TAME AND WILD PROJECTIVE CURVES AND CLASSIFICATION OF VECTOR BUNDLES

YURI A. DROZD AND GERT-MARTIN GREUEL

Abstract. We propose a new method of classifying vector bundles on projective curves, especially singular ones, according to their "representation type". In particular, we prove that the classification problem of vector bundles, respectively of torsion–free sheaves on projective curves is always either finite, or tame, or wild. We completely classify curves which are of finite, respectively tame, vector bundle type by their dual graph. Moreover, our methods yield a geometric description of all indecomposable vector bundles and torsion–free sheaves on finite and tame curves.

Introduction

Vector bundles over projective varieties, in particular, over projective curves have been widely studied. Usually, the main emphasis lies in the study of stable bundles and their moduli (cf. [33], [38], [30]). Nevertheless, not too much seems to be known about the classification of *all* vector bundles over some variety, which is a quite different problem. Compared to representation theory, stable bundles play the role of irreducible (simple) modules, as all other ones can be obtained from them by extensions. In most cases the construction of such extensions is far from being trivial or simple, even if one restricts to semi-stable bundles, which are extensions of stable ones with fixed slope [38]. On the other hand, the classification of vector bundles on projective curves is closely related to the study of Cohen–Macaulay modules on surface singularities, due to the work of Kahn [28]. Hence, from different points of view, it is important to have some ideas about the complexity of these classification problems. The most prominent results here are those of Grothendieck [26] for the projective line and of Atiyah [2] for elliptic curves. For instance, the latter result made it possible to classify Cohen–Macaulay modules on simple elliptic surface singularities [28].

This article is devoted to the study of vector bundles over projective curves, in particular, singular and reducible ones, from the point of view of representation theory. Since it could be interesting for people working in algebraic geometry as well as in representation theory, we try to explain our results in this introduction, in an informal way, such

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that it could be understood from both sides. Moreover, we relate it to some well known problems in both of these fields.

In several areas of representation theory, for instance, in studying representations of finite dimensional algebras, Cohen–Macaulay modules, etc., one usually distinguishes between three main cases of the classification problem. We propose to use analogous notions when considering curves with respect to the classification of vector bundles. Namely, these three cases are the following:

- *finite*, when indecomposable modules (respectively, vector bundles) are completely defined by some discrete parameters (this is the case for the projective line);
- *tame*, when indecomposable modules (respectively, vector bundles) form small, usually only one-parameter, families (this is the case for elliptic curves);
- *wild*, which can be defined in two ways:
  - *geometrically*, as those having families of indecomposable modules (respectively, vector bundles) depending on any prescribed number of parameters;
  - *algebraically*, as such that for any finitely generated algebra Λ there is an exact functor from the category $\Lambda\text{-}\mathsf{mod}$ of finite dimensional Λ-modules to the category of modules (respectively, vector bundles) which maps indecomposable modules to indecomposable and non-isomorphic to non-isomorphic ones (we call such a functor a *representation embedding*).

However, it is a highly non-trivial problem whether the above two definitions of wildness are equivalent. For the cases of finite dimensional algebras and Cohen-Macaulay modules, it only follows from the so called *tame-wild dichotomy* [13, 15]. For the case of vector bundles over reduced projective curves such an equivalence follows from the results of this paper (cf. Remark 1.8).

When one considers vector bundles, one has to slightly modify these notions taking into consideration the natural shifts by tensoring with line bundles of different degrees. Moreover, if the curve is reducible, one can make shifts on each of its components independently. The corresponding definitions are given in Definition 1.4.

From this point of view, the projective line is *finite*, while a smooth elliptic curve is *tame*. Note that the latter is a little different from the tame algebras in representation theory where only rational curves are used to parameterize indecomposable modules. Here one cannot avoid using the curve itself as we have to parameterize, in the first instance, the line bundles. It is not too complicated to show that all other smooth curves are *wild* (algebraically, hence, geometrically), cf.

Theorem 1.6. We suppose that it is more or less known to the experts, although we do not know any article containing this result.[1]

The aim of this article is to prove the finite–tame–wild trichotomy for vector bundles over reduced projective curves, in particular, to show that the geometrical wildness also implies the algebraic one. Note that sometimes one does not suppose that "tame" excludes "finite." We prefer to distinguish between them, following the book [20]. Moreover, there is at least one important reason. Namely, the finite case for vector bundles (just as for algebras and Cohen–Macaulay modules) is not only *discrete* in the sense that there are finitely many indecomposables, say, of given rank and degree. It is also *bounded* in the sense that all ranks of indecomposables are smaller than a prescribed number. Taking an example from the representation theory, one can easily see that the quiver of type A_∞^∞, that is

$$\dots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \dots ,$$

is representation discrete, but not bounded. It seems reasonable to call *finite* the case which is both discrete and bounded. In representation theory of finite dimensional algebras the claim that "discrete" implies "bounded" is known as the second Brauer-Thrall conjecture and its proof (a complicated one) was only given in [4].

In this article we prove that the following assertions hold, with C reduced and connected (cf. Theorem 1.6, Proposition 2.7 and Theorem 2.8):

(1) *A non–singular projective curve C is:*
  - *VB–finite if and only if it is rational;*
  - *VB–tame if and only if it is elliptic (that is, of genus 1);*
  - *VB–wild in all other cases.*

(2) *Let C be a singular projective curve, $C_1, \dots, C_s$ its irreducible components and Δ the* intersection graph *(or the* dual graph*) of C. Then C is:*
  - *VB–finite if and only if all C_i are smooth, rational and Δ is of type A_n (that is, a chain);*
  - *VB–tame if and only if all C_i are smooth, rational and Δ is of type $\widetilde{A}_n$ (that is, a cycle) or C is irreducible, rational with one simple node;*
  - *VB–wild in all other cases.*

In the wild case we construct explicitly a representation embedding of the category Λ-mod to that of *semi-stable* vector bundles. This shows that even semi-stable bundles are extremely complicated if we do not restrict to a fixed rank but allow extensions. Certainly, it is impossible that the image of a representation embedding belongs to

[1] After this article had been written, W. Scharlau informed us that he had also proved the result for smooth curves; moreover, it is also true for any smooth variety of dimension greater than 1, see [37].

the category of stable bundles, as it must preserve extensions, and extensions of stable bundles are in general not stable. On the other hand, in the finite and tame cases we give a complete description of all indecomposable vector bundles (Theorems 2.11 and 2.12) and illustrate it in a geometric way. Moreover, it happens that for VB–finite and VB–tame curves the classification of all torsion-free sheaves can also be done within the same framework.

Of course, in the smooth case there is nothing to add here. Moreover, every coherent sheaf over a smooth curve is just a direct sum of a vector bundle and a sky-scraper sheaf, and the indecomposable sky-scrapers of prescribed length are parameterized by the curve itself. But in the singular case there is an essential difference between the classification of vector bundles (or torsion free sheaves) and that of all coherent sheaves. First of all, the classification of sky-scrapers can be very complicated. It is known, for instance, that all simple plane curve singularities are *finite* with respect to the classification of torsion-free modules [24], while all of them, except of A_1, are wild with respect to the classification of modules of finite length (i.e., sky-scrapers) [12]. Secondly, in the singular case we always have also *mixed* indecomposable sheaves, i.e., neither sky-scraper nor torsion-free, and their description is also non-trivial. There is, however, some evidence that for the VB–tame projective curves a complete classification can also be done for all coherent sheaves, but we still do not have a definite result.

Our classification of vector bundles has already been used to describe Cohen–Macaulay modules over the so called *cusp surface singularities*, as well as to find out which of the *minimally elliptic* surface singularities are tame and which are wild with respect to the classification of Cohen–Macaulay modules [18].

The methods we use are well–known in representation theory. Namely, it is the techniques of *matrix problems*, which are used, for instance, to prove the tame-wild dichotomy in [13, 15] or to determine the types of some classes of classification problems. Fortunately, after eliminating the wild cases, we come to a known matrix problem (the so called "Gelfand problem" in the version due to Bondarenko [6]). This gives us the possibility to obtain a complete list of indecomposable vector bundles for the finite and tame cases.

Unfortunately, we cannot recommend any relevant textbook for this material. The only one dealing with matrix problems is [20], but it only considers a very special case of matrix problems which does not include those we use here. This is why we try to give complete definitions and include Appendix B devoted to *bunches of chains* in the sense of [6]. As we only need a special case of such bunches, we restrict Appendix B to this case, which is essentially easier than the general one. We reformulate it in terms of bimodule categories which seems to be more usual than the original matrix formulation and present the list of indecomposable objects from [6] in a form which is easier to apply in our case. On the other hand, we use the standard textbook [27] for the references concerning algebraic geometry. For more special results concerning vector bundles we refer to [30, 38], although we never use anything but some standard definitions.

The description of torsion-free sheaves in the tame singular case fits into the framework of the so called *strings and bands* which is widespread in representation theory (cf. [40, 7]). There is no *a priori* explanation why other kinds of tame matrix problem (for instance, more general *clans* [9] or bunches of *semi-chains* [6]) do not appear. Such an explanation would certainly be of interest.

It is a fact that for every VB–tame curve C the dualizing sheaf [27] coincides with the structure sheaf. Hence, Serre duality coincides with the obvious duality given by the functor $\mathcal{H}om(\_\,,\mathcal{O}_C)$. Moreover, it also follows from [3] that the *Auslander–Reiten translation* is also trivial in the category of vector bundles on such curves. This means that all indecomposable vector bundles belong to the so called *homogeneous tubes* in the sense of [35]. For elliptic curves the latter is also true for all coherent sheaves. The answer for singular tame curves can only be given from a classification of all coherent sheaves, which is not yet known (it follows from [3] that the Auslander–Reiten translation cannot be defined inside the category of torsion-free sheaves). Nevertheless, from the description of torsion-free sheaves it seems plausible that the category of all coherent sheaves in this case should look like that of modules over the so called *string algebras*. Moreover, there is a special class of string algebras which seems closely related to singular tame curves, just in the same way as the so called *canonical algebras* [35] are related to the weighted projective lines considered in [21]. We define these algebras in Appendix A which is devoted to some other open questions.

Let us give a short survey of the article. In Section 1 we define VB–types of projective curves and the result for smooth curves is proved. In Section 2 we consider singular curves, formulate the trichotomy result and give a description of torsion-free sheaves in the finite and tame cases. The following sections present the proofs of these results. Namely, Section 3 is devoted to matrix problems in a bimodule formulation. As the bimodules arising from projective curves possess a natural group of shifts, we introduce here *shifting bimodules.* Again, shifts make it necessary to modify, in an obvious way, the notions of finite and tame, which is also done in this section. Section 4 explains the relations between vector bundles over singular curves and some shifting bimodules. The latter naturally arise when one compares vector bundles on a curve and on its normalization. This procedure is very much like the one used in the study of torsion-free modules over curve singularities, for instance, in [15, 16], the main difference coming just from taking shifts into consideration. We also prove here that all irreducible components of a singular curve which is not VB–wild are rational. This leads to the consideration of *rationally composed* curves in Section 5. They give rise to a very special class of shifting bimodules. We call them *special bimodules* and consider them in Section 5, too. Finally, in Section 6 we establish representation types of special bimodules and describe their indecomposable elements in the finite and tame cases. This immediately implies the trichotomy result and the description of vector bundles from Section 2. Appendix A presents some related problems which we consider as interesting and important. Appendix B is devoted to bunches of chains.

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1. VB-type of a curve. Smooth case

Here we define the notions of finite, tame and wild curves with respect to the classification of vector bundles and prove the finite–tame–wild trichotomy for smooth curves (Theorem 1.6).

Throughout this section and further on we use the following notations:

Notations 1.1. (1) C is an algebraic curve over an algebraically closed field $\mathbf{k}$, which we suppose to be *reduced* and *connected* but usually *singular* and even *reducible.*

(2) $\mathcal{O} = \mathcal{O}_C$ denotes the structure sheaf of C and $\mathcal{K}$ denotes the sheaf of *rational functions* on C (its stalk at a point x is the full ring of quotients of $\mathcal{O}_x$).

(3) $\mathsf{VB} = \mathsf{VB}(C)$ is the category of (finite dimensional) vector bundles on C or, equivalently, that of locally free (coherent) sheaves on C. (We identify vector bundles with the corresponding locally free sheaves and in our case it is more convenient to deal with sheaves.)

(4) Let $\mathcal{M}$ be a sheaf of $\mathcal{O}$-modules. Call the *torsion part* of $\mathcal{M}$, and denote it by $\mathsf{t}(\mathcal{M})$, the kernel of the natural homomorphism $\mathcal{M} \to \mathcal{K} \otimes_{\mathcal{O}} \mathcal{M}$. The sheaf $\mathcal{M}$ is said to be *torsion-free* if $\mathsf{t}(\mathcal{M}) = 0$ and *torsion* if $\mathsf{t}(\mathcal{M}) = \mathcal{M}$. In the following we always identify a torsion-free sheaf $\mathcal{M}$ with its image in $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{M}$.

Obviously, $\mathcal{M}$ is torsion if and only if for every point $x \in C$ and for every element $t \in \mathcal{M}_x$ there is a non-zero-divisor $a \in \mathcal{O}_x$ such that $at = 0$; $\mathcal{M}$ is torsion-free if and only if, for every nonzero $t \in \mathcal{M}_x$ and for every non-zero-divisor $a \in \mathcal{O}_x$, $at \neq 0$. It is also clear that $\mathsf{t}(\mathcal{M})$ is the biggest torsion sub-sheaf of $\mathcal{M}$, while $\mathcal{M}/\,\mathsf{t}(\mathcal{M})$ is torsion-free.

We are going to define the *vector bundle type* (*VB-type*) of a curve, i.e., its type with respect to the classification of vector bundles on it. We take into consideration that such a classification involves evident discrete parameters, namely, rank and degree. However, if the curve has several irreducible components, these parameters become more complicated.

Definition 1.2. Let C be a projective curve, $C = \cup_{i=1}^t C_i$ its decomposition into irreducible components, $\mathcal{B}$ a vector bundle over C and $\mathcal{B}_i$ the restriction of $\mathcal{B}$ onto C_i. The *vector-degree* of $\mathcal{B}$ is defined as the vector $\operatorname{Deg}\mathcal{B} = (d_1, d_2, \dots, d_t)$, where $d_i = \deg \mathcal{B}_i$ (cf. [27]).

In particular, the mapping Deg defines an epimorphism $\operatorname{Pic}(C) \to \mathbb{Z}^t$. For each i choose a non-singular point $c_i \in C_i$ and put $\mathcal{O}(\mathbf{d}) = \mathcal{O}(\sum_{i=1}^t c_i)$. It gives us a section of Deg, $\omega : \mathbb{Z}^t \to \operatorname{Pic}(C)$, such that $\mathcal{O}(\mathbf{d}) = \omega(\mathbf{d})$. Thus, we define $\mathbb{Z}^t$ as a group of shifts on the category of coherent sheaves (in particular, on that of vector bundles) by setting $\mathcal{M}(\mathbf{d}) = \mathcal{O}(\mathbf{d}) \otimes_{\mathcal{O}} \mathcal{M}$. Considering representation types of categories of sheaves, we should also take into account the action of this big discrete group.

If X is an algebraic variety, there is a natural notion of a family of vector bundles on a curve C with base X. Namely, such a family is just a vector bundle $\mathcal{V}$ on $X \times C$. For our purpose, a non-commutative analogue of this notion is also important.

Definition 1.3. (1) Let Λ be a $\mathbf{k}$-algebra (not necessarily commutative). We identify Λ as well as all Λ-modules with the corresponding constant sheaves over C. Denote by $\mathsf{VB}(C, \Lambda)$ the category of sheaves over C which are coherent sheaves of $\mathcal{O} \otimes \Lambda$-modules, locally free as $\mathcal{O}$-modules and flat as Λ-modules. The objects of this category are called *families of vector bundles over C with base Λ*.

(2) Given a family $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$ and a finite dimensional[2] Λ-module N, we can construct the tensor product $\mathcal{M}(N) = \mathcal{M} \otimes_\Lambda N$, which is locally free over $\mathcal{O}$, i.e. is a vector bundle over C. We say that the modules $\mathcal{M}(N)$ *belong* to the family $\mathcal{M}$.

(3) A family $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$ is said to be *strict*[3] if the following conditions hold:

[2] "*finite dimensional*" always means finite dimensional as a vector space over $\mathbf{k}$.
[3] This notion was first introduced in [14]; see also [15, 16].

(a) If N is an indecomposable finite dimensional Λ-module, then the sheaf $\mathcal{M}(N)$ is also indecomposable.
(b) If two finite dimensional Λ-modules N and N' are non-isomorphic, then the sheaves $\mathcal{M}(N)$ and $\mathcal{M}(N')$ are also non-isomorphic.

In other words, the functor $N \to \mathcal{M}(N)$ from $\Lambda\text{-}\mathsf{mod}$ to $\mathsf{VB}(C)$ is a *representation embedding*: it is exact, maps indecomposable modules to indecomposable vector bundles and non-isomorphic to non-isomorphic ones.

For any morphism $f : C' \to C$ of curves and any family $\mathcal{M} \in \mathsf{VB}(C,\Lambda)$, the inverse image $f^*(\mathcal{M})$ belongs to $\mathsf{VB}(C',\Lambda)$. It is also quite obvious that if $\mathcal{M} \in \mathsf{VB}(C,\Lambda)$, then also $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L} \in \mathsf{VB}(C,\Lambda)$ for every invertible sheaf $\mathcal{L}$ on C; in particular, $\mathcal{M}(\mathbf{d}) \in \mathsf{VB}(C,\Lambda)$ for every vector $\mathbf{d} \in \mathbb{Z}^t$. Moreover, if $\mathcal{M}$ is strict, so is $\mathcal{M} \otimes_{\mathcal{O}} \mathcal{L}$ for each invertible sheaf $\mathcal{L}$; in particular, $\mathcal{M}(\mathbf{d})$ is strict for each $\mathbf{d}$. For every finite-dimensional Λ-module N, put $\mathcal{M}(\mathbf{d},N) = \mathcal{M}(\mathbf{d})(N)$.

If $\Lambda = \mathbf{k}[X]$ for some affine variety X, then an object from $\mathsf{VB}(C,\Lambda)$ can obviously be identified with a family of vector bundles on C with base X. However, our construction also produces families of multiple ranks that arise when one considers vector bundles $\mathcal{M}(N)$ with $\dim_{\mathbf{k}} N > 1$. Note that for two different points $p \neq q$ of X the residue fields $\mathbf{k}(p)$ and $\mathbf{k}(q)$ are non-isomorphic as $\mathbf{k}[\,X\,]$-modules. Hence, for a strict family $\mathcal{M}$ over X, the fibres over p and q, i.e., the vector bundles $\mathcal{M}(p)$ and $\mathcal{M}(q)$, are also non-isomorphic (and indecomposbale).

Definitions 1.4. (1) Call a curve C *vector bundle finite* or *VB-finite* if there is a finite set $\mathbf{M}$ of indecomposable vector bundles on C such that every indecomposable vector bundle on C is isomorphic to $\mathcal{B}(\mathbf{d})$ for some $\mathcal{B} \in \mathbf{M}$ and some vector $\mathbf{d} \in \mathbb{Z}^t$.[4]

(2) Call a curve C *VB-tame* if there is a non-empty set $\mathbf{M} = \{\,\mathcal{M}_i\,\}$ of strict sheaves $\mathcal{M}_i \in \mathsf{VB}(C,\Lambda_i)$ (note that the Λ_i may be different for different i) satisfying the following conditions:
  (a) Each Λ_i is a commutative finitely generated integral smooth $\mathbf{k}$-algebra of Krull dimension 1.
  (b) For each integer r and vector $\mathbf{d}$, the set $\mathbf{M}_{r,\mathbf{d}}$ is finite, where $\mathbf{M}_{r,\mathbf{d}} = \{\,\mathcal{M} \in \mathbf{M} \mid \operatorname{rk}(\mathcal{M}) = r\,, \operatorname{Deg}\mathcal{M} = \mathbf{d}\,\}$, where $\operatorname{Deg}\mathcal{M}$ is, by definition, $\operatorname{Deg}(\mathcal{M}/\mathbf{m}\mathcal{M})$ for some (and, hence, every) maximal ideal $\mathbf{m} \subset \Lambda_i$ (if $\mathcal{M} \in \mathsf{VB}(C,\Lambda_i)$).
  (c) For each integer r and vector $\mathbf{d}_0$, all but a finite number of locally free indecomposable sheaves on C of rank r and vector-degree $\mathbf{d}_0$ are isomorphic to those of the form

[4]We shall see later that indeed $\operatorname{rk}(\mathcal{B}) = 1$ for every indecomposable vector bundle on a VB-finite curve.

$\mathcal{M}_i(\mathbf{d}, N)$, for some $\mathcal{M}_i \in \mathbf{M}$, $\mathbf{d} \in \mathbb{Z}^t$ and some finite dimensional Λ_i-module N.

In this case call $\mathbf{M}$ a *parametrising set* for vector bundles over C. Denote by $\nu(r)$ the minimal number of sheaves in $\mathbf{M}_{r,\mathbf{d}}$, where $\mathbf{M}$ runs through all such parametrising sets and $\mathbf{d}$ runs through $\mathbb{Z}^t$, and call it the *growth function.* Then a VB-tame curve C is said to be:

- *bounded* if there is an integer m such that $\nu(r) \leq m$ for all ranks r;
- *unbounded* otherwise.

(As we have already mentioned, for representations of finite dimensional algebras, as well as for Cohen–Macaulay modules over curve singularities, only coordinate algebras of *rational* curves have occurred in the tame case. Studying vector bundles we cannot avoid, for instance, the curve C itself as it gives rise to families of line bundles. Therefore, in (2 a) we only require that Λ_i is of dimension 1.)

(3) Call a curve C *VB-wild* if, for every finitely generated $\mathbf{k}$-algebra Λ, there is a strict sheaf $\mathcal{M} \in \mathsf{VB}(C, \Lambda)$.
Hence, for wild curves, the classification of vector bundles is at least as complicated as the classification of the representations of *all* finitely generated $\mathbf{k}$-algebras, which justifies the name "wild."

Indeed, to prove wildness it is sufficient to check one typical algebra, as the following result shows.

Proposition 1.5. *A curve C is VB-wild if there is a strict sheaf $\mathcal{M} \in \mathsf{VB}(C, \Gamma)$, where Γ is one of the following algebras:*

- $\mathbf{F} = \mathbf{k}\langle z_1, z_2 \rangle$*, the free algebra in two generators* (this is one way to *define* wildness, cf. [20, 13, 15]);
- $\mathbf{k}[z_1, z_2]$, the polynomial algebra in two generators;
- $\mathbf{k}[[z_1, z_2]]$, the power series algebra in two generators.

Proof. It is well known (cf. [13]) that if Γ is one of these algebras and Λ is an arbitrary finitely generated algebra, there is a *strict representation* of Γ over Λ, i.e., a Γ-Λ-bimodule V such that:

(1) V is finitely generated and free as Λ-module.
(2) If N is an indecomposable finite dimensional Λ-module, the Γ-module $V \otimes_\Lambda N$ is also indecomposable.
(3) If N, N' are non-isomorphic finite dimensional Λ-modules, the Γ-modules $V \otimes_\Lambda N$ and $V \otimes_\Lambda N'$ are also non-isomorphic.

Therefore, if a sheaf $\mathcal{M} \in \mathsf{VB}(C, \Gamma)$ is strict, so is also $\mathcal{M} \otimes_\Lambda V \in \mathsf{VB}(C, \Lambda)$.

We recall the explicit form of a strict representation V of the free algebra $\mathbf{F}$ over any algebra Λ with generators $a_1, a_2, \dots, a_n$. As Λ-module, $V = (n+2)\Lambda$ while the action of z_1 and z_2 is given by the matrices Z_1 and Z_2, respectively, where Z_1 is a Jordan cell of dimension $n+2$ and

$$Z_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ a_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & a_2 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & a_n & 1 & 0 \end{pmatrix}.$$

□

Note also that this definition of tameness (namely, the condition $\mathbf{M} \neq \emptyset$) implies that "tame" excludes "finite," i.e., we have a real trichotomy.

First of all, consider VB-types of smooth projective curves.

Theorem 1.6. *A smooth projective curve C of genus g is:*

- *VB-finite if $g = 0$, i.e., if $C \simeq \mathbb{P}^1$,*
- *VB-tame bounded if $g = 1$, i.e., if C is an elliptic curve,*
- *VB-wild if $g > 1$.*

Proof. It is known that each indecomposable vector bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(n)$ for some n [26]. Hence, $\mathbb{P}^1$ is VB-finite. On the other hand, the classification of vector bundles on elliptic curves [2] implies that all elliptic curves are VB-tame and bounded (indeed, in this case the growth function satisfies $\nu(r) \leq 1$ for each r). So we only have to prove that any curve of genus $g > 1$ is VB-wild, i.e., to construct a strict sheaf $\mathcal{M} \in \mathsf{VB}(C, \mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle z_1, z_2 \rangle$. We shall even construct a sheaf $\mathcal{M} \in \mathsf{VB}(C, \mathbf{F})$ such that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$ for some line bundle $\mathcal{L}$ if and only if $N \simeq N'$ and $\mathcal{L} \simeq \mathcal{O}$. In other words, even the natural action of $\mathrm{Pic}(C)$ on the set of vector bundles does not simplify their classification.

For any two points $x \neq y$ of C,

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^0(C, \mathcal{O}(y-x)) = 0.$$

On the other hand,

$$\mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \simeq \mathrm{H}^1(C, \mathcal{O}(y-x)),$$

as $\mathcal{E}xt^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) = 0$. Using the Riemann–Roch theorem for the divisor $y - x$, we get

$$\dim \mathrm{H}^1(C, \mathcal{O}(y-x)) = g - 1 \geq 1.$$

We shall also use the following simple lemma.

Lemma 1.7. *If C is a smooth curve of genus $g > 0$, for any n there are n points $x_1, x_2, \dots, x_n$ on C such that $2x_i \not\sim x_j + x_k$ (as divisors on C, cf. [27]) if $i \neq j$.*

Proof. Note that since $g > 0$, the space $\mathrm{H}^0(C, \mathcal{O}(x))$ consists only of constants for any point $x \in C$: otherwise there is a non-constant function f with the unique pole at the point x and such a function defines an isomorphism $C \to \mathbb{P}^1$ [27]. On the other hand, the Riemann–Roch theorem together with the Clifford theorem [27, Theorem IV.5.4] gives that $\dim \mathrm{H}^0(C, \mathcal{O}(2z)) \le 2$. If this space is one-dimensional, i.e., consists only of constants, $x + y \sim 2z$ is impossible for $x \ne z$. Suppose that it is two-dimensional, i.e., consists of the functions $\lambda + \mu f$ for some fixed (non-constant) f and $\lambda, \mu \in \mathbf{k}$. Then f defines a two-fold surjection $C \to \mathbb{P}^1$ and the set $R = \{\, p \in \mathbb{P}^1 \mid \mathrm{card}(f^{-1}(p)) = 1 \,\}$ is finite (it is the set of the ramification points of f) [27]. Obviously, the set R does not depend on the choice of f in $\mathrm{H}^0(C, \mathcal{O}(2z))$. Hence, there are only finitely many points $y \in C$ such that $2z \sim 2y$. Moreover, since $x + y \not\sim x + y'$ for a fixed x and $y \ne y'$, an equivalence $x + y \sim 2z$ for given x, z defines y uniquely. Now the points $x_1, x_2, \dots, x_n$ can be constructed by an easy induction. □

Using this lemma, choose 5 different points $x_1, \dots, x_5$ in such a way that $2x_i \not\sim x_j + x_k$ if $i \ne j$, and consider the class of locally free sheaves $\mathcal{A}$ admitting an exact sequence:

$$0 \longrightarrow \mathcal{A}_1 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}_2 \longrightarrow 0\,, \tag{1}$$

where

$$\mathcal{A}_1 = r_1\mathcal{O}(x_1) \oplus r_2\mathcal{O}(x_2) \oplus r_3\mathcal{O}(x_3)$$

and

$$\mathcal{A}_2 = r_4\mathcal{O}(x_4) \oplus r_5\mathcal{O}(x_5)\,.$$

Let $\xi \in \mathrm{Ext}_{\mathcal{O}}(\mathcal{A}_2, \mathcal{A}_1)$ be the element corresponding to the sequence (1). As there are no homomorphisms from the sub-sheaf to the factor-sheaf, one can easily check that two elements $\xi, \xi' \in \mathrm{Ext}_{\mathcal{O}}(\mathcal{A}_2, \mathcal{A}_1)$ lead to isomorphic modules $\mathcal{A}$ and $\mathcal{A}'$ if and only if there are automorphisms $\alpha : \mathcal{A}_1 \xrightarrow{\sim} \mathcal{A}_1$ and $\beta : \mathcal{A}_2 \xrightarrow{\sim} \mathcal{A}_2$ such that $\alpha\xi = \xi'\beta$ (we mean here the Yoneda multiplication). Choose some nonzero elements $\xi_{ij} \in \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x_j), \mathcal{O}(x_i))$. Put $\mathcal{S} = \mathcal{O} \otimes \mathbf{F}$, where $\mathbf{F} = \mathbf{k}\langle\, z_1, z_2 \,\rangle$, the free $\mathbf{k}$-algebra in two generators, $\mathcal{S}(x) = \mathcal{S} \otimes_{\mathcal{O}} \mathcal{O}(x)$ for $x \in C$. Then $\mathrm{Ext}^1_{\mathcal{S}}(\mathcal{S}(x), \mathcal{S}(y)) \simeq \mathrm{Ext}^1_{\mathcal{O}}(\mathcal{O}(x), \mathcal{O}(y)) \otimes \mathbf{F}$. Consider the exact sequence of locally free $\mathcal{S}$-modules

$$0 \longrightarrow \mathcal{S}(x_1) \oplus \mathcal{S}(x_2) \oplus \mathcal{S}(x_3) \longrightarrow \mathcal{M} \longrightarrow \mathcal{S}(x_4) \oplus \mathcal{S}(x_5) \longrightarrow 0$$

corresponding to the element of the Ext-space given by the matrix

$$\begin{pmatrix} \xi_{14} & \xi_{15} \\ \xi_{24} & z_1\xi_{25} \\ \xi_{34} & z_2\xi_{35} \end{pmatrix}.$$

If N is any finite dimensional $\mathbf{F}$-module, then the locally free $\mathcal{O}$-module $\mathcal{M}(N)$ corresponds to the element of the Ext-space given by the matrix

$$\begin{pmatrix} \xi_{14}I & \xi_{15}I \\ \xi_{24}I & \xi_{25}Z_1 \\ \xi_{34}I & \xi_{35}Z_2 \end{pmatrix} .$$

Here I denotes the identity matrix of size $\dim_{\mathbf{k}} N$, while Z_1 and Z_2 are the matrices describing the action of z_1 and z_2, respectively, on the module N. Then an easy straightforward calculation shows that $\mathcal{M}(N) \simeq \mathcal{M}(N')$ if and only if $N \simeq N'$.

Suppose now that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$, where $\mathcal{L} = \mathcal{O}(D)$ for some divisor D on C. Then for each $i \in \{1,2,3\}$, there are $j,k \in \{1,2,3,4,5\}$ such that

$$\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}(x_i), \mathcal{O}(D+x_j)) = \mathrm{H}^0(C, \mathcal{O}(D+x_j-x_i)) \neq 0$$

and

$$\operatorname{Hom}_{\mathcal{O}}(\mathcal{O}(D+x_i), \mathcal{O}(x_k)) = \mathrm{H}^0(C, \mathcal{O}(-D+x_k-x_i)) \neq 0 .$$

The first inequality implies that $\deg D \geq 0$, while the second one implies $\deg D \leq 0$. Hence, $\deg D = 0$. But then both $D + x_j - x_i$ and $-D + x_k - x_i$ are equivalent to zero, whence $2x_i \sim x_k + x_j$. The choice of these points implies that $x_j = x_k = x_i$ and $D \sim 0$, i.e., we return to the case just considered. □

Remark 1.8. If C is a VB-wild curve, there are families of vector bundles on C consisting of indecomposable, pairwise non-isomorphic bundles and depending on any number of parameters. Indeed, any strict sheaf $\mathcal{M} \in \mathsf{VB}(C,\Lambda)$ for $\Lambda = \mathbf{k}[x_1, x_2, \dots, x_n]$ gives rise to such a family consisting of the vector bundles $\mathcal{M}(p)$, where $p \in \mathbb{A}^n$.

Certainly, the existence of "big families" of non-isomorphic indecomposable vector bundles for curves of genus $g > 1$ is well known and follows, for instance, from the dimension of moduli spaces of stable bundles [30, 38]. On the other hand, we could not find any paper where the VB–wildness of such curves was shown.

Just as above, in the following we give an explicit construction of strict sheaves from $\mathsf{VB}(C,\mathbf{F})$ for $\mathbf{F} = \mathbf{k}\langle x, y \rangle$ (hence, from $\mathsf{VB}(C,\Lambda)$ for any Λ) for any VB-wild curve C. This gives an explicit *representation embedding* from the category of finite dimensional Λ-modules to $\mathsf{VB}(C)$, i.e., an exact functor $\Lambda\text{-}\mathsf{mod} \to \mathsf{VB}(C)$ mapping indecomposable objects to indecomposable and non-isomorphic to non-isomorphic ones.

Note also that all vector bundles belonging to the strict families which we obtain for wild curves are *semi-stable* (cf.[33], [38]). In the proof of Proposition 1.6 this follows from the fact that such a bundle has a filtration whose factors are all of rank 1 and of degree 1 (analogous observations are also valid in the other cases considered below). As we have already mentioned, it is impossible to construct strict families such that all bundles belonging to these families are *stable*, as the category of stable vector bundles is not closed under extensions.

To cast more light on the notion of wildness, we should also mention that, if a classification problem does not involve extensions, it is not difficult to present examples where there are n-parameter families of non-isomorphic indecomposable objects for arbitrary n, but there is nothing like algebraic wildness. On the other hand, we are not aware of any classification problem including extensions where such a phenomenon appears, i.e., where algebraic and geometric wildness differ. Nevertheless, in all known cases the proof used deep investigations.

2. VB-types of singular curves

In this section we consider the case of singular curves. We formulate the trichotomy theorem (Theorem 2.8) and give an explicit description of torsion-free sheaves over VB–finite and tame singular curves (Theorems 2.11 and 2.12). The proofs of these results will be given in the following sections.

We introduce, in addition to Notations 1.1, the following

Notations 2.1. (1) Let $\pi : \widetilde{C} \to C$ denote the *normalisation* of C (cf. [27]). (Note that $\widetilde{C}$ can be reducible or, equivalently, non-connected.)

(2) $S = S(C)$ denotes the set of *singular points* of C and we put $\widetilde{S} = \pi^{-1}(S)$.

(3) Set $\widetilde{\mathcal{O}} = \pi_*(\mathcal{O}_{\widetilde{C}})$; we identify $\mathcal{O}$ with its natural image in $\widetilde{\mathcal{O}}$.

(4) Let $\mathcal{J}$ be the *conductor* of $\mathcal{O}$ in $\widetilde{\mathcal{O}}$, i.e., the biggest sheaf on C of $\widetilde{\mathcal{O}}$-ideals contained in $\mathcal{O}$.

(5) Set $\mathcal{F} = \mathcal{O}/\mathcal{J}$ and $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}/\mathcal{J}$.

(6) For any torsion-free sheaf $\mathcal{B}$ on C of $\mathcal{O}$-modules, put $\widetilde{\mathcal{B}} = \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}/\,\mathsf{t}(\widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B})$ (cf. 1.1) and $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{J}\mathcal{B}$. In particular, $\mathcal{F} = \overline{\mathcal{O}}$ and $\widetilde{\mathcal{F}} = \overline{\widetilde{\mathcal{O}}}$.
As $\mathcal{B}$ is torsion-free, the canonical map $\mathcal{B} \to \widetilde{\mathcal{B}}$ is a monomorphism and we always consider $\mathcal{B}$ as a sub-sheaf of $\widetilde{\mathcal{B}}$. Note also that if $\mathcal{B}$ is a vector bundle, then $\widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}$ has no torsion part, and hence coincides with $\widetilde{\mathcal{B}}$. Any morphism $g : \mathcal{B} \to \mathcal{B}'$ of $\mathcal{O}$-modules lifts in a unique way to a morphism $\widetilde{g} : \widetilde{\mathcal{B}} \to \widetilde{\mathcal{B}}'$ of $\widetilde{\mathcal{O}}$-modules.

Lemma 2.2. *For every torsion-free sheaf $\mathcal{B}$ of $\mathcal{O}$-modules the sheaf $\widetilde{\mathcal{B}}$ is naturally isomorphic to the $\widetilde{\mathcal{O}}$-subsheaf in $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}$ generated by $\mathcal{B}$.*
(Recall that $\mathcal{K}$ denotes the sheaf of rational functions on C.)

Proof. By definition of the torsion part, we have an exact sequence

$$0 \longrightarrow \mathsf{t}(\widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}) \longrightarrow \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B} \longrightarrow \mathcal{K} \otimes_{\mathcal{O}} (\widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}) \simeq \mathcal{K} \otimes_{\mathcal{O}} \mathcal{B}\,.$$

The image of $\widetilde{\mathcal{O}}\otimes_{\mathcal{O}}\mathcal{B}$ is hence isomorphic to $\widetilde{\mathcal{B}}$ and obviously coincides with the $\widetilde{\mathcal{O}}$-subsheaf of $\mathcal{K}\otimes_{\mathcal{O}}\mathcal{B}$ generated by $\mathcal{B}$. □

Note that $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ are sky-scraper sheaves of algebras, zero outside S and with finite dimensional stalks. Hence, we may (and will) identify them with the finite dimensional $\mathbf{k}$-algebras $\bigoplus_{x\in S}\mathcal{F}_x$ and $\bigoplus_{x\in S}\widetilde{\mathcal{F}}_x$ respectively. Just in the same way we identify the sky-scraper sheaf of modules $\overline{\mathcal{B}}$ with the $\mathcal{F}$-module $\bigoplus_{x\in S}\overline{\mathcal{B}}_x$.

When considering families of torsion-free sheaves, we have to impose some conditions, which guarantee that they are "uniformly embedded" into their $\widetilde{\mathcal{O}}$-closures. Thus, we give the following definition for such families (in [25], in a local setting, they are called *δ-constant*).

Definition 2.3. Let Λ be a $\mathbf{k}$-algebra (not necessarily commutative). Denote by $\mathsf{TF}(C,\Lambda)$ the category whose objects are coherent sheaves on C of $\mathcal{O}\otimes\Lambda$-modules $\mathcal{B}$ satisfying the following conditions:

(1) $\mathcal{B}$ is torsion-free over $\mathcal{O}$.
(2) $\widetilde{\mathcal{B}}$ is flat over $\widetilde{\mathcal{O}}\otimes\Lambda$.
(3) $\widetilde{\mathcal{B}}/\mathcal{B}$ is flat over Λ.

Such sheaves are called *families of torsion-free sheaves* on C with base Λ.

Lemma 2.4. *If $\mathcal{B}\in\mathsf{TF}(C,\Lambda)$, then it is flat over Λ and, for every Λ-module N, the sheaf $\mathcal{B}(N)=\mathcal{B}\otimes_\Lambda N$ is also torsion-free over $\mathcal{O}$; moreover, the natural homomorphism $\mathcal{B}(N)\to\widetilde{\mathcal{B}}(N)$ is an embedding and induces an isomorphism $\widetilde{\mathcal{B}(N)}\simeq\widetilde{\mathcal{B}}(N)$.*

Proof. Put $\mathcal{T}=\widetilde{\mathcal{B}}/\mathcal{B}$, which is a torsion sheaf over $\mathcal{O}$. Consider the exact sequence $0\to\mathcal{B}\to\widetilde{\mathcal{B}}\to\mathcal{T}\to 0$. As $\widetilde{\mathcal{B}}$ and $\mathcal{T}$ are both Λ-flat, so is $\mathcal{B}$. Tensoring by N over Λ, we get again an exact sequence:

$$0\longrightarrow\mathcal{B}(N)\longrightarrow\widetilde{\mathcal{B}}(N)\longrightarrow\mathcal{T}(N)\longrightarrow 0.$$

As $\widetilde{\mathcal{B}}$ is flat over $\widetilde{\mathcal{O}}\otimes\Lambda$ and $(\widetilde{\mathcal{B}}\otimes_\Lambda N)\otimes_{\widetilde{\mathcal{O}}}\mathcal{X}\simeq\widetilde{\mathcal{B}}\otimes_{\widetilde{\mathcal{O}}\otimes\Lambda}(N\otimes\mathcal{X})$ for any sheaf of $\widetilde{\mathcal{O}}$-modules $\mathcal{X}$, the sheaf $\widetilde{\mathcal{B}}(N)=\widetilde{\mathcal{B}}\otimes_\Lambda N$ is flat over $\widetilde{\mathcal{O}}$, hence, torsion free. Therefore, $\mathcal{B}(N)$ is also torsion-free. Moreover, as the image of $\mathcal{B}(N)$ obviously generates $\widetilde{\mathcal{B}}(N)$, the latter coincides with $\widetilde{\mathcal{B}(N)}$ in view of Lemma 2.2. □

Using this notion, we are able to define *TF-finite, TF-tame* and *TF-wild* curves just in the same way as we have defined the corresponding VB-types. Nevertheless, it happens that indeed the TF-type of a curve coincides with its VB-type. We formulate in this section the corresponding results; the remaining part of the article will be devoted to their proofs. First the following holds:

Proposition 2.5. *If a singular curve C is not VB-wild, then:*

(1) *All irreducible components of C are rational curves, i.e., their normalizations are isomorphic to $\mathbb{P}^1$.*

(2) *Any singular point $x \in S$ is a simple node (simple double point).*
In other words, the pre-image $\pi^{-1}(x)$ under the normalization map π consists of 2 points and $\widetilde{\mathcal{F}}_x \simeq \mathbf{k}^2$.

Proof of 1 cf. Section 4, p. 31; proof of 2 cf. Section 6, pp. 36 and 40. □

If all irreducible components of C are rational and all its singular points are simple nodes, we call C a *line configuration.* To such a configuration we associate its *dual graph* and we shall see that this graph defines the VB–type of the curve C. Recall the corresponding definition.

Definitions 2.6. If C is a line configuration, its *dual graph* is the graph $\Delta(C)$ whose vertices are the irreducible components of C, the edges are the singular points of C and an edge corresponding to the point p_j is incident to the vertex corresponding to the component C_i if and only if $p_j \in C_i$.

Note that the graph $\Delta(C)$ is non-oriented, but may have loops and multiple edges. A loop appears if a singular point p_j belongs to a unique component C_i (in this case the edge corresponding to p_j is only incident to the vertex corresponding to C_i). As we always suppose C to be connected, the graph $\Delta(C)$ is connected, as well.

It is also convenient to consider $\mathbb{P}^1$ as a line configuration. As it has only one component and no singular points, its dual graph has one vertex and no edges at all. The following result will be proved in Section 6 (Step 6.3, page 41):

Proposition 2.7. *Let C be a line configuration. Then:*

(1) *C is TF-finite (hence, VB-finite) if and only if $\Delta(C)$ is a Dynkin diagram of type A, i.e., a chain. (For instance, this is the case if $C = \mathbb{P}^1$.) Moreover, in this case all indecomposable vector bundles on C are of rank 1 and they are determined up to isomorphism by their vector-degrees.*

(2) *C is TF-tame (hence, VB-tame) if and only if $\Delta(C)$ is an extended Dynkin digram of type $\widetilde{\mathrm{A}}$, i.e., a cycle. (For instance, this is the case if C is irreducible, rational and has only one simple node.) Moreover, in this case it is VB-unbounded (hence, TF-unbounded).*

(3) *In all other cases C is VB-wild (hence, TF-wild).*

In the first, respectively, the second case, we call C a line configuration of type A, *respectively,* $\widetilde{\mathrm{A}}$.

Altogether, we obtain from Theorem 1.6 and Propositions 2.5 and 2.7 the following theorem, completely describing the VB-types of projective curves:

Theorem 2.8. *Let C be a reduced projective curve.*

1. *If C is a line configuration of type* A, *then it is TF-finite (hence, VB-finite).*
2. *If C is a smooth elliptic curve, then it is VB-tame, bounded.*
3. *If C is a line configuration of type* $\widetilde{\mathrm{A}}$, *then it is both TF-tame and VB-tame, unbounded.*
4. *In all other cases C is VB-wild (hence, TF-wild).*

Remark 2.9. Note some evident corollaries of Theorem 2.8.

1. An irreducible projective curve C is
   (a) VB-finite if and only if $C \simeq \mathbb{P}^1$;
   (b) VB-tame bounded if and only if it is smooth elliptic;
   (c) VB-tame unbounded if and only if it is rational and has only one singular point which is a simple node;
   (d) VB-wild otherwise.
2. Any deformation of a VB-finite curve is also VB-finite; any deformation of a VB-tame one is also VB-tame.
3. If a curve C is VB-finite, its arithmetic genus $\dim_{\mathbf{k}} \mathrm{H}^1(C, \mathcal{O}_C)$ (cf. [27]) is always 0; if it is VB-tame, its arithmetic genus is always 1. The converse is *not true*: any line configuration C such that its dual graph $\Delta(C)$ is a tree has arithmetic genus 0, although most of them are VB–wild.
4. In the tame case the dualizing sheaf ω_C (cf. [27]) is *trivial*, i.e., isomorphic to $\mathcal{O}_C$. Hence, Serre duality (cf. [27]) on such a curve is just given by the functor $\mathcal{H}om_{\mathcal{O}}(\_, \mathcal{O})$.

The triviality of the dualizing sheaf implies the following corollary concerning the Auslander–Reiten quiver of the category of vector bundles. We refer to [3] for the corresponding definitions.

Corollary 2.10. *Let the curve C be VB–tame and τ be the Auslander–Reiten translation in the category $\mathsf{VB}(C)$. Then $\tau\mathcal{B} \simeq \mathcal{B}$ for every indecomposable vector bundle $\mathcal{B}$. In particular, the Auslander–Reiten quiver of $\mathsf{VB}(C)$ consists only of* homogeneous tubes, *i.e., quivers of the form*

$$\cdot \leftrightarrows \cdot \leftrightarrows \ \dots \ \cdot \leftrightarrows \cdot \leftrightarrows \ \dots$$

with the identity translation.

Proof. Follows from [3, Theorem 3.3]. □

We shall now give a description of all vector bundles and torsion-free sheaves on line configurations with dual graphs of types A and $\widetilde{\mathrm{A}}$. Such a sheaf $\mathcal{B}$ can be given by its "normalization" $\widetilde{\mathcal{B}}$ and by the rule of glueing, which describes the image of $\mathcal{B}/\mathcal{J}\mathcal{B}$ in $\widetilde{\mathcal{B}}/\mathcal{J}\widetilde{\mathcal{B}}$. Recall that

$\pi : \widetilde{C} \to C$ denotes the normalization, $\widetilde{\mathcal{O}} = \pi_* \mathcal{O}_{\widetilde{C}}$, $\mathcal{J}$ the conductor of $\mathcal{O}$ in $\widetilde{\mathcal{O}}$ and $\widetilde{\mathcal{B}} = \widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B}/\mathrm{t}(\widetilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{B})$.

First, let C be a line configuration of type A, $\{ C_1, C_2, \dots, C_t \}$ its irreducible components. Each C_i is isomorphic to $\mathbb{P}^1$ and the normalization $\widetilde{C}$ of C can be identified with their disjoint union $\bigsqcup_i C_i$. Let $\{ x_1, x_2, \dots, x_{t-1} \}$ be the singular points of C; we suppose that $x_i \in C_i \cap C_{i+1}$ and denote by x_i' (resp., x_i'') the pre-image of x_i on C_i (resp., on C_{i+1}).

Consider any vector of the form $\mathbf{s} = (m; d_1, d_2, \dots, d_r)$, where $1 \le m \le t$, $r \le t - m + 1$, and define the torsion-free sheaf $\mathcal{B} = \mathcal{B}_{\mathbf{s}}$ in the following way:

- Set $\widetilde{\mathcal{B}} = \pi_* \mathcal{B}_{\widetilde{C}}$, where $\mathcal{B}_{\widetilde{C}}$ is the unique sheaf on $\widetilde{C}$ with support $\bigsqcup_{j=1}^r C_{m+j-1}$ and $\mathcal{B}_{\widetilde{C}}|_{C_{m+j-1}} = \mathcal{O}_{C_{m+j-1}}(d_j)$ for $j = 1, \dots, r$. Then $\widetilde{\mathcal{B}} \simeq \bigoplus_{j=1}^r \mathcal{O}_{C_{m+j-1}}(d_j)$ if we identify every $\mathcal{O}_{C_i}(d)$ with its direct image on C. Hence, the stalk of the sky-scraper sheaf $\widetilde{\mathcal{B}}/\mathcal{J}\widetilde{\mathcal{B}}$ at a point $x \in C$ is nonzero only if $x = x_i$ for $i = m, \dots, m+r-1$; in this case it is $\mathbf{k}(x_i') \oplus \mathbf{k}(x_i'')$.
- Let $\mathcal{B}$ be the preimage in $\widetilde{\mathcal{B}}$ of the subsheaf of the factor $\widetilde{\mathcal{B}}/\mathcal{J}\widetilde{\mathcal{B}}$ such that its stalk at the point x_i, for each $i = m, \dots, m+r-1$, is the one-dimensional subspace of $\widetilde{\mathcal{B}}_{x_i}/\mathcal{J}\widetilde{\mathcal{B}}_{x_i} = \mathbf{k}(x_i') \oplus \mathbf{k}(x_i'')$ generated by $(1,1)$.

Certainly, $\mathcal{B}_{\mathbf{s}} \not\simeq \mathcal{B}_{\mathbf{s}'}$ if $\mathbf{s} \neq \mathbf{s}'$ and the sheaf $\mathcal{B}_{\mathbf{s}}$ is a vector bundle if and only if its support coincides with all of C, i.e., we have $m = 1$ and $r = t$. In this case $\mathcal{B}$ is isomorphic to $\mathcal{O}_C(\mathbf{d})$ where $\mathbf{d} = (d_1, d_2, \dots, d_t)$.

Theorem 2.11. *If C is a line configuration of type* A*, then all torsion-free sheaves $\mathcal{B}_{\mathbf{s}}$ defined above are indecomposable and pairwise non-isomorphic and every indecomposable torsion-free sheaf on C is isomorphic to one of the sheaves $\mathcal{B}_{\mathbf{s}}$.*

In particular, an indecomposable torsion-free sheaf on C is completely determined by its vector-degree. Any indecomposable vector bundle on C is of rank 1, indeed, they are all isomorphic to the shifts of the structure sheaf $\mathcal{O}_C(\mathbf{d})$ for some $\mathbf{d}$.

The proof of this theorem is given in Section 6 (Step 6.2, case 1, page 41).

Now let C be a line configuration of type $\widetilde{\mathrm{A}}$, $\{ C_1, C_2, \dots, C_t \}$ its irreducible components. If $t > 1$, each C_i is again isomorphic to $\mathbb{P}_1$ and $\widetilde{C} \simeq \bigsqcup_i C_i$. If $t = 1$, $\widetilde{C} \simeq \mathbb{P}^1$. We denote by $\mathcal{O}_i$ the structure sheaf of the *normalization* of C_i (which coincides with C_i if $t > 1$). Let $x_1, x_2, \dots, x_t$ be the singular points of C. We suppose that $x_i \in C_i \cap C_{i+1}$ (putting $C_{s+1} = C_1$) and denote, if $t > 1$, its pre-image on C_i (resp., on C_{i+1}) by x_i' (resp., x_i''). If $t = 1$, x_1 has two pre-images on $\widetilde{C}$ and we denote them by x'_1 and x''_1 too. Put also $\mathcal{F}_i = \widetilde{\mathcal{O}}_{x_i}/\mathcal{J}_{x_i}$ and identify it with $\mathbf{k}(x'_i) \oplus \mathbf{k}(x''_i)$.

Call a *band datum* a triple $\mathbf{b} = (\mathbf{d}, n, \lambda)$, where m is a positive integer, λ is a nonzero element of the field $\mathbf{k}$ and $\mathbf{d}$ is a sequence of integers $(d_1, d_2, \dots, d_{tr})$, which is t*-aperiodic*, i.e., cannot be obtained by a repetition of a shorter sequence whose length is also a multiple of t. For every band datum $\mathbf{b}$ define the vector bundle $\mathcal{B} = \mathcal{B}_\mathbf{b}$ as follows ($n\mathcal{M}$ always denotes the n-fold direct sum of the sheaf $\mathcal{M}$):

- Set $\mathcal{A}^j = n\mathcal{O}_j(d_j)$, where $\mathcal{O}_j = \mathcal{O}_k$ for $j \equiv k \pmod t$, and set $\mathcal{A} = \oplus_{j=1}^{tr}\mathcal{A}^j$.
  Note that
  $$(\mathcal{A}^j/\mathcal{J}\mathcal{A}^j)_{x_i} \simeq \begin{cases} n\mathbf{k}(x'_i) & \text{if } j \equiv i \pmod t \\ n\mathbf{k}(x''_i) & \text{if } j \equiv i+1 \pmod t \\ 0 & \text{otherwise.} \end{cases}$$
  We identify a nonzero factor $(\mathcal{A}^j/\mathcal{J}\mathcal{A}^j)_{x_i}$ with the vector space $n\mathbf{k}(x'_i)$ or $n\mathbf{k}(x''_i)$ and denote by $\{\, \mathbf{e}'_{jk} \mid 1 \le k \le n \,\}$, respectively by $\{\, \mathbf{e}''_{jk} \mid 1 \le k \le n \,\}$, its canonical basis (consisting of the vectors $(0, \dots, 1, \dots, 0)$).
- Define $\mathcal{B} = \mathcal{B}_\mathbf{b}$ as the vector bundle on C such that $\widetilde{\mathcal{B}}$ coincides with the sheaf $\mathcal{A}$ defined above and the image of $\mathcal{B}/\mathcal{J}\mathcal{B}$ in $\mathcal{A}/\mathcal{J}\mathcal{A}$ coincides with the subspace spanned by the following vectors:
  $$\begin{aligned} \mathbf{e}'_{jk} + \mathbf{e}''_{j+1,k} & \qquad \text{for all } j \ne rt \text{ and all } k; \\ \mathbf{e}'_{rt,1} + \lambda\mathbf{e}''_{1,1}; & \\ \mathbf{e}'_{rt,k} + \lambda\mathbf{e}''_{1,k} + \mathbf{e}''_{1,k} & \qquad \text{for } 1 < k \le n. \end{aligned}$$

One easily sees that $\mathcal{B}_\mathbf{b} \simeq \mathcal{B}_{\mathbf{b}'}$ if $\mathbf{b}' = (\mathbf{d}', m, \lambda)$, where $\mathbf{d}'$ is obtained from $\mathbf{d} = (d_1, d_2, \dots, d_{tr})$ by a t*-cyclic permutation*, i.e., $\mathbf{d}' = (d'_1, d'_2, \dots, d'_{tr})$, where $d'_i = d_{tl+i}$ for some l (putting $d_{j+tr} = d_j$ for all j). In this case we say that the band datum $\mathbf{b}'$ is obtained from $\mathbf{b}$ by a t-cyclic permutation.

Now call a *string datum* a sequence $\mathbf{s} = (m; d_1, d_2, \dots, d_r)$, where $1 \le m \le t$ and d_i are integers (r being any positive integer). Define the torsion-free sheaf $\mathcal{B}_\mathbf{s}$ on C as follows:

- Put $\mathcal{A}^j = \mathcal{O}_{m+j-1}(d_j)$ for $1 \le j \le r$ and $\mathcal{A} = \oplus_{j=1}^{r}\mathcal{A}^j$.
  Again we identify here a nonzero factor $(\mathcal{A}^j/\mathcal{J}\mathcal{A}^j)_{x_i}$ with $\mathbf{k}(x'_i)$ or $\mathbf{k}(x''_i)$ and denote, respectively, by $\mathbf{e}'_j$ or by $\mathbf{e}''_j$ its basis vector.
- Define $\mathcal{B} = \mathcal{B}_\mathbf{s}$ as the torsion-free sheaf on C such that $\widetilde{\mathcal{B}}$ coincides with the sheaf $\mathcal{A}$ defined above and the image of $\mathcal{B}/\mathcal{J}\mathcal{B}$ in $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{J}\mathcal{A}$ coincides with the subspace $\overline{\mathcal{B}}$ spanned by the vectors $\mathbf{e}''_1$, $\mathbf{e}'_j + \mathbf{e}''_{j+1}$ ($1 \le j < r$) and $\mathbf{e}'_r$.

Note that the sheaf $\mathcal{B} = \mathcal{B}_{\mathbf{s}}$ is *never* locally free. This follows from Proposition 4.2 as the $\mathcal{F}$-submodule $\overline{\mathcal{B}} \subset \overline{\mathcal{A}}$ is *not correct* in the sense of Definition 4.1.1. One can also see it from Figure 4 below, as the rank varies on a certain component.

Theorem 2.12. *If C is a line configuration of type $\widetilde{\mathrm{A}}$, then all torsion-free sheaves $\mathcal{B}_{\mathbf{b}}$ and $\mathcal{B}_{\mathbf{s}}$, where $\mathbf{b}$ (respectively $\mathbf{s}$) runs through all possible band (respectively string) data, are indecomposable and every indecomposable torsion-free sheaf on C is isomorphic to one of the sheaves $\mathcal{B}_{\mathbf{b}}$ or $\mathcal{B}_{\mathbf{s}}$. The only possible isomorphisms between these sheaves are $\mathcal{B}_{\mathbf{b}} \simeq \mathcal{B}_{\mathbf{b}'}$, where $\mathbf{b}'$ is obtained from $\mathbf{b}$ by an s-cyclic permutation; the sheaves $\mathcal{B}_{\mathbf{s}}$ are pairwise non-isomorphic.*

In particular, $\mathcal{B}_{\mathbf{b}}$ are all indecomposable vector bundles, while $\mathcal{B}_{\mathbf{s}}$ are all indecomposable torsion-free sheaves which are not vector bundles.

Note that in the latter case there are only discrete sets of sheaves: there are no non-trivial families of torsion-free sheaves, which are not vector bundles.

The proof of Theorem 2.12 is also given in Section 6 (Step 6.2, case 2, page 42).

We should like to illustrate the above classification by some pictures in order that the structure of these sheaves becomes more clear.

Let $t = 3$, C_1, C_2, C_3 the three components of C, $C_i = \mathbb{P}^1$ and x_1, x_2, x_3 the three intersection points as in Figure 1:

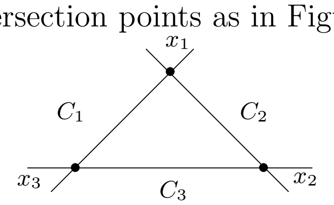

Figure 1

According to the description in Theorem 2.11, a vector bundle $\mathcal{B} = \mathcal{B}_{\mathbf{b}}$ is given by a t-aperiodic band datum $\mathbf{b} = (m, \lambda, \mathbf{d})$, where $\mathbf{d} = (d_1, \dots, d_{tr}) \in \mathbb{Z}^{tr}$, $\mathrm{rk}(\mathcal{B}) = mr$, $m \in \mathbb{N}$, $\lambda \in \mathbf{k}^*$. (In our example $t = 3$.) The sequence $\mathbf{d}$ describes $\pi^*\mathcal{B}$ on $\widetilde{C}$ where $\pi : \widetilde{C} \to C$ is the normalization.

Since we consider bundles together with a trivialization in neighbourhoods of singular points, we have fixed a basis of each vector space $\mathcal{B}(x_i)$ and also of $\pi^*\mathcal{B}$ at the two pre-images of x_i. Hence, any vector bundle $\mathcal{B}$ on C is completely described by $\pi^*\mathcal{B}$, where $\pi^*\mathcal{B}|_{\widetilde{C}_j} \equiv \mathcal{B}|_{C_j}$ is a direct sum of $\mathcal{O}_{C_j}(d_i)$ for certain d_i, and by the glueing of $\mathcal{B}|_{C_j}$ with $\mathcal{B}|_{C_{j+1}}$ at x_j with respect to the given bases. The glueing of $\mathcal{B}|_{C_j}$ with $\mathcal{B}|_{C_{j+1}}$ can be trivialized for $1 \le j < t$, that is, given by the identity matrix, but the glueing of $\mathcal{B}|_{C_s}$ with $\mathcal{B}|_{C_1}$ at x_1 perhaps not. Indeed, if $m \ge 1$ and $\mathcal{B}$ is of rank mr, this glueing can be described as being the identity on the first $m(r-1)$ basis vectors, and the Jordan cell of rank d with eigenvalue λ on the last m basis vectors (in particular, multiplication with λ if $m = 1$).

In Figure 2 and Figure 3, a thick line with label d_i at the component C_j corresponds to the sheaf $\mathcal{O}_{Cj}(d_i)$, a thin line corresponds to the trivial glueing and a dotted line to a non-trivial glueing, described by the matrix A. The marked points symbolize a basis of the fibres over the corresponding intersection points:

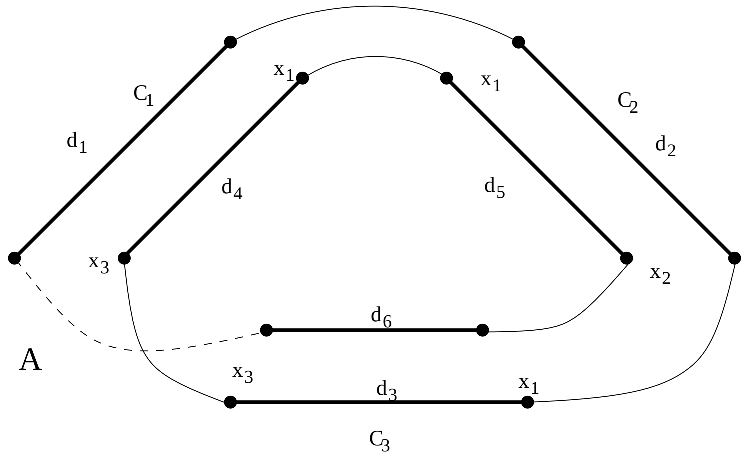

Figure 2
Bundle of rank 2, $\mathbf{b} = (1, \lambda; d_1, \dots, d_6)$,
$A = (\lambda)$

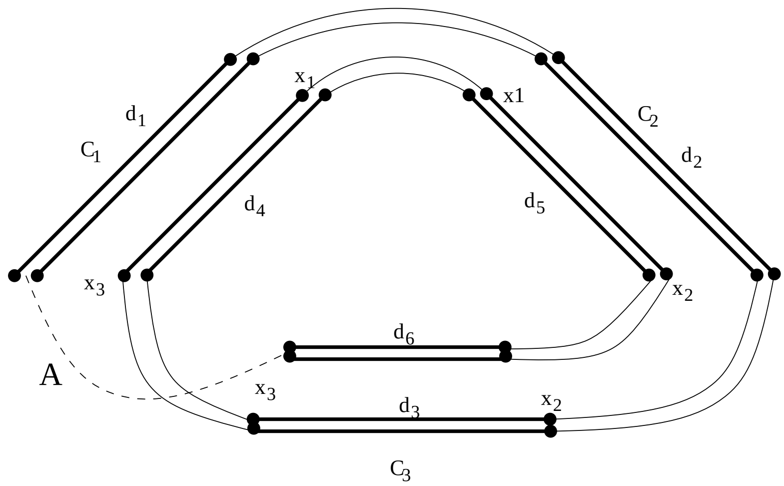

Figure 3
Bundle of rank 4, $\mathbf{b} = (2, \lambda; d_1, \dots, d_6)$,
$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

A torsion-free sheaf $\mathcal{M}$ on C which is not locally free is given by a string datum which can be coded as $\mathbf{s} = (m; \mathbf{d})$, $\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{Z}^r$, $1 \leq m \leq s$.

Here, $\mathbf{d}$ describes $\pi^*\mathcal{M}$ as before, while C_m is the component where the glueing starts. Again we do cyclic glueing, although the cycle does not close but has two free ends corresponding to the basis elements which are not glued (encircled in Fig. 4). Hence all glueings are trivial and we see that (not locally free) torsion-free sheaves on C are characterized by discrete data without moduli.

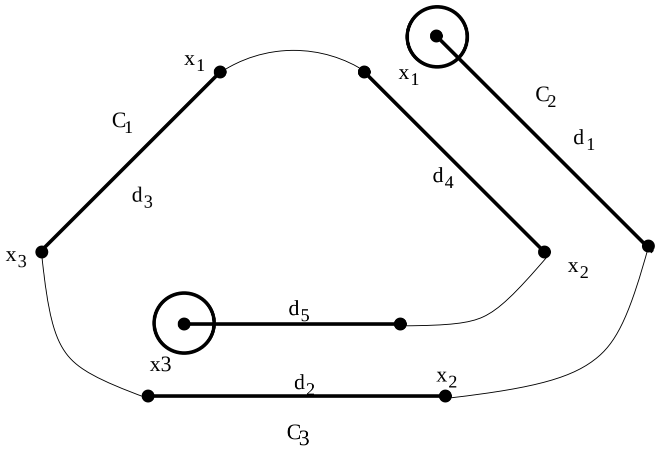

Figure 4
Torsion-free sheaf of rank 1 on C_1, rank 2 on C_2 and C_3,
$\mathbf{s} = (2; d_1, \dots, d_5)$.

3. Shifting bimodules

The study of vector bundles on singular curves is closely related to *bimodule problems* considered in [11, 13, 9]. Bimodules appear during the description of the glueing necessary to obtain a sheaf on such a curve from a sheaf on its normalization. This relation will be studied in the next section. Here we recall and make precise some of the corresponding definitions and, in addition, modify them to take into account more complicated sets of discrete parameters. Namely, our bimodules are endowed with a group of shifts. As we shall see later, these shifts reflect the natural shifts $\mathcal{B} \mapsto \mathcal{B}(\mathbf{d})$ in the category of vector bundles considered in the previous section. Hence, we have to change slightly the definitions of representation types, taking into account these shifts, just as we have done for VB–types of curves.

As the definitions of this section are somewhat abstract and there is no appropriate textbook for references, we try to explain them by giving some simple examples.

In the following "*category*" usually means a category over an algebraically closed field $\mathbf{k}$. This means that all Hom-spaces are vector spaces over $\mathbf{k}$ and the product of morphisms is $\mathbf{k}$-bilinear. Given two categories A_1 and A_2, an A_1-A_2-*bimodule* is, by definition, a functor $\mathsf{U} : \mathsf{A}_1\o \times \mathsf{A}_2 \to \mathsf{Vect}$, the category of $\mathbf{k}$-vector spaces. If $\mathsf{A}_1 = \mathsf{A}_2 = \mathsf{A}$, U is called an A-*bimodule*. Usually we suppose that our categories are additive. Whenever some category C is not additive, we consider its *additive hull*, i.e., the smallest additive category $\mathsf{A} = \operatorname{add}\mathsf{C}$ containing C, and identify C-bimodules with A-bimodules.

Let U be an A-bimodule. Define the category $\mathsf{El}(\mathsf{U})$ of *elements of the bimodule* U as follows. Its object set is

$$\operatorname{Ob}\mathsf{El}(\mathsf{U}) = \bigcup_{A\in\operatorname{Ob}\mathsf{A}} \mathsf{U}(A,A)$$

and morphisms from $u \in \mathsf{U}(A,A)$ to $v \in \mathsf{U}(B,B)$ are morphisms $f \in \mathsf{A}(A,B)$ such that $fu = vf$. Here (and later on) we write fu instead of $\mathsf{U}(1,f)u$ and vf instead of $\mathsf{U}(f,1)v$. Note that both of these elements belong to $\mathsf{U}(A,B)$.

Elements of a bimodule U are often called "matrices over U" [13, 20], since, given a decomposition of an object A into a direct sum of indecomposbales: $A = \bigoplus_{i=1}^n A_i$, one can consider an element of $\mathsf{U}(A,A)$ as an $n\times n$ matrix (u_{ij}), where $u_{ij} \in \mathsf{U}(A_j,A_i)$.

Example 3.1. One of the main examples arises when $\mathsf{A} = \mathsf{Pr}\,\Lambda$ is the category of (finitely generated) projective modules over a $\mathbf{k}$-algebra Λ. Suppose that Λ is finite dimensional and *basic*, i.e., $\Lambda/\operatorname{rad}\Lambda \simeq \mathbf{k}^n$. Then an A-bimodule U can be completely determined by the value $\mathsf{U}_0 = \mathsf{U}(\Lambda,\Lambda)$, which is a Λ-bimodule. Namely, let $1 = e_1 + e_2 + \dots + e_n$ be a decomposition of the unit element of Λ into a sum of primitive idempotents. Put $P_i = \Lambda e_i$. Any projective Λ-module P is isomorphic to a direct sum $\bigoplus_{i=1}^n k_iP_i$ [19]. (We denote by kM the direct sum of k copies of M.) It is known that $\mathsf{A}(P_i,P_j) \simeq \Lambda_{ij} = e_i\Lambda e_j$ and $\mathsf{A}(P,P')$, where $P' = \bigoplus_{i=1}^n l_iP_i$ can be identified with the set of matrices of the form

$$X = \begin{pmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \dots & \dots & \dots & \dots \\ X_{n1} & X_{n2} & \dots & X_{nn} \end{pmatrix}, \tag{2}$$

where each X_{ij} is an $l_i \times k_j$-matrix with entries from Λ_{ij}. In the same way, one can see that $\mathsf{U}(P_i,P_j)$ is naturally isomorphic to $\mathsf{U}_{ij} = e_i\mathsf{U}_0e_j$ and $\mathsf{U}(P,P)$ can be considered as the set of matrices of the form

$$Y = \begin{pmatrix} Y_{11} & Y_{12} & \dots & Y_{1n} \\ Y_{21} & Y_{22} & \dots & Y_{2n} \\ \dots & \dots & \dots & \dots \\ Y_{n1} & Y_{n2} & \dots & Y_{nn} \end{pmatrix}, \tag{3}$$

where each Y_{ij} is a $k_j \times k_i$-matrix with entries from U_{ij}. The multiplication of morphisms, as well as the action of A on U, coincide under this identification with the usual multiplication of matrices. In particular, a morphism in $\mathsf{El}(\mathsf{U})$ from the matrix (3) to another matrix $Y' \in \mathsf{U}(P',P')$ of the same shape is a matrix (2) such that $XY = Y'X$.

Call a *shift* in a category A any auto-equivalence $\mathsf{A} \xrightarrow{\sim} \mathsf{A}$. Define now a *shifting category* to be a triple $(\mathsf{A}, \Sigma, \rho)$ where A is a category, Σ is a group and ρ is a homomorphism from Σ to the group of shifts in A. As usual, we write $\sigma(a)$ instead of $\rho(\sigma)(a)$, where a is an object or a morphism of A. We call Σ *a group of shifts* in A.

Let U be an A-bimodule and σ be some shift in A. A *shift in* U *compatible with* σ is, by definition, an isomorphism of A-bimodules $\sigma : \mathsf{U} \xrightarrow{\sim} \mathsf{U}^\sigma$, where U^σ denotes the bimodule obtained from U via pullback by σ, i.e. $\mathsf{U}^\sigma(a_1, a_2) = \mathsf{U}(\sigma(a_1), \sigma(a_2))$.

Given a shifting category $(\mathsf{A}, \Sigma, \rho)$, define a *shifting* A*-bimodule* as a pair (U, ρ_u), where ρ_u maps each element $\sigma \in \Sigma$ to a shift of U compatible with $\rho(\sigma)$ and $\rho_u(\sigma\tau) = \rho_u(\sigma)\rho_u(\tau)$ for each $\sigma, \tau \in \Sigma$. Again we write $\sigma(u)$ instead of $\rho_u(\sigma)(u)$. Note that in this case the category $\mathsf{El}(\mathsf{U})$ also becomes a shifting category with the same group Σ of shifts.

Example 3.2. Let A_0 be the category with the set of objects $\mathbb{Z}$ and the set of morphisms generated by morphisms $x_n : n \to n+1$. In other words, it is the category of paths of the graph

$$\dots \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \dots$$

(the quiver of type A_∞^∞). Denote by $\mathsf{A} = \operatorname{add} \mathsf{A}_0$, the aditive hull of A_0. There is a natural shift σ in A mapping n to $n+1$, so one can consider A as a shifting category with the group of shifts $\mathbb{Z} \simeq \{\, \sigma^k \mid k \in \mathbb{Z} \,\}$. Note that $\mathsf{A}(n,m)$ is one-dimensional if $n \le m$ and zero otherwise. An object $A \in \mathsf{A}$ is a (formal) finite direct sum $\bigoplus_{n\in\mathbb{Z}} k_n \cdot n,\ k_n \ge 0$, and elements of $\mathsf{A}(A,B)$, where $B = \bigoplus_{n\in\mathbb{Z}} l_n \cdot n$, can be considered as matrices

$$X = \begin{pmatrix} \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dots & X_{-1,-1} & 0 & 0 & \dots \\ \dots & X_{0,-1} & X_{0,0} & 0 & \dots \\ \dots & X_{1,-1} & X_{1,0} & X_{1,1} & \dots \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \end{pmatrix}, \tag{4}$$

where X_{mn} is an $l_m \times k_n$ matrix with entries from $\mathbf{k}$, zero if $m < n$. The shift σ just replaces X_{mn} by $X_{m-1,n-1}$.

Consider the *regular* A-bimodule, i.e., such that $\mathsf{U}(A,B) = \mathsf{A}(A,B)$. Then an element from $\mathsf{U}(A,A)$ is also a matrix Y of the form (4) (with $l_i = k_i$). A morphism in $\mathsf{El}(\mathsf{U})$ from Y to $Y' \in \mathsf{U}(B,B)$ is a matrix X of the form (4) such that $XY = Y'X$.

We need the notion of the *tensor product* of two categories $\mathsf{A} \otimes \mathsf{C}$. By definition, the objects of this category are formal direct sums of formal products $A \otimes P$, where $A \in \mathsf{A}$, $P \in \mathsf{C}$. The space of morphisms $(\mathsf{A} \otimes \mathsf{C})(A \otimes P, B \otimes Q)$ is defined as the tensor product of vector spaces $\mathsf{A}(A,B) \otimes \mathsf{C}(P,Q)$. If U is an A-bimodule, we define the $\mathsf{A} \otimes \mathsf{C}$-bimodule $\mathsf{U} \otimes \mathsf{C}$ by putting $(\mathsf{U} \otimes \mathsf{C})(A \otimes P, B \otimes Q) = \mathsf{U}(A,B) \otimes \mathsf{C}(P,Q)$.

Hence, the category $\mathsf{El}(\mathsf{U}\otimes\mathsf{C})$ is well defined. We say that the elements of $(\mathsf{U}\otimes\mathsf{C})(A\otimes P, A\otimes P)$ are *based on* A. We are mostly interested in the case when $\mathsf{C}=\mathsf{Pr}\,\Lambda$ is the category of finitely generated (right) projective modules over some $\mathbf{k}$-algebra Λ. Then we write U^Λ instead of $\mathsf{U}\otimes\mathsf{C}$ and $\mathsf{El}(\mathsf{U},\Lambda)$ instead of $\mathsf{El}(\mathsf{U}\otimes\mathsf{C})$. If, moreover, Λ is a commutative domain and $u\in\mathsf{U}^\Lambda(A\otimes P, A\otimes P)$, where $\mathrm{rk}_\Lambda P=r$, we say that u is an *element of rank* r *based on* A.

If every projective Λ-module is free, one can identify $\mathsf{A}\otimes\mathsf{Pr}\,\Lambda$ with $\mathsf{A}\otimes\Lambda$ and U^Λ with $\mathsf{U}\otimes\Lambda$, since every object from $\mathsf{A}\otimes\mathsf{Pr}\,\Lambda$ is of the form $A\otimes r\Lambda\simeq rA\otimes\Lambda$ for some $r\ge 0$.

Example 3.3. Suppose that U is the bimodule of Example 3.2 and $\Lambda=\mathbf{k}[x]$. Then an element $u\in\mathsf{U}^\Lambda(P\otimes M, P\otimes M)$, where $\mathrm{rk}_\Lambda M=r$, is again given by a matrix Y of the form (4), but this time its components Y_{ij} are of size $rk_i\times rk_j$ and with entries from Λ. If N is a finite dimensional Λ-module, we can consider it as Λ-$\mathbf{k}$-bimodule. In order to obtain $u(N)$ one has to replace every entry f of the corresponding matrix Y (which is a polynomial from $\mathbf{k}[x]$) by the matrix defining the multiplication with u in the module N. In particular, if $\dim_{\mathbf{k}} N=1$, hence, $N\simeq\Lambda/(x-\lambda)$, the entry f is replaced by $f(\lambda)$.

Note that any functor $\theta:\mathsf{C}\to\mathsf{C}'$ induces the functor $\theta_*=1\otimes\theta:\mathsf{El}(\mathsf{U}\otimes\mathsf{C})\to\mathsf{El}(\mathsf{U}\otimes\mathsf{C}')$. In particular, given a Λ'-Λ-bimodule N (i.e., a left Λ-module and right Λ'-module), which is finitely generated and projective over Λ', we get the functor $\mathsf{El}(\mathsf{U},\Lambda)\to\mathsf{El}(\mathsf{U},\Lambda')$ induced by the tensor product $\_\otimes_\Lambda N$. The image of an element $u\in\mathsf{El}(\mathsf{U},\Lambda)$ under this functor will be denoted by $u(N)$.

Denote by vect the category of finite dimensional vector spaces over $\mathbf{k}$. Then $\mathsf{A}\otimes\mathsf{vect}\simeq\mathsf{A}$ for each additive category A and we always identify these categories. Hence, any functor $N:\mathsf{C}\to\mathsf{vect}$ gives rise to the functor $N_*:\mathsf{El}(\mathsf{U}\otimes\mathsf{C})\to\mathsf{El}(\mathsf{U})$. In particular, if $\mathsf{C}=\mathsf{Pr}\,\Lambda$, such a functor is given by some finite dimensional Λ-module and we shall identify this module with the functor N. For this reason, in the general case, the functors $\mathsf{C}\to\mathsf{vect}$ are also called C-*modules* (more precisely, they should be called *finite dimensional modules*, but we never deal with other ones). Denote the category of C-modules by $\mathsf{C}\text{-}\mathsf{mod}$.

Definition 3.4. Let U be a shifting bimodule with group of shifts Σ. Call an element $u\in\mathsf{El}(\mathsf{U}\otimes\mathsf{C})$ *strict* if it satisfies the following conditions:

(1) The element $u(N)$ is indecomposable in $\mathsf{El}(\mathsf{U})$ for each indecomposable C-module N.
(2) For any two C-modules N, N' and for each shift $\sigma\in\Sigma$, the elements $u(N)$ and $\sigma(u)(N')$ are isomorphic in $\mathsf{El}(\mathsf{U})$ if and only if $\sigma=1$ and $N\simeq N'$ (as Λ-modules).

For instance, in Example 3.3, the element $u\in(\mathsf{U}\otimes\Lambda)(n\otimes\Lambda, n\otimes\Lambda)$ given by the 1×1 matrix (x) is obviously strict.

Definition 3.5. Let U be a shifting bimodule with the group Σ of shifts. Suppose we have given, for each $\mathbf{k}$-algebra Λ, a full sub-category $\mathsf{El}'(\mathsf{U},\Lambda) \subseteq \mathsf{El}(\mathsf{U},\Lambda)$ satisfying the following conditions:

(1) $u(N) \in \mathsf{El}'(\mathsf{U},\Lambda')$ for each element $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ and for each Λ'-Λ-bimodule N which is finitely generated and projective over Λ'.

(2) $\sigma(u) \in \mathsf{El}'(\mathsf{U},\Lambda)$ for each shift $\sigma \in \Sigma$ and for each element $u \in \mathsf{El}'(\mathsf{U},\Lambda)$.

Then call the family of sub-categories $\{\, \mathsf{El}'(\mathsf{U},\Lambda) \mid \Lambda \,\}$ a *correct family* and the elements $u \in \mathsf{El}'(\mathsf{U},\Lambda)$ *correct elements* (with respect to this correct family).

For instance, in Example 3.2 one can define a correct family of elements $\{\, \mathsf{El}'(\mathsf{U},\Lambda) \,\}$ by taking only matrices X of the form (4) such that all diagonal components X_{nn} are invertible.

Correct families will appear later from some conditions that are to be imposed on the elements of bimodules in order that they correspond to vector bundles (cf. page 29).

Definitions 3.6. The *representation type* of a shifting A-bimodule U (with the group Σ of shifts) supplied by a correct family of sub-categories is defined as follows. The bimodule is said to be:

- *Correctly finite* if there exists a finite set of indecomposable correct elements $\mathbf{M} \subseteq \mathsf{El}'(\mathsf{U})$ such that each indecomposable correct element is isomorphic to $\sigma(u)$ for some $\sigma \in \Sigma$ and some $u \in \mathbf{M}$.
- *Correctly tame* if there exists a set $\mathbf{M}$ consisting of strict elements $u \in \mathsf{El}'(\mathsf{U},\Lambda_u)$ such that:
  (1) Each Λ_u is a commutative domain, finitely generated as $\mathbf{k}$-algebra and of Krull dimension 1 (note that it may depend on u).
  (2) For each object $A \in \operatorname{Ob}\mathsf{A}$ and for each natural number r, the set $\mathbf{M}_{A,r} = \{\, u \in M \mid u$ is an element of rank r based on $A \,\}$ is finite.
  (3) For each object $A \in \operatorname{Ob}\mathsf{A}$, all indecomposable correct elements from $\mathsf{U}(A,A)$, except possibly for a finite number, are isomorphic to $\sigma(u)(N)$ for some element $u \in \mathbf{M}$, some shift $\sigma \in \Sigma$ and some (finite dimensional) Λ_u-module N.

  In this case we call $\mathbf{M}$ a *parametrising set* for correct elements of U.

  Moreover, if U is correctly tame, call it:
  – *bounded* if there exists a parametrising set $\mathbf{M}$ for correct elements of U such that all cardinalities $|\mathbf{M}_{A,r}|$ are not greater than a constant c for all possible A and r;
  – *unbounded* if there is no such parametrising set.

- *Correctly wild* if there exists a correct strict element $u \in \mathsf{El}'(\mathsf{U}, \Lambda)$ for each finitely generated $\mathbf{k}$-algebra Λ.

In the case that *all* elements are considered to be correct, we omit the word "correct" and speak about *finite*, *tame* (bounded or unbounded) or *wild* shifting bimodules. Sometimes, to stress it, we say that the bimodule is "*absolutely*" finite, tame or wild.

Recall once more that to prove U to be correctly wild, we only have to find a correct strict element in $\mathsf{El}'(\mathsf{U}, \mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle z_1, z_2 \rangle$ is a free (non-commutative) $\mathbf{k}$-algebra with 2 generators.

In most cases we deal with so called *bipartite bimodules* [11]. They are defined as follows. If U is an A_1-A_2-bimodule, we can consider it as a bimodule over the direct product $\mathsf{A} = \mathsf{A}_1 \times \mathsf{A}_2$ by setting $\mathsf{U}((a_1, a_2), (b_1, b_2)) = \mathsf{U}(a_1, b_2)$ for $a_i, b_i \in \mathsf{A}_i$. Call this bimodule a *bipartite* A_1-A_2-bimodule. In this case an element of U based on a pair (A_1, A_2), where $A_i \in \operatorname{Ob} \mathsf{A}_i$, is indeed an element $u \in \mathsf{U}(A_1, A_2)$. A morphism from u to another element $u' \in \mathsf{U}(A'_1, A'_2)$ is a pair (f, g), where $f : A_1 \to A'_1$, $g : A_2 \to A'_2$, such that $gu = u'f$ (both these elements are from $\mathsf{U}(A_1, A'_2)$). Note that in [13] and [20] only bipartite bimodules were considered.

Bipartite bimodules correspond to the class of matrix problems which were called *separated* in [20]. We prefer the word "bipartite," since "separated" is too widely used (and sometimes in quite different senses).

4. Relation between vector bundles and bimodules

We are now going to apply the notions of the preceding section to the study of torsion-free sheaves on a singular curve C. Namely, we connect with such a curve a shifting bimodule which describes the correspondence between vector bundles on this curve and on its normalization. Moreover, we give here the proof of the first assertion of Proposition 2.5 (page 31).

We use Notations 2.1. In particular, $\pi : \widetilde{C} \to C$ denotes the normalization of C, $\mathcal{O}$ the structure sheaf of C, $\widetilde{\mathcal{O}} = \pi_* \mathcal{O}_{\widetilde{C}}$ and $\mathcal{J}$ the conductor of $\mathcal{O}$ in $\widetilde{\mathcal{O}}$; finally, $\mathcal{F} = \mathcal{O}/\mathcal{J}$ and $\widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}/\mathcal{J}$. To describe the glueing, necessary to obtain sheaves of $\mathcal{O}$-modules from $\widetilde{\mathcal{O}}$-modules, the following notions are convenient.

Definitions 4.1. (1) Let Λ be a $\mathbf{k}$-algebra, $\mathcal{A}$ a coherent flat sheaf on C of $\widetilde{\mathcal{O}} \otimes \Lambda$-modules and $\mathcal{M}$ a coherent $\mathcal{F} \otimes \Lambda$-subsheaf of $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{J}\mathcal{A}$. Call this subsheaf *correct* if it satisfies the following conditions:

(a) $\overline{\mathcal{A}}/\mathcal{M}$ is flat over Λ.
(b) $\mathcal{M}$ is flat over $\mathcal{F} \otimes \Lambda$.
(c) The natural homomorphism $\widetilde{\mathcal{F}} \otimes_{\mathcal{F}} \mathcal{M} \to \overline{\mathcal{A}}$ is an isomorphism.

If only condition (a) holds and $\widetilde{\mathcal{F}}\mathcal{M} = \overline{\mathcal{A}}$, call $\mathcal{M}$ *semi-correct.*

(2) Define the category $\mathsf{C}^s = \mathsf{C}^s(C,\Lambda)$ as follows:
   (a) its objects are pairs $(\mathcal{A},\mathcal{M})$, where $\mathcal{A}$ is a flat coherent sheaf of $\widetilde{\mathcal{O}}\otimes\Lambda$-modules and $\mathcal{M}$ is a semi-correct submodule of $\overline{\mathcal{A}}$;
   (b) a morphism $(\mathcal{A},\mathcal{M}) \to (\mathcal{A}',\mathcal{M}')$ is a morphism $f:\mathcal{A}\to\mathcal{A}'$ such that the induced mapping $\overline{f}:\overline{\mathcal{A}}\to\overline{\mathcal{A}}'$ maps $\mathcal{M}$ to $\mathcal{M}'$.

(3) Let $\mathsf{C} = \mathsf{C}(C,\Lambda)$ be the full sub-category of $\mathsf{C}^s(C,\Lambda)$ consisting of all pairs $(\mathcal{A},\mathcal{M})$ with a correct submodule $\mathcal{M}$.
   We write $\mathsf{C}^s(C)$ and $\mathsf{C}(C)$ instead of $\mathsf{C}^s(C,\mathbf{k})$ and $\mathsf{C}(C,\mathbf{k})$ correspondingly.

(4) Define a functor $F:\mathsf{TF}(C,\Lambda)\to\mathsf{C}^s(C,\Lambda)$, mapping any sheaf $\mathcal{B}\in\mathsf{TF}(C,\Lambda)$ to the pair $(\widetilde{\mathcal{B}},\overline{\mathcal{B}})$ and any morphism $g:\mathcal{B}\to\mathcal{B}'$ to the morphism $\widetilde{g}:\widetilde{\mathcal{B}}\to\widetilde{\mathcal{B}}'$. (Recall that $\widetilde{\mathcal{B}} = \widetilde{\mathcal{O}}\otimes_{\mathcal{O}}\mathcal{B}/\mathrm{t}(\widetilde{\mathcal{O}}\otimes_{\mathcal{O}}\mathcal{B})$ and $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{J}\mathcal{B}$.)

One can easily check that $\overline{\mathcal{B}}$ is indeed a semi-correct submodule of $\widetilde{\mathcal{B}}/\mathcal{J}\widetilde{\mathcal{B}}$ and $\overline{\widetilde{g}}(\overline{\mathcal{B}}) \subseteq \overline{\mathcal{B}}'$. Note only that the definitions imply immediately that $\mathcal{J}\widetilde{\mathcal{B}} = \mathcal{J}\mathcal{B}$. Moreover, if $\mathcal{B}$ is a flat sheaf, the submodule $\overline{\mathcal{B}}$ is indeed correct. This construction leads to the folloing statement.

Proposition 4.2. *The functor F establishes an equivalence between the categories $\mathsf{TF}(C,\Lambda)$ and $\mathsf{C}^s(C,\Lambda)$. Moreover, the restriction of F to $\mathsf{VB}(C,\Lambda)$ establishes an equivalence of $\mathsf{VB}(C,\Lambda)$ and $\mathsf{C}(C,\Lambda)$.*

Proof. Define the inverse functor G as follows: for any object $\mathcal{P} = (\mathcal{A},\mathcal{M})$ of C^s let $\mathcal{B} = G\mathcal{P}$ be the pre-image of $\mathcal{M}\subseteq\overline{\mathcal{A}}$ in $\mathcal{A}$. It is a coherent subsheaf in $\mathcal{A}$ such that $\widetilde{\mathcal{O}}\mathcal{B} = \mathcal{A}$. In particular, $\mathcal{B}$ is torsion-free and $\widetilde{\mathcal{B}}\simeq\mathcal{A}$. As $\mathcal{A}/\mathcal{B}\simeq\overline{\mathcal{A}}/\mathcal{M}$ is flat over Λ, $\mathcal{B}\in\mathsf{TF}(C,\Lambda)$. If $\mathcal{P}' = (\mathcal{A}',\mathcal{M}')$ is another pair and $f:\mathcal{P}\to\mathcal{P}'$ is a morphism from $\mathsf{C}^s(C,\Lambda)$, then, by construction, $f(G\mathcal{A})\subseteq G\mathcal{A}'$. Therefore, we obtain a functor $G:\mathsf{C}^s\to\mathsf{VB}$, inverse to F.

Let now the pair $\mathcal{P}$ be correct, that is, $\mathcal{M}\simeq\mathcal{B}/\mathcal{J}\mathcal{B}$ is flat over $\mathcal{F}\otimes\Lambda$. As it is also coherent, it is a projective $\mathcal{F}\otimes\Lambda$-module. Fix a point $x\in S$ and put $M = \mathcal{M}_x/\mathfrak{m}\mathcal{M}_x$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{F}_x$. M is a flat finitely generated, hence projective Λ-module. Then $P = \mathcal{O}_x\otimes M$ is a projective $\mathcal{O}_x\otimes\Lambda$-module such that $P/\mathfrak{m}P\simeq\mathcal{B}_x/\mathfrak{m}\mathcal{B}_x$. Hence, there is a homomorphism $f:P\to\mathcal{B}_x$ such that $\operatorname{Im} f + \mathfrak{m}\mathcal{B}_x = \mathcal{B}_x$. Since $\mathcal{B}_x$ is finitely generated as $\mathcal{O}_x\otimes\Lambda$-module, f is an epimorphism. Moreover, as $\overline{\mathcal{A}}_x\simeq\widetilde{\mathcal{F}}\otimes_{\mathcal{F}}\mathcal{M}_x$, also $\mathcal{A}_x/\mathfrak{m}\mathcal{A}_x\simeq\widetilde{P}/\mathfrak{m}\widetilde{P}$, where $\widetilde{P} = \widetilde{\mathcal{O}}_x\otimes M = \widetilde{\mathcal{O}}_x\otimes_{\mathcal{O}_x}P$. Since $\widetilde{P}$ and $\mathcal{A}_x$ are both finitely generated projective $\widetilde{\mathcal{O}}_x\otimes\Lambda$-modules, they are isomorphic and the commutative diagram

$$\begin{array}{ccc} \mathcal{O}_x \otimes M & \xrightarrow{f} & \mathcal{B}_x \\ \big\downarrow & & \big\downarrow \\ \widetilde{\mathcal{O}}_x \otimes M & \xrightarrow{\sim} & \mathcal{A}_x \end{array}$$

shows that f is a monomorphism, hence, an isomorphism. Therefore, $\mathcal{B}_x$ is a projective (thus flat) $\mathcal{O}_x \otimes \Lambda$-module for every point $x \in S$. For all other points y, $\mathcal{B}_y = \mathcal{A}_y$ is also flat, therefore, the whole sheaf $\mathcal{B}$ is flat over $\mathcal{O} \otimes \Lambda$, that is, belongs to $\mathsf{VB}(C, \Lambda)$. □

For each vector bundle $\widetilde{\mathcal{A}}$ of constant rank r on $\widetilde{C}$, we can always choose an open affine sub-variety $C' \subset \widetilde{C}$ such that $\widetilde{S} \subset C'$ and the restriction of $\widetilde{\mathcal{A}}$ on C' is trivial: $\widetilde{\mathcal{A}}|_{C'} \simeq r\mathcal{O}_{\widetilde{C}}|_{C'}$. Using this, we can (and do) always suppose that $\widetilde{\mathcal{A}}|_{C'} = r\mathcal{O}_{\widetilde{C}}|_{C'}$. Therefore, setting $\mathcal{A} = \pi_*\widetilde{\mathcal{A}}$, we get $\mathcal{A}|_{\pi(C')} = r\widetilde{\mathcal{O}}|_{\pi(C')}$ and hence $\overline{\mathcal{A}} = r\widetilde{\mathcal{F}}$. This identification is compatible with tensor products if we identify $\widetilde{\mathcal{O}} \otimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{O}}$ with $\widetilde{\mathcal{O}}$ via the natural isomorphism. A correct subsheaf of $\overline{\mathcal{A}}$ is then given by r elements $v_1, v_2, \dots, v_r$ of $r\widetilde{\mathcal{F}}$ linearly independent over $\mathcal{F}$, namely, $\mathcal{M} = \sum_{i=1}^r \mathcal{F} v_i$. We often write $(\mathcal{A}, v_1, v_2, \dots, v_r)$ instead of $(\mathcal{A}, \mathcal{M})$ for objects from $\mathsf{C}^s(C)$.

For instance, a line bundle $\mathcal{L}$ over C is given by a line bundle $\widetilde{\mathcal{L}}$ over $\widetilde{C}$ and an invertible element v of the algebra $\widetilde{\mathcal{F}}$. If $\mathcal{B}$ is the torsion-free sheaf corresponding to a semi-correct pair $\mathcal{P} = (\mathcal{A}, \mathcal{M})$ then their tensor product $\mathcal{B} \otimes_{\mathcal{O}} \mathcal{L}$ is given by the pair $\mathcal{P}^{\mathcal{L}} = (\mathcal{A} \otimes_{\widetilde{\mathcal{O}}} \mathcal{L}, v\mathcal{M})$. This is how the Picard group $\operatorname{Pic}(C)$ acts on the category $\mathsf{C}^s(C)$. Of course, if $\mathcal{P} \in \mathsf{C}$, also $\mathcal{P}^{\mathcal{L}} \in \mathsf{C}$.

In particular, we get a rule for tensor products of line bundles. Note that the bundles corresponding to the pairs $(\widetilde{\mathcal{L}}, v)$ and $(\widetilde{\mathcal{L}}', v')$ are isomorphic if and only if $\widetilde{\mathcal{L}} \simeq \widetilde{\mathcal{L}}'$ and $v' = \theta v$ for some invertible element $\theta \in \mathcal{F}$ (take into account that both $\widetilde{\mathcal{L}}|_{C'}$ and $\widetilde{\mathcal{L}}'|_{C'}$ coincide with $\pi_*\mathcal{O}_{\widetilde{C}}|_{C'}$). Note also that any isomorphism $\widetilde{\mathcal{L}} \xrightarrow{\sim} \widetilde{\mathcal{L}}'$ is locally constant (as the curve $\widetilde{C}$ is projective). Denote by $\mathcal{C}$ the image of all locally constant functions in $\widetilde{\mathcal{F}}$. Then the preceding considerations immediately give the following corollary.

Corollary 4.3. $\operatorname{Pic}(C) \simeq \operatorname{Pic}(\widetilde{C}) \times (\widetilde{\mathcal{F}}^*/\mathcal{C}^*\mathcal{F}^*)$.

If C is irreducible, $\mathcal{C} = \mathbf{k} \subseteq \mathcal{F}$, so the latter factor is nothing but $\widetilde{\mathcal{F}}^*/\mathcal{F}^*$.

The following result is quite obvious.

Proposition 4.4. *If $\widetilde{C}$ is VB-wild, so is also C.*

Proof. Indeed, let $\mathcal{A} \in \mathsf{VB}(\widetilde{C}, \mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle x, y \rangle$, be a strict sheaf. For any point $x \in \widetilde{S}$, $\mathcal{A}_x/\mathcal{J}\mathcal{A}_x$ is a projective $\mathbf{F}$-module, hence, it is free (cf. [8]). It is evident that all these factors are of the same rank r. Fix an $\mathbf{F}$-bases $\mathbf{e}_1^x, \mathbf{e}_2^x, \dots, \mathbf{e}_r^x$ of these factors for all points $x \in \widetilde{S}$ and consider the $\mathcal{F}\otimes\mathbf{F}$-subsheaf $\mathcal{M} \subset \mathcal{A}/\mathcal{J}\mathcal{A}$ generated over $\mathcal{F} \otimes \mathbf{F}$ by the set

$$\left\{ \sum_{\pi(x)=y} \mathbf{e}_i^x \mid y \in S,\, 1 \le i \le r \right\}.$$

Clearly, this subsheaf is correct, hence, defines a family $\mathcal{B} \in \mathsf{VB}(C, \mathbf{F})$. But this family is strict as so is $\mathcal{A}$ and evidently $\widetilde{\mathcal{B}(N)} \simeq \mathcal{A}(N)$ for every N. □

The vector-degree Deg defines a homomorphism $\operatorname{Pic}(C) \to \mathbb{Z}^t$ and it is evident that $\operatorname{Deg}\mathcal{B} = \operatorname{Deg}\widetilde{\mathcal{B}}$. Recall that we have chosen a section $\omega : \mathbb{Z}^t \to \operatorname{Pic}(C)$ in such a way that $\omega(\mathbf{e}_i) = \mathcal{O}(p_i)$ for some smooth point p_i, where $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots 0)$ (1 at the i-th place), and put $\mathcal{O}(\mathbf{d}) = \omega(\mathbf{d})$ and $\mathcal{A}(\mathbf{d}) = \mathcal{A}\otimes_{\mathcal{O}}\mathcal{O}(\mathbf{d})$ for every sheaf $\mathcal{A}$ of $\mathcal{O}$-modules (cf. page 7). Thus $\mathsf{VB}(C)$ becomes a shifting category with the group $\Sigma = \mathbb{Z}^t$ of shifts. The same is true for C and C^s: if $\mathcal{P} = (\mathcal{A}, \mathcal{M})$, put $\mathcal{P}(\mathbf{d}) = (\mathcal{A}(\mathbf{d}), \mathcal{M})$. Note that under the identification $\widetilde{\mathcal{A}}|_{C'} = r\mathcal{O}_{\widetilde{C}}|_{C'}$ imposed above, $\mathcal{A}(\mathbf{d})|_{\pi(C')} = \mathcal{A}|_{\pi(C')}$, hence, $\overline{\mathcal{A}}(\mathbf{d}) = \overline{\mathcal{A}}$. Certainly, the equivalence $\mathsf{VB}(C) \simeq \mathsf{C}(C)$ preserves these shifts.

The category C^s has the advantage that it can be easily reinterpreted with the help of some bimodule category.

Denote by A the category of locally free (coherent) $\widetilde{\mathcal{O}}$-sheaves on C and by B the category of projective (finitely generated) $\mathcal{F}$-modules. Define a bipartite B-A-bimodule U by setting, for $B \in \mathsf{B}$ and $\mathcal{A} \in \mathsf{A}$,

$$\mathsf{U}(B, \mathcal{A}) = \operatorname{Hom}_{\mathcal{F}}(B, \overline{\mathcal{A}}).$$

This is also a shifting bimodule with the same group $\Sigma = \mathbb{Z}^t$ of shifts. Namely, shifting by $\mathbf{d}$ acts on A as above, acts on B trivially and maps an element $u \in \mathsf{U}(B, \mathcal{A})$, i.e., a homomorphism $B \to \overline{\mathcal{A}}$, to the same element considered as a homomorphism $B \to \overline{\mathcal{A}}(\mathbf{d}) = \overline{\mathcal{A}}$ (we denote it by $u(\mathbf{d})$).

Define two correct families of elements of the bimodule U. The first one is denoted by El_c and called the family of *correct* elements consisting of all elements $u \in \mathsf{U}^\Lambda(B, \mathcal{A}) = \operatorname{Hom}_{\mathcal{F}\otimes\Lambda}(B, \overline{\mathcal{A}})$ satisfying the following conditions:

(1) $B \simeq F \otimes P$ for some free $\mathcal{F}$-module F and some projective Λ-module P.
(2) $\operatorname{Ker} u \subseteq (\operatorname{rad}\mathcal{F})B$.
(3) $\operatorname{Coker} u$ is flat as Λ-module.
(4) The induced map $u_{\widetilde{\mathcal{F}}} : \widetilde{\mathcal{F}} \otimes_{\mathcal{F}} B \to \overline{\mathcal{A}}$ is an isomorphism.

The second family is denoted by El_{sc} and called the family of *semi-correct* elements. It consists of all elements satisfying conditions 2 and 3 above and the following one:

4′. $u_{\widetilde{\mathcal{F}}}$ is an epimorphism.

Proposition 4.5. *For each correct (respectively semi-correct) element* $u \in \mathsf{U}^\Lambda(P,\mathcal{A})$ *, the image* $\operatorname{Im} u$ *is a correct (respectively semi-correct) submodule in* $\overline{\mathcal{A}}$ *and the mapping* $u \to (\mathcal{A}, \operatorname{Im} u)$ *induces a functor* $F : \mathsf{El}_c(\mathsf{U},\Lambda) \to \mathsf{C}(C,\Lambda)$ *(respectively* $\mathsf{El}_{sc}(\mathsf{U},\Lambda) \to \mathsf{C}^s(C,\Lambda)$ *) having the following properties:*

1. *It is* dense*, i.e., each object from* C *(respectively* C^s *) is isomorphic to* Fu *for some object* u *from* El_c *(respectively from* El_{sc} *).*
2. *It is* full*, i.e., all induced maps of morphism spaces* $\mathsf{El}_{sc}(u,u') \to \mathsf{C}^s(Fu,Fu')$ *are surjective.*
3. *It* reflects isomorphisms*, i.e.,* $u \simeq u'$ *if and only if* $Fu \simeq Fu'$ *.*
4. *It* preserves indecomposability*, i.e.,* u *is indecomposable if and only if* Fu *is indecomposable.*
5. *It is* compatible with shifts*, i.e.,* $F(u(\mathbf{d})) \simeq \sigma(F(u))(\mathbf{d})$ *for each* $\sigma \in \mathbb{Z}^n$ *.*

Proof. For each pair $(\mathcal{A},\mathcal{M}) \in \mathsf{C}^s$ the $\mathcal{F}$-module $U = \mathcal{M}/(\operatorname{rad}\mathcal{F})\mathcal{M}$ is semi-simple, hence, isomorphic to a direct sum of $U_i \otimes P_i$, where U_i are simple $\mathcal{F}$-modules and P_i are projective Λ-modules. Then $U_i \simeq B_i/(\operatorname{rad}\mathcal{F})B_i$, where B_i is a projective $\mathcal{F}$-module, and there is an epimorphism $u : B = \oplus_i B_i \otimes P_i \to \mathcal{M}$ such that $\operatorname{Ker} u \subseteq (\operatorname{rad}\mathcal{F})B$. Obviously, u is a semi-correct (respectively correct if $\mathcal{M}$ is a correct submodule) element of $\mathsf{U}^\Lambda(B,\mathcal{A})$ and $F(u) \simeq (\mathcal{A},\mathcal{M})$. Moreover, as any homomorphism can be lifted to projective covers, the functor F is full, and as $\operatorname{Ker} u \subseteq (\operatorname{rad}\mathcal{F})B$, it reflects isomorphisms. The compatibility with shifts follows immediately from the definition of F. □

Due to condition *2* of the definition of correct elements, if an element $u \in \mathsf{U}(B,\mathcal{A})$ is correct, the number of indecomposable summands in B is at most $\dim_{\mathbf{k}} \overline{\mathcal{A}}$. Hence, if $\mathcal{A}$ is fixed, there are only finitely many possibilities for B. On the other hand, if $\mathcal{A} \in \mathsf{A}$ is of rank r, then $\operatorname{Deg}\mathcal{A}(\mathbf{e}_i) = \operatorname{Deg}\mathcal{A} + r\mathbf{e}_i$. Hence, any correct element u has a shift lying in $\mathsf{U}(B,\mathcal{A})$ with $0 \le \operatorname{Deg}\mathcal{A} < (r,r,\dots,r)$. Here an inequality for vectors means inequality for all components. Together with Proposition 4.5 this implies the following corollary.

Corollary 4.6. *A curve* C *is VB-finite, VB-tame or VB-wild if and only if the bimodule* $\mathsf{U}(C)$ *is correctly finite, correctly tame or correctly wild, respectively.*

The same is true for TF-types of curves and semi-correct types of the corresponding bimodules.

We accomplish this section with the PROOF OF PROPOSITION 2.5 (*1*):

If a singular curve C is not VB-wild, then all its irreducible components are rational curves.

Proof. Accordingly to Proposition 4.4 and Theorem 1.6, whenever a curve C is not VB-wild, the irreducible components of $\widetilde{C}$ are either rational or elliptic curves. We show that the latter case is still impossible.

Let $C_1, C_2, \dots, C_t$ be the irreducible components of $\widetilde{C}$, $\mathcal{O}_k = \mathcal{O}_{C_k}$. Suppose there is a component C_1 which is elliptic. As C is singular and connected, there is a singular point $e \in C$ which lies on $\pi(C_1)$. Consider the case when also $e \in \pi(C_2)$ for another component C_2 (the other cases are even simpler to handle). Using Lemma 1.7, find 4 different points $x_1, \dots, x_4$ on $C_1 \setminus \widetilde{S}$ such that $2x_i \not\sim x_j + x_k$ for $i \neq j$, and a point $y \in C_2 \setminus \widetilde{S}$. Consider the element u from $\mathsf{El}(\mathsf{U}, \mathbf{F})$, where $\mathbf{F} = \mathbf{k}\langle\, z_1, z_2 \,\rangle$, defined as follows:

$$
\begin{aligned}
&u \in \mathsf{U}(B, A), \quad \text{where} \\
&A = \oplus_{i=1}^4 A_i \otimes \mathbf{F}, \quad \text{where} \quad A_i = \widetilde{\mathcal{O}}(x_i + iy), \\
&B = 4\mathcal{F} \otimes \mathbf{F}.
\end{aligned}
$$

In this case $u : B \to A$ can be given by a set of 4×4 matrices $u_{p,k}$ with entries from $(\mathcal{O}_{kp}/\mathcal{J}) \otimes \mathbf{F}$, as such a matrix defines a homomorphism $B_p \to A_p$. Here p runs through $S(C)$ and $k = 1, 2, \dots t$. We set all components equal to the identity matrices except $u_{e,2}$ which is

$$
\begin{pmatrix}
z_1 & z_2 & 1 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
$$

Obviously, $u \in \mathsf{El}_c(\mathsf{U}, \mathbf{F})$. We shall prove that this element is strict.

The Riemann–Roch theorem [27] for the components C_i of $\widetilde{C}$ implies that $\mathrm{Hom}_{\widetilde{\mathcal{O}}}(A_i, A_i) \simeq \mathbf{k}^t$ (each direct factor reflects the multiplication by scalars on the corresponding component of $\widetilde{C}$). If $i \neq j$, homomorphisms from $\mathrm{Hom}_{\widetilde{\mathcal{O}}}(A_i, A_j)$ are zero on the component C_1 and can be nonzero on C_2 only if $i < j$ and $C_2 \simeq \mathbb{P}^1$. If $N \in \mathbf{F}\text{-}\mathsf{mod}$, then $A \otimes_{\mathbf{F}} N \simeq \oplus_{i=1}^4 A_i \otimes N$ and $B \otimes_{\mathbf{F}} N \simeq 4\mathcal{F} \otimes N$. An element $u(N)$ can be obtained from u if we replace z_k by the matrix Z_k which defines the multiplication with z_k in N ($k = 1, 2$) and replace 1 and 0 everywhere by the identity and zero matrices, respectively.

Suppose that N' is another $\mathbf{F}$-module given by matrices Z'_1, Z'_2. A morphism from $u(N)$ to $u(N')$ is a pair of homomorphisms (f, g), where $f : 4\mathcal{F} \otimes N \to 4\mathcal{F} \otimes N'$ and $g : \oplus_{i=1}^4 A_i \otimes N \to \oplus_{i=1}^4 A_i \otimes N'$, such that $gu(N) = u(N')f$. Since the component $u_{e,1}$ is the identity matrix, the component f_p coincides with the restriction of g onto C_1, which is a block-diagonal matrix of the form

$$G_1 = \begin{pmatrix} G_1 & 0 & 0 & 0 \\ 0 & G_2 & 0 & 0 \\ 0 & 0 & G_2 & 0 \\ 0 & 0 & 0 & G_2 \end{pmatrix},$$

where the blocks correspond to the direct summands $A_i \otimes N$ and $A_i \otimes N'$. The restriction of g onto C_2 is a block-triangular matrix

$$G_2 = \begin{pmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}$$

such that $G_2 u_{e,2}(N) = u_{e,2}(N')G_1$. The last equality implies that $A_{ij} = 0$ if $i \neq j$, all matrices G_i and A_{jj} $(i,j = 1,2,3,4)$ are equal, and if G denotes their common value, then $GZ_k = Z'_k G$ for $k = 1,2$. In particular, if $u(N) \simeq u(N')$, then also $N \simeq N'$, and if N is indecomposable, i.e., has no nontrivial idempotent endomorphisms, so is $u(N)$.

Moreover, if $\mathcal{M} \in \mathsf{VB}(C, \mathbf{F})$ is the corresponding sheaf, one can verify, just as in the proof of Theorem 1.6, that $\mathcal{M}(N) \simeq \mathcal{M}(N') \otimes_{\mathcal{O}} \mathcal{L}$ for some line bundle $\mathcal{L}$ if and only if $N \simeq N'$ and $\mathcal{L} \simeq \mathcal{O}$. Hence, the element u is strict and the curve C is VB–wild. □

5. Rationally composed curves and special bimodules

From now on we suppose that all components of $\widetilde{C}$ are rational curves (i.e., isomorphic to $\mathbb{P}^1$). In this case we say that the curve C itself is *rationally composed*. Note that a rationally composed curve is a *line configuration* if and only if all its singular points are simple nodes. To find the VB-types of rationally composed curves, we introduce a specific class of bipartite bimodules (called *special bimodules*).

First of all, consider the category L_0 such that $\operatorname{Ob}\mathsf{L}_0 = \mathbb{Z}$ (the integers) and the set of morphisms is generated by morphisms $x_n : n \to n+1$ and $y_n : n \to n+1$ subject to the relations: $x_{n+1}y_n = y_{n+1}x_n$ for all $n \in \mathbb{Z}$. Let L be its additive hull. It is well known that L is equivalent to the category of vector bundles over the projective line $\mathbb{P}^1$: the object n corresponds to the sheaf $\mathcal{O}_{\mathbb{P}^1}(n)$ and the morphisms x_n, respectively y_n, correspond to the multiplication with x, respectively y, the homogeneous coordinates on $\mathbb{P}^1$. There is a natural shift σ on L mapping n to $n+1$ and we consider L as a shifted category with the group of shifts $\mathbb{Z} \simeq \{\, \sigma^k \mid k \in \mathbb{Z} \,\}$.

Definition 5.1. *Special data* consist of the following components:

(1) *A finite dimensional commutative algebra* $\mathbf{R}$ *and a sub-algebra* $\mathbf{S} \subset \mathbf{R}$.

Let $\mathbf{R} = \prod_{k=1}^{r} \mathbf{R}_k$ and $\mathbf{S} = \prod_{j=1}^{s} \mathbf{S}_j$ be their decompositions into direct product of local algebras. Denote by $\mathbf{J}$ the set of all pairs (k, j) such that $\mathbf{S}_j \mathbf{R}_k \neq 0$.

(2) *An equivalence relation* $\sim$ *on the set of local components of* $\mathbf{R}$ *(indeed on the index set* $\{1, 2, \dots, r\}$ *).*
Let $\mathbf{C}$ be the set of equivalence classes of the relation $\sim$, $\mathbf{C} = \{c_1, c_2, \dots, c_t\}$. For each class c_i put $\mathbf{R}(i) = \prod_{k \in c_i} \mathbf{R}_k$. Denote also by $\overline{\mathbf{J}}$ the set of all pairs (i, j) such that $(k, j) \in \mathbf{J}$ for some $k \in c_i$.

(3) *For each class* $c_i \in \mathbf{C}$*, two elements,* $\xi_i, \eta_i \in \mathbf{R}(i)$.
If $k \in c_i$, denote by ξ_{ik}, respectively η_{ik}, the image of ξ_i, respectively η_i in the field $\mathbf{K}$ considered as the residue field $\mathbf{R}_k / \operatorname{rad} \mathbf{R}_k$.

We impose the following restrictions on these data:

(1) $\mathbf{S}$ *contains no nonzero ideal of* $\mathbf{R}$.
(2) *For each* $k = 1, 2, \dots, r$ *there exists a* j *such that* $\mathbf{S}_j \mathbf{R}_k \neq 0$.
(3) *If* $k \in c_i$*, then* $(\xi_{ik}, \eta_{ik}) \neq (0, 0)$*, so the point* $\kappa(i, k) = (\xi_{ik} : \eta_{ik}) \in \mathbb{P}^1$ *is well defined, and if* $k' \in c_i$, $k' \neq k$*, then* $\kappa(i, k) \neq \kappa(i, k')$.
(4) *If* $\overline{\mathbf{J}} = \mathbf{J}' \cup \mathbf{J}''$ *such that* $(i, j) \in \mathbf{J}'$ *implies* $(i, j') \notin \mathbf{J}''$ *for each* j' *as well as* $(i', j) \notin \mathbf{J}''$ *for each* i'*, then either* $\mathbf{J}'$ *or* $\mathbf{J}''$ *is empty.*[5]
We denote such data (somewhat ambiguously) by $[\mathbf{R}, \mathbf{S}, \xi, \eta]$.

To special data $[\mathbf{R}, \mathbf{S}, \kappa]$, we associate a shifting bimodule $\mathsf{U}[\mathbf{R}, \mathbf{S}, \kappa]$ (called a *special bimodule*) in the following way.

Consider the category $\mathsf{A} = \mathsf{L}^t$, L as defined above. Its indecomposable objects are in one-to-one correspondence with the pairs (n, i), where $n \in \mathbb{Z}$, $1 \leq i \leq t$ and we identify them with such pairs. If some object of A decomposes into a direct sum: $A = \oplus_k (n_k, i_k)$, put $|A| = \oplus_k \mathbf{R}(i_k)$ considered as a projective module over the algebra $\mathbf{R}$. Note that the endomorphism ring of the object (n, i) coincides with $\mathbf{k}$ and the complete set of morphisms in A is generated by the morphisms $x_{ni}, y_{ni} : (n, i) \to (n+1, i)$ originated in the morphisms x_n, y_n of L. Then A is a shifting category with the group of shifts $\Sigma = \mathbb{Z}^t = \langle \sigma_i \,|\, 1 \leq i \leq t \rangle$: the shift σ_i maps (n, i') to $(n + \delta_{ii'}, i')$.

Now let $\mathsf{B} = \mathsf{Pr}\, \mathbf{S} \simeq \prod_{j=1}^{s} \mathsf{Pr}\, \mathbf{S}_j$, the category of finitely generated projective $\mathbf{S}$-modules. Its indecomposable objects are in one-to-one correspondence with the indices $j = 1, \dots, s$ and we identify them. Here the endomorphism ring of the object j is $\mathbf{S}_j$ and there are no morphisms between different indecomposable objects. We consider B

[5] This condition, some sort of connectedness, is not restrictive indeed, but we prefer to impose it to simplify the definitions. In any case, we never need "non-connected" data.

as a shifting category with the same groups of shifts Σ but with all shifts acting trivially.

Definition 5.2. Define the *special* B-A*-bimodule* $\mathsf{U} = \mathsf{U}[\mathbf{R},\mathbf{S},\kappa]$, shifted and bipartite, with the same group Σ of shifts in the following way:

$$\begin{aligned} \mathsf{U}(j,(n,i)) &= \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j,\mathbf{R}(i)) \ \text{ for each } \ n; \\ x_{ni}f(n) &= \xi_i f(n+1)\,; \\ y_{ni}f(n) &= \eta_i f(n+1)\,; \\ \sigma_{i'}f(n) &= f(n+\delta_{ii'})\,. \end{aligned}$$

Here $f(n)$ denotes a homomorphism $f \in \operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j,\mathbf{R}(i))$ considered as an element of $\mathsf{U}(j,(n,i))$ (and we regard such homomorphisms with different indices n as different elements of U). Note that if $(i,j) \notin \overline{\mathbf{J}}$, we have $\mathsf{U}(j,(n,i)) = 0$.

One can easily check that replacing the pair (ξ_i,η_i) with $(a\xi_i + b\eta_i, c\xi_i + d\eta_i)$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an invertible matrix, leads to an isomorphic shifting bimodule. Especially, if the algebra $\mathbf{R}$ (hence, also $\mathbf{S}$) is semi-simple, this bimodule, up to isomorphism, only depends on the equivalence relation $\sim$, the set $\mathbf{J}$ and the points $\kappa(i,k) \in \mathbb{P}^1$, the latter defined up to a collineation of the projective line.

Now let Λ be a $\mathbf{k}$-algebra. The objects from $\mathsf{A}\otimes\Lambda$ are direct sums of the form $A = \bigoplus_k A_k \otimes P_k$, where $A_k \in \operatorname{Ob}\mathsf{A}$, $P_k \in \mathsf{Pr}\,\Lambda$. Put $|A| = \bigoplus_k |A_k| \otimes P_k$ and consider it as a projective $\mathbf{R}\otimes\Lambda$-module. On the other hand, objects from $\mathbf{B}\otimes\Lambda$ are indeed projective $\mathbf{S}\otimes\Lambda$-modules. Hence, to each element $u \in \mathsf{U}^\Lambda(B,A)$ we can associate an homomorphism $|u| \in \operatorname{Hom}_{\mathbf{S}\otimes\Lambda}(B,|A|)$.

Call an element $u \in \mathsf{U}^\Lambda(B,A)$ *correct* if it satisfies the following conditions:

1. $B \simeq F\otimes P$ *for some* free S*-module* F *and some* $P \in \mathsf{Pr}\,\Lambda$*.*
2. $\operatorname{Ker}|u| \subseteq (\operatorname{rad}\mathsf{S})B$.
3. $\operatorname{Coker}|u|$ *is flat as* Λ*-module.*
4. *The induced homomorphism* $|u|_{\mathbf{R}} : \mathbf{R}\otimes_{\mathbf{S}} B \to |A|$ *is an isomorphism.*

If u satisfies the conditions 2 and 3 above and the following one:

$4'$. $|u|_{\mathbf{R}}$ *is an epimorphism,*

we call it *semi-correct.* Note that condition 3 is empty if $\Lambda = \mathbf{k}$ (i.e., for elements of U themselves).

Denote by $\mathsf{El}_c(\mathsf{U},\Lambda)$ and $\mathsf{El}_{sc}(\mathsf{U},\Lambda)$ respectively the full sub-categories of $\mathsf{El}(\mathsf{U},\Lambda)$ consisting of all correct and of all semi-correct elements. Evidently, both families are correct families in the sense defined in Section 3. Therefore, we have for the bimodule U the notion of *correctly* (or *semi-correctly*) *finite*, *tame* (*bounded* or *unbounded*) or *wild.*

Special bimodules do naturally arise when we apply the procedure of Section 4 to rationally composed curves. Namely, let C be such a curve, $\widetilde{C}$ be its normalization and $C_1, C_2, \dots, C_t$ be the irreducible components of $\widetilde{C}$. Fix isomorphisms $C_k \simeq \mathbb{P}^1$. Then $\mathsf{VB}(\widetilde{C}) \simeq \mathsf{L}^t$. Put $\mathbf{R} = \widetilde{\mathcal{F}} = \widetilde{\mathcal{O}}/\mathcal{J}$ and $\mathbf{S} = \mathcal{F} = \mathcal{O}/\mathcal{J}$. Let $S = \{ p_1, p_2, \dots, p_s \}$ be the set of singular points of C and $\widetilde{S} = \{ q_1, q_2, \dots, q_r \}$ the pre-image of S in $\widetilde{C}$. Then $\mathbf{S} = \prod_{j=1}^{s} \mathbf{S}_j$, where $\mathbf{S}_j = \mathcal{F}_{p_j}$, $\mathbf{R} = \prod_{k=1}^{r} \mathbf{R}_k$, where $\mathbf{R}_k = \widetilde{\mathcal{F}}_{q_k}$, and all algebras $\mathbf{S}_j$, $\mathbf{R}_k$ are local. Note that in this case $(k,j) \in \mathbf{J}$ means that $\pi(q_k) = p_j$. Define the equivalence relation on the indices $1, 2, \dots, r$ by putting $k \sim l$ if and only if q_k and q_l belong to the same component of $\widetilde{C}$. We fix homogeneous coordinates $(x_i : y_i)$ on each component C_i, consider them as global sections of the sheaf $\mathcal{O}_{C_i}(1)$ and take for ξ_i, η_i their images in $\mathbf{R}(i) = \mathcal{O}_{C_i}/\mathcal{J}\mathcal{O}_{C_i} \simeq \mathcal{O}_{C_i}(1)/\mathcal{J}\mathcal{O}_{C_i}(1)$ (then, in particular, $\kappa(i,k) = q_k$). Now the following result is quite obvious.

Proposition 5.3. *In the situation above* $[\mathbf{R}, \mathbf{S}, \kappa]$ *are special data, the special bimodule* $\mathsf{U}[\mathbf{R}, \mathbf{S}, \kappa]$ *is isomorphic to the bimodule* $\mathsf{U}(C)$ *corresponding to the curve* C *via Proposition 4.5 and the notions of correct and semi-correct elements for these bimodules coincide.*

Example 5.4. Suppose that C is a rational irreducible curve with one simple node $p = p_1$. Then $\widetilde{C} = \mathbb{P}^1$, $\widetilde{S} = \{ q_1, q_2 \}$ and we may suppose that the homogeneous coordinates are chosen such that $q_1 = (1:0)$, $q_2 = (0:1)$. The normalization $\pi : \widetilde{C} \to C$ is an isomorphism outside $\widetilde{S}$ and $\pi(q_1) = \pi(q_2) = p$. In this case $\mathbf{S} = \mathcal{F}_p = \mathbf{k}(p)$, $\mathbf{R} = \widetilde{\mathcal{F}}_p = \mathbf{k}(q_1) \times \mathbf{k}(q_2)$, $\mathbf{J} = \{ (1,1), (1,2) \}$ and $1 \sim 2$ under the equivalence relation $\sim$. Hence, there is only one equivalence class c_1 under $\sim$ with $\mathbf{R}(1) = \mathbf{R}$, so $\mathsf{A} = \mathsf{L}$, $\mathsf{B} = \mathsf{vect}$ and $\mathsf{U}(j,n) \simeq \mathbf{R} \simeq \mathbf{k} \times \mathbf{k}$ (we write n instead of $(n,1)$ for the objects of A, since $t = 1$). Thus, if $A = \oplus_n k_n \cdot n$ and B is a vector space of dimension k, then $\mathsf{U}(B, A) \simeq \bigoplus_n \operatorname{Hom}(k \cdot \mathbf{k}(p), k_n \cdot \mathbf{k}(q_1)) \times \operatorname{Hom}(k \cdot \mathbf{k}(p), k_n \cdot \mathbf{k}(q_2))$ and an element $u \in \mathsf{U}(B, A)$ can be considered as a set of matrices $\{ X_n, Y_n \,|\, n \in \mathbb{Z} \}$, where X_n and Y_n are both of size $k_n \times k$. It is convenient to identify u with the pair of matrices (X, Y), where

$$X = \begin{pmatrix} \vdots \\ X_{-1} \\ X_0 \\ X_1 \\ \vdots \end{pmatrix}, \qquad Y = \begin{pmatrix} \vdots \\ Y_{-1} \\ Y_0 \\ Y_1 \\ \vdots \end{pmatrix}. \tag{5}$$

Such an element is *correct* if and only if both matrices X, Y are invertible. It is *semi-correct* if and only if $\operatorname{rk} X = \operatorname{rk} Y = \sum_n k_n$ and $\operatorname{rk} \begin{pmatrix} X \\ Y \end{pmatrix} = k$.

Let u' be another element described by the set of matrices $\{X'_n, Y'_n\}$ of size $l_n \times l$ and let X', Y' be the corresponding matrices of the form (5). A morphism φ in $\mathsf{El}(\mathsf{U})$ from u to u' is given by three matrices, T^X, T^Y, R, where R is of size $l \times k$ and T^X, T^Y are lower triangular of the form

$$T^X = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \dots \\ \dots & T_{-1,-1} & 0 & 0 & \dots \\ \dots & T^X_{0,-1} & T_{00} & 0 & \dots \\ \dots & T^X_{1,-1} & T^X_{10} & T_{11} & \dots \\ \dots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and

$$T^Y = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \dots \\ \dots & T_{-1,-1} & 0 & 0 & \dots \\ \dots & T^Y_{0,-1} & T_{00} & 0 & \dots \\ \dots & T^Y_{1,-1} & T^Y_{10} & T_{11} & \dots \\ \dots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where T^X_{mn} and T^Y_{mn} are both of size $l_m \times k_n$, such that $T^X X = X'R$ and $T^Y Y = Y'R$. Note that the *diagonal* blocks of T^X and T^Y are equal. Of course, φ is an isomorphism if and only if R and all diagonal blocks T_{nn} are invertible.

6. Representation type of special bimodules

Now we are going to describe the representation types of special bimodules and hence the VB-types of rationally composed curves. This will accomplish the proofs of Theorems 2.8, 2.11 and 2.12. In what follows, we always use Definitions 5.1 and 5.2, as well as the notations introduced there.

Step 6.1. *If the algebra* $\mathbf{R}$ *is non-semi-simple, then the shifting bimodule* $\mathsf{U} = \mathsf{U}[\mathbf{R}, \mathbf{S}, \kappa]$ *is correctly (hence also semi-correctly) wild.*

For a rationally composed curve C this means, whenever C is not VB-wild, that for every singular point p, $\widetilde{\mathcal{F}}_p \simeq \mathbf{k}^m$ where m is the cardinality of the preimage $\pi^{-1}(p)$. In other words, p is a simple k-fold point, where k is the number of branches passing through p.

Proof. Note first that there are no morphisms in A from (i,n) to (i,n') if $n > n'$. Hence, verifying the second condition of the definition of a strict element from Section 3 we may always suppose that $\sigma = 1$.

We consider in details the most complicated case when $r = 1$ (i.e., $\mathbf{R}$ and thus $\mathbf{S}$ are both local). In other cases the corresponding constructions are either analogous or even easier and we omit them. We write (n) for the object $(n,1) \in \mathsf{A}$. In this case $\mathsf{U}(\mathbf{S}, (n)) \simeq \mathbf{R}$ and we write $\hat{a}$, where $a : (n) \to (m)$, for $a \cdot 1$. Choose an element

$\alpha \in \operatorname{rad}\mathbf{R}$ such that $\alpha\mathbf{R}$ is a minimal ideal. Then $\alpha \operatorname{rad}\mathbf{R} = 0$, and $\dim_{\mathbf{k}} \alpha\mathbf{R} = 1$, therefore, $\alpha \notin \mathbf{S}$. Decompose $\mathbf{R} = \mathbf{k} \oplus \alpha\mathbf{R} \oplus W$, $\mathbf{S} = \mathbf{k} \oplus J$, where $J = \operatorname{rad}\mathbf{S}$, $W \subset \operatorname{rad}\mathbf{R}$. W need not be an ideal, but $W\alpha = 0$.

We are going to construct a correct and strict element from U^Γ, where Γ is the path algebra of the following quiver:

$$\begin{array}{ccccc} 5 & \xrightarrow{z_5} & 4 & \xrightarrow{z_4} & 3 \\ & & {\scriptstyle z_3}\big\downarrow & & \big\downarrow{\scriptstyle z_2} \\ & & 2 & \xrightarrow[z_1]{} & 1 \end{array}$$

which is wellknown to be wild [10, 31]. Indeed, a strict representation M of Γ over the free algebra $\mathbf{F} = \mathbf{k}\langle x_1, x_2 \rangle$ can be given as follows:

$$M(1) = M(5) = 2\mathbf{F},\ M(2) = M(3) = 3\mathbf{F},\ M(4) = 5\mathbf{F};$$

$$M(z_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ M(z_2) = \begin{pmatrix} x_1 & x_2 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

$$M(z_3) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, M(z_4) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M(z_5) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We denote by e_k the idempotent in Γ corresponding to the vertex k (the empty path starting and ending at this vertex), $P_k = e_k\Gamma$. Any arrow $z : k \to l$ defines a homomorphism $P_k \to P_l$, namely, the left multiplication by z. We denote this homomorphism by the same letter z. Set:

$$\begin{aligned} A =& (0,1) \otimes (P_3 \oplus P_1) \oplus (1,1) \otimes (P_1 \oplus P_2) \oplus \\ & \oplus (2,1) \otimes (P_3 \oplus P_4) \oplus (3,1) \otimes P_5, \\ B =& \mathbf{S} \otimes (P_3 \oplus P_1 \oplus P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5), \\ u \in& \mathsf{U}^\Gamma(B, A) \quad \text{given by the matrix:} \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 1 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \alpha z_1 & \alpha z_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha z_3 & 0 \\ 0 & 0 & 0 & 0 & 1 & \alpha z_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha z_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(we write γb for the tensor product $\gamma \otimes b$, where $\gamma : \mathbf{S} \to \mathbf{R}$, $b : P_k \to P_l$). It is obviously correct since $|u|$ coincides modulo radical with the natural embedding $\mathbf{S}^2 \otimes P \to \mathbf{R}^2 \otimes P$, where $P = P_3 \oplus P_1 \oplus P_1 \oplus P_2 \oplus P_3 \oplus P_4 \oplus P_5$ We shall prove that it is strict.

Let N be a Γ-module, $N_k = e_k N$, $n_k = \dim_{\mathbf{k}} N_k$ and Z_m be the matrices describing the multiplication by $z_m : k \to l$ as a linear mapping $N_k \to N_l$. Then

$$\begin{aligned} A \otimes_\Gamma N \simeq &(0,1)^{n_3+n_1} + (1,1)^{n_1+n_2} + (2,1)^{n_3+n_4} + (3,1)^{n_5}, \\ B \otimes_\Gamma N \simeq &\mathbf{S}^{n_3+n_1+n_1+n_2+n_3+n_4+n_5} \end{aligned}$$

and $u(N)$ can be identified with the matrix

$$\begin{pmatrix} I & 0 & 0 & 0 & \alpha I & 0 & 0 \\ 0 & I & \alpha I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & \alpha Z_1 & \alpha Z_2 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & \alpha Z_3 & 0 \\ 0 & 0 & 0 & 0 & I & \alpha Z_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & \alpha Z_5 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

Suppose that another Γ-module N' is given by the matrices Z'_m and $\varphi : u(N) \to u(N')$ is given by a pair of block matrices, $X = (X_{kl})$ and $Y = (Y_{kl})$, $k, l = 1, \dots, 7$: X defines a homomorphism $A \otimes_\Gamma N \to A \otimes_\Gamma N'$, Y defines a homomorphism $B \otimes_\Gamma N \to B \otimes_\Gamma N'$, so that $Xu(N) = u(N')Y$. Since $\mathsf{A}((n,1),(n',1)) = 0$ if $n > n'$, $X_{kl} = 0$ if $l > k+1$ or k is even and $l > k$. Moreover, if $k = l$ or $\{k, l\} \in \{\{1,2\}, \{3,4\}, \{5,6\}\}$, X_{kl} has entries from $\mathbf{k}$.

Decompose $\hat{X}_{kl} = X^0_{kl} + \alpha X'_{kl} + X''_{kl}$ and $Y_{kl} = Y^0_{kl} + Y'_{kl}$, where $X^0_{kl}, X'_{kl}, Y^0_{kl}$ are with entries from $\mathbf{k}$, X''_{kl} from W and Y_{kl} from J. Since $u(N) \equiv u(N') \equiv I \pmod{\alpha}$, $X^0_{kl} = Y^0_{kl}$. Consider the equalities of the following blocks from $Xu(N)$ and $u(N')Y$:

$$\begin{aligned}
(14): &\quad 0 = Y_{14} + \alpha Y_{54}; \\
(24): &\quad 0 = Y_{24} + \alpha Y_{34}; \\
(16): &\quad 0 = Y_{16} + \alpha Y_{56}; \\
(21): &\quad X^0_{21} = Y_{21} + \alpha Y_{31}; \\
(34): &\quad \alpha X^0_{33} Z_1 + X^0_{34} = Y_{34} + \alpha Z_1 Y_{44} + \alpha Z_2 Y_{54}; \\
(35): &\quad \alpha X_{31} + \alpha X^0_{33} Z_2 = Y_{35} + \alpha Z_1 Y_{45} + \alpha Z_2 Y_{55}; \\
(46): &\quad \alpha X^0_{44} Z_3 = Y_{46} + \alpha Z_3 Y_{66}; \\
(56): &\quad \alpha X_{54} Z_3 + \alpha X^0_{55} Z_4 + X^0_{56} =; \\
(67): &\quad \alpha X^0_{66} Z_5 = Y_{67} + \alpha Z_5 Y_{77}.
\end{aligned}$$

Remind that $\alpha W = \alpha J = 0$ and $\alpha \mathbf{R} \cap W = \alpha \mathbf{R} \cap J = \{0\}$. Therefore, the (14)-equation implies that $Y^0_{54} = X^0_{54} = 0$, the (24)-equation implies that $Y^0_{34} = X^0_{34} = 0$, the (16)-equation implies that $Y^0_{56} = X^0_{56} = 0$, and the (21)-equation implies that $Y^0_{31} = X^0_{31} = 0$. Now the other equations give:

$$\begin{aligned}
(34):&\quad X^0_{33}Z_1 = Z_1X^0_{44};\\
(35):&\quad X^0_{33}Z_2 = Z_2X^0_{55};\\
(46):&\quad X^0_{44}Z_3 = Z_3X^0_{66};\\
(56):&\quad X^0_{55}Z_4 = Z_4X^0_{66};\\
(67):&\quad X^0_{66}Z_5 = Z_5X^0_{77}.
\end{aligned}$$

It means that the set of matrices X^0_{kk} with $k = 3, 4, 5, 6, 7$ defines a Γ-homomorphism $\psi : N \to N'$. If φ is isomorphism, all these matrices must be invertible, hence, $N \simeq N'$; if φ is a non-trivial idempotent, so is ψ, hence, $u(N)$ and N decompose simultineously. Therefore, the element u is strict. Since the algebra Γ is wild, the same observation as in Proposition 1.5 shows that the bimodule U is correctly wild. □

For the rest of the section suppose $\mathbf{R}$ (and hence $\mathbf{S}$) to be semi-simple. In this case condition 2 from the definition of correct or semi-correct elements means that $|u|$ is a monomorphism. For each $i = 1, \dots, t$, let m_i be the number of such j that $(k, j) \in \mathbf{J}$ for some $k \in c_i$. On the other hand, for each $j = 1, \dots, s$, let l_j be the number of such k that $(k, j) \in \mathbf{J}$. Note that we always have $l_j > 1$. Otherwise, there is a unique index k such that $(k, j) \in \mathbf{J}$ with $\dim \mathbf{R}_k = 1$. But then $\mathbf{S}_j \subseteq \mathbf{R}_k$, hence, $\mathbf{S}_j = \mathbf{R}_k$, which contradicts condition 1 of the definition of special data. On the other hand, as $\mathbf{S}_j\mathbf{S}_{j'} = 0$ for $j \neq j'$, $(k, j) \in \mathbf{J}$ implies $(k, j') \notin \mathbf{J}$ for $j \neq j'$.

Step 6.2. *If $l_{j_0} > 2$ for some j_0, the shifting bimodule U is correctly (hence also semi-correctly) wild.*

For a rationally composed curve C this means that, whenever C is not wild, $\pi^{-1}(p)$ consists of two points for every singular point p. In other words, p is a simple node, so it accomplishes the PROOF *of* Proposition 2.5.

Proof. Suppose there are 3 indices k_1, k_2, k_3 such that $(k_q, j_0) \in \mathbf{J}$ for $q = 1, 2, 3$. Put $B = \mathbf{S}^4 \otimes \mathbf{F}$ and

$$A = \left(\bigoplus_{i=1}^{t} \oplus_{m=0}^{3} (i, n + m) \right) \otimes \mathbf{F}$$

for some (arbitrary) $n \in \mathbb{Z}$. Note that now $\operatorname{Hom}_{\mathbf{S}}(\mathbf{S}_j, \mathbf{R}_k) \simeq \mathbf{k}$ for $(k, j) \in \mathbf{S}$. Hence, the elements from $\mathsf{U}^{\mathbf{F}}(B, A)$ can be regarded as sets of 4×4 matrices $\{ u_{kj} \mid (k, j) \in \mathbf{J} \}$, u_{kj} having entries in $\mathbf{R}_k \otimes \mathbf{F}$.

Now take u such that all its components are identity matrices except for the next two:

$$u_{k_1 j_0} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$u_{k_2 j_0} = \begin{pmatrix} 1 & 1 & z_1 & z_2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Just as in Step 6.1, some easy straightforward calculation, which we omit, shows that u is indeed a strict element. □

Steps 1 and 2 above accomplish the proof of Proposition 2.5. Indeed, if the special bimodule $\mathsf{U}[\mathbf{R},\mathbf{S},\kappa]$ corresponds to a rationally composed curve C as described in Section 5, page 35, then $\mathbf{R}$ is semi-simple if and only if $\widetilde{\mathcal{O}}_q/\mathcal{J}_q = \mathbf{k}$ for every preimage $q \in \widetilde{C}$ of every singular point $p \in C$. Furthermore, the condition $l_j \le 2$ for all j holds if and only if every singular point $p \in C$ has at most 2 preimages in $\widetilde{C}$. But altogether it means that all singular points are indeed simple double points.

Suppose now that $l_j \le 2$ (hence $l_j = 2$) for each j. Then the pair $\mathbf{S} \subset \mathbf{R}$, together with the equivalence relation $\sim$, can be completely described by its *diagram* $\Delta = \Delta[\mathbf{R},\mathbf{S},\sim]$. The vertices of Δ are just the indices $i = 1,\dots,t$, its edges are the indices $j = 1,\dots,s$ and an edge j is incident to a vertex i if and only if $(k,j) \in \mathbf{J}$ for some $k \in c_i$. In the case where j is incident to a unique i (then, of course, $\dim \mathbf{R}_i = 2$), consider j as a *loop* at the vertex i.

Of course, given any graph Δ, non-oriented but possibly with loops and (or) multiple edges, we can restore some pair $\mathbf{S} \subset \mathbf{R}$ and an equivalence relation $\sim$ (with semi-simple $\mathbf{R}$ and all $l_j = 2$) such that Δ is just their diagram. So, to obtain a special data of this kind, we need only a graph and a function κ. Therefore, we call such data and the corresponding bimodule *graphical*.

As we have already noted, graphical data correspond to *line configurations* (cf. Section 2). Moreover, if a graphical bimodule U with the graph Δ corresponds to a line configuration C, then Δ coincides with the dual graph of C. So the next Step accomplishes the proof of Proposition 2.7 and hence of Theorem 2.8. Moreover, the description of all indecomposable elements in Cases *1* and 2 below also gives a description of indecomposable sheaves on the corresponding curves. One can easily see that this description coincides with Theorems 2.11 and 2.12.

Step 6.3. *Graphical data with the diagram Δ are:*

(1) *Finite (hence also semi-correctly and correctly finite) if Δ is a Dynkin diagram of type* A *, i.e., a chain.*
(2) *Tame (hence also semi-correctly and correctly tame) if Δ is an extended Dynkin diagram of type* $\widetilde{\mathrm{A}}$ *, i.e., a cycle (possibly, one vertex with one loop). Moreover, in this case they are correctly (hence, semi-correctly and "absolutely"[6]) unbounded.*
(3) *Correctly (hence also semi-correctly and "absolutely") wild otherwise.*

Note that in the first two cases $m_i \le 2$ for each $i = 1, 2, \dots, r$ (in particular, each equivalence class c_i consists of at most two elements). Hence, the graph Δ determines the bimodule U up to isomorphism (as any pair of points of a projective line can be moved to any other pair by a collineation).

Proof. Case *1* is very simple. Put $\mathbf{R}_{kj} = \mathbf{R}_k\mathbf{S}_j$ for $(k, j) \in \mathbf{J}$. All these spaces are one-dimensional, hence, we may identify them with $\mathbf{k}$. Moreover, $s = t - 1$ and we can arrange the indices in such a way that the edge i is incident to the vertices i and $i + 1$ for each $i = 1, \dots, t-1$. Then one checks immediately that the indecomposable elements of $\mathsf{El}_{sc}(\mathsf{U})$ are in one-to-one correspondence with the finite sequences $\mathbf{s}$ of integers of the form:

$$\mathbf{s} = (m, r; \delta_0, \delta_1; d_1, d_2, \dots, d_r)\,,$$

where $1 \le m \le t$, $0 \le r \le t - m + 1$ and both δ_0 and δ_1 are either 0 or 1 , while d_i is arbitrary; moreover, if $m = 1$, then $\delta_0 = 1$, if $r = t - m + 1$, then $\delta_1 = 0$, and if $r = 0$, then $\delta_0 = \delta_1 = 1$. Namely, the element $u = u(\mathbf{s})$ corresponding to such a sequence lies in $\mathsf{U}(B, A)$, where

$$\begin{aligned} A &= \oplus_{i=1}^{r}(m + i - 1, d_i)\,, \\ B &= \oplus_{i=\delta_0}^{r+\delta_1}\mathbf{S}_{m+i-1}\,, \end{aligned} \tag{6}$$

and all components of u are equal to 1 . Certainly, such an element equals $\sigma_m^{d_0}\sigma_{m+1}^{d_1}\dots\sigma_{m+r-1}^{d_{m+r-1}}u'$, where u' corresponds to the sequence $\mathbf{s}'$ with the same values of m, r, δ_0, δ_1 but with all $n_i' = 0$. So the bimodule U is finite. Note that the element $u(\mathbf{s})$ is semi-correct if and only if either $\delta_0 = 0$ or $m = 1$ and, moreover, either $\delta_1 = 1$ or $r = t - m + 1$. This element is correct if and only if $m = 1$ and $r = t$.

Case *2*. Here $s = t$ and we can arrange the indices in such a way that the edge i is incident to the vertices i and $i + 1$ (we define the vertex $t + k$ to be the same as the vertex k). This case fits into the framework of "bunches of chains" (cf. [6] or Appendix A).

[6] i.e., without any restrictions, cf. Definition 3.6, page 25

Namely, in our case the underlying index set $\mathbf{I}$ is just the set of pairs $\{ (i,i),(i,i+1) \,|\, i=1,2,\dots,t \}$. We set, for each pair $(i,j)\in\mathbf{I}$, $E_{ij}=\mathbb{Z}$ (with natural ordering) and $F_{ij}=\{ o_{ij} \}$ (a single element). To distinguish the elements from different E_{ij} we write them in the form $n(i,j)$ with $n\in\mathbb{Z}$, $(i,j)\in\mathbf{I}$. The equivalence relation on the union of all E_{ij} and F_{ij} is given by the rule: $n(i,j)\sim n(i,j')$ and $o_{ij}\sim o_{i'j}$ for all possible values of i,i',j,j'. Then one can verify that the bimodule corresponding to this bunch of chains coincides with the graphical bimodule corresponding to the graph Δ. Hence, we can use the results of [6] (cf. also Appendix B) to obtain a complete list of indecomposable elements. Taking into account the shape of this bunch, we can rearrange strings and bands defined in Appendix B as follows.

String representations are very similar to the representations of the previous case. They correspond to *string data*, i.e., sequences of integers:

$$\mathbf{s}=(m,r;\delta_0,\delta_1;d_1,d_2,\dots,d_r)\,,$$

where $1\le m\le t$, $r\ge 0$ and both δ_0 and δ_1 is either 0 or 1, while d_i are arbitrary. The corresponding element $u(\mathbf{s})$ lies in $\mathsf{U}(A,B)$, where A and B are defined by formulae (6) if we put $(nt+j,k)=(j,k)$ and $\mathbf{S}_{nt+j}=\mathbf{S}_j$ for each n. The nonzero components of u are only those belonging to $\mathsf{U}(\mathbf{S}_{m+i-1},(m+i-1,d_i))$, except for $i=1$ if $\delta_0=1$, and $\mathsf{U}(\mathbf{S}_{m+i},(m+i-1,d_i))$, except for $i=r$ if $\delta_1=0$. All these components are equal to 1. A string element is semi-correct if and only if $r>0$ and $\delta_0=0,\delta_1=1$. It is never correct.

Band representations correspond to *band data*, which are triples $\mathbf{b}=(\mathbf{d},n,\lambda)$, where n is a positive integer, $\lambda\in\mathbf{k}^*$ and $\mathbf{d}$ is a sequence of integers $(d_1,d_2,\dots,d_{tr})$ which is *t-aperiodic*, i.e., is not a multiple self-concatenation of a shorter sequence whose length is also divisible by t.

The corresponding element $u_{\mathbf{b}}=u(\mathbf{d},n,\lambda)$ lies in $\mathsf{U}(A,B)$ for

$$A=\oplus_{i=1}^{tr}n(i,d_i)\,,$$
$$B=\oplus_{i=1}^{tr}n\mathbf{S}_i\,,$$

Its nonzero components are those belonging to $\mathsf{U}(\mathbf{S}_i,(i,d_i))$ and $\mathsf{U}(\mathbf{S}_{i+1},(i,d_i))$, which are given by unit matrices of dimension n, and the component belonging to $\mathsf{U}(\mathbf{S}_1,(tr,d_{tr}))$, which is given by the Jordan cell of dimension n with the eigenvalue λ. All band elements are correct.

All string elements are pairwise non-isomorphic. The isomorphic band elements are those corresponding to the triples $(\mathbf{d},n,\lambda)$ and $(\mathbf{d}',n,\lambda)$, where $\mathbf{d}'$ is a *t-cyclic* permutation of $\mathbf{d}$, i.e., $\mathbf{d}'=(d_{tl+1},d_{tl+2},\dots,d_{tl})$.

As string and band elements exhaust all indecomposable elements of U, this bimodule is tame. Moreover, the band elements $u(\mathbf{d},1,T)\in$

$\mathsf{El}(\mathsf{U}, \mathbf{k}[T, T^{-1}])$ form a parametrising family of elements of U. But their number grows with r. For instance, consider the band elements corresponding to sequences $\mathbf{d}_{r,l}$ of length rt having all components 0 except the first and the $(tl+1)$-st ones, which are equal to 1. If $l \le [\frac{r}{2}]$ these band elements are pairwise non-isomorphic even up to shift and there are $[\frac{r}{2}]$ of them. Hence, U is correctly unbounded.

Example 6.4. We present here the explicit view of string and band elements for the case $s = 1$, which corresponds to the irreducible rational curve with one simple node, see Example 5.4. We use the notations of matrices X_n, Y_n, X, Y from the latter example (page 35). To simplify the shape, it is convenient to make common permutations of rows of the matrices X, Y, even if they belong to different blocks X_n, Y_n. A *semi-correct string element* is completely defined by a sequence $(d_1, d_2, \dots, d_r)$, since always $m = 1$ and $\delta_0 = \delta_1 = 1$ for a semi-correct string. Then the corresponding matrices are the following:

$$X = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

where the ith row of X or Y belongs to X_{d_i} or Y_{d_i}, respectively.

The matrix X corresponding to a *band* $(\mathbf{d}, n, \lambda)$ is the identity matrix of size $nr \times nr$ and the matrix Y is the following:

$$Y = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & J \\ I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & I & 0 \end{pmatrix},$$

where I is the identity matrix of size $n \times n$ and J is the Jordan matrix of the same size with eigenvalue λ. The rows of the ith block of Y and the corresponding rows of X belong to Y_{d_i} and X_{d_i}, respectively.

Case *3*. Suppose that Δ is neither a chain nor a cycle. Then it contains a vertex i_0 incident either to at least three edges or to a loop and to at least one more edge. We consider the former situation (the latter can be treated in the same way). Let j_k ($k = 1, 2, 3$) be three edges incident to the vertex i_0. Denote the second end of the edge j_k by i_k (some of i_k may be equal). We put again $B = \mathbf{S} \otimes \mathbf{F}$,

$$A = \left(\bigoplus_{i=1}^{t} \oplus_{k=0}^{3} (i, n+k) \right) \otimes \mathbf{F}$$

for some (arbitrary) n and take u, all of whose components are identity matrices except for the next three:

$$u_{i_0j_1} = u_{i_0j_2} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$u_{i_0j_3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & z_1 \\ 1 & 1 & 0 & z_2 \end{pmatrix}.$$

Again some easy straightforward calculation, which we omit, shows that u is a strict element. □

Appendix A. Some problems

A.1. **Classification of coherent sheaves.** The question naturally arises, whether it is possible to classify *all coherent sheaves* on VB–finite and VB–tame curves, including both sky-scrapers and mixed sheaves (i.e., non-trivial extensions of sky-scrapers with torsion-free ones). It seems possible, as all singularities are just simple nodes, thus the classification of sky-scrapers can be obtained from [23, 32]. It is clear that no curve can be of finite type with respect to the classification of all coherent sheaves, since support of a sky-scraper sheaf can vary along the curve, which gives rise to a one-dimensional family of non-isomorphic sheaves.

After this article was prepared for publication, I. Burban obtained a description of the *derived categories* of coherent sheaves on line configurations of types A and $\widetilde{A}$. It includes, in particular, information on coherent sheaves and on the question mentioned above.

A.2. **Simple and stable vector bundles.** Another question is to distinguish *simple vector bundles* $\mathcal{B}$, i.e., such that $\operatorname{End}\mathcal{B} = \mathbf{k}$. It is important, for instance, as such bundles provide new solutions of the Yang–Baxter equations (cf. [34]). For VB–finite curves, all indecomposable vector bundles are obviously simple. On the other hand, for VB–tame singular curves, only vector bundles $\mathcal{B}_{(\mathbf{d},1,\lambda)}$ can be (although not all of them are) simple: if $n > 1$, the vector bundle $\mathcal{B}_{(\mathbf{d},n,\lambda)}$ has non-trivial endomorphisms corresponding to matrices commuting with the Jordan cell.

If a vector bundle $\mathcal{B}$ is *stable*, it must be simple. Indeed, otherwise there is a nonzero, non-invertible endomorphism α of $\mathcal{B}$. Then $\mathcal{B}' = \operatorname{Im}\alpha$ is both a subsheaf and a factor-sheaf of $\mathcal{B}$, hence, both inequalities $\operatorname{slope}\mathcal{B}' \geq \operatorname{slope}\mathcal{B}$ and $\operatorname{slope}\mathcal{B}' < \operatorname{slope}\mathcal{B}$ are impossible if $\mathcal{B}$ is stable. Here $\operatorname{slope}\mathcal{B} = \dfrac{\deg\mathcal{B}}{\operatorname{rk}\mathcal{B}}$ (cf. [30, 38]). We do not know whether all simple vector bundles are stable for VB–tame curves, nor which vector bundles are stable. Some recent results on this problem can be found in [39].

For $n > 1$ vector bundles $\mathcal{B}_{(\mathbf{d},n,\lambda)}$ are multiple extensions of $\mathcal{B}_{(\mathbf{d},1,\lambda)}$. Therefore, they are *semi-stable* if and only if the latter is stable. As was mentioned before, $\mathcal{B}_{(\mathbf{d},n,\lambda)}$ is never stable if $n > 1$.

A.3. **Semi-continuity.** Let C be a projective curve, r a positive integer and $\mathbf{d}$ some vector-degree of vector bundles over C (cf. Definition 1.2). Consider a family of vector bundles $\mathcal{M} \in \mathsf{VB}(C,\Lambda)$ with $\operatorname{Deg}\mathcal{M} = \mathbf{d}$ and $\operatorname{rk}\mathcal{M} = r$, where Λ is the coordinate ring of an affine variety V. Denote by $\mathsf{VB}(C,r,\mathbf{d})$ the set of all such families (for all possible V). For any point $v \in V$, $\mathcal{M}(v) = \mathcal{M} \otimes_\Lambda \mathbf{k}(v)$ is a vector bundle of rank r and vector-degree $\mathbf{d}$ over C. Set, for $v \in V$,

$$V(v,\mathcal{M}) = \{\, w \in V \mid \mathcal{M}(w) \simeq \mathcal{M}(v) \,\}, \qquad \text{where } v \in V.$$

It can be shown that $V(v,\mathcal{M})$ is a constructible set, hence, the following definitions make sense:

$$V_k(\mathcal{M}) = \{\, v \in V \mid \dim V(v,\mathcal{M}) = k \,\};$$
$$\operatorname{par}(\mathcal{M}) = \max \{\, \dim V_k(\mathcal{M}) - k \mid k \geq 0 \,\};$$

and

$$\operatorname{par}(C,r,\mathbf{d}) = \max \{\, \operatorname{par}(\mathcal{M}) \mid \mathcal{M} \in \mathsf{VB}(C,r,\mathbf{d}) \,\}.$$

The latter number, the *number of parameters*, can be considered as the maximal dimension of a family of non-isomorphic vector bundles of rank r and vector-degree $\mathbf{d}$ over the given curve C.

The question arises, how the number of parameters varies if we deform the curve.

Conjecture. *Suppose that $\mathcal{C}$ is a* family of projective curves *with base X. Then the function $x \to \operatorname{par}(\mathcal{C}(x),r,\mathbf{d})$ is* semi-continuous *on X, i.e., all sets $\{\, x \in X \mid \operatorname{par}(\mathcal{C}(x),r,\mathbf{d}) \geq k \,\}$ are closed in X.*

Recall that such a semi-continuity was proved in [29] for Cohen–Macaulay modules over curve singularities, in [17, 22] for representations of finite dimensional algebras and in [17] for Cohen–Macaulay modules over non-commutative Cohen–Macaulay algebras of Krull dimension 1. The solution of this problem for vector bundles over projective curves is of interest, for instance, for investigation of Cohen–Macaulay modules over surface singularities, in view of [28, 18].

A.4. **Relation to finite dimensional algebras.** There exists an amazing correspondence between rationally composed curves and some finite dimensional algebras. Suppose C is a rationally composed curve such that the algebra $\widetilde{\mathcal{F}}$ is semi-simple, i.e., all branches at every singular point have different tangents. Then C can be completely described by its normalization $\widetilde{C}$, the set S of its singular points (just a finite set), its pre-image $\widetilde{S} \subset \widetilde{C}$ and the projection $\pi : \widetilde{S} \to S$. Define the corresponding algebra $\mathbf{A} = \mathbf{A}(C)$ by its diagram $\Gamma = \Gamma(C)$ (quiver) and relations as follows.

Let $\widetilde{C} = \cup_{i=1}^{t} C_i$, where $C_i \simeq \mathbb{P}^1$, $S = \{ p_1, p_2, \dots, p_s \}$ and $\pi^{-1}(p_j) \cap C_i = \{ q_{ijk} \mid 1 \le k \le m_{ij} \}$ for each j. Then Γ has $2t + s$ vertices $\{ a_i, b_i, c_j \mid 1 \le i \le t,\ 1 \le j \le s \}$. There are two arrows $x_i, y_i : a_i \to b_i$ and m_{ij} arrows $z_{ijk} : c_j \to a_i$. The defining relations for these arrows are:

$$\xi_{ijk} x_i z_{ijk} = \eta_{ijk} y_i z_{ijk}, \text{ where } q_{ijk} = (\xi_{ijk} : \eta_{ijk}).$$

Then the following theorem holds.

Theorem A.1. *The algebra $\mathbf{A}(C)$ is tame (wild) if and only if the curve C is VB-tame (VB-wild).*

The proof is quite easy and straightforward, so we only sketch it. For each $i = 1, 2, \dots, t$ the sub-algebra generated by x_i, y_i is a Kronecker algebra, for which all representations are known. So we can reduce all these arrows and then get a bimodule problem for the remainder. The observation is that this bimodule problem "almost coincides" to that corresponding to the curve C, as in Section 5. At least, it is not simpler, which implies the "only if" part of the theorem. But if C is not VB-wild, the points q_{ijk} can always be chosen as $(1 : 0)$ or $(0 : 1)$. In this case the resulting algebra $\mathbf{A}$ is a *string algebra*, so its representations are described in [40, 7].

Indeed, in the tame case the algebra $\mathbf{A}$ is a *gentle algebra* in the sense of [1]. So, its *derived category* of modules is also tame as it has been shown in [36, 5]. It is very plausible that there is some relation between this derived category and the derived category of coherent sheaves over the curve C, although these derived categories cannot be equivalent, since the global dimension of the category $\mathbf{A}\text{-}\mathsf{mod}$ is finite, while that of $\mathrm{Coh}(C)$ is infinite. Perhaps further investigation of them could give more a intrinsic explanation of Theorem A.1, which does not involve explicit calculations.

Appendix B. Bunches of chains

Here we recall some definitions and results related to the *bunches of chains* considered by Bondarenko in [6]. We rearrange the definitions to make them more convenient for our purpose and consider only the case of *chains* (not semi-chains) as we need only this one and it is technically much easier. Note that almost the same class of matrix problems was considered in [9] as "representations of clans," though both the encoding of the problem and the form of the answer from [6] is more convenient for our purpose.

Definition B.1. A *bunch of chains* $\mathbf{C} = \{ \mathbf{I}, E_i, F_i, \sim \}$ is defined by the following data:

(1) A set $\mathbf{I}$ of indices.

(2) Two chains (i.e., linear ordered sets) E_i and F_i given for each $i \in \mathbf{I}$.
Put $\mathbf{E} = \bigcup_{i\in\mathbf{I}} E_i$, $\mathbf{F} = \bigcup_{i\in\mathbf{I}} F_i$ and $|\mathbf{C}| = \mathbf{E} \cup \mathbf{F}$.
(3) An equivalence relation $\sim$ on $|\mathbf{C}|$ such that each equivalence class consists of at most 2 elements.

We also write $a - b$ if $a \in E_i$, $b \in F_i$ or vice versa (with the same index i). Moreover, we consider the ordering on $|\mathbf{C}|$, which is just the union of all orderings on E_i and F_i (i.e., $a < b$ means that a, b belong to the same chain E_i or F_i and $a < b$ in this chain).

If a bunch of chains $\mathbf{C} = \{\mathbf{I}, E_i, F_i, \sim\}$ is given, define the corresponding category $\mathsf{A} = \mathsf{A}(\mathbf{C})$ and the corresponding A-bimodule $\mathsf{U} = \mathsf{U}(\mathbf{C})$ as follows:

- The objects of A are the equivalence classes of $|\mathbf{C}|$ with respect to $\sim$.
- If x, y are two such equivalence classes, a basis of the morphism space $\mathsf{A}(x, y)$ consists of elements p_{ab} with $b \in x$, $a \in y$, $b < a$ and, if $x = y$, the identity morphism 1_x.
- The multiplication is given by the rule: $p_{ab}p_{bc} = p_{ac}$ if $c < b < a$, while all other possible products are zeros.
- A basis of $\mathsf{U}(x, y)$ consists of elements u_{ab} with $a \in y \cap \mathbf{E}$, $b \in x \cap \mathbf{F}$.
- The action of A on U is given by the rule: $p_{ca}u_{ab} = u_{cb}$ if $a < c$; $u_{ab}p_{bd} = u_{ad}$ if $d < b$, while all other possible products are zeros.

The category of representations of the bunch $\mathbf{C}$ is then defined as the category $\mathsf{El}(\mathsf{U})$ of the elements of this bimodule. In other words, a representation is a set of block matrices

$$M_i = \begin{pmatrix} \dots\dots\dots\dots \\ \dots \quad M_{ab} \quad \dots \\ \dots\dots\dots\dots \end{pmatrix}, \; i \in \mathbf{I},\; a \in E_i,\; b \in F_i,\; M_{ab} \in \mathrm{Mat}(n_a \times n_b, \mathbf{k}).$$

such that $x \sim y$ implies $n_x = n_y$. Two representations are isomorphic if and only if they can be obtained from one another by a sequence of the following *elementary transformations*:

- elementary transformations of rows (columns) in each horizontal (vertical) stripe; it means that they are performed simultineously in all matrices M_{ab} with fixed a (b); moreover, if $x \sim y$, the transformations of the x-stripe must be the same as those of y-stripe (if one of them is horizontal and the other is vertical, then "the same" certainly means "contragredient");
- if $x < y$, then scalar multiples of rows (columns) of the x-stripe can be added to rows (columns) of the y-stripe.

For instance, the matrix problem arising from the rational curve with one simple double point (Example 5.4) coincides with the representations of the following bunch of chains:

$\mathbf{I} = \{1, 2\}$,
$E_1 = \{ a_n \mid n \in \mathbf{Z} \}$, $E_2 = \{ b_n \mid n \in \mathbf{Z} \}$,
the order in both E_i coincides with the natural order for indices,
$F_1 = \{1\}$, $F_2 = \{2\}$,
$a_i \sim b_i$ for all $i \in \mathbf{Z}$, $\quad 1 \sim 2$.

In particular, this definition coincide with that from [6]. Note that in [6] a more general situation was investigated, but we need only this case, which is essentially simpler than the general one. The following result is the specialization of the description of the representations given in [6] to our case, although it can be obtained directly using the same recursive procedure. First define some combinatorial objects called "strings" and "bands."

Definitions B.2. Let $\mathbf{C} = \{ \mathbf{I}, E_i, F_i, \sim \}$ be a bunch of chains.

(1) A $\mathbf{C}$-*word* is a sequence $w = a_0 r_1 a_1 r_2 a_2 \dots r_m a_m$, where $a_k \in |\mathbf{C}|$ and each r_k is either $\sim$ or $-$, such that, for all possible values of k:
  (a) $a_{k-1} r_k a_k$ in $|\mathbf{C}|$.
  (b) $a_k \neq a_{k+1}$ and $r_k \neq r_{k+1}$.
  Possibly $m = 0$, i.e., $w = a$ for some $a \in \mathbf{C}$.
(2) If $a_m = a_0$, $r_1 = \sim$ and $r_m = -$ call the word w a $\mathbf{C}$-*cycle.* Note that in this case m is always even.
(3) Call a $\mathbf{C}$-word *full* if, whenever a_0 is not a unique element in its equivalence class, then $r_1 = \sim$ and whenever a_m is not a unique element in its equivalence class, then $r_m = \sim$.
(4) Call a $\mathbf{C}$-cycle $w = a_0 r_1 a_1 r_2 a_2 \dots r_m a_m$ *aperiodic* if the sequence $a_0 r_1 a_1 r_2 a_2 \dots r_m$ cannot be written as a multiple self-concatenation $vv \dots v$ of a shorter sequence v.
(5) We say that an equivalence class x *occurs* in a word w if w contains a sub-word a in case $x = \{a\}$ is a singleton, or either a sub-word $a \sim b$ or a sub-word $b \sim a$ in case $x = \{a, b\}$ with $a \neq b$. In the former case we say that this occurrence corresponds to the occurrence of a, while in the latter case we say that it corresponds to both the occurrence of a and to the occurrence of b. Denote by $\nu(x, w)$ the number of occurrences of x in w.

Definition B.3. For a $\mathbf{C}$-word $w = a_0 r_1 a_1 r_2 a_2 \dots r_m a_m$ call its $\sim$-*sub-word* any sub-word of the form $v = a \sim b$ as well as that of the form $v = a$, where $a \in \mathbf{C}$ is unique in its equivalence class. In the latter case put $|v| = \{a\}$, while in the former case put $|v| = \{a, b\}$. Denote by $[w]$ the collection of all $\sim$-sub-words of w.

Note that if w is a cycle it contains no entries $a \in \mathbf{C}$ such that a is unique in its equivalence class.

Definition B.4. For any full $\mathbf{C}$-word $w = a_0 r_1 a_1 r_2 a_2 \dots r_m a_m$ define the corresponding *string representation* $u = u_s(w)$ of the bunch $\mathbf{C}$ as follows.

(1) $u \in \mathsf{U}(A, A)$, where $A = \bigoplus_{v \in [w]} |v|$.
(2) Suppose there is a sub-word $v_1 - v_2$ in w with $v_i \in [w]$. Let a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand $\mathsf{U}(|v_1|, |v_2|) \oplus \mathsf{U}(|v_2|, |v_1|)$ and we define the corresponding components of u to be $(0, u_{ab})$ if $a \in \mathbf{E}$ and $(u_{ba}, 0)$ if $a \in \mathbf{F}$.
(3) All other components of u are defined to be zero.

Definition B.5. For any triple (w, d, λ), where w is an aperiodic $\mathbf{C}$-cycle, d is a positive integer and $\lambda \in \mathbf{k}^* = \mathbf{k} \setminus \{0\}$, define the corresponding *band representation* $u = u_b(w, d, \lambda)$ of the bunch $\mathbf{C}$ as follows.

(1) $u \in \mathsf{U}(A, A)$, where $A = \bigoplus_{v \in [w]} d|v|$.
(2) Suppose there is a sub-word $v_1 - v_2$ in w with $v_i \in [w]$. Let a be the right end of the word v_1 and b be the left end of the word v_2. Then $\mathsf{U}(A, A)$ has a direct summand

$$\mathsf{U}(d|v_1|, d|v_2|) \oplus \mathsf{U}(d|v_2|, d|v_1|) \simeq \\ \operatorname{Mat}(d \times d, \mathsf{U}(|v_1|, |v_2|)) \oplus \operatorname{Mat}(d \times d, \mathsf{U}(|v_2|, |v_1|))$$

and we define the corresponding components of u to be $(0, u_{ab} I)$ if $a \in \mathbf{E}$ and $(u_{ba} I, 0)$ if $a \in \mathbf{F}$, where I denotes the identity matrix.
(3) Now let v_1 be the last and v_2 be the first $\sim$-sub-word in w (they may coincide), a be the right end of the word v_1 and b be the left end of the word v_2. Again $\mathsf{U}(A, A)$ has a direct summand

$$\mathsf{U}(d|v_1|, d|v_2|) \oplus \mathsf{U}(d|v_2|, d|v_1|) \simeq \\ \operatorname{Mat}(d \times d, \mathsf{U}(|v_1|, |v_2|)) \oplus \operatorname{Mat}(d \times d, \mathsf{U}(|v_2|, |v_1|))$$

and we define the corresponding components of u to be $(0, u_{ab} J)$ if $a \in \mathbf{E}$ and $(u_{ba} J, 0)$ if $a \in \mathbf{F}$, where J denotes the Jordan cell of dimension d with the eigenvalue λ.
(4) All other components of u are defined to be zero.

Now the main result of [6] is the following.

Theorem B.6. (1) *All representations $u_s(w)$ and $u_b(w, d, \lambda)$ defined above are indecomposable and each indecomposable representation of $\mathbf{C}$ is isomorphic to one of these representations.*
(2) *The only possible isomorphisms between these representations are the following:*

(a) $u_s(w) \simeq u_s(w')$ *if* $w = a_0 r_1 a_1 \dots r_m a_m$ *and* $w' = a_m r_m a_{m-1} \dots r_1 a_0$ *, the reversed word.*

(b) $u_b(w,d,\lambda) \simeq u_b(w',d,\lambda')$ *if* $w = a_0 r_1 a_1 \dots r_m a_m$ *,* $w' = a_{2k}\ r_{2k+1} a_{2k+2} \dots r_{2k} a_{2k}$ *is a cyclic permutation of* w*, and* $\lambda' = \lambda$ *for* k *even, while for* k *odd* $\lambda' = \lambda^{-1}$ *.*

(c) $u_b(w,d,\lambda) \simeq u_b(w',d,\lambda')$ *if* $w = a_0 r_1 a_1 \dots r_m a_m$ *,* $w' = a_{2k+1}\ r_{2k+1} a_{2k} \dots r_{2k+2} a_{2k+1}$ *is a cyclic permutation of the reversed word, and* $\lambda' = \lambda$ *for* k *odd, while for* k *even* $\lambda' = \lambda^{-1}$ *.*

Corollary B.7. *For any bunch of chains* $\mathbf{C}$ *the bimodule* $\mathsf{U}(\mathbf{C})$ *is tame (finite if there are no* $\mathbf{C}$*-bands at all). Moreover, a parametrising set for its elements consists of all band representations* $u_b(w,1,T) \in \mathsf{El}(\mathsf{U}, \mathbf{k}[T,T^{-1}])$ *.*

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Department of Mechanics and Mathematics, Kyiv Taras Shevchenko University, 01033 Kyiv, Ukraine
E-mail address: unialg@ln.com.ua

Fachbereich Mathematik, Universität Kaiserslautern, 67663 Kaiserslautern, Deutschland
E-mail address: greuel@mathematik.uni-kl.de