ON EXPONENTIAL MODEL OF POISSON SPACES

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Abstract. We generalize N.Privault's interpretation of Poisson space over $\mathbb{R}$ as the space of sequences with exponential product measure. The proposed model of Poisson spaces enables to reformulate the recent results in analysis and geometry in Poisson spaces in the classical framework of infinite dimensional analysis with respect to product measure.

1.Introduction

The well-known result of theory of stochastic processes states that the inter-jumping times of a homogeneous Poisson process form a sequence of independent random variables having identical exponential distribution (see, e.g., [9]). This fact enables to connect a Poisson space over the right semiaxis with an infinite dimensional space equipped with an exponential product measure and apply methods of the classical infinite dimensional analysis (IDA) in Poisson stochastic analysis (see [13]).

The recent progress in analysis and differential geometry on Poisson spaces (see [1] and references therein) makes the interpretation of Poisson analysis and geometry in the framework of classical IDA particularly interesting. To obtain such an interpretation we introduce an exponential model of Poisson space $\Gamma_{\mathbb{R}}$ over $\mathbb{R}$, as a space X with exponential product measure and corresponding mapping $T : \Gamma_{\mathbb{R}} \longmapsto X$. Using it we obtain new differential operators in Poisson spaces as well as new results on quasiinvariance of the exponential product measures with respect to some group of transformations connected with the famous diffeomorphisms group (see [4], [1] and references therein).

The paper is organized as follows. In Sections 2, 3 we recall basic notions of Poisson analysis and classical IDA respectively. The exponential model of Poisson space is introduced and discussed in Section 4. In Section 5 we apply this model to construct some new riggings of Poisson space in terms of spaces of test and generalized functions. The quasiinvariance of the exponential product measures is investigated in Section 6. The connections between differential operators of IDA and Poisson analysis are considered in Section 7. The analogue of the exponential model for compound Poisson space is introduced in the final Section 8.

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2. Poisson measure on configuration space.

The configuration space $\Gamma_{\mathbb{R}} \equiv \Gamma$ over $\mathbb{R}$ is defined as the set of all locally finite subsets (configurations) in $\mathbb{R}$

$$\Gamma = \{\gamma \subset \mathbb{R} \mid |\gamma \cap K| < \infty \text{ for any } K \in \mathcal{B}_c(\mathbb{R})\}.$$

Here $|A|$ denotes the cardinality of the set A, $\mathcal{B}_c(\mathbb{R})$ is the system of all Borel sets in $\mathbb{R}$, which have compact closures.

Let $K \in \mathcal{B}_c(\mathbb{R}), n \in \mathbb{Z}_+$. The set

$$C(K;n) := \{\gamma \in \Gamma \mid |\gamma \cap K| = n\} \tag{2.1}$$

is called a cylinder set. It is clear, that for any nonempty $K \in \mathcal{B}_c(\mathbb{R})$,

$$\Gamma = \bigcup_{n=0}^{\infty} C(K;n),$$

where $C(K;n_1) \cap C(K;n_2) = \emptyset, n_1 \neq n_2$. We denote by $\mathcal{C}_\sigma(\Gamma)$ the σ-algebra, generated by the collection of all cylinder sets $\{C(K;n) \mid K \in \mathcal{B}_c(\mathbb{R}), n \in \mathbb{Z}_+\}$.

In order to obtain another description of the measurable space $(\Gamma, \mathcal{C}_\sigma(\Gamma))$ consider a one-to-one correspondence between Γ and the set $\mathcal{M}_p(\mathbb{R})$ of all non-negative integer-valued Radon measures on $\mathbb{R}$,

$$\Gamma \ni \gamma \longmapsto \gamma(\cdot) := \sum_{x\in\gamma} \varepsilon_x(\cdot) \in \mathcal{M}_p(\mathbb{R}) \subset \mathcal{M}(\mathbb{R}), \tag{2.2}$$

where $\varepsilon_x(\cdot)$ denotes Dirac measure at the point x, $\mathcal{M}(\mathbb{R})$ — the set of all non-negative Radon measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, $\sum_{x\in\emptyset} \varepsilon_x(\cdot) :=$ zero measure. The space $\mathcal{M}_p(\mathbb{R})$ is endowed with the relative topology as a subset of $\mathcal{M}(\mathbb{R})$ with the vague topology, i.e. the weakest topology on $\mathcal{M}_p(\mathbb{R})$ such that the maps

$$\mathcal{M}_p(\mathbb{R}) \ni \gamma(\cdot) \longmapsto \langle f, \gamma\rangle := \int_{\mathbb{R}} f(x)\,\gamma(dx) = \sum_{x\in\gamma} f(x) \tag{2.3}$$

are continuous for any $f \in C_0(\mathbb{R})$ (the space of the continuous functions with compact support).

Using correspondence (2.2) one can endow Γ with topology and construct the corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$. It follows from (2.1) and (2.3) that

$$C(K;n) = \{\gamma \in \Gamma \mid \langle 1_K, \gamma\rangle = n\},$$

there 1_K stands for the indicator of the set K. This equality yields that any cylinder set belongs to $\mathcal{B}(\Gamma)$, so $\mathcal{C}_\sigma(\Gamma) \subset \mathcal{B}(\Gamma)$. Moreover, the Borel σ-algebra $\mathcal{B}(\Gamma)$ coincides with $\mathcal{C}_\sigma(\Gamma)$ (see, e.g., [8,12]).

We recall two equivalent definitions of Poisson measure on a configuration space. In the sequel, σ denotes a non-atomic infinite Radon measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$; it is taken to be the intensity measure of the Poisson measure π_σ on $(\Gamma, \mathcal{B}(\Gamma))$.

1. Values of π_σ on cylinder sets are defined by

$$\pi_\sigma (C (K; n)) := \frac{(\sigma (K))^n}{n!} e^{-\sigma(K)}, \quad K \in \mathcal{B}_c (\mathbb{R}), \quad n \in \mathbb{Z}_+. \tag{2.4}$$

Then one can apply an extension procedure and obtain the Poisson measure π_σ with the intensity σ on $(\Gamma, \mathcal{B}(\Gamma))$ (see, e.g., [8, 12, 4]) .

2. The measure π_σ on a Schwartz space $\mathcal{D}' (\mathbb{R})$ of generalized functions is determined by its Laplace transform

$$l_{\pi_\sigma} (\varphi) := \int_\Gamma \exp \{\langle \varphi, \gamma\rangle\} \pi_\sigma (d\gamma) = \exp \left\{ \int_{\mathbb{R}} \left(e^{\varphi(x)} - 1\right) \sigma (dx) \right\}, \varphi \in \mathcal{D} (\mathbb{R}) \tag{2.5}$$

via Minlos theorem (see [5]). An additional analysis shows that π_σ is concentrated on the generalized functions corresponding to measures from $\mathcal{M}_p (\mathbb{R})$. Therefore π_σ can be considered as a measure on $(\Gamma, \mathcal{B}(\Gamma))$.

Remark 2.1. In the sequel, we will need the probabilistic interpretation of π_σ. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be the probability space. A Poisson process on $\mathbb{R}$ with the intensity σ is a random countable subset Π_σ in $\mathbb{R}$, i.e. the mapping

$$\Omega \ni \omega \longmapsto \Pi_\sigma (\omega) \in \mathbb{R}^\infty$$

from Ω into the set $\mathbb{R}^\infty$ of all countable subsets in $\mathbb{R}$, such that the random variables (r.v.'s)

$$N_\sigma (A) := |\Pi_\sigma \cap A| , A \in \mathcal{B} (\mathbb{R})$$

satisfy the following conditions

(i) for any disjoint sets $A_1, ..., A_n \in \mathcal{B}(\mathbb{R})$ r.v.'s $N_\sigma (A_1) , ..., N_\sigma (A_n)$ are independent;

(ii) r.v. $N_\sigma (A)$ has the Poisson distribution with the parameter $\sigma (A)$, i.e. for any $n \in \mathbb{Z}_+$ $\mathrm{P}(\{\omega \in \Omega | N_\sigma (A) = n\}) = \frac{(\sigma(A))^n}{n!} e^{-\sigma(A)}$, if $0 < \sigma (A) < \infty$; $\Pi_\sigma \cap A$ is countably infinite with probability 1, if $\sigma (A) = \infty$; and $\Pi_\sigma \cap A = \emptyset$, if $\sigma (A) = 0$ (see, e.g., [9]).

Setting $(\Omega, \mathcal{F}, \mathrm{P}) = (\Gamma, \mathcal{B}(\Gamma) , \pi_\sigma)$ we obtain the following (so called direct) representation of the Poisson process Π_σ:

$$\Gamma \ni \gamma \longmapsto \Pi_\sigma (\gamma) = \gamma \in \Gamma \subset \mathbb{R}^\infty. \tag{2.6}$$

It is worth noting that the non-negative integer random variable

$$\Gamma \ni \gamma \longmapsto N_\sigma (\cdot; \gamma) := |\cdot \cap \gamma| \in \mathcal{M}_p (\mathbb{R})$$

is called a Poisson random measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ with the intensity σ.

The strong law of large numbers for Π_σ yields the following property of π_σ.

Proposition 2.2. *The set*

$$\Gamma_\sigma := \left\{ \gamma \in \Gamma \left| \frac{|\gamma \cap \Lambda|}{\sigma (\Lambda)} \right. \to 1, \Lambda \in \mathcal{B}_c (\mathbb{R}) , \Lambda \uparrow \mathbb{R} \right\} \tag{2.7}$$

has full π_σ-measure.

Proof. By (2.5) and (2.6) we have for any $u \in \mathbb{R}$ and $\Lambda \in \mathcal{B}_c(\mathbb{R})$ that

$$\int_\Omega \exp\left\{iu\frac{N_\sigma(\Lambda;\omega)}{\sigma(\Lambda)}\right\} \mathrm{P}(d\omega) = \int_\Gamma \exp\left\{\frac{iu}{\sigma(\Lambda)}\langle 1_\Lambda, \gamma\rangle\right\}\pi_\sigma(d\gamma)$$
$$= \exp\left\{\sigma(\Lambda)\left(e^{i\frac{u}{\sigma(\Lambda)}} - 1\right)\right\}.$$

The last expression converges to $\exp\{iu\}$, when $\sigma(\Lambda) \to +\infty$. That's why the ratio $\frac{\mathbb{N}_\sigma(\Lambda;\gamma)}{\sigma(\Lambda)}$ converges in measure π_σ to 1, when $\sigma(\Lambda) \to +\infty$.

To establish the π_σ-a.e. convergence one can apply Kolmogorov strong law of large numbers (see, e.g., [11]). Namely, any increasing sequence $\{\Lambda_k, k \geq 1\} \subset \mathcal{B}_c(\mathbb{R}), \Lambda_k \to \mathbb{R}$ contains the subsequence $\left\{\Lambda_{k(n)}, n \geq 1\right\}$, such that

$$\frac{\sigma\left(\Lambda_{k(n-1)}\right)}{\sigma\left(\Lambda_{k(n)}\right)} \to 1, n \to \infty; \qquad \sum_{n=1}^{\infty} \frac{1}{\sigma\left(\Lambda_{k(n)}\right)} < \infty.$$

By the mentioned above Kolmogorov law the sequence of r.v.'s $\left\{\frac{N_\sigma\left(\Lambda_{k(n)};\cdot\right)}{\sigma\left(\Lambda_{k(n)}\right)}, n \geq 1\right\}$ converges π_σ-a.e. to 1. Let $n_m \in \mathbb{N}$ be such that

$$k(n_m) \leq m < k(n_m + 1), \quad m \in \mathbb{N}.$$

Then we have

$$\frac{\sigma\left(\Lambda_{k(n_m)}\right)}{\sigma(\Lambda_m)} \cdot \frac{N_\sigma\left(\Lambda_{k(n_m)};\cdot\right)}{\sigma\left(\Lambda_{k(n_m)}\right)} \leq \frac{N_\sigma(\Lambda_m;\cdot)}{\sigma(\Lambda_m)} \leq \frac{\sigma\left(\Lambda_{k(n_m+1)}\right)}{\sigma(\Lambda_m)} \cdot \frac{N_\sigma\left(\Lambda_{k(n_m+1)};\cdot\right)}{\sigma\left(\Lambda_{k(n_m+1)}\right)},$$

whence the required convergence follows. □

3. Exponential product measure on $\left(\mathbb{R}_+^\infty, \mathcal{B}\left(\mathbb{R}_+^\infty\right)\right)$.

At first we recall definitions of a measurable space $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ and the product measure $\times_{k=1}^\infty \mu_k$ (for more details see, e.g., [14, 16, 3]).

One can equip the set $\mathbb{R}^\infty := \mathbb{R} \times \mathbb{R} \times ...$ with the metric of the coordinate-wise convergence,

$$d(t,s) = \sum_{n=0}^{\infty} 2^{-n} \frac{|t_n - s_n|}{1 + |t_n - s_n|}, \quad t = \{t_k, k \geq 1\} \in \mathbb{R}^\infty, \quad s = \{s_k, k \geq 1\} \in \mathbb{R}^\infty.$$

It is known that $(\mathbb{R}^\infty, d)$ is a Polish space, and it's Borel σ-algebra is generated by the algebra $\mathcal{C}(\mathbb{R}^\infty)$ of the cylinder sets

$$C(\delta_n) = \{t \in \mathbb{R}^\infty | (t_1, ..., t_n) \in \delta_n\}, \quad \delta_n \in \mathcal{B}(\mathbb{R}^n), \quad n \in \mathbb{N}.$$

Let $\{\mu_k, k \in \mathbb{N}\}$ be probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The product measure $\times_{k=1}^\infty \mu_k$ is defined on $\mathcal{C}(\mathbb{R}^\infty)$ by

$$\left(\times_{k=1}^\infty \mu_k\right)(C(\delta_n)) := \left(\times_{k=1}^n \mu_k\right)(\delta_n)$$

and then extended onto $\mathcal{B}(\mathbb{R}^\infty)$.

In what follows we will consider the product measure on the measurable space $\left(\mathbb{R}_+^\infty, \mathcal{B}\left(\mathbb{R}_+^\infty\right)\right)$, where $\mathbb{R}_+ := (0;\infty)$, $\mathcal{B}\left(\mathbb{R}_+^\infty\right) := \left\{\Delta \cap \mathbb{R}_+^\infty | \Delta \in \mathcal{B}\left(\mathbb{R}^\infty\right)\right\}$. Namely, let the measure ν_k on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$ be a probability distribution of the r.v. τ_{k,λ_k} exponentially distributed with the parameter $\lambda_k > 0$. In other words, ν_k is absolutely continuous with respect to (w.r.t.) Lebesgue measure m and the Radon-Nikodym derivative has the form

$$\frac{d\nu_k}{dm}(t) = \lambda_k e^{-\lambda_k t} 1_{\mathbb{R}_+}(t). \tag{3.1}$$

The product measure $\times_{k=1}^\infty \nu_k$ on $\left(\mathbb{R}_+^\infty, \mathcal{B}\left(\mathbb{R}_+^\infty\right)\right)$ is a joint probability distribution of the sequence of the independent r.v. $\{\tau_{k,\lambda_k}, k \geq 1\}$. The following property of $\times_{k=1}^\infty \nu_k$ is an analogue of Kolmogorov-Khinchine criterion for the standard Gauss measure on $(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty))$ (see, e.g., [14]).

To formulate this property we denote

$$l_1(\alpha) := \left\{ t \in \mathbb{R}^\infty \left| \sum_{n=1}^\infty \alpha_n |t_n| < \infty \right.\right\},$$

where the weight $\alpha = \{\alpha_k, k \geq 1\} \in \mathbb{R}_+^\infty$. (If $\alpha_k = 1$ for any $k \geq 1$, then $l_1(\alpha) \equiv l_1$). Obviously, $l_1(\alpha)$ is a Borel set and moreover it belongs to the tail σ-algebra

$$\mathcal{B}_\infty(\mathbb{R}^\infty) := \bigcap_{m=1}^\infty \{\mathbb{R}^m \times \Delta | \Delta \in \mathcal{B}(\mathbb{R}^\infty)\}.$$

Then by Kolmogorov "0 or 1" law the $\times_{k=1}^\infty \nu_k$-measure of $l_1(\alpha) \cap \mathbb{R}_+^\infty =: l_{1,+}(\alpha)$ is equal to either 0 or 1 (see, e.g., [16]). More exactly, the following statement is true.

Lemma 3.1. *The set $l_{1,+}(\alpha)$ has zero $\times_{k=1}^\infty \nu_k$-measure iff the series $\sum_{k=1}^\infty \frac{\alpha_k}{\lambda_k}$ diverges.*

Proof. The following arguments are nothing but a corresponding modification of the proof of Kolmogorov–Khinchine criterion (see [14, pp. 54–56]).

Given $\varepsilon > 0$ consider the decreasing sequence of functions

$$\mathbb{R}_+^\infty \ni t \longmapsto f_{\varepsilon,n}(t) := \exp\left\{-\varepsilon \sum_{k=1}^n \alpha_k t_k\right\} \in (0;1] \tag{3.2}$$

and its point-wise limit $f_\varepsilon(t) = \exp\left\{-\varepsilon \sum_{k=1}^\infty \alpha_k t_k\right\} \in [0;1]$. Obviously, $f_\varepsilon(t) > 0$, if $t \in l_{1,+}(\alpha)$, and $f_\varepsilon(t) = 0$ otherwise. Moreover, for any $t \in \mathbb{R}_+^\infty$,

$$f_\varepsilon(t) \to 1_{l_{1,+}(\alpha)}(t), \quad \varepsilon \to 0. \tag{3.3}$$

On the other hand, by Lebesgue monotone convergence theorem and (3.2), we have

$$\begin{aligned} \int_{\mathbb{R}_+^\infty} f_\varepsilon(t) \left(\times_{k=1}^\infty \nu_k\right)(dt) &= \lim_{n\to\infty} \int_{\mathbb{R}_+^\infty} f_{\varepsilon,n}(t) \left(\times_{k=1}^\infty \nu_k\right)(dt) \\ &= \left(\prod_{k=1}^\infty \left(1 + \varepsilon \frac{\alpha_k}{\lambda_k}\right)\right)^{-1}. \end{aligned} \tag{3.4}$$

Suppose that

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{\lambda_k} = \infty.$$

Then by (3.4)

$$\int_{\mathbb{R}_+^{\infty}} f_{\varepsilon}(t) \left(\times_{k=1}^{\infty} \nu_k\right)(dt) = 0,$$

whence if follows that $\left(\times_{k=1}^{\infty} \nu_k\right)\left(l_{1,+}(\alpha)\right) = 0$.

If the above series converges, then for any $\varepsilon > 0$ the infinite product in the l.h.s. of (3.4) does the same. Therefore for any $\varepsilon > 0$,

$$0 < \int_{\mathbb{R}_+^{\infty}} f_{\varepsilon}(t) \left(\times_{k=1}^{\infty} \nu_k\right)(dt) < \infty.$$

By Lebesgue bounded convergence theorem and (3.3), the r.h.s. of (3.4) converges to $\left(\times_{k=1}^{\infty} \nu_k\right)\left(l_{1,+}(\alpha)\right)$, when $\varepsilon \to 0$. On the other hand, $\prod_{k=1}^{\infty}\left(1 + \varepsilon \frac{\alpha_k}{\lambda_k}\right) \to 1, \varepsilon \to 0$, whence the claim follows. □

Corollary 3.2. *Denote by ν_λ^{∞} the product measure $\times_{k=1}^{\infty} \nu_k$, when $\lambda_k = \lambda$. The set*

$$\widetilde{\mathbb{R}_+^{\infty}} := \mathbb{R}_+^{\infty} \backslash l_{1,+}, \tag{3.5}$$

has full ν_λ^{∞}-measure.

Remark 3.3. The measure ν_λ^{∞} is a joint probability distribution of i.i.d. r.v.'s $\{\tau_{k,\lambda_k}, k \geq 1\}$. One can apply the strong law of large numbers and obtain that

$$\nu_\lambda^{\infty}\left(\left\{ t \in \mathbb{R}_+^{\infty} \,\middle|\, \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} t_k = \lambda^{-1} \right\}\right) = 1. \tag{3.6}$$

4. Exponential model of Poisson space.

Consider in detail the Poisson space $(\Gamma, \mathcal{B}(\Gamma), \pi_{\lambda m}), \lambda > 0$. Any configuration $\gamma \in \Gamma$ is either a finite set or countably infinite one without finite limit points. It follows from (2.7) (see also Remark 2.1) that for any $A \in \mathcal{B}(\mathbb{R}), m(A) = \infty$ and $\pi_{\lambda m}$-a.e. $\gamma \in \Gamma \, |\gamma \cap A| = \infty$. In particular, the sets $\gamma \cap (0; +\infty)$ and $\gamma \cap (-\infty; 0)$ are infinite, so that their points can be written in an order so that

$$0 < x_1 < x_2 < \ldots \quad \text{and} \quad \ldots\ x_{-2} < x_{-1} < 0 \tag{4.1}$$

respectively. These exhaust the points of $\pi_{\lambda m}$-a.e. $\gamma \in \Gamma$, since by (2.1) and (2.4),

$$\pi_{\lambda m}\left(\{\gamma \in \Gamma | 0 \in \gamma\}\right) = \pi_{\lambda m}\left(C\left(\{0\}; 1\right)\right) = 0.$$

Lemma 4.1. *The set*

$$\tilde{\Gamma} := \{\gamma \in \Gamma|\, |\gamma| = \infty; 0 \notin \gamma\} \tag{4.2}$$

belongs to $\mathcal{B}(\Gamma)$ *and has full* $\pi_{\lambda m}$ *-measure.*

Proof. The complement of $\tilde{\Gamma}$ coincides with the union of $C(\{0\};1)$ and $\{\gamma \in \Gamma|\, |\gamma| < \infty\}$. One can represent the latter set as

$$\bigcup_{p=0}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=0}^{p} C((-N;N);k),$$

whence the required measurability of $\tilde{\Gamma}$ follows. The equality $\pi_{\lambda m}\left(\tilde{\Gamma}\right) = 1$ was explained above. □

Definition 4.2. The triple

$$\left(\tilde{\Gamma}, \tilde{\Gamma} \cap \mathcal{B}(\Gamma) =: \mathcal{B}\left(\tilde{\Gamma}\right), \pi_{\lambda m} \upharpoonright \tilde{\Gamma}\right) \tag{4.3}$$

is called a standard Poisson space.

To introduce the exponential model of a standard Poisson space notice that any configuration may be written in the form $\gamma = \gamma_+ \cup \gamma_-$, where

$$\gamma_+ := \gamma \cap (0;+\infty) = \{x_k, k \in \mathbb{N}\}, \qquad \gamma_- := \gamma \cap (-\infty;0) = \{x_{-k}, k \in \mathbb{N}\}$$

(see (4.1)). The sequences $\{x_k, k \in \mathbb{N}\}$ and $\{-x_{-k}, k \in \mathbb{N}\}$ are monotonically increasing and tend to $+\infty$. Therefore the difference sequences

$$t_1^+ := x_1, \quad t_k^+ := x_k - x_{k-1}; \quad t_1^- := -x_{-1}, \quad t_k^- := x_{-k+1} - x_{-k} \tag{4.4}$$

belong to $\widetilde{\mathbb{R}_+^\infty}$. Indeed, for any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} t_k^+ = x_n \to +\infty, \qquad \sum_{k=1}^{n} t_k^- = -x_{-n} \to +\infty.$$

We introduce the mapping

$$\tilde{\Gamma} \ni \gamma = \gamma_+ \cup \gamma_- \longmapsto T\gamma = \{T\gamma_+, T\gamma_-\} = \left\{t^+, t^-\right\} \in \widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}. \tag{4.5}$$

It is almost obvious that T is a one-to-one correspondence between $\tilde{\Gamma}$ and $\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}$. Indeed, for any $\{t,s\} \in \widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}$, the set

$$T^{-1}\{t,s\} := \left\{\sum_{k=1}^{n} t_k, n \in \mathbb{N}\right\} \bigcup \left\{-\sum_{k=1}^{n} s_k, n \in \mathbb{N}\right\} \tag{4.6}$$

is a configuration from $\tilde{\Gamma}$ and (4.6) is nothing but the definition of $T^{-1} : \widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty} \longmapsto \tilde{\Gamma}$.

Recall that $\tilde{\Gamma}$ and $\widetilde{\mathbb{R}_+^\infty}$ are endowed with the vague topology (see (2.3)) and the product topology respectively. It follows from (4.4) – (4.6) that both mappings T and $T^{-1}$are continuous and hence Borel measurable. That's why T is a measurable one-to-one correspondence between the measurable spaces $\left(\tilde{\Gamma}, \mathcal{B}\left(\tilde{\Gamma}\right)\right)$ and $\left(\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}, \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right) \times \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right)\right)$.

So we can consider the images

$$T^*\pi_{\lambda m} := \pi_{\lambda m} \circ T^{-1}, \qquad \left(T^{-1}\right)^* \nu_\lambda^\infty \times \nu_\lambda^\infty := \left(\nu_\lambda^\infty \times \nu_\lambda^\infty\right) \circ T$$

of the measures $\pi_{\lambda m}$ and $\nu_\lambda^\infty \times \nu_\lambda^\infty$ under the transformations T and T^{-1} respectively. It is easy to verify that

$$\pi_{\lambda m} = \left(T^{-1}\right)^* \nu_\lambda^\infty \times \nu_\lambda^\infty.$$

This equality follows from

$$\pi_{\lambda m}\left(C\left((a;b);n\right)\right) = \nu_\lambda^\infty \times \nu_\lambda^\infty\left(T\left(C\left((a;b);n\right)\right)\right),$$

where $-\infty < a < b < +\infty, n \in \mathbb{Z}_+$. At first we put $a = 0$. It is clear that

$$T\left(C\left((0;b);n\right)\right) = \left\{ t \in \widetilde{\mathbb{R}_+^\infty} \,\middle|\, \sum_{k=1}^{n} t_k < b, \sum_{k=1}^{n+1} t_k \geq b \right\} =: A_n(b),$$

and a simple computation shows that

$$\begin{aligned}\nu_\lambda^\infty\left(A_n(b)\right) = \lambda^{n+1} &\int_{\left\{\sum_{k=1}^n t_k < b\right\}} \exp\left\{-\lambda \sum_{k=1}^{n} t_k\right\} \\ &\times \int_{b-\sum_{k=1}^n t_k}^{\infty} \exp\left\{-\lambda t_{n+1}\right\} dt_{n+1} dt_1 ... dt_n \\ &= \frac{(\lambda b)^n}{n!} e^{-\lambda b} = \pi_{\lambda m}\left(C\left((0;b);n\right)\right).\end{aligned}$$

Let $a > 0$. Then the required equality follows by

$$C\left((0;b);m\right) = \bigcup_{k=0}^{m} \left(C\left((0;a);k\right) \cap C\left((a;b);m-k\right)\right)$$

from the one just proved. Analogously one can consider the case $a < b < 0$.

At last for $a < 0 < b$ one has:

$$\begin{aligned}C\left((a;b);n\right) &= \left\{\gamma = \gamma_+ \cup \gamma_- \in \tilde{\Gamma} \,|\, |\gamma \cap (a;b)| = n\right\} \\ &= \bigcup_{k=0}^{n} \left(\left\{|\gamma_- \cap (a;0)| = n-k\right\} \cup \left\{|\gamma_+ \cap (0;b)| = k\right\}\right),\end{aligned}$$

so that

$$\begin{aligned}\nu_\lambda^\infty \times \nu_\lambda^\infty\left(T\left(C\left((a;b);n\right)\right)\right) &= \sum_{k=0}^{n} \frac{(\lambda b)^k}{k!} \frac{(-\lambda a)^{n-k}}{(n-k)!} e^{-\lambda(b-a)} \\ &= \frac{\left(\lambda(b-a)\right)^n}{n!} e^{-\lambda(b-a)} = \pi_{\lambda m}\left(C\left((a;b);n\right)\right).\end{aligned}$$

We have just proved the following statement

Theorem 4.3. *Given by (4.5) and (4.6) the transformations T and T^{-1} realize a measurable one-to-one correspondence between the standard Poisson space (4.3) and the probability space*

$$\left(\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}, \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right) \times \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right), \nu_\lambda^\infty \times \nu_\lambda^\infty\right) \tag{4.7}$$

such that

$$\pi_{\lambda m} = \left(T^{-1}\right)^* \nu_\lambda^\infty \times \nu_\lambda^\infty, \qquad \nu_\lambda^\infty \times \nu_\lambda^\infty = T^* \pi_{\lambda m}. \tag{4.8}$$

Corollary 4.4. *By Proposition 2.2 the set*

$$\tilde{\Gamma}_{\lambda m} := \Gamma_{\lambda m} \backslash C\left(\{0\}, 1\right)$$

has the full $\pi_{\lambda m}$-measure. The restriction $T \upharpoonright \tilde{\Gamma}_{\lambda m}$ realizes a one-to-one correspondence between $\tilde{\Gamma}_{\lambda m}$ and the Cartesian square of the set (3.6) of full ν_λ^∞-measure.

Remark 4.5. In fact, Theorem 4.3 is nothing but a corollary of the well-known interval theorem of Poisson theory (see, e.g., [9, pp. 39–40]). Namely, let the points of a Poisson process $\Pi_{\lambda m}$ on $(0;\infty)$ be written in an ascending order as in (4.1). Then the random variables $y_1(\gamma_+) = x_1, y_n(\gamma_+) = x_n - x_{n-1}, n \geq 2$, are independent and each has exponential density with the parameter λ.

Definition 4.6. The pair consisting of the probability space (4.7) and the mapping (4.6) is called an exponential model of a standard Poisson space. And vice versa, the pair consisting of the Poisson space (4.3) and the mapping (4.5) is called a Poisson model of the probability space (4.7). Of course, one can consider the homogeneous Poisson process $\Pi_{\lambda m}$ on $\mathbb{R}_+$ or, in other words, its direct representation (2.6) via the standard Poisson space

$$\left(\tilde{\Gamma}_{\mathbb{R}_+}, \mathcal{B}\left(\tilde{\Gamma}_{\mathbb{R}_+}\right), \pi_{\lambda m} \upharpoonright \mathbb{R}_+\right). \tag{4.9}$$

To obtain an exponential model of this space one must replace the Cartesian square in (4.7) by the probability space $\left(\widetilde{\mathbb{R}_+^\infty}, \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right), \nu_\lambda^\infty\right)$ and T in (4.5) by its restriction $T \upharpoonright \tilde{\Gamma}_{\mathbb{R}_+}$.

Remark 4.7. N. Privault [13] introduced an interpretation of the Poisson space as the space of sequences

$$\mathcal{B} := \left\{ t \in \mathbb{R}^\infty \,\middle|\, \sup\left\{ \frac{|t_k|}{k+1}, k \in \mathbb{N} \right\} < \infty \right\}$$

equipped with the exponential product measure ν_λ^∞. This interpretation were used for investigation of multiple stochastic integrals w.r.t. Poisson martingale, Wiener-Poisson chaotic decomposition etc.

Remark 4.8. Suppose that the Radon measure σ is equivalent to m but doesn't coincide with λm. In this case the exponential model of the corresponding Poisson space

$$\left(\tilde{\Gamma}_\sigma, \mathcal{B}\left(\tilde{\Gamma}_\sigma\right), \pi_\sigma\right) \tag{4.10}$$

can be constructed via the so-called homogenization procedure (see, e.g., [9, pp. 50 – 52]).

Namely, any $\sigma \in \mathcal{M}(\mathbb{R})$ is a Lebesgue-Stieltjes measure generated by the non-decreasing function

$$\varphi_\sigma(x) = -\sigma([x;0))\, 1_{(-\infty;0)}(x) + \sigma([0;x))\, 1_{[0;\infty)}(x).$$

Since $\sigma(\mathbb{R}) = \infty$, at least one of the limit values $\varphi_\sigma(\pm\infty) := \lim_{x\to\pm\infty} \varphi_\sigma(x)$ is infinite. By the non-atomicity of σ, the function φ_σ is continuous, so that $\varphi_\sigma(\mathbb{R}) = (\varphi_\sigma(-\infty); \varphi_\sigma(+\infty))$. The equivalence $\sigma \sim m$ yields that φ_σ is a increasing function and the map

$$\mathbb{R} \ni x \longmapsto \varphi_\sigma(x) \in \varphi_\sigma(\mathbb{R}) \tag{4.11}$$

is a one-to-one correspondence.

It is easy to see that the image of the Poisson space (4.10) under the transformation (4.11) coincides with

$$\left(\tilde{\Gamma}_{\varphi_\sigma(\mathbb{R})}, \mathcal{B}\left(\tilde{\Gamma}_{\varphi_\sigma(\mathbb{R})}\right), \pi_{m\restriction\varphi_\sigma(\mathbb{R})}\right). \tag{4.12}$$

An obvious modification of above construction gives an exponential model of Poisson space (4.12). Replacing here the corresponding mapping (4.5) T_σ by $T_\sigma \circ \varphi_\sigma$ we obtain an exponential model of non-homogeneous Poisson space (4.10).

Notice that (4.10) and (4.12) realizes a direct representation of non-homogeneous Poisson process Π_σ and homogeneous one Π_m respectively. That is why the transition from (4.10) to (4.12) is called homogenization.

5. Riggings of Poisson space via exponential model.

Consider the Poisson space (4.3) with $\lambda = 1$ and its exponential model. The transformations (4.5) and (4.6) determine a unitary isomorphism of the Hilbert spaces

$$L^2(\pi_m) := L^2\left(\tilde{\Gamma}, \mathcal{B}\left(\tilde{\Gamma}\right), \pi_m\right)$$

and

$$L^2(\nu^\infty \times \nu^\infty) := L^2\left(\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}, \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right) \times \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right), \nu^\infty \times \nu^\infty\right).$$

Namely,

$$\begin{aligned} L^2(\pi_m) \ni F \longmapsto UF = F \circ T^{-1} \in L^2(\nu^\infty \times \nu^\infty), \\ L^2(\nu^\infty \times \nu^\infty) \ni f \longmapsto U^{-1}f = f \circ T \in L^2(\pi_m) \end{aligned} \tag{5.1}$$

(here and in the sequel on we omit the index 1). Isomorphism (5.1) enables to interpret the results of analysis and differential geometry in Poisson space (4.3) in the framework of the classical infinite dimensional analysis w.r.t. the product measures. And vice versa, one can reformulate the results of analysis on $\mathbb{R}_+^\infty \times \mathbb{R}_+^\infty$ w.r.t. $\nu^\infty \times \nu^\infty$ in Poisson space.

We start from a simple example. Laguerre polynomials $\{L_n(t), n \in \mathbb{Z}_+\}$ form an orthonormal basis in $L^2(\mathbb{R}_+, \nu)$. Recall that they are determined by the generating function

$$l(t,z) = \sum_{n=0}^{\infty} L_n(t) z^n = (1-z)^{-1} \exp\left\{\frac{tz}{z-1}\right\}, \quad |z| < 1.$$

Then the collection of the Laguerre polynomials on $\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}$,

$$L_{\alpha,\beta}(t,s) = \prod_{k=1}^{\nu(\alpha)} L_{\alpha_k}(t_k) \prod_{j=1}^{\nu(\beta)} L_{\beta_j}(s_j), \quad \alpha,\beta \in \mathbb{Z}_{+,0}^\infty \tag{5.2}$$

forms an orthonormal basis in $L^2(\nu^\infty \times \nu^\infty)$. (Here the multiindex $\alpha = \left(\alpha_1, \ldots, \alpha_{\nu(\alpha)}, 0, \ldots\right), \alpha_k \in \mathbb{Z}_+$). The family (5.2) is an orthogonal basic in the weighted space

$$H(p) = \left\{ f = \sum f_{\alpha,\beta} L_{\alpha,\beta} \,\middle|\, \sum p_{\alpha,\beta} |f_{\alpha,\beta}|^2 < \infty \right\}. \tag{5.3}$$

Here $\sum$ denotes summation over $\alpha,\beta \in \mathbb{Z}_{+,0}^\infty$, the weight $p := \left\{p_{\alpha,\beta}, \alpha,\beta \in \mathbb{Z}_{+,0}^\infty\right\} \subset \mathbb{R}_+^\infty$. The projective and inductive limits of the proper families of the spaces (5.3) form riggings of $L^2(\nu^\infty \times \nu^\infty)$ with Köthe spaces (see, e.g., [3]).

Applying U^{-1} in (5.2) one can obtain an orthonormal basis in $L^2(\pi_m)$ of the form

$$L_{\alpha,\beta}(T\gamma) = \prod_{k=1}^{\nu(\alpha)} L_{\alpha_k}(x_k - x_{k-1}) \prod_{j=1}^{\nu(\beta)} L_{\beta_j}(x_{-j+1} - x_{-j}), \tag{5.4}$$
$$\alpha,\beta \in \mathbb{Z}_{+,0}^\infty, \quad x_0 := 0.$$

It is worth noting that the basis (5.4) differs from the usual orthonormal basis in $L^2(\pi_m)$ constituted by the generalized Charlier polynomials (see, e.g., [7], [19]). The latter polynomials are defined via the generating function

$$c(\gamma;\psi) = \exp\left\{ \langle \gamma, \log(1+\psi)\rangle - \int_{\mathbb{R}} \psi(x)\,dx \right\}, \tag{5.5}$$
$$\gamma \in \tilde{\Gamma}, \quad \psi \in L^1 \cap L^2(\mathbb{R}), \quad \psi > -1.$$

Namely, the generalized Charlier polynomials are given by

$$C_n\left(\gamma; \varphi_1 \hat{\otimes} \ldots \hat{\otimes} \varphi_n\right) := \frac{\partial^n}{\partial u_1 \ldots \partial u_n} c\left(\gamma; \sum_{k=1}^n u_k \varphi_k\right)\Bigg|_{u_1 = \cdots = u_n = 0},$$
$$n \in \mathbb{N}, \quad C_0(\gamma) := 1,$$

where $\varphi_1, \ldots, \varphi_n \in L^1 \cap L^2(\mathbb{R})$, $\hat{\otimes}$ denotes the symmetric tensor product.

Let $\{e_k, k \in \mathbb{Z}_+\} \subset L^1 \cap L^2(\mathbb{R})$ be an orthonormal basis in $L^2(\mathbb{R})$. The collection of the generalized Charlier polynomials

$$\left\{ C_n\left(\cdot; e_{\alpha_1} \hat{\otimes} \ldots \hat{\otimes} e_{\alpha_p}\right); p, \alpha_1, \ldots, \alpha_p \in \mathbb{Z}_+ \right\} \tag{5.6}$$

forms an orthonormal basis in $L^2(\pi_m)$ (see [7]).

The bases (5.4) and (5.6) are different (for a detailed discussion and probabilistic interpretation, see [13]). Therefore applying U^{-1} in (5.3) we obtain riggings of $L^2(\pi_m)$ with $U^{-1}H(p)$ and corresponding Köthe spaces. These spaces of test and generalized functions on $\tilde{\Gamma}$ differ from the spaces constructed in [19] via generalized Charlier polynomials and corresponding Appell system $(\mathbb{P}^{\pi_m}, \mathbb{Q}^{\pi_m})$ (see also [10]).

6. The group of diffeomorphisms and exponential model.

Denote the group of all diffeomorphisms of $\mathbb{R}$ by $\mathrm{Diff}(\mathbb{R})$, and by $\mathrm{Diff}_0(\mathbb{R})$ — the subgroup of all diffeomorphisms $\phi : \mathbb{R} \longmapsto \mathbb{R}$ which are equal to the identity outside a compact set depending on ϕ, say, K_ϕ. Since the mapping $\phi : \mathbb{R} \longmapsto \mathbb{R}$ is one-to-one we have $\phi K_\phi = K_\phi$.

Any $\phi \in \mathrm{Diff}_0(\mathbb{R})$ pointwise defines a transformation of any subset in $\mathbb{R}$ and, in particular, of any configuration:

$$\Gamma \ni \gamma \longmapsto \phi(\gamma) = \{\phi(x) \,|x \in \gamma\} \in \Gamma. \tag{6.1}$$

Note that $\phi(\gamma)$ coincides with γ for all but a finite number of points, namely

$$\gamma \cap K_\phi \neq \phi(\gamma) \cap K_\phi.$$

The mapping (6.1) is obviously measurable and one can define the image $\phi^*\pi_\sigma$ of the measure π_σ under ϕ, as usual, by

$$(\phi^*\pi_\sigma)(\cdot) = \pi_\sigma\left(\phi^{-1}(\cdot)\right).$$

It is well-known that this transformation is nothing but a change of the intensity measure, namely

$$\phi^*\pi_\sigma = \pi_{\phi^*\sigma} \tag{6.2}$$

(for more details see [1]).

Further on we suppose that $\sigma \sim m$ and the density $\rho = \frac{d\sigma}{dm}$ is continuous on $\mathbb{R}$. This density is positive m-a.e. whence, by continuity, it follows that $\rho(x) > 0, x \in \mathbb{R}$. By $\sigma \sim m$ the measures σ and $\phi^*\sigma$ are equivalent for any $\phi \in \mathrm{Diff}_0(\mathbb{R})$, and the Radon-Nikodym density has the form

$$p_\phi^\sigma(x) := \frac{d(\phi^*\sigma)}{d\sigma}(x) = \frac{\rho\left(\phi^{-1}(x)\right)}{\rho(x)}\phi'(x), \quad x \in \mathbb{R} \tag{6.3}$$

(this function is well-defined by $\rho(x) > 0, x \in \mathbb{R}$).

It is worth noting that $p_\phi^\sigma(x) = 1, x \notin K_\phi$, whence it follows that

$$p_\phi^\sigma - 1 \in L^1(\mathbb{R}, \sigma) \tag{6.4}$$

and

$$\begin{aligned} \int_{\mathbb{R}} \left(1 - p_\phi^\sigma(x)\right)\rho(x) &= \int_{K_\phi} \left(\rho(x) - \rho\left(\phi^{-1}(x)\right)\phi'(x)\right) dx \\ &= \sigma(K_\phi) - \sigma(K_\phi) = 0. \end{aligned} \tag{6.5}$$

The following fact is a direct consequence of (6.2) – (6.5) and Skorokhod's theorem on absolute continuity of Poisson measures (see [17], [18], [15]). We give a sketch of the proof for the reader's convenience.

Proposition 6.1. *Assume that $\sigma \sim m$ and $\rho = \frac{d\sigma}{dm} \in C(\mathbb{R})$. The Poisson measure π_σ is quasi-invariant w.r.t. the group* $\mathrm{Diff}_0(\mathbb{R})$, *and for any $\phi \in \mathrm{Diff}_0(\mathbb{R})$ we have*

$$\frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\gamma) = \prod_{x\in\gamma} p_\phi^\sigma(x). \tag{6.6}$$

Proof. It follows from $\phi^*\sigma \sim \sigma$ and (6.4) that for any $\phi \in \mathrm{Diff}_0(\mathbb{R})$ $\pi_{\phi^*\sigma} \sim \pi_\sigma$ (see [18]). By (6.2) the latter equivalence means the required quasi-invariance of π_σ. For the proof of equality (6.6) we refer to [1, Proposition 2.2]. It is worth noting that the r.h.s. of (6.6) coincides with the same of (2.12) from [1] by (6.5). □

Corollary 6.2. *Let $\sigma = \lambda m$; then for any $\phi \in \mathrm{Diff}_0(\mathbb{R})$*

$$\frac{d(\phi^*\pi_{\lambda m})}{d\pi_{\lambda m}}(\gamma) = \prod_{x\in\gamma} \phi'(x). \tag{6.7}$$

Consider the Poisson space (4.3). It is clear that, generally speaking, $\phi\left(\tilde{\Gamma}\right) \neq \tilde{\Gamma}$, $\phi \in \mathrm{Diff}_0(\mathbb{R})$. That's why we introduce the subgroup

$$\mathrm{Diff}_{0,0}(\mathbb{R}) := \{\psi \in \mathrm{Diff}_0(\mathbb{R})\,|\,\psi(0) = 0\}. \tag{6.8}$$

Obviously, the subgroup (6.8) is the maximal one preserving $\tilde{\Gamma}$.

For any $\psi \in \mathrm{Diff}_{0,0}(\mathbb{R})$ consider the transformation $g_\psi := T\psi T^{-1}$ of the space $\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}$; let

$$G := \{g_\psi | \psi \in \mathrm{Diff}_{0,0}(\mathbb{R})\} \tag{6.9}$$

be the corresponding group. It follows from (4.5) and (4.6) that for any $\{t, s\} \in \widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}$

$$\begin{aligned} g_\psi\{t,s\} &= T\left\{\psi\left(-\sum_{k=1}^n s_k\right), \psi\left(\sum_{j=1}^m t_j\right); n, m \in \mathbb{N}\right\} \\ &= \left\{\left\{\psi\left(-\sum_{k=1}^{n-1} s_k\right) - \psi\left(-\sum_{k=1}^{n} s_k\right); n \in \mathbb{N}\right\},\right. \\ &\qquad \left.\left\{\psi\left(\sum_{j=1}^{m} t_j\right) - \psi\left(\sum_{j=1}^{m-1} t_j\right); m \in \mathbb{N}\right\}\right\} \end{aligned} \tag{6.10}$$

(here $\sum_{k=1}^0 a_k := 0$).

The following statement is a direct consequence of Proposition 6.1 and general results on absolute continuity of measures.

Theorem 6.3. *The exponential product measure $\nu_\lambda^\infty \times \nu_\lambda^\infty$ is quasi-invariant w.r.t. the group (6.9), and for any $g_\psi \in G$, the Radon-Nikodym density has the form*

$$\frac{d\left(g_\psi^*\nu_\lambda^\infty \times \nu_\lambda^\infty\right)}{d\nu_\lambda^\infty \times \nu_\lambda^\infty}(t,s) = \prod_{n=1}^\infty \psi'\left(-\sum_{k=1}^n s_k\right) \cdot \prod_{m=1}^\infty \psi'\left(\sum_{j=1}^m t_j\right). \tag{6.11}$$

Proof. At first we note that the r.h.s. of (6.11) is well defined because all but a finite number of factors of the infinite products are equal to 1.

The measure $\pi_{\lambda m}$ on $\left(\tilde\Gamma, \mathcal{B}\left(\tilde\Gamma\right)\right)$ is quasiinvariant w.r.t. the group $\mathrm{Diff}_{0,0}(\mathbb{R})$. By Theorem 4.3 the exponential product measure $\nu_\lambda^\infty \times \nu_\lambda^\infty$ coincides with $T^*\pi_{\lambda m}$, where T is a one-to-one correspondence (see (4.5)). Taking into account the general facts on absolute continuity of measures under transformations (see, e.g., [6]) we obtain that $\nu_\lambda^\infty \times \nu_\lambda^\infty$ is quasiinvariant w.r.t. the group G and the Radon-Nikodym density has the form

$$\frac{d\left(g_\psi^*\nu_\lambda^\infty \times \nu_\lambda^\infty\right)}{d\nu_\lambda^\infty \times \nu_\lambda^\infty}(t,s) = \frac{d\left(\psi^*\pi_{\lambda m}\right)}{d\pi_{\lambda m}}\left(T^{-1}\{t,s\}\right). \tag{6.12}$$

By (6.7) and (4.6) the equality (6.12) yields (6.11). □

Remark 6.4. Let $\varphi \in \mathrm{Diff}_{0,0}(\mathbb{R}), \varphi \neq \mathrm{Id}$, be such that $K_\varphi \subset (0;+\infty)$. Then a non-linear transformation (6.10) has the form

$$g_\varphi(t) = \left\{\varphi\left(\sum_{k=1}^n t_k\right) - \varphi\left(\sum_{k=1}^{n-1} t_k\right); n \in \mathbb{N}\right\}, \tag{6.13}$$

and therefore differs from the identity mapping on the set

$$A_\varphi = \bigcup_{n=1}^\infty \left\{\sum_{k=1}^n t_k \in K_\varphi\right\} \supseteq \{t_1 \in K_\varphi\}$$

of positive $\nu_\lambda^\infty-$measure. Namely, $\nu_\lambda^\infty(A_\varphi) \geq \nu_\lambda(K_\varphi) > 0$. The corresponding Radon-Nikodym density

$$\frac{d\left(g_\varphi^*\nu_\lambda^\infty\right)}{d\nu_\lambda^\infty}(t) = \prod_{n=1}^\infty \varphi'\left(\sum_{k=1}^n t_k\right) \tag{6.14}$$

differs from 1 on A_φ. By (6.13) and (6.14) neither the transformed measure $g_\varphi^*\nu_\lambda^\infty$ nor all its restrictions onto $\mathbb{R}_+^n \times \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty}\right)$ are product measures. That's why one can't apply Kakutani criterion of equivalence of product measures to prove theorem 6.3.

Of course, there exists an independent on Poissonian analysis proof of this result. Namely, one can use the restrictions of $g_\varphi^*\nu_\lambda^\infty$ and ν_λ^∞ onto $\mathcal{B}(\mathbb{R}^n) \times \widetilde{\mathbb{R}_+^\infty}$, $n \in \mathbb{N}$. These restrictions are mutually equivalent, the sequence of densities $\rho_n(t) := \prod_{m=1}^n \varphi'\left(\sum_{k=1}^m t_k\right), n \in \mathbb{N}$, converges ν_λ^∞- a.e. and $g_\varphi^*\nu_\lambda^\infty \sim \nu_\lambda^\infty$ iff

$$\int_{\widetilde{\mathbb{R}_+^\infty}} \lim \rho_n(t)\,\nu_\lambda^\infty(dt) = \int_{\widetilde{\mathbb{R}_+^\infty}} \prod_{m=1}^\infty \varphi'\left(\sum_{k=1}^m t_k\right)\nu_\lambda^\infty(dt) = 1 \tag{6.15}$$

(see, e.g., [6]). So it remains to prove the latter rather non-obvious equality. (By the way, (6.15) is a corollary of Theorem 6.3).

7. Operators in Poisson space and exponential model.

At first we recall the definition of directional derivative in Poisson space [1]. For any $v \in C_0^\infty(\mathbb{R})$ and $x \in \mathbb{R}$ the curve $\mathbb{R} \ni t \longmapsto \phi_t^v(x) \in \mathbb{R}$ is defined as a solution of the following Cauchy problem

$$\frac{d}{dt}\phi_t^v(x) = v(\phi_t^v(x)), \quad \phi_0^v(x) = x. \tag{7.1}$$

The mappings $\{\phi_t^v, t \in \mathbb{R}\}$ form a one-parameter subgroup of diffeomorphisms in the group $\mathrm{Diff}_0(\mathbb{R})$.

Assume that the function u belongs to

$$C_{0,0}^\infty(\mathbb{R}) := \{u \in C_0^\infty(\mathbb{R}) \,|u(0) = 0\}.$$

Putting $x = 0$ in (7.1) one can easily obtain that for any $u \in C_{0,0}^\infty(\mathbb{R})$

$$\phi_t^u(0) = 0, t \in \mathbb{R}, \tag{7.2}$$

i.e. $\{\phi_t^v, t \in \mathbb{R}\}$ is a one-parameter subgroup of the group $\mathrm{Diff}_{0,0}(\mathbb{R})$.

Let us fix $u \in C_{0,0}^\infty(\mathbb{R})$. Having the group $\{\phi_t^v, t \in \mathbb{R}\}$ we can consider, for any $\gamma \in \tilde{\Gamma}$, the curve

$$\mathbb{R} \ni t \longmapsto \phi_t^u(\gamma) \in \tilde{\Gamma}$$

(the latter inclusion follows from (7.2)). For a function $F : \tilde{\Gamma} \to \mathbb{R}$ the directional derivative along the direction u is defined by

$$\left(\nabla_u^\Gamma F\right)(\gamma) := \frac{d}{dt} F(\phi_t^u(\gamma))\,|_{t=0}, \tag{7.3}$$

provided the r.h.s. exists. The class $\mathfrak{D} = \mathcal{F}C_b^\infty\left(\mathcal{D}, \tilde{\Gamma}\right)$ of the smooth cylinder functions on $\tilde{\Gamma}$ is defined as the set of all functions $F : \tilde{\Gamma} \to \mathbb{R}$ of the form

$$F(\gamma) = g_F(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle), \quad \gamma \in \tilde{\Gamma}, \quad N \in \mathbb{N}, \tag{7.4}$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D}(\mathbb{R}) \equiv \mathcal{D}$ and $g_F \in C_b^\infty\left(\mathbb{R}^N\right)$. By the chain rule the directional derivative (7.3) of (7.4) has the following form

$$\left(\nabla_u^\Gamma F\right)(\gamma) = \sum_{j=1}^N \frac{\partial g_F}{\partial q_j}(\langle \varphi_1, \gamma\rangle, \ldots, \langle \varphi_N, \gamma\rangle)\left\langle \varphi_j' \cdot u, \gamma\right\rangle. \tag{7.5}$$

Further on we use $\mathfrak{D}$ as the domain of differential operators of Poissonian analysis. Namely, consider in $L^2(\pi_{\lambda m})$ on $\mathfrak{D}$ the operator ∇_u^Γ of directional derivative, its adjoint $\nabla_u^{\Gamma *}$ and self-adjoint generator $J_{\pi_{\lambda m}}(u)$ of a one-parameter unitary group

$$\left(V_{\pi_{\lambda m}}(\phi_t^u) F\right)(\gamma) = F(\phi_t^u(\gamma))\sqrt{\frac{d\phi_t^{u*}\pi_{\lambda m}}{d\pi_{\lambda m}}(\gamma)}, \quad F \in L^2(\pi_{\lambda m}), \quad \gamma \in \tilde{\Gamma}. \tag{7.6}$$

By Theorem 3.1 and Proposition 3.3 from [1] the following operator equalities hold on $\mathfrak{D}$

$$\nabla_u^{\Gamma *} = -\nabla_u^\Gamma - \langle u', \cdot\rangle, \tag{7.7}$$

$$J_{\pi_{\lambda m}}(u) = \frac{1}{i}\nabla_u^\Gamma + \frac{1}{2i}\langle u', \cdot\rangle. \tag{7.8}$$

Let us consider the images of these operators in the space $L^2(\nu_\lambda^\infty \times \nu_\lambda^\infty)$. The following statement describes an action of the image $U\nabla_u^\Gamma U^{-1} =: \hat{\nabla}_u^\Gamma$ on the domain $\hat{\mathfrak{D}} := U\mathfrak{D}$. To obtain the corresponding formulas for the operators (7.7), (7.8) it suffices to note that the multiplication by a function G in $L^2(\pi_{\lambda m})$ transforms into the same by $\hat{G} := UG$ in $L^2(\nu_\lambda^\infty \times \nu_\lambda^\infty)$.

Proposition 7.1. *For any $\hat{F} := UF \in \mathfrak{D}$ with F of the form (7.4) the following equality holds*

$$\begin{aligned}&\left(\hat{\nabla}_u^{\Gamma}\hat{F}\right)(t,s)\\ &=\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}\left(\sum_n\left[\varphi_1\left(-\sum_{k=1}^n s_k\right)+\varphi_1\left(\sum_{k=1}^n t_k\right)\right],\ \ldots,\right.\\ &\left.\sum_n\left[\varphi_N\left(-\sum_{k=1}^n s_k\right)+\varphi_N\left(\sum_{k=1}^n t_k\right)\right]\right)\\ &\times\sum_n\left[\left(u\cdot\varphi_j'\right)\left(-\sum_{k=1}^n s_k\right)+\left(u\cdot\varphi_j'\right)\left(\sum_{k=1}^n t_k\right)\right],\quad t,s\in\widetilde{\mathbb{R}_+^{\infty}}. \qquad (7.9)\end{aligned}$$

Proof. Since $\varphi_1,\ldots,\varphi_N\in\mathcal{D}$, all but a finite number terms in the sums $\sum_n$ are equal to zero, so that the r.h.s. of (7.9) is well defined. The unitary isomorphism of $L^2(\pi_{\lambda m})$ and $L^2(\nu_\lambda^\infty\times\nu_\lambda^\infty)$ is given by

$$L^2(\pi_{\lambda m})\ni F\longmapsto UF=F\circ T^{-1}\in L^2(\nu_\lambda^\infty\times\nu_\lambda^\infty)$$

(see Section 5). Therefore one can deduce from (7.4) and (7.5) that

$$\begin{aligned}&\left(\hat{\nabla}_u^{\Gamma}\hat{F}\right)(t,s)\\ &=\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}\left(\left\langle\varphi_1,T^{-1}(t,s)\right\rangle,\ldots,\left\langle\varphi_N,T^{-1}(t,s)\right\rangle\right)\\ &\times\left\langle u\cdot\varphi_j',T^{-1}(t,s)\right\rangle,\end{aligned}$$

whence by (4.6) the required equality (7.9) follows. □

Remark 7.2. Further on we suppose that

$$\operatorname{supp}u,\operatorname{supp}\varphi_1,\ldots,\operatorname{supp}\varphi_N\subset\mathbb{R}_+. \qquad (7.10)$$

Then the function $\hat{F}=F\circ T^{-1}$ with F of the form (7.4) as well as $\hat{\nabla}_u^{\Gamma}\hat{F}$ depend on the variable $t\in\widetilde{\mathbb{R}_+^{\infty}}$ only. This yields that the operator $\hat{\nabla}_u^{\Gamma}$ acts in $L^2(\nu_\lambda^\infty\times\nu_\lambda^\infty)$ w.r.t. variable t and by (7.7), (7.8) $\hat{\nabla}_u^{\Gamma*}$, $\hat{J}_{\pi_{\lambda m}}(u)$ do the same. In particular, taking into account (7.8) one can rewrite (7.8) on $\hat{\mathfrak{D}}$ as follows

$$\begin{aligned}&\left(\hat{J}_{\pi_{\lambda m}}(u)\hat{F}\right)(t)\\ &=\frac{1}{i}\left[\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}\left(\sum_n\varphi_1\left(\sum_{k=1}^n t_k\right),\ldots,\sum_n\varphi_N\left(\sum_{k=1}^n t_k\right)\right)\right.\\ &\left.\times\left(\sum_n u\left(\sum_{k=1}^n t_k\right)\varphi_j'\left(\sum_{k=1}^n t_k\right)\right)\right]\\ &+\frac{1}{2i}\left[\sum_n u'\left(\sum_{k=1}^n t_k\right)\right.\\ &\left.\times g_F\left(\sum_n\varphi_1\left(\sum_{k=1}^n t_k\right),\ldots,\sum_n\varphi_N\left(\sum_{k=1}^n t_k\right)\right)\right]. \qquad (7.11)\end{aligned}$$

(Remark, that by (7.10) the function F of the form (7.4) depends on $\gamma_+ = \gamma \cap \mathbb{R}_+$ only).

Notice, that $L^2(\nu_\lambda^\infty)$ is a separable subspace of an infinite tensor product of the Hilbert spaces $L^2(\nu_\lambda(dt_n)), n \in \mathbb{N}$ (see, e.g., [2]). The most investigated selfadjoint differential operators in the infinite tensor products are the functions $\mathcal{F}(A_1, A_2, \dots)$ of the operators $\{A_n, n \geq 1\}$ acting w.r.t. different variables and their perturbations by multiplication operators. The equality (7.11) shows that the operator $\hat{J}_{\pi_{\lambda m}}(u)$ doesn't belong to these classes. So the images of the operators of Poisson analysis and geometry form a new class of the differential operators in IDA w.r.t. exponential product measure.

It is worth noting that the converse is also true. Namely, the images of the familiar differential operators in $L^2(\nu_\lambda^\infty)$ form a new class of operators in Poisson spaces. The following example shows this.

Example 7.3. Denote by $D_m, m \in \mathbb{N}$, a minimal operator generated in $L^2(\nu_\lambda^\infty)$ by the selfadjoint differential expression

$$\mathcal{L}_m := \frac{1}{i}\left(\frac{\partial}{\partial t_m} - \frac{\lambda}{2}\right);$$

the operators $D_1, D_2, \dots$ act w.r.t. different variables in $L^2(\nu_\lambda^\infty)$ (for more details see [2,20]). A direct calculation shows that for any $\hat{F} \in \hat{\mathfrak{D}}$ with F of the form (7.4)

$$\begin{aligned}&\left(D_m\hat{F}\right)(t)\\ &= \frac{1}{i}\left[\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}\left(\sum_n \varphi_1\left(\sum_{k=1}^n t_k\right), \dots, \sum_n \varphi_N\left(\sum_{k=1}^n t_k\right)\right)\right.\\ &\times\left.\left(\sum_{n\geq m}\varphi_j'\left(\sum_{k=1}^n t_k\right)\right) - \frac{\lambda}{2}\cdot g_F\left(\sum_n \varphi_1\left(\sum_{k=1}^n t_k\right), \dots, \sum_n \varphi_N\left(\sum_{k=1}^n t_k\right)\right)\right].\end{aligned}$$

Therefore the image $\check{D}_m := U^{-1}D_m U$ of D_m acts in $L^2(\nu_\lambda^\infty)$ on $F \in \mathfrak{D}$ as follows

$$\begin{aligned}&\left(\check{D}_m F\right)(\gamma)\\ &= \frac{1}{i}\left[\sum_{j=1}^{N}\frac{\partial g_F}{\partial q_j}\left(\langle\varphi_1,\gamma\rangle, \dots, \langle\varphi_N,\gamma\rangle\right)\cdot\left\langle\varphi_j', S_m\gamma\right\rangle - \frac{\lambda}{2}F(\gamma)\right],\end{aligned} \tag{7.12}$$

where $S_m\gamma = S_m\{x_1, x_2, \dots\} := \{x_m, x_{m+1}, \dots\}$.

Comparing (7.5) and (7.12) we stress that the factor $\left\langle\varphi_j', S_m\gamma\right\rangle$ cannot be presented in the form $\left\langle\varphi_j' u, \gamma\right\rangle$ with the function u depending only on x. Namely, we have for any $\varphi \in \mathcal{D}$

$$\langle\varphi, S_m\gamma\rangle = \sum_{n\geq m}\varphi(x_n) = \int_{\mathbb{R}}\varphi(x)\,1_{[x_m(\gamma);\infty)}(x)\,\gamma(dx) =: \langle\phi_m(\cdot;\gamma), \gamma\rangle,$$

where $\phi_m(\cdot;\gamma) := \varphi(\cdot)\,1_{[x_m(\gamma);\infty)}(\cdot)$. That's why the operators $\{\check{D}_m, m \in \mathbb{N}\}$ and the functions $\mathcal{F}(\check{D}_1, \check{D}_2, \dots)$ of them form a new class of differential operators in the Poisson space $L^2(\pi_{\lambda m})$. To construct such operators in the framework of Poissonian analysis and differential geometry one needs, say the least, the derivatives along the directions $v(\gamma; x)$ depending on $\gamma \in \tilde{\Gamma}$.

8. Exponential model of compound Poisson space.

Let σ be a non-atomic infinite Radon measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and τ — a probability measure on Borel sets of $\mathbb{R}_\times := (-\infty; 0) \cup (0; +\infty)$ such that

$$\int_{\mathbb{R}_\times} \min\left(1, u^2\right) \tau(du) < \infty.$$

The compound Poisson measure π_σ^τ on $(\mathcal{D}'(\mathbb{R}), \mathcal{B}(\mathcal{D}'(\mathbb{R})))$ is determined by the Laplace transform

$$l_{\pi_\sigma^\tau}(\varphi) = \exp\left\{\int_{\mathbb{R}\times\mathbb{R}_\times} \left(e^{u\varphi(x)} - 1\right) \tau(du)\, \sigma(dx)\right\}, \quad \varphi \in \mathcal{D}$$

via Minlos theorem (see, e.g., [4,5]).

The additional analysis shows that this measure coincides with the image $\Sigma^* \pi_{\sigma\times\tau}$ of the Poisson measure $\pi_{\sigma\times\tau}$ on $\Gamma_{\mathbb{R}\times\mathbb{R}_\times}$ under the transformation

$$\mathcal{M}_p(\mathbb{R}\times\mathbb{R}_\times) \ni \hat{\gamma} = \sum \varepsilon_{(x_k,u_k)} \longmapsto \Sigma\hat{\gamma} = \omega := \sum u_k \varepsilon_{x_k} \in \mathcal{D}'(\mathbb{R})$$

(see [10]). That's why π_σ^τ is concentrated on the set Ω of the generalized functions corresponding to Radon signed measure of the form

$$\omega(\cdot) = \sum_{x_k \in \gamma_\omega} u_k \varepsilon_{x_k}(\cdot), \quad \gamma_\omega \in \tilde{\Gamma}, \quad u_k \in \operatorname{supp}\tau, \quad k \in \mathbb{N}. \tag{8.1}$$

This enables to introduce an analogue of the exponential model for the compound Poisson space

$$\left(\Omega, \mathcal{B}(\Omega), \pi_\sigma^\tau\right). \tag{8.2}$$

Let for brevity $\sigma = \lambda m$ (see Remark 4.8 for the general σ's). Consider the product

$$\left(\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty} \times \mathbb{R}_\times^\infty, \mathcal{B}\left(\widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty}\right) \times \mathcal{B}\left(\mathbb{R}_\times^\infty\right), \nu_\lambda^\infty \times \nu_\lambda^\infty \times \tau^\infty\right) \tag{8.3}$$

of the probability spaces (4.7) and $\left(\mathbb{R}_\times^\infty, \mathcal{B}\left(\mathbb{R}_\times^\infty\right), \tau^\infty\right)$ and the transformation

$$\Omega \ni \omega = \sum_{x_k \in \gamma_\omega} u_k \varepsilon_{x_k} \longmapsto \Upsilon\omega = \{T\gamma_\omega, \{u_k, k \in \mathbb{N}\}\} \in \widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty} \times \mathbb{R}_\times^\infty \tag{8.4}$$

(see (8.1)). Theorem 4.3 and the mentioned above equality

$$\pi_{\lambda m}^\tau = \Sigma^* \pi_{\lambda m\times\tau}$$

from [10] yield that

$$\Upsilon\Omega = \widetilde{\mathbb{R}_+^\infty} \times \widetilde{\mathbb{R}_+^\infty} \times \mathbb{R}_\times^\infty, \qquad \Upsilon^* \pi_{\lambda m}^\tau = \nu_\lambda^\infty \times \nu_\lambda^\infty \times \tau^\infty.$$

That's why we can call the pair consisting of the probability space (8.3) and the transformation (8.4) the exponential model of the compound Poisson space (8.2). Finally, it is worth noting that the above construction is based on familiar representation of the homogeneous compound Poisson process as a random sum of independent identically τ -distributed random variables.

References

1. S.Albeverio, Yu.G.Kondratiev, M.Röckner, *Analysis and geometry on configuration spaces*, J. Func. Anal. **154** (1998), 444–500.
2. Yu.M.Berezansky, *Selfadjoint operators in spaces of functions of infinitely many variables*, AMS, Providence (in Russian, Naukova Dumka, Kiev, 1978), 1986.
3. Yu.M.Berezansky, Yu.G.Kondratiev, *Spectral Methods in Infinite-Dimensional Analysis*, Kluwer Academic Publishers, Dordrecht (in Russian, Naukova Dumka, Kiev, 1988), 1995.
4. I.M.Gel'fand, M.I.Graev, A.M.Vershik, *Representation of the group of diffeomorphisms*, Russian Math. Surveys **30** (1975), 3–50.
5. I.M.Gel'fand, N.Ya.Vilenkin, *Generalized Functions, volume IV*, Academic Press, New York, London (in Russian, Nauka, Moscow, 1961), 1968.
6. I.I.Gikhman, A.V.Skorokhod, *On densities of probability measures in functional spaces*, Russian Math. Surveys **21** (1966), 83–152.
7. Y.Ito, I.Kubo, *Calculus on Gaussian and Poisson white noises*, Nagoya Math. J. **111** (1988), 41–84.
8. O.Kallenberg, *Random measures*, Academic Press, London, New York, 1976.
9. J.F.C.Kingman, *Poisson Processes*, Clarendon Press, Oxford, 1993.
10. Yu.G.Kondratiev, J.L.Silva, L.Streit, G.F.Us, *Analysis on Poisson and gamma spaces*, Infinite Dimensional Analysis, Quantum Probabilistics and Related Topics **1** (1998), 91–117.
11. M.Loève, *Probability Theory*, D.van Nostrand, Toronto, New York, London, 1960.
12. K.Matthes, J.Kerstan, J.Mekke, *Infinitely Divisible Point Processes*, John Viley and Sons, New York, 1978.
13. N.Privault, *Chaotic and variational calculus in discrete and continuous time for the Poisson process*, Stochastic&Stochastics Reports **51** (1994), 83–109.
14. G.E.Shilov, Fan Dyk Tin', *Integral, measure and derivative on linear spaces*, Nauka, Moscow (in Russian), 1967.
15. H.Shimomura, *Poisson measures on the configuration space and unitary representations of the group of diffeomorphisms*, J. Math. Kyoto Univ. **34** (1994), 599–614.
16. A.N.Shiryaev, *Probability*, Nauka, Moscow (in Russian), 1989.
17. A.V.Skorokhod, *On the differentiability of measures which corresponds to stochastic processes. I. Processes with independent increments*, Theory Probab. Appl. **2** (1957), 407–423.
18. Y.Takanashi, *Absolute continuity of Poisson random fields*, Publ. RIMS Kyoto Univ. **26** (1990), 629–647.
19. G.F.Us, *Dual Appell systems in Poissonian analysis*, Methods Func. Anal. Topol. **1** (1995), 93–108.
20. G.F.Us, *Functions of countable families of commuting selfadjoint operators*, Methods Func. Anal. Topol. **3, 1** (1997), 89–97.

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