamma) \int_{\mathbb{R}^d} c(x) \, \sigma(dx) \right) G(\gamma) \, \pi_\sigma(d\gamma).
\end{aligned}
$$

So, by (3.13) the proof is fulfilled. □

Proposition 3.6. *For any $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_\Lambda)$ the following formula holds*

$$
\begin{aligned}
\left(H_{A_\sigma}^{P,\Lambda} F \right)(\gamma) &= \langle (A_\sigma)_\cdot F(\gamma), \gamma \rangle - \langle F(\gamma - \varepsilon_\cdot) \, c(\cdot), \gamma \rangle \\
&- \int_{\Lambda} F(\gamma + \varepsilon_x) \, c(x) \, \sigma(dx) + F(\gamma) \int_{\Lambda} c(x) \, \sigma(dx) \\
&- \int_{\partial\Lambda} \frac{\partial}{\partial n^a} F(\gamma + \varepsilon_s) \, \tilde{\sigma}(ds).
\end{aligned} \tag{3.15}
$$

Proof. The proof is analogous to that of (3.14). □

4 Green Formulas

We start with the so-called second Green formula:

Proposition 4.1. *For any $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_\Lambda)$ the following formula holds*

$$
\int_{\Gamma_\Lambda} \left(\left(H_{A_\sigma}^P F \right)(\gamma) \cdot G(\gamma) - F(\gamma) \cdot \left(H_{A_\sigma}^P G \right)(\gamma) \right) \pi_\sigma^\Lambda(d\gamma) \tag{4.1}
$$

$$
\begin{aligned}
&= \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \left(\frac{\partial}{\partial n^a} F(\gamma + \varepsilon_s) \cdot G(\gamma + \varepsilon_s) - F(\gamma + \varepsilon_s) \cdot \frac{\partial}{\partial n^a} G(\gamma + \varepsilon_s) \right) \tilde{\sigma}(ds) \, \pi_\sigma^\Lambda(d\gamma) \\
&- \int_{\Gamma_\Lambda} \int_{\mathbb{R}^d \setminus \Lambda} \Big(F(\gamma + \varepsilon_x) \, G(\gamma) - G(\gamma + \varepsilon_x) \, F(\gamma) \Big) c(x) \, \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma).
\end{aligned}
$$

Proof. By (3.14), (3.12) and the Mecke identity we have

$$
\begin{aligned}
&\int_{\Gamma_\Lambda} \left(H_{A_\sigma}^P F \right)(\gamma) \cdot G(\gamma) \, \pi_\sigma^\Lambda(d\gamma) \\
&\qquad = \int_{\Gamma_\Lambda} \langle (A_\sigma)_\cdot F(\gamma), \gamma \rangle \cdot G(\gamma) \, \pi_\sigma^\Lambda(d\gamma) \\
&\qquad - \int_{\Gamma_\Lambda} \langle F(\gamma - \varepsilon_\cdot) \, c(\cdot), \gamma \rangle \cdot G(\gamma) \, \pi_\sigma^\Lambda(d\gamma) \\
&\qquad - \int_{\Gamma_\Lambda} \left(\int_{\mathbb{R}^d} F(\gamma + \varepsilon_x) \, c(x) \, \sigma(dx) \right) \cdot G(\gamma) \, \pi_\sigma^\Lambda(d\gamma) \\
&\qquad + \int_{\Gamma_\Lambda} \left(F(\gamma) \int_{\mathbb{R}^d} c(x) \, \sigma(dx) \right) \cdot G(\gamma) \, \pi_\sigma^\Lambda(d\gamma) \\
&\qquad = \int_{\Gamma_\Lambda} \int_{\Lambda} A_\sigma F(\gamma + \varepsilon_x) \cdot G(\gamma + \varepsilon_x) \, \sigma(dx) \, \pi_\sigma^\Lambda(d\gamma)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\Gamma_\Lambda} \int_{\Lambda} F(\gamma)\, c(x)\, G(\gamma + \varepsilon_x)\, \sigma(dx)\, \pi_\sigma^\Lambda(d\gamma) \\
&- \int_{\Gamma_\Lambda} \int_{\mathbb{R}^d} F(\gamma + \varepsilon_x)\, G(\gamma)\, c(x)\, \sigma(dx)\, \pi_\sigma^\Lambda(d\gamma) \\
&+ \int_{\Gamma_\Lambda} \int_{\mathbb{R}^d} F(\gamma)\, G(\gamma)\, c(x)\, \sigma(dx)\, \pi_\sigma^\Lambda(d\gamma),
\end{aligned}
$$

and from the corresponding formula for $\int_{\Gamma_\Lambda} F(\gamma) \cdot \left(H_{A_\sigma}^P G\right)(\gamma)\, \pi_\sigma^\Lambda(d\gamma)$ one has

$$
\begin{aligned}
&\int_{\Gamma_\Lambda} \left(\left(H_{A_\sigma}^P F\right)(\gamma) \cdot G(\gamma) - F(\gamma) \cdot \left(H_{A_\sigma}^P G\right)(\gamma) \right) \pi_\sigma^\Lambda(d\gamma) \\
&= \int_{\Gamma_\Lambda} \int_{\Lambda} \left(A_\sigma F(\gamma + \varepsilon_x) \cdot G(\gamma + \varepsilon_x) - F(\gamma + \varepsilon_x) \cdot A_\sigma G(\gamma + \varepsilon_x) \right) \sigma(dx)\, \pi_\sigma^\Lambda(d\gamma) \\
&\quad - \int_{\Gamma_\Lambda} \int_{\Lambda} \Big(F(\gamma)\, G(\gamma + \varepsilon_x) - G(\gamma)\, F(\gamma + \varepsilon_x) \Big)\, c(x)\, \sigma(dx)\, \pi_\sigma^\Lambda(d\gamma) \\
&\quad - \int_{\Gamma_\Lambda} \int_{\mathbb{R}^d} \Big(F(\gamma + \varepsilon_x)\, G(\gamma) - G(\gamma + \varepsilon_x)\, F(\gamma) \Big)\, c(x)\, \sigma(dx)\, \pi_\sigma^\Lambda(d\gamma).
\end{aligned}
$$

So, (3.12) and (2.8) imply the statement of proposition. □

Let us consider for any $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_\Lambda)$ (F has the form (3.9) , G has the analogous form $G(\gamma) := g_G(\langle \psi_1, \gamma \rangle, ..., \langle \psi_M, \gamma \rangle)$) the bilinear form

$$
a^\Gamma(F, G)(\gamma) := \sum_{j=1}^{N} \sum_{k=1}^{M} \hat{F}_j(\gamma)\, \hat{G}_k(\gamma)\, \langle a(\varphi_j, \psi_k), \gamma \rangle \tag{4.2}
$$

and the corresponding (pre-)Dirichlet form:

$$
\mathcal{E}_{a, \pi_\sigma^\Lambda}^\Gamma(F, G) := \int_{\Gamma_\Lambda} a^\Gamma(F, G)(\gamma)\, \pi_\sigma^\Lambda(d\gamma). \tag{4.3}
$$

In the following we always suppose, that

$$
c \equiv 0 \text{ on } \mathbb{R}^d \setminus \Lambda. \tag{4.4}
$$

Using this condition we can prove the so-called first Green formula.

Proposition 4.2. *For any $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_\Lambda)$*

$$
\begin{aligned}
\mathcal{E}_{a, \pi_\sigma^\Lambda}^\Gamma(F, G) &= \int_{\Gamma_\Lambda} \left(H_{A_\sigma}^P F\right)(\gamma) \cdot G(\gamma)\, \pi_\sigma^\Lambda(d\gamma) \\
&\quad - \int_{\Gamma_\Lambda} \left(H_c^{P,\Lambda} F\right)(\gamma) \cdot G(\gamma)\, \pi_\sigma^\Lambda(d\gamma) \\
&\quad - \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \frac{\partial}{\partial n^a} F(\gamma + \varepsilon_s)\, G(\gamma + \varepsilon_s)\, \tilde{\sigma}(ds)\, \pi_\sigma^\Lambda(d\gamma),
\end{aligned} \tag{4.5}
$$

where we understand c as the operator of the multiplication on the function $c(\cdot)$ in $L^2(\Lambda, \sigma)$.

Proof. We have for any $F, G \in \mathcal{F}C_p^\infty(\mathcal{D}, \Gamma_\Lambda)$

$$
\begin{aligned}
&\int_{\Gamma_\Lambda} \left(H^P_{A_\sigma} F\right)(\gamma) \cdot G(\gamma)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad = \int_{\Gamma_\Lambda} \int_\Lambda A_\sigma F(\gamma + \varepsilon_x) \cdot G(\gamma + \varepsilon_x)\, \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad - \int_{\Gamma_\Lambda} \int_\Lambda F(\gamma)\, c(x)\, G(\gamma + \varepsilon_x)\, \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad - \int_{\Gamma_\Lambda} \int_\Lambda F(\gamma + \varepsilon_x)\, G(\gamma)\, c(x)\, \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad + \int_{\Gamma_\Lambda} \int_\Lambda F(\gamma)\, G(\gamma)\, c(x)\, \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad = \int_{\Gamma_\Lambda} \int_\Lambda a\left(F(\gamma + \varepsilon_x), G(\gamma + \varepsilon_x)\right) \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad + \int_{\Gamma_\Lambda} \int_\Lambda c(x)\, \nabla^P F(\gamma, x)\, \nabla^P G(\gamma, x)\, \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad + \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \frac{\partial}{\partial n^a} F(\gamma + \varepsilon_s)\, G(\gamma + \varepsilon_s)\, \tilde{\sigma}(ds)\, \pi^\Lambda_\sigma(d\gamma).
\end{aligned}
$$

Next, by Mecke identity one has

$$
\int_{\Gamma_\Lambda} \int_\Lambda a\left(F(\gamma + \varepsilon_x), G(\gamma + \varepsilon_x)\right) \sigma(dx)\, \pi^\Lambda_\sigma(d\gamma) = \int_{\Gamma_\Lambda} a^\Gamma(F, G)(\gamma)\, \pi^\Lambda_\sigma(d\gamma),
$$

so, by (3.8) we have

$$
\begin{aligned}
&\int_{\Gamma_\Lambda} \left(H^P_{A_\sigma} F\right)(\gamma) \cdot G(\gamma)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad = \int_{\Gamma_\Lambda} a^\Gamma(F, G)(\gamma)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad + \int_{\Gamma_\Lambda} \left(H^{P,\Lambda}_c F\right)(\gamma) \cdot G(\gamma)\, \pi^\Lambda_\sigma(d\gamma) \\
&\qquad + \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \frac{\partial}{\partial n^a} F(\gamma + \varepsilon_s)\, G(\gamma + \varepsilon_s)\, \tilde{\sigma}(ds)\, \pi^\Lambda_\sigma(d\gamma).
\end{aligned}
\tag{4.6}
$$

□

5 The symmetric extensions of the minimal operator

Let us consider the minimal operator:

$$
H_{\min} := \left(H^P_{A_\sigma}, \mathcal{F}C_p^\infty(\mathcal{D}(\Lambda), \Gamma_\Lambda)\right). \tag{5.1}
$$

By the first Green formula the operator $H_{\min}$ is symmetric in $L^2(\Gamma_\Lambda)$. We define the maximal operator by the standard way:

$$H_{\max} := (H_{\min})^* . \tag{5.2}$$

We can formulate the following proposition

Proposition 5.1.

$$\mathcal{F}C_p^\infty(\mathcal{D},\Gamma_\Lambda) \subset D(H_{\max})$$

and for any $G \in \mathcal{F}C_p^\infty(\mathcal{D},\Gamma_\Lambda)$

$$H_{\max} G = H_{A_\sigma}^{P,\Lambda} G.$$

Proof. For any $F \in \mathcal{F}C_p^\infty(\mathcal{D}(\Lambda),\Gamma_\Lambda), G \in FC_p^\infty(\mathcal{D},\Gamma_\Lambda)$ we have

$$\int_{\Gamma_\Lambda} \left((H_{\min}F)(\gamma)\cdot G(\gamma) - F(\gamma)\cdot \left(H_{A_\sigma}^P G\right)(\gamma)\right)\pi_\sigma^\Lambda(d\gamma)$$

$$= \int_{\Gamma_\Lambda}\int_{\partial\Lambda}\left(\frac{\partial}{\partial n^a}F(\gamma+\varepsilon_s)\cdot G(\gamma+\varepsilon_s) - F(\gamma+\varepsilon_s)\cdot\frac{\partial}{\partial n^a}G(\gamma+\varepsilon_s)\right)\tilde{\sigma}(ds)\,\pi_\sigma^\Lambda(d\gamma)$$

$$= -\int_{\Gamma_\Lambda} F(\gamma)\int_{\partial\Lambda}\frac{\partial}{\partial n^a}G(\gamma+\varepsilon_s)\,\tilde{\sigma}(ds)\,\pi_\sigma^\Lambda(d\gamma).$$

Hence $G \in D(H_{\max})$ and by (4.4)

$$H_{\max}G = \left(H_{A_\sigma}^P G\right)(\gamma) - \int_{\partial\Lambda}\frac{\partial}{\partial n^a}G(\gamma+\varepsilon_s)\,\tilde{\sigma}(ds) = H_{A_\sigma}^{P,\Lambda}G(\gamma).$$

□

Therefore we have the following inclusion

$$H_{\min} \subset H_{A_\sigma}^{P,\Lambda} \subset H_{\max}. \tag{5.3}$$

Suppose that $\mathcal{A}$ be an algebra, such that $\mathcal{D}(\Lambda) \subset \mathcal{A} \subset \mathcal{D}$.

Remark 5.2. If the operator $(A_\sigma, \mathcal{A})$ is symmetric in $L^2(\Lambda)$, then $\left(H_{A_\sigma}^{P,\Lambda}, \mathcal{FP}(\mathcal{A},\Gamma_\Lambda)\right)$ is the symmetric extension of $H_{\min}$ in $L^2(\Gamma_\Lambda)$.

To describe symmetric extensions of $H_{\min}$, which are defined by the differential expression $H_{A_\sigma}^P$, we start with the following simple proposition.

Proposition 5.3. *Suppose that*

$$\mathcal{F}C_p^\infty(\mathcal{D}(\Lambda),\Gamma_\Lambda) \subset \mathcal{F} \subset \mathcal{F}C_p^\infty(\mathcal{D},\Gamma_\Lambda)$$

and $\left(H_{A_\sigma}^P, \mathcal{F}\right)$ *be a symmetric extensions of* $H_{\min}$. *Then for any* $F \in \mathcal{F}$

$$\int_{\partial\Lambda}\frac{\partial}{\partial n^a}F(\gamma+\varepsilon_s)\,\tilde{\sigma}(ds) = 0. \left(\text{mod } \pi_\sigma^\Lambda\right)$$

Proof. It's easy to see from (5.3) and the fact, that the symmetric extension of $H_{\min}$ is the symmetric restriction of $H_{\max}$. □

Now we can formulate our main result.

Theorem 5.4. *The operator* $\left(H^P_{A_\sigma}, \mathcal{FP}(\mathcal{A}, \Gamma_\Lambda)\right)$ *is symmetric in* $L^2(\Gamma_\Lambda)$ *if and only if the operator* $(A_\sigma, \mathcal{A})$ *is symmetric in* $L^2(\Lambda)$ *and for any* $\varphi \in \mathcal{A}$

$$\int_{\partial\Lambda} \frac{\partial}{\partial n^a} \varphi(s)\, \tilde{\sigma}(ds) = 0. \tag{5.4}$$

Moreover, in this case

$$\left(H^P_{A_\sigma}, \mathcal{FP}(\mathcal{A}, \Gamma_\Lambda)\right) = \left(H^{P,\Lambda}_{A_\sigma}, \mathcal{FP}(\mathcal{A}, \Gamma_\Lambda)\right).$$

Proof. Suppose that the operator $(A_\sigma, \mathcal{A})$ is symmetric in $L^2(\Lambda)$ and for any $\varphi \in \mathcal{A}$ the condition (5.4) is valid. For $\varphi, \psi \in \mathcal{A}$ consider $F(\gamma) = Q_{n,\Lambda}(\varphi^{\otimes n}, \gamma)$, $G(\gamma) = Q_{m,\Lambda}(\psi^{\otimes m}, \gamma)$. By (4.1) and condition (4.4) one has

$$\begin{aligned}
&\left(H^P_{A_\sigma} F, G\right)_{L^2(\pi^\Lambda_\sigma)} - \left(F, H^P_{A_\sigma} G\right)_{L^2(\pi^\Lambda_\sigma)} \\
&= \int_{\Gamma_\Lambda} \int_{\partial\Lambda} \Big[n \frac{\partial}{\partial n^a} \varphi(s)\, Q_{n-1,\Lambda}\left(\varphi^{\otimes(n-1)}, \gamma\right) \\
&\qquad \times \left(Q_{m,\Lambda}\left(\psi^{\otimes m}, \gamma\right) + m\psi(s)\, Q_{m-1,\Lambda}\left(\psi^{\otimes(m-1)}, \gamma\right)\right) \\
&\qquad - \left(Q_{n,\Lambda}\left(\varphi^{\otimes n}, \gamma\right) + n\varphi(s)\, Q_{n-1,\Lambda}\left(\varphi^{\otimes(n-1)}, \gamma\right)\right) \\
&\qquad \times m \frac{\partial}{\partial n^a} \psi(s)\, Q_{m-1,\Lambda}\left(\psi^{\otimes(m-1)}, \gamma\right) \Big] \tilde{\sigma}(ds)\, \pi^\Lambda_\sigma(d\gamma) \\
&= n \int_{\Gamma_\Lambda} Q_{n-1,\Lambda}\left(\varphi^{\otimes(n-1)}, \gamma\right) Q_{m,\Lambda}\left(\psi^{\otimes m}, \gamma\right) \pi^\Lambda_\sigma(d\gamma) \\
&\qquad \times \int_{\partial\Lambda} \frac{\partial}{\partial n^a} \varphi(s)\, \tilde{\sigma}(ds) \\
&+ nm \int_{\Gamma_\Lambda} Q_{n-1,\Lambda}\left(\varphi^{\otimes(n-1)}, \gamma\right) Q_{m-1,\Lambda}\left(\psi^{\otimes(m-1)}, \gamma\right) \pi^\Lambda_\sigma(d\gamma) \\
&\qquad \times \int_{\partial\Lambda} \frac{\partial}{\partial n^a} \varphi(s) \cdot \psi(s)\, \tilde{\sigma}(ds) \\
&- m \int_{\Gamma_\Lambda} Q_{n,\Lambda}\left(\varphi^{\otimes n}, \gamma\right) Q_{m-1,\Lambda}\left(\psi^{\otimes(m-1)}, \gamma\right) \pi^\Lambda_\sigma(d\gamma) \\
&\qquad \times \int_{\partial\Lambda} \frac{\partial}{\partial n^a} \psi(s)\, \tilde{\sigma}(ds) \\
&- nm \int_{\Gamma_\Lambda} Q_{n-1,\Lambda}\left(\varphi^{\otimes(n-1)}, \gamma\right) Q_{m-1,\Lambda}\left(\psi^{\otimes(m-1)}, \gamma\right) \pi^\Lambda_\sigma(d\gamma) \\
&\qquad \times \int_{\partial\Lambda} \varphi(s) \cdot \frac{\partial}{\partial n^a} \psi(s)\, \tilde{\sigma}(ds) \\
&= 0,
\end{aligned}$$

by (2.8) and condition (5.4).

Next, we have by (3.14), (3.15) and (4.4)

$$\left(H^P_{A_\sigma}F\right)(\gamma)-\left(H^{P,\Lambda}_{A_\sigma}F\right)(\gamma)=\int_{\partial\Lambda}\frac{\partial}{\partial n^a}F\left(\gamma+\varepsilon_s\right)\tilde{\sigma}\left(ds\right)$$
$$=nQ_{n-1,\Lambda}\left(\varphi^{\otimes(n-1)},\gamma\right)\int_{\partial\Lambda}\frac{\partial}{\partial n^a}\varphi\left(s\right)\tilde{\sigma}\left(ds\right)=0$$

by condition (5.4).

Conversely, let the operator $\left(H^P_{A_\sigma},\mathcal{FP}\left(\mathcal{A},\Gamma_\Lambda\right)\right)$ be symmetric in $L^2\left(\Gamma_\Lambda\right)$. For any $\varphi,\psi\in\mathcal{A}$ consider the functions $F\left(\gamma\right)=Q_{1,\Lambda}\left(\varphi,\gamma\right)=\langle\varphi,\gamma\rangle-\langle\varphi\rangle_{\sigma,\Lambda}$ and $G\left(\gamma\right)=Q_{1,\Lambda}\left(\psi,\gamma\right)=\langle\psi,\gamma\rangle-\langle\psi\rangle_{\sigma,\Lambda}$. Then $F,G\in\mathcal{FP}\left(\mathcal{A},\Gamma_\Lambda\right)$. Clearly,

$$\int_{\Gamma_\Lambda}Q_{1,\Lambda}\left(\varphi,\gamma\right)\pi^\Lambda_\sigma\left(d\gamma\right)=\int_{\Gamma_\Lambda}Q_{1,\Lambda}\left(\psi,\gamma\right)\pi^\Lambda_\sigma\left(d\gamma\right)=0.$$

By (4.1) and (4.4) we have

$$0=\int_{\Gamma_\Lambda}\int_{\partial\Lambda}\left(\frac{\partial}{\partial n^a}\varphi\left(s\right)\cdot\left(Q_{1,\Lambda}\left(\psi,\gamma\right)+\psi\left(s\right)\right)\right.$$
$$\left.-\left(Q_{1,\Lambda}\left(\varphi,\gamma\right)+\varphi\left(s\right)\right)\cdot\frac{\partial}{\partial n^a}\psi\left(s\right)\right)\tilde{\sigma}\left(ds\right)\pi^\Lambda_\sigma\left(d\gamma\right)$$
$$=\int_{\partial\Lambda}\left(\frac{\partial}{\partial n^a}\varphi\left(s\right)\cdot\psi\left(s\right)-\varphi\left(s\right)\cdot\frac{\partial}{\partial n^a}\psi\left(s\right)\right)\tilde{\sigma}\left(ds\right)$$
$$=\int_{\Lambda}\left(\left(A_\sigma\varphi\right)\left(x\right)\cdot\psi\left(x\right)-\varphi\left(x\right)\cdot\left(A_\sigma\psi\right)\left(x\right)\right)\sigma\left(dx\right).$$

Now set for any $\varphi\in\mathcal{A}$ $F\left(\gamma\right)=Q_{1,\Lambda}\left(\varphi,\gamma\right),G\left(\gamma\right)=Q_{0,\Lambda}\equiv1$. Then by (4.1) and (4.4) one has

$$0=\int_{\Gamma_\Lambda}\int_{\partial\Lambda}\frac{\partial}{\partial n^a}\varphi\left(s\right)\tilde{\sigma}\left(ds\right)\pi^\Lambda_\sigma\left(d\gamma\right)=\int_{\partial\Lambda}\frac{\partial}{\partial n^a}\varphi\left(s\right)\tilde{\sigma}\left(ds\right).$$

□

Example 5.5. The simple example of such algebra $\mathcal{A}$ is the algebra $\mathcal{D}_\mathcal{N}$ of functions, which satisfy Neumann-type boundary condition:

$$\frac{\partial}{\partial n^a}\varphi\left(s\right)=0,\quad s\in\partial\Lambda,\quad\varphi\in\mathcal{D}_\mathcal{N}.\tag{5.5}$$

It's easy to see that for any $F\in\mathcal{F}C^\infty_p\left(\mathcal{D}_\mathcal{N},\Gamma_\Lambda\right),G\in\mathcal{F}C^\infty_p\left(\mathcal{D},\Gamma_\Lambda\right)$

$$\mathcal{E}^\Gamma_{a,\pi^\Lambda_\sigma}\left(F,G\right)=\int_{\Gamma_\Lambda}\left(H^P_{A_\sigma}F\right)\left(\gamma\right)\cdot G\left(\gamma\right)\pi^\Lambda_\sigma\left(d\gamma\right)$$
$$-\int_{\Gamma_\Lambda}\left(H^{P,\Lambda}_cF\right)\left(\gamma\right)\cdot G\left(\gamma\right)\pi^\Lambda_\sigma\left(d\gamma\right).$$

Corollary 5.6. *Suppose that* $(A_\sigma, \mathcal{A})$ *is the essentially self-adjoint operator in* $L^2(\Lambda)$. *Then* $\left(H^P_{A_\sigma}, \mathcal{FP}(\mathcal{A}, \Gamma_\Lambda)\right)$ *is the essentially self-adjoint operator in* $L^2(\Gamma_\Lambda)$ *if and only if for any* $\varphi \in \mathcal{A}$ *the condition* (5.4) *is fulfilled.*

Proof. The result follows immediately from Corollary 5.4 and the fact that the second quantization of an essentially self-adjoint operator is the essentially self-adjoint operator in the corresponding Hilbert space. □

Now we find the explicit formula for the action of the Friedrichs extension of $H_{\min}$ on the smooth cylinder functions. Denote by $\mathcal{D}_D$ the set of all functions from $\mathcal{D}$ satisfying the Dirichlet boundary condition on $\partial\Lambda$ and let $A_{\sigma,D}$ be the Friedrichs extension of $(A_\sigma, \mathcal{D}(\Lambda))$. The following proposition gives a formula for the action of $H^P_{A_{\sigma,D}}$ on the smooth cylinder functions.

Theorem 5.7. *Suppose that* $\mathcal{D}_D$ *is an essential domain for* $A_{\sigma,D}$. *Then the closure of the operator* $\left(H^P_{A_{\sigma,D}}, \mathcal{FP}(\mathcal{D}_D, \Gamma_\Lambda)\right)$ *defined by the differential expression*

$$\begin{aligned}(H^P_{A_{\sigma,D}}F)(\gamma) &:= (H^P_{A_\sigma}F)(\gamma) \\ &\quad - \sum_{j=1}^N \frac{\partial g_F}{\partial q_j}\left(\langle\varphi_1,\gamma\rangle, \ldots, \langle\varphi_N,\gamma\rangle\right) \cdot \int_{\partial\Lambda} \frac{\partial}{\partial n^a}\varphi_j(s)\,\tilde{\sigma}(ds),\end{aligned} \tag{5.6}$$

coincides with the Friedrichs extension of $H_{\min}$ *in* $L^2(\Gamma_\Lambda)$. *Moreover, for any* $F, G \in \mathcal{FC}^\infty_p(\mathcal{D}_D, \Gamma_\Lambda)$

$$\begin{aligned}\mathcal{E}^\Gamma_{a,\pi^\Lambda_\sigma}(F,G) &= \int_{\Gamma_\Lambda} \left(H^P_{A_{\sigma,D}}F\right)(\gamma) \cdot G(\gamma)\,\pi^\Lambda_\sigma(d\gamma) \\ &\quad - \int_{\Gamma_\Lambda} \left(H^{P,\Lambda}_c F\right)(\gamma) \cdot G(\gamma)\,\pi^\Lambda_\sigma(d\gamma).\end{aligned}$$

Proof. First, we recall that, for $F \in \mathcal{FP}(\mathcal{D}(\Lambda), \Gamma_\Lambda)$

$$H^P_{A_\sigma}F = H^{P,\Lambda}_{A_\sigma}F$$

and we can consider $H^{P,\Lambda}_{A_\sigma}$ as the image of the second quantization of the symmetric (in $L^2(\Lambda)$) operator $(A_\sigma, \mathcal{D}(\Lambda))$. Therefore, the Friedrichs extension $H^P_{A_{\sigma,D}}$ of the minimal operator $H_{\min}$ is the image of the second quantization of $A_{\sigma,D}$. In particular, $\mathcal{FP}(\mathcal{D}_D, \Gamma_\Lambda)$ is an essential domain of $H^P_{A_{\sigma,D}}$. So, the assertion directly follows from Proposition 5.1, the operator inclusion $H_{\min} \subset H^P_{A_{\sigma,D}} \subset H_{\max}$ and the fact that for $F \in \mathcal{FP}(\mathcal{D}_D, \Gamma_\Lambda)$

$$H^P_{A_{\sigma,D}}F = H^{P,\Lambda}_{A_\sigma}F = H_{\max}F.$$

The last statement is a direct consequence of (4.5). □

Remark 5.8. Note that for any $F \in \mathcal{F}C_p^{\infty}(\mathcal{D}, \Gamma_\Lambda)$, $G \in FC_p^{\infty}(\mathcal{D}_D, \Gamma_\Lambda)$

$$\begin{aligned}\mathcal{E}^{\Gamma}_{a,\pi^{\Lambda}_{\sigma}}(F,G) = &\int_{\Gamma_\Lambda} (H_{\max} F)(\gamma) \cdot G(\gamma)\, \pi^{\Lambda}_{\sigma}(d\gamma) \\ &- \int_{\Gamma_\Lambda} \left(H_c^{P,\Lambda} F\right)(\gamma) \cdot G(\gamma)\, \pi^{\Lambda}_{\sigma}(d\gamma).\end{aligned}$$

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