COVERINGS OF TAME BOXES AND ALGEBRAS

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Abstract. We consider Galois coverings of algebras with torsion free Galois groups and prove that such a covering is tame if and only if so is the original algebra; moreover, the indecomposable representations of the latter consist of the images of those of the covering and of several 1-parametre rational families obtained from some infinite dimensional representations of the covering. The proof uses the techniques of "matrix problems" (representations of boxes), in particular, the reduction algorithms for such representations.

Introduction

Coverings of algebras were first considered by Riedtmann [22] and proved their efficiency in plenty of questions of the representation theory (cf., e.g., [5, 15, 4] and others). In particular, they played a crucial role in the criterion for an algebra to be representation-finite (cf. [4]), as it had been known that an algebra is representation-finite if and only if so is its covering [15]. Certainly, one would like to apply coverings in the representation-infinite case too. It is well known that among the representation-infinite algebras there are two quite different types: tame and wild (cf. [10, 11, 7]), and the question naturally arises whether an algebra is tame if and only if so is its covering. Unfortunately, rather simple examples shows that it is not always the case (cf. [17]). Nevertheless, there was some evidence that this claim would remain valid under some conditions imposed on the covering; for instance, if one supposed that the Galois group of the covering were torsion free (for partial results in this direction cf. [8, 9]).

Our article is devoted to the proof of this conjecture both for finite dimensional algebras and for a wide class of "matrix problems" (representations of boxes or bimodules). Certainly, dealing with coverings, one should consider not only algebras but also the so called *locally bound categories*; they are, in some sense, "finite dimensional algebras with infinitely many objects." The main tool is, just as in [11, 7], a reduction algorithm allowing to "simplify" representations and to prove some basic results by induction. As it often happens, to construct such an algorithm we had to enlarge the considered class of boxes ("quasi-triangular boxes," cf. Section 8), though we have no idea whether the new class could be useful anywhere but in inductive proofs.

Moreover, this procedure gives a rather complete description of the relations between the representation categories of an algebra (or a box) and of its Galois covering. Remind that in the representation-finite case all representations of the covered algebra are "direct images" of those of the covering (cf. [5]). In the tame case it is seldom so; maybe, the unique exception is the case of *locally support finite* coverings (cf. [8]). Usually, there are some infinite dimensional representations of the covering which produce 1-parametre families of representations of the covered algebra. So, one has at least two sorts of representations: those obtained from finite dimensional representations of the covering and those belonging to the abovementioned families (we denote them $ind_0\,\Lambda$ and $ind_1\,\Lambda$). The main theorem can be stated as follows (cf. Theorem 9.1 and Corollary 9.7 for more details)

Let $\Pi : \widetilde{\Lambda} \to \Lambda$ be a Galois covering of locally finite dimensional categories (over an algebraically closed field) with a torsion free Galois group G. Then:

(1) *Λ is tame if and only if so is $\widetilde{\Lambda}$.*
(2) *If Λ and $\widetilde{\Lambda}$ are tame, then:*
   - *Every indecomposable representation of Λ belongs either to $ind_0\,\Lambda$ or to $ind_1\,\Lambda$.*
   - *$ind\,\widetilde{\Lambda}$ is a Galois covering of $ind_0\,\Lambda$ with the same Galois group G.*
   - *The Auslander-Reiten quiver of Λ is a disjoint union of that of $ind_0\,\Lambda$ (which is the orbit quiver of G on the Auslander-Reiten quiver of $\widetilde{\Lambda}$) and that of $ind_1\,\Lambda$ (which is a disjoint union of homogeneous tubes).*

(Note that the proof of this theorem announced in [13] was incomplete.) One can conjecture that the first claim of this theorem also remains valid when G does not contain elements of the order equal to the characteristic of the ground field, though the relations between indecomposable modules become more sofisticated in this case.

1. *Categories*

Through the whole paper we suppose that all our categories and algebras are categories and algebras over some fixed field $\mathbf{k}$ and all functors and homomorphisms are $\mathbf{k}$-linear. We shall write Hom , $\otimes$, dim, etc.

instead of $\mathrm{Hom}_{\mathbf{k}}$, $\otimes_{\mathbf{k}}$, $\dim_{\mathbf{k}}$, etc. For any subset $\mathsf{S} \subseteq \mathrm{Mor}\,\mathcal{A}$ and any two objects $x, y \in \mathrm{Ob}\,\mathcal{A}$ denote $\mathsf{S}(x, y) = \mathsf{S} \cap \mathcal{A}(x, y)$.

Remind that a category $\mathcal{A}$ is said to be *fully additive* [11] if it is additive and any idempotent in $\mathcal{A}$ splits (i.e., corresponds to a direct decomposition). On the other hand, call a category $\mathcal{A}$ *basic* if it satisfies the following conditions:

- all its objects are pairwise non-isomorphic;
- for each object x there are no non-trivial idempotents in $A(x, x)$.

A full subcategory $\mathcal{S} \subseteq \mathcal{A}$ is called a *skeleton* of $\mathcal{A}$ if it is basic and each object $x \in \mathcal{A}$ is isomorphic to a direct summand of a (finite) direct sum of some objects of $\mathcal{S}$.

A fully additive category *add* $\mathcal{A}$ is called a *fully additive hull* of the category $\mathcal{A}$ if $\mathcal{A}$ is a full subcategory of *add* $\mathcal{A}$ and each object of *add* $\mathcal{A}$ is isomorphic to a direct summand of a direct sum of objects of $\mathcal{A}$. It is known (and quite obvious) that each category $\mathcal{A}$ has a fully additive hull. Moreover, all such hulls are equivalent and for each functor $F : \mathcal{A} \to \mathcal{C}$, where the category $\mathcal{C}$ is fully additive, there exists a unique (up to isomorphism) functor *add* $\mathcal{A} \to \mathcal{C}$ extending F. We denote this extension by F too. We also write $\mathcal{A}(x, y)$ instead of $(\mathit{add}\,\mathcal{A})(x, y)$ for objects x, y from *add* $\mathcal{A}$.

Call a category $\mathcal{A}$ *rigid* if *add* $\mathcal{A}$ is a *unique direct decomposition* category, i.e., any object of it can be decomposed into a direct sum of indecomposable ones and if

$$x_1 \oplus x_2 \oplus \ldots \oplus x_n \simeq y_1 \oplus y_2 \oplus \ldots \oplus y_m$$

with indecomposable objects x_i and y_j then $n = m$ and $x_i \simeq y_i$ (up to a permutation of indices). It is evident that if A is a rigid category then *add* A has a skeleton and the last one is unique up to isomorphism; thus, we may (and will) denote it by $\mathcal{S}k\,\mathcal{A}$. Obviously, $\mathcal{S}k\,\mathcal{A}$ is also rigid. Call the objects of a skeleton of a rigid category $\mathcal{A}$ its *vertices* and denote their set by $\mathrm{Ver}\,\mathcal{A}$.

There are two important examples of rigid basic categories. The first one is that of *local categories*. Namely, a category $\mathcal{A}$ is said to be *local* if its objects are pairwise non-isomorphic and for each object $x \in \mathrm{Ob}\,\mathcal{A}$ its endomorphism algebra $\mathcal{A}(x, x)$ is local. The well-known Adzumaya-Krull-Schmidt theorem implies that any local category is rigid (cf. e.g. [2]). Suppose now that $\mathcal{A}$ is *locally finite dimensional*, i.e., $\mathcal{A}(x, y)$ is finite dimensional vector space for each x, y. Then, in particular, $\mathcal{A}(x, x)$ is a finite dimensional algebra. Hence, in *add* $\mathcal{A}$ the object x splits into a direct sum of objects with local endomorphism algebras.

Thus, $\mathit{add}\,\mathcal{A}$ has a local skeleton $\mathcal{S}k\,\mathcal{A}$ and $\mathcal{A}$ is rigid. In particular, any finite dimensional algebra is a rigid category.

For a local category $\mathcal{A}$ define its *radical* $\operatorname{rad}\mathcal{A}$ as the set of all non-invertible morphisms. It is an ideal in $\mathcal{A}$. Denote by $\operatorname{rad}^\infty\mathcal{A}$ the intersection $\bigcap_{k=1}^\infty(\operatorname{rad}\mathcal{A})^k$. If a category $\mathcal{A}$ has a local skeleton one can also define its radical: by definition, $a : x \to y$ belongs to $\operatorname{rad}(x,y)$ if and only if its components with respect to some (hence, to any) decompositions $x \simeq \oplus_i x_i$, $y \simeq \oplus_j y_j$, where $x_i, y_j \in \mathcal{S}k\,\mathcal{A}$, belong to $\operatorname{rad}(x_i, y_j)$.

Another example of rigid categories is that of *free categories.* Let Γ be a graph (in the sense of [19], i.e., oriented and, maybe, with loops and multiple arrows). Define the *free category* $\mathbf{k}\Gamma$ *generated by* Γ in the following way. Its object set coincides with the set $\operatorname{Ver}\Gamma$ of vertices of Γ. The morphisms $f : x \to y$ are defined as formal (finite) sums

$$\sum_i \lambda_i p_i$$

where $\lambda_i \in \mathbf{k}$ and p_i are some *paths* starting at x and ending at y. Remind that such a path is a formal product $a_1a_2\dots a_l$, where $a_k : x_k \to x_{k-1}$ are some arrows of Γ and $x_l = x$, $x_0 = y$. The number l is called *the length* of this path. If $x = y$, it is allowed that $l = 0$ (the *empty path* at the vertex x). To define the products of morphisms, one has only to do it for paths. But in this case it can be defined as their concatenation.

Note that a free category in this sense is linear. We shall not consider free non-linear categories defined, e.g., in [19]. Surely, if Γ has only 1 vertex, the free category $\mathbf{k}\Gamma$ coincides with the free algebra generated by the set of arrows of Γ. It is known that any free category is rigid (and, of course, basic), cf., e.g., [24, 14], or [2, 6] for the case of free algebras. Moreover, the free categories generated by two graphs are isomorphic if and only if these graphs are isomorphic. A category of the form $\mathit{add}\,\mathbf{k}\Gamma$ will be called a *free additive category* (remark that the unique decomposition implies that it is indeed the least additive category containing $\mathbf{k}\Gamma$). The set of arrows of the graph Γ is often called *the set of free generators* for $\mathbf{k}\Gamma$ (or for $\mathit{add}\,\mathbf{k}\Gamma$).

A basic category $\mathcal{B}$ is said to be *trivial* if $\mathcal{B}(x,x) = \mathbf{k}$ for any object $x \in \operatorname{Ob}\mathcal{B}$ and $\mathcal{B}(x,y) = 0$ for any two different objects $x \neq y$. A category $\mathcal{A}$ is said to be *trivial* if it possesses a trivial skeleton. Surely, any trivial category $\mathcal{A}$ is rigid and its additive hull $\mathit{add}\,\mathcal{A}$ is also trivial. For any basic category $\mathcal{B}$ one can define its *trivial part* as the basic trivial category $\mathcal{B}^\emptyset$ having the same objects as $\mathcal{B}$. If $\mathcal{A}$ is any rigid category with a skeleton $\mathcal{A}_0$, fix for every $x \in \operatorname{Ob}\mathcal{A}$ an isomorphism

$\phi_x : x \xrightarrow{\sim} \bigoplus_k x_k$, where $x_k \in \operatorname{Ob}\mathcal{A}_0$, and define the *trivial part* of $\mathcal{A}$ to be the trivial category $\mathcal{A}^\emptyset$ with the same objects as $\mathcal{A}$, with the skeleton $\mathcal{A}_0^\emptyset$ and such that all isomorphisms ϕ_x belong to $\mathcal{A}^\emptyset$ too. As $\mathcal{A}$ is rigid, its trivial part does not depend (up to isomorphism) on the choice of the skeleton $\mathcal{A}_0$ and the decompositions ϕ_x.

A basic category $\mathcal{B}$ is said to be *minimal* if $\mathcal{B}(x,y) = 0$ for any two distinct objects x, y, while $\mathcal{B}(x,x)$ either coincide with $\mathbf{k}$ or is isomorphic to a *rational algebra*, i.e., a $\mathbf{k}$-algebra of the form $\mathbf{k}[t, f(t)^{-1}]$ for some non-zero polynomial $f(t)$. In the latter case the morphism $a : x \to x$ corresponding to the element t will be called *the loop at the vertex* x and the polynomial $f(t)$ will be denoted by f_a. If this polynomial is non-constant, call the vertex x and the loop *a marked vertex* and *marked loop* respectively. A category $\mathcal{A}$ is said to be *minimal* if it possesses a minimal skeleton. The marked vertices and the marked loops of this skeleton are also called the marked vertices and marked loops of the category $\mathcal{A}$.

Let $\mathcal{V}ec$ be the category of vector spaces over $\mathbf{k}$. The functors $M : \mathcal{A} \to \mathcal{V}ec$ are called $\mathcal{A}$*-modules*, or, more precisely, *left* $\mathcal{A}$*-modules*. If x is an object of $\mathcal{A}$, the elements of $M(x)$ are called *the elements of the module* M *at the object* x. We shall write au instead of $M(a)(u)$ for $u \in M(x)$, $a \in \mathcal{A}(x,y)$ (then $au \in M(y)$). The category of all $\mathcal{A}$-modules is denoted by $\mathcal{A}$-$\mathcal{M}od$. We also define *right* $\mathcal{A}$*-modules* as $\mathcal{A}^\circ$-modules, where $\mathcal{A}^\circ$ is the category opposite (or dual) to $\mathcal{A}$. In this case we write va for an element $v \in N(y)$ of a right module N at the object y and a morphism $a \in \mathcal{A}(x,y) = \mathcal{A}^\circ(y,x)$. The category of right $\mathcal{A}$-modules is denoted by $\mathcal{M}od$-$\mathcal{A}$. Given a module M, the set $\bigsqcup_{x \in \operatorname{Ob}\mathcal{A}} M(x)$ is called *the set of elements* of M and denoted by $\operatorname{El} M$. We may, of course, identify the categories of modules over $\mathcal{A}$ and over $add\,\mathcal{A}$, and we will usually do so.

Important examples of left (right) $\mathcal{A}$-modules are the modules $\mathcal{A}^x = \mathcal{A}(x, \_)$ (respectively, $\mathcal{A}_x = \mathcal{A}(\_, x)$). We call this module the *principal left (right) module corresponding to the object* x (as the usual name for these modules – "*representable*" – is not very convenient in the representation theory). The direct sums of principal modules are called *free modules* (if $\mathcal{A}$ is an algebra, i.e., a category with only 1 object, they are usual free $\mathcal{A}$-modules).

If $\mathcal{A}$ and $\mathcal{B}$ are two categories, a (bilinear) bifunctor $\mathcal{T} : \mathcal{A}^\circ \times \mathcal{B} \to \mathcal{V}ec$ is called an $\mathcal{A}$-$\mathcal{B}$-bimodule. If $t \in \mathcal{T}(x,y)$, we say that t is an *element of* $\mathcal{T}$ *with the source* x *and the target* y and write bta instead of $\mathcal{T}(a,b)(t)$ for $a \in \mathcal{A}(x',x)$, $b \in \mathcal{B}(y,y')$ (then $bta \in \mathcal{T}(x',y')$). As before, $\mathcal{A}$-$\mathcal{B}$-bimodules are the same as $(add\,\mathcal{A})$-$(add\,\mathcal{B})$-bimodules. If $\mathcal{A} = \mathcal{B}$ we say

"$\mathcal{A}$-bimodule" instead of "$\mathcal{A}$-$\mathcal{A}$-bimodule". Again we can consider *the set of elements* of a bimodule: $\operatorname{El}\mathcal{T} = \bigsqcup_{x\in\operatorname{Ob}\mathcal{A},\, y\in\operatorname{Ob}\mathcal{B}} \mathcal{T}(x,y)$. Surely, any category $\mathcal{A}$ can be considered as $\mathcal{A}$-bimodule mapping a pair of objects (x,y) to $\mathcal{A}(x,y)$. Call it the *regular* $\mathcal{A}$-bimodule. On the other hand, having any right $\mathcal{A}$-module N and any left $\mathcal{A}$-module M, we can construct the bimodule $N\otimes M$ putting $(N\otimes M)(x,y) = N(x)\otimes M(y)$. In particular, the bimodule of the form $\mathcal{A}_x^y = \mathcal{A}_x \otimes \mathcal{A}^y$ will be called the *principal bimodule with the source* y *and the target* x. The direct sums of principal bimodules are called *free bimodules*. Note that the regular bimodule, as a rule, is not free. On the other hand, in contrast with algebras, there is no natural notion of the regular module over a category having more than 1 object. If $\mathcal{V} = \bigoplus_j \mathcal{A}_{x_j}^{y_j}$, then the set of its elements $e_j = 1_{x_j}\otimes 1_{y_j}$ generates $\mathcal{V}$ as $\mathcal{A}$-bimodule. Moreover, each element of $\mathcal{V}$ can be uniquely written as $\sum_{ij} a_{ij}e_jb_{ij}$ for $a_{ij}, b_{ij} \in \mathcal{A}$. That is why this set is called *the set of free generators* for the bimodule $\mathcal{V}$.

To any $\mathcal{A}$-bimodule $\mathcal{T}$ one can associate a new category $\mathcal{A}[\mathcal{T}]$, called the *tensor category* of this bimodule, in the following way. First, construct the *tensor powers* $\mathcal{T}^{\otimes m}$ of this bimodule putting

$$\mathcal{T}^{\otimes 0} = \mathcal{A}\,,\ \ \mathcal{T}^{\otimes 1} = \mathcal{T} \ \text{ and } \ \mathcal{T}^{\otimes m+1} = \mathcal{T}\otimes_{\mathcal{A}}\mathcal{T}^{\otimes m}\,.$$

Now define $\mathcal{A}[\mathcal{T}]$ as the category whose object set coincides with that of $\mathcal{A}$, but

$$\mathcal{A}[\mathcal{T}](x,y) = \coprod_{m=0}^{\infty} \mathcal{T}^{\otimes m}(x,y)$$

with the obvious multiplication. If $\mathcal{T}$ is a free bimodule with a set S of free generators, call $\mathcal{A}[\mathcal{T}]$ *freely generated over* $\mathcal{A}$ (by the set S). The last set is called *a set of free generators for* $\mathcal{A}[\mathcal{T}]$ (over $\mathcal{A}$). In particular, if $\mathcal{A}$ is a trivial category, then $\mathcal{A}[\mathcal{T}]$ is a free category. If $\mathcal{A}$ is a minimal category, call $\mathcal{A}' = \mathcal{A}[\mathcal{T}]$ a *semi-free category with the set of semi-free generators* $\mathsf{S}\cup\mathsf{L}$, where L is the set of loops of $\mathcal{A}$. Usually, the set $\mathsf{S}\cup\mathsf{L}$ is denoted by $\operatorname{Arr}\mathcal{A}'$ and called *the set of arrows* of $\mathcal{A}'$. The *marked vertices* and *marked loops* of this category coincide, by definition, with those of the minimal category $\mathcal{A}$. The set of marked objects will be denoted by $\operatorname{Obm} A$ and the set of marked loops by $\operatorname{Lom} A$. The polynomial corresponding to a marked loop a will be denoted by $\operatorname{mk} a$. Hence, the semi-free category $\mathcal{A}'$ is also given by a graph $\Gamma = \Gamma(\mathcal{A}')$ (whose vertices are those of $\mathcal{A}$ and whose set of arrows is $\operatorname{Arr}\mathcal{A}'$) equipped with the subset $\operatorname{Lom}\Gamma \subseteq \operatorname{Arr}\Gamma$ of marked loops (at most one at each vertex) and the map $\operatorname{mk} : \operatorname{Lom}\Gamma \to \mathbf{k}[t]$. Call the triple $(\Gamma, \operatorname{Lom}\Gamma, \operatorname{mk})$ the *diagram* of the semi-free category

$\mathcal{A}'$ and the function mk its *marking function*. One can easily see that $\mathcal{A}' \simeq \mathcal{A} \coprod^{\mathcal{A}^{\emptyset}} \mathcal{A}_1$, where $\mathcal{A}_1$ is the free category with the set of free generators S and $\coprod$ denotes the *amalgamation* (or the *free product*) over the trivial subcategory $\mathcal{A}^{\emptyset}$ (cf. [16]).

Define the *degree* of a morphism from a semi-free category with a fixed set of arrows as follows. Such a morphism is a linear combination of *paths* of the form $p = a_l \dots a_2 a_1$, where each a_i either belong to the minimal category $\mathcal{A}$ or is an arrow, and if $a_k \in \mathcal{A}$, then neither a_{k+1} nor a_{k-1} belongs to $\mathcal{A}$. The degree of $a \in \mathcal{A}$ can be defined just as that of a rational function: the degree of the numerator minus that of the denominator. The degree of an arrow, by definition, is 1. Now, $\deg p$ is defined as $\sum_k \deg a_k$ and $\deg a$ as the maximum of degrees of the paths occurring in a.

Define the *category of representations* $\mathcal{R}ep(\mathcal{A}, \mathcal{B})$ of a category $\mathcal{A}$ over another category $\mathcal{B}$ to be the functor category $\mathcal{F}unc(\mathcal{A}, add\,\mathcal{B})$. Any representation $M : \mathcal{A} \to add\,\mathcal{B}$ defines an $\mathcal{A}$-$\mathcal{B}$-bimodule ${}_M\mathcal{B}$ mapping a pair (x, y), where $x \in \operatorname{Ob} \mathcal{B}$, $y \in \operatorname{Ob} \mathcal{A}$, to $\mathcal{B}(x, My)$. On the contrary, if an $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{T}$ is such that the $\mathcal{B}$-module $T(\_, y)$ is finitely generated and projective for any $y \in \operatorname{Ob} \mathcal{A}$, then one can easily construct a functor $M : \mathcal{A} \to add\,\mathcal{B}$ such that the corresponding bimodule ${}_M\mathcal{B}$ is isomorphic to $\mathcal{T}$. Thus, we will sometimes identify M with ${}_M\mathcal{B}$. In the same way, the $\mathcal{B}$-$\mathcal{A}$-bimodule $\mathcal{B}_M$ and the $\mathcal{A}$-bimodule ${}_M\mathcal{B}_M$ are defined. Sometimes we will also write $\mathcal{B}$ instead of any of these three bimodules if the real sense of this notation is clear from the context. Remark that, for instance, $\mathcal{R}ep(\mathcal{A}, \mathbf{k})$ is equivalent to the full subcategory of $\mathcal{A}$-$\mathcal{M}od$ consisting of such modules M that $M(x)$ is finite dimensional for any $x \in \operatorname{Ob} \mathcal{A}$. Of course, the categories $\mathcal{R}ep(\mathcal{A}, \mathcal{B})$ and $\mathcal{R}ep(add\,\mathcal{A}, \mathcal{B})$ are equivalent and we will often identify them.

2. *Boxes*

A *coalgebra* over a category $\mathcal{A}$ is, by definition, an $\mathcal{A}$-bimodule $\mathcal{V}$ equipped with two bimodule homomorphisms: *comultiplication* $\mu : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V}$ and *counit* $\varepsilon : \mathcal{V} \to \mathcal{A}$, subject to the usual conditions (cf. [19]):

- $(\mu \otimes 1)\mu = (1 \otimes \mu)\mu$ (co-associativity);
- $(\varepsilon \otimes 1)\mu = \iota_l$ and $(1 \otimes \varepsilon)\mu = \iota_r$ (counit properties),

where $\iota_l : \mathcal{V} \xrightarrow{\sim} \mathcal{A} \otimes_{\mathcal{A}} \mathcal{V}$ and $\iota_r : \mathcal{V} \xrightarrow{\sim} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{A}$ are the natural isomorphisms. For instance, the regular $\mathcal{A}$-bimodule has the natural coalgebra structure with $\varepsilon = 1$ and μ being the natural isomorphism. Call it the *regular $\mathcal{A}$-coalgebra.* The kernel $\overline{\mathcal{V}} = \ker \varepsilon$ of the counit is also called *the kernel of the coalgebra.*

Now a *box* is defined as a pair $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ consisting of a category $\mathcal{A}$ and an $\mathcal{A}$-coalgebra $\mathcal{V}$. (More usual term is "bocs", but this word seems not existing in any language. Thus, we propose to replace it by its existing homonym.) In particular, the pair $(\mathcal{A}, \mathcal{A})$ (the second component being the regular $\mathcal{A}$-coalgebra) will be called the *regular A-box.* (Remark that earlier ([23] or [11]) they used here the term "principal box", while the term "regular" was used for other purposes, but we think that the new terminology is more convenient.) The kernel $\overline{\mathcal{V}}$ of the coalgebra $\mathcal{V}$ is also called *the kernel of the box* $\mathfrak{A}$.

A *representation* of the box $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ over a category $\mathcal{C}$ is defined as a functor $M : \mathcal{A} \to add\,\mathcal{C}$. If M' is another representation, then a morphism $\varphi : M \to M'$ is, by definition, an homomorphism of the $\mathcal{A}$-$\mathcal{C}$-bimodules $\varphi : \mathcal{V} \otimes_{\mathcal{A}} M \to M'$, where the functors are identified with bimodules as above. The set of such morphisms will be denoted by $\mathrm{Hom}_{\mathfrak{A}-\mathcal{C}}(M, M')$. For another morphism $\psi : M' \to M''$ their product $\psi\varphi$ is defined as the composition

$$\mathcal{V} \otimes_{\mathcal{A}} M \xrightarrow{\mu\otimes 1} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} M \xrightarrow{1\otimes\varphi} \mathcal{V} \otimes_{\mathcal{A}} M' \xrightarrow{\psi} M'' .$$

Thus, we obtain the *category of representations* $\mathcal{R}ep(\mathfrak{A}, \mathcal{C})$ of the box $\mathfrak{A}$ over the category $\mathcal{C}$. The unit morphism 1_M of this category is the composition

$$\mathcal{V} \otimes_{\mathcal{A}} M \xrightarrow{\varepsilon\otimes 1} \mathcal{A} \otimes_{\mathcal{A}} M \xrightarrow{\iota_l^{-1}} M .$$

In particular, if $\mathcal{C} = \mathcal{V}ec$ we call these representations $\mathfrak{A}$-*modules*, write $\mathrm{Hom}_{\mathfrak{A}}(M, N)$ for the set of morphisms of two $\mathfrak{A}$-modules and $\mathfrak{A}$-$\mathcal{M}od$ for the corresponding category. It is quite obvious that if $\mathfrak{A} = (\mathcal{A}, \mathcal{A})$ is the regular $\mathcal{A}$-box then the representation category $\mathcal{R}ep(\mathfrak{A}, \mathcal{C})$ coincides with the functor category $\mathcal{F}unc(\mathcal{A}, add\,\mathcal{C})$. Note also that if C is an algebra (i.e. a category with 1 object), then $add\,C$ can be considered as the category of finitely generated projective C-modules. In particular, $add\,\mathbf{k}$ can be identified with the category *vec* of finite dimensional vector spaces over $\mathbf{k}$.

It is often convenient to apply the natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{V} \otimes_{\mathcal{A}} M,\, M') \simeq \mathrm{Hom}_{\mathcal{A}-\mathcal{A}}(\mathcal{V},\, \mathrm{Hom}(M, N))$$

and consider a morphism of boxes as a function on $\mathrm{El}\,\mathcal{V}$. In other words, to define a morphism $M \to N$ one has to define for every $\gamma \in \mathcal{V}(x, y)$ the linear mapping $\varphi(\gamma) : M(x) \to N(y)$ such that $\varphi(a\gamma b) = N(a)\varphi(\gamma)M(b)$ for all $a : y \to y'$, $b : x' \to x$.

A *system of generators* for a box $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ is, by definition, a union $\mathrm{A} = \mathrm{A}_0 \cup \mathrm{A}_1$, where A_0 is a system of generators of the category $\mathcal{A}$ and A_1 that of the kernel $\overline{\mathcal{V}}$ of the box. If the category $\mathcal{A}$ has a skeleton $\mathcal{S}k\,\mathcal{A}$ denote by $\mathcal{S}k\,\mathcal{V}$ the restriction of the bimodule $\mathcal{V}$ on $\mathcal{S}k\,\mathcal{A}$ and by $\mathcal{S}k\,\mathfrak{A}$ the pair $(\mathcal{S}k\,\mathcal{A}, \mathcal{S}k\,\mathcal{V})$. Certainly, $\mathcal{S}k\,\mathfrak{A}$ is again a box, which we call the *skeleton* of the box $\mathfrak{A}$. Call $\mathcal{A}$ *locally finitely generated* if its skeleton has a system of generators A such that $\mathrm{A}(x, y)$ is finite for each pair of objects of $\mathcal{S}k\,\mathcal{A}$. If, moreover, the set of objects of $\mathcal{S}k\,\mathcal{A}$ is finite, call the box $\mathfrak{A}$ *finitely generated.*

Suppose now that the category $\mathcal{A}$ is rigid. Denote then by $\mathit{rep}(\mathfrak{A}, \mathcal{C})$ the full subcategory of $\mathcal{R}ep(\mathfrak{A}, \mathcal{C})$ consisting of the representations M such that $M(x) = 0$ for almost all objects $x \in \mathrm{Ver}\,\mathcal{A}$ (i.e., all but a finite number of them). In particular, the category of *finite dimensional* $\mathfrak{A}$*-modules* $\mathfrak{A}\text{-}\mathit{mod} = \mathit{rep}(\mathfrak{A}, \mathbf{k})$, is defined. The *support* of a representation $M \in \mathcal{R}ep(\mathfrak{A}, \mathcal{C})$ is defined as the set

$$\mathrm{Supp}(M) = \{\, x \in \mathrm{Ver}\,\mathcal{A} \mid M(x) \neq 0 \,\}\,.$$

In the case when C is an algebra such that any projective C-module is free of unique rank, define the *dimension* of a representation $M \in \mathit{rep}(\mathfrak{A}, C)$ as the function

$$\mathsf{dim}(M) : \mathrm{Ver}\,\mathcal{A} \to \mathbb{N} : \quad \mathsf{dim}(M)(x) = \mathrm{rk}_C\, M(x)\,.$$

In the case when $\mathrm{Ver}\,\mathcal{A} = \{\, x_1, x_2, \dots, x_n \,\}$ is finite, this function can be identified with the vector $(d_1, d_2, \dots, d_n)$, where $d_i = \mathrm{rk}_C\, M(x_i)$.

We call any function $\mathsf{d} : \mathrm{Ver}\,\mathcal{A} \to \mathbb{N}$ with a finite support *a dimension of representations of the box* $\mathfrak{A}$ and denote $\mathrm{Dim}(\mathfrak{A})$ the set of all such dimensions. For any $\mathsf{d} \in \mathrm{Dim}(\mathfrak{A})$ put $|\mathsf{d}| = \sum_{x \in \mathrm{Ver}\,\mathcal{A}} \mathsf{d}(x)$.

Fixing a representative of free C-modules of each given rank r (say, rC), we are able to consider the set $\mathit{rep}_{\mathsf{d}}(\mathfrak{A}, C)$ of all representations of $\mathfrak{A}$ over C of the prescribed dimension d. In particular, if $\mathfrak{A}$ is locally finitely generated then the set $\mathit{rep}_{\mathsf{d}}(\mathfrak{A}) = \mathit{rep}_{\mathsf{d}}(\mathfrak{A}, \mathbf{k})$ can be considered as an algebraic variety over the field $\mathbf{k}$.

A *morphism* of boxes $\Phi : \mathfrak{A} \to \mathfrak{B}$, where $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ and $\mathfrak{B} = (\mathcal{B}, \mathcal{W})$, is, by definition, a pair $\Phi = (\Phi_0, \Phi_1)$ where $\Phi_0 : \mathcal{A} \to \mathcal{B}$ is a functor and $\Phi_1 : \mathcal{V} \to \mathcal{W}$ is an homomorphism of $\mathcal{A}$-coalgebras ($\mathcal{W}$ is considered as $\mathcal{A}$-coalgebra via its $\mathcal{B}$-coalgebra structure and the functor Φ_0). Usually we will write Φ for both Φ_0 and Φ_1. Given such morphism we can evidently construct the *inverse image* functor

$$\Phi^* : \mathcal{R}ep(\mathfrak{B}, \mathcal{C}) \to \mathcal{R}ep(\mathfrak{A}, \mathcal{C})$$

for every category $\mathcal{C}$.

In particular, each representation $M \in \mathcal{R}ep(\mathfrak{A},\mathcal{C})$ can be considered as a morphism from $\mathfrak{A}$ to the regular box over the category $add\,\mathcal{C}$, whose value on $\mathcal{V}$ is the composition $M \circ \varepsilon$. In this case the inverse image functor is isomorphic to the tensor product functor $M \otimes_{\mathcal{C}} \_$ and we shall often identify them.

Call Φ an *equivalence* of boxes provided Φ_0 is an equivalence of categories and Φ_1 is an isomorphism of $\mathcal{A}$-bimodules. Then Φ^* is an equivalence of their categories of representations over each category$\mathcal{C}$.

Suppose that a box $\mathfrak{A} = (\mathcal{A},\mathcal{V})$ and a functor $F : \mathcal{A} \to \mathcal{B}$ are given. Define a new box $\mathfrak{A}^F = (\mathcal{B},\mathcal{V}^F)$, where $\mathcal{V}^F = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B}$. The comultiplication in $\mathcal{V}^F$ is defined as the composition

$$\begin{aligned} \mathcal{V}^F &\longrightarrow \mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{A} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \longrightarrow \\ &\longrightarrow \mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \simeq \mathcal{V}^F \otimes_{\mathcal{B}} \mathcal{V}^F , \end{aligned}$$

the first arrow being induced by the comultiplication in $\mathcal{V}$ and the second one by the functor F. Of course, the pair (F, F'), where $F' : \mathcal{V} \to \mathcal{V}^F$ is the natural homomorphism, is a morphism of boxes $\mathfrak{A} \to \mathfrak{A}^F$, which we also denote by F. The following evident "change-of-base theorem" shows the advantage of boxes and is the main tool in the so called "reduction processes".

Theorem 2.1. *In the above situation the functor $F^* : \mathcal{R}ep(\mathfrak{A}^F,\mathcal{C}) \to \mathcal{R}ep(\mathfrak{A},\mathcal{C})$ is fully faithful for each category $\mathcal{C}$. Hence, it induces an equivalence of $\mathcal{R}ep(\mathfrak{A}^F,\mathcal{C})$ and the full subcategory in $\mathcal{R}ep(\mathfrak{A},\mathcal{C})$ consisting of all functors $\mathcal{A} \to add\,\mathcal{C}$ which can be factored through F.*

Proof. We may suppose that $\mathcal{C} = add\,\mathcal{C}$. For any two representations $M, N \in \mathcal{R}ep(\mathfrak{A}^F,\mathcal{C})$ we have the natural isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathcal{B}-\mathcal{C}}(\mathcal{V}^F \otimes_{\mathcal{B}} M, N) = \operatorname{Hom}_{\mathcal{B}-\mathcal{C}}(\mathcal{B} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{B}} M, N) \simeq \\ \simeq \operatorname{Hom}_{\mathcal{A}-\mathcal{C}}(\mathcal{V} \otimes_{\mathcal{A}} \mathcal{B} \otimes_{\mathcal{B}} M, \operatorname{Hom}_{\mathcal{B}}(\mathcal{B}, N)) \simeq \operatorname{Hom}_{\mathcal{A}-\mathcal{C}}(\mathcal{V} \otimes_{\mathcal{A}} M, N) . \end{aligned}$$

But it is just what we need. □

Remark. It is important to notice that even if $\mathfrak{A} = (\mathcal{A},\mathcal{A})$ is a regular box, the box $\mathfrak{A}^F$ is, as a rule, no more regular.

Call any set $\boldsymbol{\omega} = \{\, \omega_x \in \mathcal{V}(x,x) \,|\, x \in \operatorname{Ob}\mathcal{A},\ \varepsilon(\omega_x) = 1_x \,\}$ *a section* of the box $\mathfrak{A}$. If, moreover, $\mu(\omega_x) = \omega_x \otimes \omega_x$ for each x, call this section *normal.* A box having a normal section is also called *normal.*

If a section $\boldsymbol{\omega}$ in $\mathcal{S}k\,\mathfrak{A}$ is fixed, one can define the $\mathcal{A}^\emptyset$-sub-bimodule $\mathcal{V}^\emptyset \subseteq \mathcal{V}$. Namely, if $x, y \in \mathcal{S}k\,\mathcal{A}$ put $\mathcal{V}^\emptyset(x,x) = \mathbf{k}\omega_x$ and $\mathcal{V}^\emptyset(x,y) = 0$ if $x \neq y$. Then expand this definition to the whole category $\mathcal{A}$ just as it has been done for $\mathcal{A}^\emptyset$ above. If the section $\boldsymbol{\omega}$ is normal, $\mathfrak{A}^\emptyset = (\mathcal{A}^\emptyset, \mathcal{V}^\emptyset)$ is even a sub-box in $\mathfrak{A}$.

One can easily check that given any section $\boldsymbol{\omega}$ we have $\partial a = a\omega_x - \omega_y a \in \overline{\mathcal{V}}$ for each $a \in \mathcal{A}(x,y)$ and $\partial v = \mu(v) - v\otimes\omega_x - \omega_y\otimes v \in \overline{\mathcal{V}}\otimes_{\mathcal{A}}\overline{\mathcal{V}}$ for each $v \in \overline{\mathcal{V}}(x,y)$. We call the mapping ∂ *the differential* of the box $\mathfrak{A}$ (with respect to the given section). We prolong ∂ to a mapping $\overline{\mathcal{V}}^{\otimes 2} \to \overline{\mathcal{V}}^{\otimes 3}$ which maps $v\otimes w$ to $\partial v\otimes w - v\otimes\partial w$; then one can verify the following rules:

$$\partial^2(a) = a\delta_x - \delta_x a \quad \text{and} \quad \partial^2(v) = v\otimes\delta_x + \delta_x\otimes v$$

where $\delta_x = \mu(\omega_x) - \omega_x\otimes\omega_x$. In particular, if the section is normal, $\partial^2 = 0$. Usually one only deals with normal boxes, but this time we need to involve some non-normal ones in the proof.

A box $\mathfrak{A} = (\mathcal{A},\mathcal{V})$ is said to be *semi-free* if $\mathcal{A}$ is a semi-free category, the kernel $\overline{\mathcal{V}}$ is a free $\mathcal{A}$-bimodule and $\partial a = 0$ for all marked loops from $\mathcal{A}$. In this case its *set of arrows* $\operatorname{Arr}\mathfrak{A}$ (or a *set of semi-free generators* of $\mathfrak{A}$) is, by definition, the union $\operatorname{Arr}\mathcal{A}\cup\operatorname{Arr}\mathcal{V}$ where $\operatorname{Arr}\mathcal{V}$ is the set of free generators for $\overline{\mathcal{V}}$. The sets of *marked objects*, *marked loops* and the function mk for the box $\mathfrak{A}$ are, by definition, those for the category $\mathcal{A}$. Call $\mathfrak{A}$ *triangular* if there exists a function $\nu : \operatorname{Arr}\mathfrak{A} \to \mathbb{N}$ (called *triangular height*) such that, for each $b \in \operatorname{Arr}\mathfrak{A}$ ∂b lies in the semi-free sub-box generated by all elements $a \in \operatorname{Arr}\mathfrak{A}$ with $\nu(a) < \nu(b)$. A normal semi-free box $\mathfrak{A}$ is usually given by its *bigraph* $\Gamma = \Gamma(\mathfrak{A})$ having two sorts of arrows: *solid*, corresponding to $\operatorname{Arr}\mathcal{A}$, and *dotted*, corresponding to $\operatorname{Arr}\mathcal{V}$. To define $\mathfrak{A}$ we also have to choose the set $\operatorname{Lom}\Gamma$ of marked loops together with the marking map $\operatorname{Lom}\Gamma \to \mathbf{k}[t]$ and to prescribe the differential ∂a for each arrow a in such way that $\partial^2 = 0$ if we calculate it using the Leibniz rule. Note that while the bigraph Γ is uniquely determined by the semi-free box it is no more true for its differential. If we choose another system of semi-free generators their differentials usually change. Moreover, sometimes we need to choose a "good" system of semi-free generators with respect to the differential, i.e., such one that the formulae for the differential were the simplest (or the most convenient). In particular, the triangularity strongly depends on the choice of generators. So, when we speak about a triangular semi-free box we always mean a fixed choice. Then the definition of *degree* given above for semi-free categories can be evidently extended to the elements of the bimodule $\mathcal{V}$ putting $\deg v = 1$ for any $a \in \operatorname{Arr}\overline{\mathcal{V}}$ and $\deg\omega_x = 0$ for any object x.

If we fix a set of arrows $\mathrm{A} = \operatorname{Arr}\mathfrak{A}$ for a semi-free normal box $\mathfrak{A}$ we can associate to it a 2-dimensional cell complex $\operatorname{Com}(\mathfrak{A},\mathrm{A})$. Its set of vertices is that of the bigraph $\Gamma(\mathfrak{A})$, the 1-dimensional cells (edges) are the elements of A, and 2-dimensional cells are constructed as follows.

Let $a \in \mathrm{A}$ and $\partial a = \sum_p \lambda_p p$ where p are some paths in Γ. Then for each path p such that $\lambda_p \neq 0$ we attach to $\mathrm{Com}(\mathfrak{A}, \mathrm{A})$ a 2-dimensional cell with the border pa^{-1} where a^{-1} denotes the same edge a but directed oppositely to the corresponding arrow. Now we can also define *the fundamental group* $\mathsf{Gr}(\mathfrak{A}, \mathrm{A}, x)$ of the pair $(\mathfrak{A}, \mathrm{A})$ with respect to a vertex x as that of the complex $\mathrm{Com}(\mathfrak{A}, \mathrm{A})$. Simple examples show that this complex and its fundamental group depend on the choice of the system of semi-free generators A and not only on the box $\mathfrak{A}$ itself. If this complex is connected, then we may omit the vertex and write $\mathsf{Gr}(\mathfrak{A}, \mathrm{A})$. Surely, this is the case if and only if the corresponding bigraph is connected, and then we call the box $\mathfrak{A}$ connected too.

A box $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ is said to be *so-trivial* if $\mathcal{A}$ is a trivial category. If, moreover, it is regular (i.e., $\mathcal{V} = \mathcal{A}$, hence, $\overline{\mathcal{V}} = 0$), it is called *trivial*. For a so-trivial box one can easily reconstruct the module category $\mathfrak{A}$-$\mathcal{M}od$. Obviously, its indecomposable objects are of the form S_x where x runs through $\mathrm{Ver}\,\mathcal{A}$ and S_x is determined by the formulae:

$$S_x(x) = \mathbf{k} \ \text{ and } \ S_x(y) = 0 \ \text{ for } \ y \in \mathrm{Ver}\,\mathcal{A}\,, \ y \neq x\,.$$

Any $\mathfrak{A}$-module can be uniquely decomposed into a direct sum of S_x. Moreover, $\mathrm{Hom}_{\mathfrak{A}}(S_x, S_y) \simeq \mathrm{D}\mathcal{V}(x, y)$, where D denotes the *duality functor* for vector spaces over $\mathbf{k}$: $\mathrm{D}X = \mathrm{Hom}(X, \mathbf{k})$, while the multiplication of the homomorphisms is dual to the comultiplication in the coalgebra $\mathcal{V}$.

Analogously, call a box $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ *so-minimal* (respectively, *minimal*) provided $\mathcal{A}$ is a minimal category (respectively, $\mathcal{A}$ is minimal and the box is regular).

3. *Reduction algorithms*

Let $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ be a box and $\mathfrak{A}' = (\mathcal{A}', \mathcal{V}')$ be its sub-box. Suppose that a functor $F' : \mathcal{A}' \to \mathcal{B}'$ is given. Construct the *amalgamation* $\mathcal{B} = \mathcal{A} \coprod^{\mathcal{A}'} \mathcal{B}'$, or, the same, the couniversal square:

$$\begin{array}{ccc} \mathcal{A}' & \xrightarrow{F'} & \mathcal{B}' \\ \downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

Consider now the box $\mathfrak{A}^F = (\mathcal{B}, \mathcal{V}^F)$. Then Theorem 2.1 implies the following corollary.

Corollary 3.1. *In the situation above, for each category $\mathcal{C}$ the functor $F^* : \mathcal{R}ep(\mathfrak{A}^F, \mathcal{C}) \to \mathcal{R}ep(\mathfrak{A}, \mathcal{C})$ induces an equivalence between the category $\mathcal{R}ep(\mathfrak{A}^F, \mathcal{C})$ and the full subcategory $\mathcal{R}ep(\mathfrak{A}, \mathcal{C} \mid F') \subseteq \mathcal{R}ep(\mathfrak{A}, \mathcal{C})$*

consisting of those representations $M : \mathcal{A} \to \operatorname{add}\mathcal{C}$ *whose restrictions on* $\mathcal{A}'$ *can be factored through* F'.

We shall use the following "elementary cases" of this procedure considered in [11] (cf. also [18, 7]).

Example 3.2. (1) Suppose that $\mathcal{A}'$ is freely generated by the unique solid arrow $a : x \to y$ with $x \neq y$ and $\partial a = 0$. Then take for $\mathcal{B}'$ the trivial category with 3 vertices, $\{\, x', y', xy \,\}$, and for F' the functor which maps:

$$x \mapsto x' \oplus xy\,,\ y \mapsto xy \oplus y'\,,\ a \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

(later on we usually write x and y instead of x' and y'). They say that the box $\mathfrak{A}^F$ is obtained from $\mathfrak{A}$ *by reducing the edge* a. In this case one can easily see that $\mathcal{R}ep(\mathfrak{A}, \mathcal{V}ec \mid F') = \mathcal{R}ep(\mathfrak{A}, \mathcal{V}ec)$. Thus, F^* is an equivalence of the module categories $\mathfrak{A}^F\text{-}\mathcal{M}od \simeq \mathfrak{A}\text{-}\mathcal{M}od$.

(2) Suppose now that $\mathfrak{A}'$ is freely generated by one solid arrow $a : x \to y$ and one dotted arrow $\alpha : x \to y$ such that $\partial a = \alpha$ (never mind whether $x = y$ or not). Then put $\mathcal{B}' = \mathcal{A}'/(a)$ and $F' : \mathcal{A}' \to \mathcal{B}'$ being the natural projection. In this case we have $\mathcal{R}ep(\mathfrak{A}, \mathcal{C} \mid F') = \mathcal{R}ep(\mathfrak{A}, \mathcal{C})$ for any category $\mathcal{C}$. Hence, $F^* : \mathcal{R}ep(\mathfrak{A}^F, \mathcal{C}) \xrightarrow{\sim} \mathcal{R}ep(\mathfrak{A}, \mathcal{C})$. They say that the box $\mathfrak{A}^F$ is obtained from $\mathfrak{A}$ *by deleting the arrow* a. We call such an arrow a a *superfluous arrow* (in [18, 11] it was called *non-regular*).

(3) At last, let $\mathcal{A}'$ be generated by the unique solid *loop* (maybe, marked) $a : x \to x$ with $\partial a = 0$. Take for $\mathcal{B}'$ the minimal category containing $n+1$ vertices $\{\, x_*, x_1, x_2, \dots, x_n \,\}$ and one marked loop $a_* : x_* \to x_*$ with $\operatorname{mk}(a_*) = (t-\lambda)\operatorname{mk}(a)$, and put

$$F'(x) = x_* \oplus \bigoplus_{k=1}^{n} kx_k \ \text{ and }\ F'(a) = a_* \oplus \bigoplus_{k=1}^{n} J_k\,,$$

where J_k is the Jordan cell of size k with the eigenvalue λ and λ is an element of $\mathbf{k}$ such that $\operatorname{mk}(a)(\lambda) \neq 0$. Then the Jordan theorem implies that $rep(\mathfrak{A}, \mathbf{k} \mid F')$ consists of all representations M such that $A^n = 0$, where A denotes the nilpotent part of $M(a-\lambda)$. In this case they say that the box $\mathfrak{A}^F$ is obtained from $\mathfrak{A}$ *by reducing the* λ*-part of the loop* a *up to degree* n. (Note that even if a was not marked, a_* is still marked with $\operatorname{mk}(a_*) = t - \lambda$.)

Note that in all these examples, if the box $\mathfrak{A}$ was semi-free and normal, so is also the box $\mathfrak{A}^F$. Moreover, if $\mathfrak{A}$ was triangular, $\mathfrak{A}^F$ is triangular too.

Later we need one more "elementary step".

Example 3.3. Call *a line* in the category $\mathcal{A}$ (or in the box $\mathfrak{A}$) any subcategory $\mathcal{L}$ generated by a set of its (distinct) vertices and arrows $\{\, x_k,\, a_k : x_k \to x_{k+1} \,|\, k \in \mathbb{Z} \,\}$. (We often denote this set itself by $\mathcal{L}$ and call it a line too.) For any interval $I \subseteq \mathbb{Z}$ (maybe, infinite), denote by L_I the representation of the line $\mathcal{L}$ such that

$$\begin{aligned} L_I(x_k) &= \begin{cases} \mathbf{k} & \text{if } k \in I \\ 0 & \text{otherwise} \end{cases} \\ L_I(a_k) &= 1 \text{ if } k, k+1 \in I \end{aligned}$$

The following proposition is an easy exercise in linear algebra.

Proposition 3.4. *Any representation of the line $\mathcal{L}$ (not necessarily finite dimensional) is a direct sum of representations isomorphic to some of L_I.*

Suppose now that $\partial a_k = 0$ for all $k \in \mathbb{Z}$ (such a line is said to be *minimal*). Put $\mathcal{A}' = \mathcal{L}$, take for $\mathcal{B}'$ the trivial category with the set of vertices $\{\, x_{ij} \,|\, i,j \in \mathbb{Z},\ 0 \le j-i < n \,\}$, and define the functor $F' : \mathcal{A}' \to \mathcal{B}'$ as follows:

$F'(x_k) = \bigoplus_{i \le k \le j} x_{ij}$ and $F'(a_k)$ is given by the matrix $(a^{ij}_{i'j'})$ in which $a^{ij}_{ij} = 1$ for all possible values of i, j (all other entries of this matrix are automatically 0).

Proposition 3.4 implies that in this case the subcategory $\mathcal{R}ep(\mathfrak{A}, \mathcal{V}ec \,|\, F')$ consists of all representations M of $\mathfrak{A}$ such that $\prod_{i=0}^{n-1} a_{k+i} = 0$ for each $k \in \mathbb{Z}$. Again, if the box $\mathfrak{A}$ was semi-free, normal, triangular, so is also $\mathfrak{A}^F$. We say that $\mathfrak{A}^F$ is obtained from $\mathfrak{A}$ *by reducing the line $\mathcal{L}$ up to degree n.*

All these algorithms "improve" the representations in the following sense. Let $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ be a semi-free box, $M \in \mathit{rep}(\mathfrak{A}, \mathcal{C})$, $\mathsf{d} = \operatorname{Dim} M$. Denote by $n(x,y)$ the number of elements in $\operatorname{Arr} \mathcal{A}(x,y)$ and put

$$||M|| = ||\mathsf{d}|| = \sum_{x,y \in \operatorname{Ver} \mathcal{A}} n(x,y)\mathsf{d}(x)\mathsf{d}(y)\,.$$

Then the following proposition is immediate.

Proposition 3.5. *Suppose that the box $\mathfrak{B} = \mathfrak{A}^F$ is obtained from $\mathfrak{A}$ by one of the following operations:*

(i) *reduction of an edge* $a : x \to y$ *;*
(ii) *deleting an arrow* $a : x \to y$ *;*
(iii) *reduction of the* λ*-part of a loop* $a : x \to x$ *;*
(iv) *reduction of a line* $\mathcal{L} = \{ x_k, a_k \}$ *.*

Let $M \in rep(\mathfrak{A}, \mathcal{C})$ *,* $N \in rep(\mathfrak{B}, \mathcal{C})$ *be such that* $M \simeq F^*N$ *. Then* $||N|| \le ||M||$ *. Moreover,* $||N|| < ||M||$ *if the following conditions hold (in accordance with the numeration of the cases above):*

(i) $M(x) \neq 0$ *and* $M(y) \neq 0$ *;*
(ii) $M(x) \neq 0$ *and* $M(y) \neq 0$ *;*
(iii) $M(a - \lambda)$ *is not invertible (in particular,* $M(x) \neq 0$ *);*
(iv) $M(x_k) \neq 0$ *and* $M(x_{k+1} \neq 0$ *for at least one* k *.*

4. *Tame and wild boxes*

From now on we suppose that all boxes are rigid and locally finitely generated; a semi-free box is always supposed to be triangular.

Definition 4.1. (1) We say that a functor $F : \mathcal{C} \to \mathcal{D}$

- *reflects isomorphisms* if $F(x) \simeq F(y)$ implies that $x \simeq y$ for any objects $x, y \in \mathcal{C}$;
- *reflects decomposability* if an object $x \in \mathcal{C}$ is decomposable (into a non-trivial direct sum) if and only if $F(x)$ is decomposable in $\mathcal{D}$.
- *is strict* if it reflects isomorphisms and indecomposability.

(2) A morphism of boxes $\Phi : \mathfrak{A} \to \mathfrak{B}$ is called *strict* if the inverse image functor Φ^* is strict. (In particular, we have the notion of a *strict representation* of a box over a category, e.g., over an algebra).

(3) A box $\mathfrak{A}$ is called *wild* if it has a strict representation over each finitely generated $\mathbf{k}$-algebra. (In particular, we have the notion of a *wild category* (e.g., wild algebra) if we identify it with its regular box.)

Remind some standard well-known examples of wild categories and boxes (cf. [11]).

Proposition 4.2. *The following categories, algebras and boxes are wild:*

(1) *Free algebra* $\mathbf{k}\langle x, y \rangle$ *in 2 generators.*
(2) *Polynomial algebra* $\mathbf{k}[x, y]$ *in 2 generators.*
(3) *Formal power series algebra* $\mathbf{k}[[x, y]]$ *in 2 generators.*
(4) *The free category* $\mathbf{k}(x|y)$ *generated by the graph with 2 vertices* $1, 2$ *and two arrows* $x : 1 \to 1$ *and* $y : 1 \to 2$*, as well as its opposite category* $\mathbf{k}(x|y)^\circ$*.*

(5) *The free category* $\mathcal{C}_5$ *generated by the graph with 6 vertices* $0,1,2,3,\ 4,5$ *and 5 arrows* $a_i : 0 \to i$ *for* $i = 1,2,\dots,5$, *as well as the opposite category* $\mathcal{C}_5^\circ$.
(6) *Any semi-free normal box* $\mathfrak{W}_1$ *whose bigraph has one vertex and two solid arrows* a, b *such that* $\partial a = 0$ *and* b *is not superfluous.*
(7) *Any semi-free normal box* $\mathfrak{W}_2$ *whose diagram has two vertices* $1,2$ *and three solid arrows* $a : 1 \to 1\,,\ c : 2 \to 2\,,\ b : 1 \to 2$ *(or* $b : 2 \to 1$*) such that* $\partial a = 0\,,\ \partial c = 0$ *and* b *is not superfluous.*

Definition 4.3. (1) A *rational algebra* is, by definition, an algebra of the form $R = \mathbf{k}[t, f(t)^{-1}]$ where $f(t)$ is a non-zero polynomial.
(2) A strict representation over a rational algebra will be called *a rational family* of representations.
(3) The *dimension* of a rational family M is defined as the dimension $\mathsf{dim}(M)$.
(4) For any R-module L put $M(L) = M \otimes_R L$ and denote $[M] = \{\, M(L) \mid L \in R\text{-}mod\,\}$. We say that the modules isomorphic to some $M(L)$ *belong* to the family M.
(5) If $\mathcal{F}$ is any set of rational families, put $[\mathcal{F}] = \bigcup_{M\in\mathcal{F}}[M]$. Again, we say that the modules isomorphic to those from $[\mathcal{F}]$ *belong* to this set of rational families. For any dimension d of representations of the box $\mathfrak{A}$ denote by $\mathcal{F}_{\mathsf{d}}$ the set of representations from $\mathcal{F}$ having dimension d.

Definition 4.4. A box (in particular, a category or an algebra) $\mathfrak{A}$ is said to be *tame* if there is a set $\mathcal{F}$ of rational families of its representations such that:

(1) $\mathcal{F}$ is *locally finite*, i.e., $\mathcal{F}_{\mathsf{d}}$ is finite for any prescribed dimension $\mathsf{d} \in \mathrm{Dim}(\mathcal{A})$.
(2) $\mathcal{F}$ is *almost exhaustive*, i.e., for any prescribed dimension d the set $[\mathcal{F}] \cap rep_{\mathsf{d}}(\mathfrak{A})$ intersects almost all isomorphism classes of indecomposable $\mathfrak{A}$-modules of this dimension (i.e., all but a finite number of them).

Note that local finiteness implies that there is only finitely many $M \in \mathcal{F}$ such that $[M] \cap rep_{\mathsf{d}}(\mathfrak{A}) \neq \emptyset$.

It was proved in [11] (cf. also [7]) that any (locally finitely generated, triangular) semi-free box as well as any finite dimensional algebra (indeed, any locally finite dimensional category) over an algebraically closed field is either wild or tame. It implies some simple but rather useful geometrical criteria for tameness and wildness based on the following fact concerning the properties of "geometrical dimensions" (cf. [10]).

Remind that a *constructible subset* Y of an algebraic variety X is, by definition, a finite union of its locally closed subsets. The *dimension* of Y is defined as the maximum of dimensions of these locally closed subsets. An equivalence relation E on X is said to be *constructible* if it is constructible as subset of $X \times X$. Then, of course, the equivalence class $\mathrm{E}(x)$ of each element $x \in X$ is also constructible.

Proposition 4.5. (cf. [10].) *Let X be an algebraic variety (over an algebraically closed field), E be a constructible equivalence relation on X , and Y, Z be two constructible subsets of X such that:*

- $\dim Y \cap \mathrm{E}(x) \geq m$ for each $x \in X$;
- $\dim Z \cap \mathrm{E}(x) \leq n$ for each $x \in X$.

Then $\dim Z - n \leq 2\dim X - \dim \mathrm{E} \leq \dim Y - m$.

Corollary 4.6. *Suppose that the filed* $\mathbf{k}$ *is algebraically closed. For any semi-free box $\mathfrak{A}$ (or locally finite dimensional category) the following conditions are equivalent:*

(1) $\mathfrak{A}$ *is not wild.*
(2) $\mathfrak{A}$ *is tame.*
(3) *For each* $\mathsf{d} \in \mathrm{Dim}(\mathfrak{A})$ *there is a constructible subset* $Y \subseteq rep_{\mathsf{d}}(\mathfrak{A})$ *such that* $\dim Y \leq 1$ *and Y intersects each isomorphism class of indecomposable representations from* $rep_{\mathsf{d}}(\mathfrak{A})$.
(4) *For each* $\mathsf{d} \in \mathrm{Dim}(\mathfrak{A})$ *there is a constructible subset* $Y \subseteq rep_{\mathsf{d}}(\mathfrak{A})$ *such that* $\dim Y \leq |\mathsf{d}|$ *and Y intersects each isomorphism class from* $rep_{\mathsf{d}}(\mathfrak{A})$.
(5) *If* $Z \subseteq rep_{\mathsf{d}}(\mathfrak{A})$ *is a constructible subset with finite intersections with each isomorphism class, then* $\dim Z \leq |\mathsf{d}|$.

Corollary 4.7. *Suppose that* $\mathfrak{A}, \mathfrak{B}$ *are semi-free boxes (or locally finite dimensional categories) and* $F : \mathfrak{A}\text{-}mod \to \mathfrak{B}\text{-}mod$ *is a functor having the following properties:*

(1) *For each* $\mathsf{d} \in \mathrm{Dim}(\mathfrak{A})$ *the set*
$$F(\mathsf{d}) = \{\, \mathsf{dim}(F(M)) \mid M \in rep_{\mathsf{d}}(\mathfrak{A}) \,\}$$
is finite.
(2) *For each* $\mathsf{d}' \in F(\mathsf{d})$ *the subset*
$$F^{-1}(\mathsf{d}', \mathsf{d}) = \{\, M \in rep_{\mathsf{d}}(\mathfrak{A}) \mid \mathsf{dim}(F(M)) = \mathsf{d}' \,\} \subseteq rep_{\mathsf{d}}(\mathfrak{A})$$
is constructible and the induced map $F^{-1}(\mathsf{d}', \mathsf{d}) \to rep_{\mathsf{d}'}(\mathfrak{B})$ *is locally regular.*

(3) *For each* $L \in \mathfrak{B}$*-mod the set* $F^{-1}(L) = \{\, M \mid F(M) \simeq L \,\}$ *consists of a finite number of isomorphism classes.*

Then, if $\mathfrak{A}$ *is wild, so is also* $\mathfrak{B}$*.*

5. *Galois coverings*

Definition 5.1. An *action* T *of a group* G *on a box* $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ is a mapping $T : g \mapsto T(g)$ where $g \in \mathsf{G}$ and $T(g)$ are automorphisms (i.e., invertible morphisms of boxes) $\mathfrak{A} \to \mathfrak{A}$ such that $T(gh) = T(g)T(h)$ for any $g, h \in \mathsf{G}$. (We do not consider here more general notion of action with a system of factors like in [13, 12].) If the action T is fixed, we say that $\mathfrak{A}$ is a G-*box* and often write gx instead of $T(g)x$.

From now on we suppose that the category $\mathcal{A}$ is rigid.

Definition 5.2. The action T is said to be *free* if $gx \neq x$ for any object $x \in \operatorname{Ver} \mathcal{A}$ and any non-unit element $g \in \mathsf{G}$ (then also $ga \neq a$ for any $a \in \mathcal{A}(x, y)$ or $a \in \mathcal{V}(x, y)$ with $x, y \in \operatorname{Ver} \mathcal{A}$).

Given a G-box $\mathfrak{A}$ we can define a new box $\mathsf{G} \setminus \mathfrak{A} = (\mathsf{G} \setminus \mathcal{A},\ \mathsf{G} \setminus \mathcal{V})$ in the following way. The object set $\operatorname{Ob}(\mathsf{G} \setminus \mathcal{A})$ is the orbit set $\mathsf{G} \setminus (\operatorname{Ob} \mathcal{A})$. For two objects $X, Y \in \operatorname{Ob}(\mathsf{G} \setminus \mathcal{A})$ put

$$(\mathsf{G} \setminus \mathcal{A})(X, Y) = \bigoplus_{x \in X,\, y \in Y} \mathcal{A}(x, y) \,/ U_{\mathcal{A}}$$

and

$$(\mathsf{G} \setminus \mathcal{V})(X, Y) = \bigoplus_{x \in X,\, y \in Y} \mathcal{V}(x, y) \,/ U_{\mathcal{V}}$$

where $U_{\mathcal{A}}$ and $U_{\mathcal{V}}$ denote the subspaces generated by all differences $a - ga$ from the corresponding space. If $a \in \mathcal{A}(x, y)$ and $b \in \mathcal{A}(gy, z)$, the product of the orbits $\mathsf{G}b$ and $\mathsf{G}a$ is defined as the orbit $\mathsf{G}(b(ga))$. One can immediately check that this definition does not depend on the choice of representatives inside the orbits. Quite analogously, the bimodule structure on $\mathsf{G} \setminus \mathcal{V}$ is defined. Moreover, the action of G can be evidently prolonged to $\mathcal{V} \otimes_{\mathcal{A}} \mathcal{V}$ in such way that $\mathsf{G} \setminus (\mathcal{V} \otimes_{\mathcal{A}} \mathcal{V}) \simeq (\mathsf{G} \setminus \mathcal{V}) \otimes_{\mathsf{G} \setminus \mathcal{A}} \mathsf{G} \setminus \mathcal{V}$. Thus, we are able to define a coalgebra structure on $\mathsf{G} \setminus \mathcal{V}$ putting

$$\varepsilon(\mathsf{G}v) = \mathsf{G}\varepsilon(v) \ \text{ and } \ \mu(\mathsf{G}v) = \mathsf{G}\mu(v) \,.$$

So, $\mathsf{G} \setminus \mathfrak{A}$ is indeed a box. Call it the *factor of the* G-*box* $\mathfrak{A}$. The *factoring morphism* of boxes $\Pi : \mathfrak{A} \to \mathsf{G} \setminus \mathfrak{A}$ mapping x to $\mathsf{G}x$ is well-defined and the following properties are immediate.

Proposition 5.3. (1) *Given any morphism of boxes* $\Phi : \mathfrak{A} \to \mathfrak{B}$ *such that* $\Phi(gx) = \Phi(x)$ *for any* $x \in \operatorname{Ob}\mathcal{A} \cup \operatorname{Mor}\mathcal{A} \cup \operatorname{El}\mathcal{V}$ *and* $g \in \mathsf{G}$*, there exists (unique up to isomorphism) a morphism of boxes* $\Psi : \mathsf{G} \setminus \mathfrak{A} \to \mathfrak{B}$ *such that* $\Phi = \Psi\Pi$.

(2) *If the action is free, then for any objects* $x \in \operatorname{Ob}\mathcal{A}$ *and* $Y \in \operatorname{Ob}(\mathsf{G} \setminus \mathcal{A})$ *the morphism* Π *induces isomorphisms:*

$$\bigoplus_{\Pi y = Y} \mathcal{A}(x,y) \simeq (\mathsf{G} \setminus \mathcal{A})(\Pi x, Y)\,;$$
$$\bigoplus_{\Pi y = Y} \mathcal{A}(y,x) \simeq (\mathsf{G} \setminus \mathcal{A})(Y, \Pi x)\,;$$
$$\bigoplus_{\Pi y = Y} \mathcal{V}(x,y) \simeq (\mathsf{G} \setminus \mathcal{V})(\Pi x, Y)\,;$$
$$\bigoplus_{\Pi y = Y} \mathcal{V}(y,x) \simeq (\mathsf{G} \setminus \mathcal{V})(Y, \Pi x)\,.$$

Definition 5.4. Suppose that two boxes $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$, $\mathfrak{B} = (\mathcal{B}, \mathcal{W})$ and a morphism of boxes $\Phi : \mathfrak{A} \to \mathfrak{B}$ are given. Call Φ a *Galois covering of the box* $\mathfrak{B}$ *with Galois group* G if the following conditions hold:

(1) Both $\mathcal{A}$ and $\mathcal{B}$ are rigid categories.
(2) The group G acts freely on the box $\mathfrak{A}$.
(3) The morphism $\Phi : \mathfrak{A} \to \mathfrak{B}$ can be decomposed as $\Psi\Pi$ where $\Pi : \mathfrak{A} \to \mathsf{G} \setminus \mathfrak{A}$ is the factoring morphism and $\Psi : \mathsf{G} \setminus \mathfrak{A} \to \mathfrak{B}$ is an isomorphism.

In this situation we also call the box $\mathfrak{A}$ *a Galois covering of* $\mathfrak{B}$ (with Galois group G).

The notion of Galois coverings of categories (cf. [5]) is a partial case of Galois coverings of boxes: one only has to consider regular boxes. As usually, the advantage of boxes is the possibility of reduction procedures. The following proposition is an immediate consequence of the universality properties of amalgamations and factors.

Proposition 5.5. *Suppose that the following data are given:*

- *a Galois covering of boxes* $\Pi : \widetilde{\mathfrak{A}} \to \mathfrak{A}$ *with Galois group* G*;*
- *a sub-box* $\mathfrak{A}' \subseteq \widetilde{\mathfrak{A}}$ *and its pre-image* $\widetilde{\mathfrak{A}}' = \Pi^{-1}(\mathfrak{A}')$*;*
- *a Galois covering of categories* $\Theta' : \widetilde{\mathcal{B}}' \to \mathcal{B}'$ *with the same Galois group* G*;*

- *a commutative diagram of functors:*

$$\begin{array}{ccc} \widetilde{\mathcal{A}}' & \xrightarrow{\widetilde{F}'} & \widetilde{\mathcal{B}}' \\ \Pi\downarrow & & \downarrow\Theta' \\ \mathcal{A}' & \xrightarrow{F'} & \mathcal{B}' \end{array}$$

such that $\widetilde{F}'(gx) = g\widetilde{F}'(x)$ *for all* x.

Consider the amalgamations $\widetilde{\mathcal{B}} = \widetilde{\mathcal{A}} \coprod^{\widetilde{\mathcal{A}}'} \widetilde{\mathcal{B}}'$ *and* $\mathcal{B} = \mathcal{A} \coprod^{\mathcal{A}'} \mathcal{B}'$ *and the corresponding functors*

$$\widetilde{F} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}} \quad \text{and} \quad F : \mathcal{A} \to \mathcal{B}.$$

Then they induce a commutative diagram of morphisms of boxes:

$$\begin{array}{ccc} \widetilde{\mathfrak{A}} & \xrightarrow{\widetilde{F}} & \widetilde{\mathfrak{A}}^{\widetilde{F}} \\ \Pi\downarrow & & \downarrow\Theta \\ \mathfrak{A} & \xrightarrow{F} & \mathfrak{A}^{F} \end{array}$$

and Θ *is again a Galois covering with the Galois group* G.

Suppose that $\Pi : \widetilde{\mathfrak{A}} \to \mathfrak{A}$ is a Galois covering of semi-free boxes with the Galois group G. We call this covering *degree preserving* if, for some choice of semi-free generators in $\widetilde{\mathfrak{A}}$, $\deg ga = \deg a$ for all morphisms of $\widetilde{\mathcal{A}}$ and elements of $\widetilde{\mathcal{V}}$. We will only consider degree preserving coverings.

Proposition 5.6. *Let* $\Pi : \widetilde{\mathfrak{A}} \to \mathfrak{A}$ *be a degree preserving Galois covering of semi-free boxes with the Galois group* G. *Then:*

1. *There is a set of semi-free generators* $\widetilde{\Sigma}$ *of* $\widetilde{\mathfrak{A}}$ *such that* $\mathsf{G}\widetilde{\Sigma} = \widetilde{\Sigma}$.
2. *The set of the orbits* $\Sigma = \mathsf{G} \setminus \widetilde{\Sigma}$ *is a set of semi-free generators of the box* $\mathfrak{A}$.
3. *For any object* $x \in \operatorname{Ob}\mathfrak{A}$ *the functor* Π *induces an homomorphism* $\pi : \mathsf{Gr}(\mathfrak{A}, \Sigma, x) \to \mathsf{G}$, *which is an epimorphism if* $\widetilde{\mathfrak{A}}$ *is connected.*

Proof. Note first that any degree preserving automorphism of $\mathbf{k}(t)$ maps t to $\alpha t + \beta$ for some $\alpha, \beta \in \mathbf{k}$. Hence, for any $g \in \mathsf{G}$, if a is a marked loop from $\widetilde{\mathcal{A}}$, so is $\alpha\gamma + \beta$ for some $\alpha, \beta \in \mathbf{k}$. Consider an arbitrary set of semi-free generators Σ' of $\widetilde{\mathfrak{A}}$ and a set of representatives X of the orbits of G on $\operatorname{Ob}\widetilde{\mathcal{A}}$. Put $\Sigma_X = \bigcup \widetilde{\mathcal{A}}(y, x)$, where x runs through the objects from X and y through all objects from $\widetilde{\mathcal{A}}$. Let $\widetilde{\Sigma} = \mathsf{G}\Sigma'$. For every element a one can find $g \in G$ such that the source of $g^{-1}a$ belongs to X. Hence, $g^{-1}a = \sum_i a_i b_i$, where $a_i \in \widetilde{\Sigma}$

and $\deg b_i < \deg a$. Now an evident induction shows that $\widetilde{\Sigma}$ generates $\widetilde{\mathfrak{A}}$. Just in the same way one shows that this set of generators is semi-free.

As any element from $\mathfrak{A}$ is a linear combination of elements of the form $\Pi_* a$ with $a \in \mathfrak{A}$, the set $\Sigma = \Pi\widetilde{\Sigma} = \Pi\Sigma_X$ generates $\mathfrak{A}$. Moreover, due to Proposition 5.3(2) any relation between these elements imply a relation between elements from $\widetilde{\Sigma}$. Hence, it is a set of semi-free generators. Now, one can construct a natural homomorphism $\pi : \mathsf{Gr}(\mathfrak{A}, \Sigma, x) \to \mathsf{G}$ (the fundamental group of the pair $(\mathfrak{A}, \Sigma)$). Indeed, let w be a circuit at the point x of the complex $\mathrm{Com}(\mathfrak{A}, \Sigma)$. Evidently, we may suppose that it consists of the arrows of the corresponding bigraph and their inverses. But, for every arrow $s \in \Sigma$, $s : x \to y$ and any object $\widetilde{x} \in \mathrm{Ob}\,\widetilde{A}$ with $\Pi(\widetilde{x}) = x$ there is a unique arrow $\widetilde{s} \in \widetilde{\Sigma}$ such that $\Pi(\widetilde{s}) = s$. Hence, there is a way $\widetilde{w}$ in the complex $\mathrm{Com}(\widetilde{\mathfrak{A}}, \widetilde{\Sigma})$ starting at $\widetilde{x}$ such that $\Pi(\widetilde{w}) = w$. (We extend the mapping Π to the complex $\mathrm{Com}(\widetilde{\mathfrak{A}}, \widetilde{\Sigma})$ in the obvious way.) Let the vertex $y \in \mathrm{Ob}\,\widetilde{A}$ be its end. Then $\Pi(y) = x$, hence, $y = g\widetilde{x}$ for some (uniquely defined) $g \in \mathsf{G}$. Moreover, one can easily see that g only depends on the homotopy class of w and if g' corresponds to another way w' the product $g'g$ corresponds to the concatenation of the ways $w'w$. Therefore, the homomorphism $\pi : \mathsf{Gr}(\mathfrak{A}, \Sigma, x) \to \mathsf{G}$ is well defined. If, moreover, $\widetilde{\mathfrak{A}}$ is connected, then π is epimorphism as for any $g \in \mathsf{G}$ there is a way in $\widetilde{\Sigma}$ starting at x and ending at gx. □

Remark 5.7. One can conjecture that Proposition 5.6 remains valid even if we do not suppose that the action preserves degree. We cannot prove this conjecture, but we never need actions which are not degree preserving. In any case, one has the following evident complement to Proposition 5.5.

Proposition 5.8. *In the situation of Proposition 5.5, suppose that* $\widetilde{\mathfrak{A}}$ *is semi-free, the covering* Π *is degree preserving, the sub-category* $\widetilde{\mathcal{A}}'$ *is generated by a part of semi-free generators of* $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}'$ *is also semi-free, and* Θ' *is degree preserving too. Then* $\widetilde{\mathcal{B}}$ *is also semi-free and* Θ *is degree preserving too.*

It shows that all reduction procedures described above are compatible with the degree preserving coverings.

6. *ind*$_0$ *and* *ind*$_1$

From now on we assume that the field $\mathbf{k}$ is algebraically closed. We denote by $\mathit{ind}\,\mathfrak{A}$ a skeleton of the category of modules $\mathfrak{A}$-*mod*. In other words, it is a full subcategory of $\mathfrak{A}$-*mod* consisting of some representatives of the isomorphism classes of all indecomposable modules.

Proposition 6.1. *Assume that a group* G *acts freely on a box* $\widetilde{\mathfrak{A}}$ *and* $\mathfrak{A} = \mathsf{G} \setminus \widetilde{\mathfrak{A}}$. *Then if* $\widetilde{\mathfrak{A}}$ *is wild, so is* $\mathfrak{A}$.

(For the case of algebras it has been proved in [8].)

Proof. Consider a strict representation $M \in \mathit{rep}(\widetilde{\mathfrak{A}}, \mathbf{k}\langle\, x, y\,\rangle)$ and put $\overline{M} = \Pi_* M$, $\mathsf{S} = \mathsf{St}(\operatorname{Supp} M)$. As $\operatorname{Supp} M$ is finite and the action is free, S is finite. Suppose that $\overline{M} \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L \simeq \overline{M} \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L'$ for some $L, L' \in \mathit{ind}\, \mathbf{k}\langle\, x, y\,\rangle$. Taking inverse images we obtain: $\bigoplus_{g\in\mathsf{G}}(M \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L)^g \simeq \bigoplus_{g\in\mathsf{G}}(M \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L')^g$. As all these direct summands are finite dimensional and indecomposable, we can use the unique decomposition theorem. Taking into account the supports we get $M \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L \simeq (M \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L')^g$ for some $g \in \mathsf{S}$. As M is strict the last formula defines L' up to isomorphism. Hence, there can be only finitely many $L' \in \mathit{ind}\, \mathbf{k}\langle\, x, y\,\rangle$ such that $\overline{M} \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L \simeq \overline{M} \otimes_{\mathbf{k}\langle\, x,y\,\rangle} L'$. Then $\mathfrak{A}$ is wild by Corollary 4.7.

□

Note that if G is torsion free the same consideration shows that if M is strict, $\overline{M}$ is also strict. Moreover, the following proposition holds.

Proposition 6.2. *Let* $\Pi : \widetilde{\mathfrak{A}} \to \mathfrak{A}$ *be a Galois covering of boxes with a torsion free Galois group* G. *Put* $\mathit{ind}_0\, \mathfrak{A} = \Pi_*(\mathit{ind}\, \widetilde{\mathfrak{A}})$ *Then* $\mathit{ind}_0\, \mathfrak{A} \subseteq \mathit{ind}\, \mathfrak{A}$, *and* Π_* *induces an equivalence* $\mathsf{G} \setminus \mathit{ind}\, \widetilde{\mathfrak{A}} \simeq \mathit{ind}_0\, \mathfrak{A}$.

The proof is quite analogous to that given for algebras by Gabriel [15], so we omit it.

Definition 6.3. A representation $N \in \mathcal{R}ep(\widetilde{\mathfrak{A}})$ is called a $\mathbb{Z}$-*representation* if the following conditions hold:

1. N is indecomposable.
2. All spaces $N(x)$, where $x \in \operatorname{Ob} \mathcal{A}$, are finite dimensional.
3. $\mathsf{St}\, N = \{\, g \in \mathsf{G} \mid N^g \simeq N \,\}$ is an infinite cyclic group.
4. $\mathsf{St}\, N \setminus \operatorname{Supp} N$ is finite.

Denote by $\mathit{ind}_{\mathbb{Z}}\, \widetilde{\mathfrak{A}}$ a set of representatives of all isomorphism classes of $\mathbb{Z}$-representation of $\widetilde{\mathfrak{A}}$.

As $\mathsf{St}\, N \simeq \mathbb{Z}$ we can choose isomorphisms $\varphi_g : N \xrightarrow{\sim} N^g$ in such way that $\varphi_{gh} = \varphi_g^h \varphi_h$. Therefore, if N is a $\mathbb{Z}$-representation of $\widetilde{\mathfrak{A}}$ its direct image $\Pi_* N$ can be considered as a representation of $\mathsf{G} \setminus \widetilde{\mathfrak{A}}$ over the group ring $\mathbf{k}[\mathbb{Z}] \simeq \mathbf{k}[T, T^{-1}]$. Moreover, as the group $\mathsf{St}\, N$ acts freely on $\operatorname{Supp} N$, $\Pi_* N$ is free as $\mathbf{k}[\mathbb{Z}]$ -module. Hence, for each $\mathbf{k}[\mathbb{Z}]$-module L we can define the representation $\Pi_* N \otimes_{\mathbf{k}[\mathbb{Z}]} L$ of $\mathfrak{A}$ over $\mathbf{k}$.

Definition 6.4. For any $\mathbb{Z}$-representation of the box $\widetilde{\mathfrak{A}}$ denote by $\Pi_\downarrow N$ the set $\{\, \Pi_* N \otimes_{\mathbf{k}[\mathbb{Z}]} J \mid J \in \mathit{ind}\, \mathbf{k}[\mathbb{Z}] \,\} \subseteq \mathit{rep}\, \mathfrak{A}$.

Lemma 6.5. *Let* G *act freely on a box* $\widetilde{\mathfrak{A}}$ *and* $N \in \mathit{ind}_{\mathbb{Z}} \widetilde{\mathfrak{A}}$. *Then* $\operatorname{End}_{\widetilde{\mathfrak{A}}} N$ *is local.*

Proof. We may (and will) assume that the representation N is *strict*, i.e., $\operatorname{Supp} N = \operatorname{Ver} \widetilde{\mathfrak{A}}$. Denote by d the maximal dimension $\dim N(x)$, by S the stabilizer of N, by s the generator of S, and by D a fundamental domain for S on $\operatorname{Supp} N$. For each arrow a (solid or dotted) let $l(a)$ be the maximal length of all solid paths occurring in ∂a. As D is finite and there are only finitely many arrows starting or ending at each vertex, $l = \sup \{\, l(a) \mid a \in \operatorname{Supp} N \,\}$ is finite. Just in the same way, $\nu = \sup \{\, \nu(a) \mid a \in \operatorname{Supp} N \,\}$ is also finite, $\nu(a)$ being the triangular height of a.

Consider all arrows starting or ending in D. Find such an integer m that all of them have both sources and targets in $D_1 = \bigcup_{i=-m}^{m} s^i D$. Then any arrow having one end in D_1 has the second one in $D_1 \cup s^m D_1 \cup s^{-m} D_1$. Put $d_1 = |D_1|$.

Let $\alpha \in \operatorname{End}(N)$ and $\alpha_x = \alpha(\omega_x)$. Every α_x is a linear mapping in the finite dimensional vector space $N(x)$, hence, there is a non-zero polynomial annihilating α_x. If there is a common annihilating polynomial $f(T)$ for all α_x then α is either invertible or nilpotent. Indeed, otherwise $f(T) = f_1(T) f_2(T)$ for coprime polynomials f_1, f_2. We can suppose that $f(T)$ is of the minimal possible degree. There are such polynomials $u_i(T)$ that $u_2(T) f_1(T) + u_1(T) f_2(T) = 1$ and $\deg u_i < \deg f_i$. Therefore, $u_2(\alpha_x) f_1(\alpha_x)$ are all idempotents and not all of them are 0 or 1. As it was shown in [18] then there is a non-trivial idempotent in $\operatorname{End}(N)$, that is impossible.

Suppose that there is no polynomial annihilating all α_x. As all dimensions $\dim N(x)$ are bounded, there should be infinitely many elements $\lambda \in \mathbf{k}$ occurring as eigenvalues of α_x. Find a subset $Q \subset \operatorname{Supp} N$ such that the mappings α_x for $x \in Q$ have at least $2r + 1$ distinct eigenvalues where $r = 2 d d_1 (2\nu - 1)(2l + 1)$. We can suppose that

$Q = \cup_{i=0}^{q} s^i(D_1)$ for some q. Put

$$\begin{aligned} D_- &= \bigcup_{i=0}^{l} s^{im} D_1\,, \\ D_+ &= \bigcup_{i=l}^{2l} s^{im} D_1\,, \\ D_\pm^i &= \begin{cases} s^{q+(2l+1)im} D_\pm & \text{if } i > 0 \\ s^{(2l+1)im} D_\pm & \text{if } i < 0 \end{cases} \\ D_2^i &= D_-^i \cup D_+^i\,, \\ P &= \bigcup_{i=1}^{2\nu-1} (D_2^i \cup D_2^{-i})\,, \end{aligned}$$

$$\begin{aligned} Q_1 &= Q \cup (\bigcup_{i=1}^{\nu} (D_2^i \cup D_2^{-i})\,, \\ Q' &= Q_1 \setminus (D_+^\nu \cup D_-^{-\nu})\,. \end{aligned}$$

Then there are at most $2r$ distinct eigenvalues of all α_x with $x \in P$; hence, they do not exhaust all eigenvalues of α_x with $x \in Q$. Therefore, we can find a polynomial $f(T)$ such that $f(\alpha_x) = 0$ for $x \in P$ and $f(\alpha_x)$ is non-nilpotent for some $x \in Q$. Put $\beta = f(\alpha)^\nu$. Then $\beta(\varphi) = 0$ for each dotted arrow φ having both target and source in $D_2^{\pm\nu}$.

Define an homomorphism $\gamma : N \to N$ in the following way:

$$\begin{aligned} \gamma_x &= \beta_x \ \text{ for } \ x \in Q_1\,; \\ \gamma_x &= 0 \ \text{ otherwise}\,; \\ \gamma(\varphi) &= \beta(\varphi) \ \text{ if both source and target of } \varphi \text{ are in } \ Q_1\,; \\ \gamma(\varphi) &= 0 \ \text{ otherwise}\,. \end{aligned}$$

Certainly, we have to verify that γ is indeed an homomorphism, i.e., to check for each solid arrow $a : x \to y$ with $\partial a = \sum_j p_j \varphi_j q_j$ that

$$N(a)\gamma_x - \gamma_y N(a) = \sum_j N(p_j)\gamma(\varphi_j)N(q_j)\,. \tag{1}$$

Suppose first that either x or y, or the source or the target of at least one of φ_j is in Q'. Then $x, y \in Q_1$ and all φ_j have their targets and sources in Q_1. Hence, all values of γ occurring in (1) coincide with the corresponding values of β and (1) holds for γ as it holds for β. Otherwise, all values of γ occurring in (1) are zeroes as their sources and targets are either in $D_2^{\pm\nu}$ or outside Q_1. Hence, (1) holds trivially.

Thus, we have obtained an endomorphism of N which is neither nilpotent nor invertible and is annihilated by some polynomial. As we have seen before it is impossible. □

Lemma 6.6. *Suppose that a group* G *acts freely on a semi-free box* $\widetilde{\mathfrak{A}}$, $\mathfrak{A} = \mathsf{G} \setminus \widetilde{\mathfrak{A}}$ *and* $N \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$. *Then:*

(1) $\Pi_* N$ *is strict, i.e., is a rational family of* $\widetilde{\mathfrak{A}}$*-modules.*
(2) *No module from* $ind_0 \mathfrak{A}$ *belongs to the family* $\Pi_* N$.
(3) *If* $N' \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$ *is such that* $N' \not\simeq N^g$ *for all* $g \in \mathsf{G}$ *then no* $\widetilde{\mathfrak{A}}$*-module belonging to the family* $\Pi_* N$ *belongs to* $\Pi_* N'$.

We will prove a generalization of this lemma in Section 8 (Lemma 8.5).

7. *Main Theorem for Boxes*

Now we can state the main theorem of this article for the case of boxes. Remind that the ground field is supposed to be algebraically closed and all semi-free boxes are supposed triangular.

Theorem 7.1. *Let* $\Pi : \widetilde{\mathfrak{A}} \to \mathfrak{A}$ *be a degree preserving Galois covering of semi-free boxes with a torsion free Galois group* G; $\widetilde{\mathfrak{A}} = (\widetilde{\mathcal{A}}, \widetilde{\mathcal{V}})$, $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$. *Then:*

(1) $\mathfrak{A}$ *is tame if and only if so is* $\widetilde{\mathfrak{A}}$.
(2) *If* $\widetilde{\mathfrak{A}}$ *is tame (hence,* $\mathfrak{A}$ *is also tame), then:*
  (a) $ind\, \mathfrak{A} = ind_0 \mathfrak{A} \sqcup ind_1 \mathfrak{A}$.
  (b) *If* $M \in ind_0 \mathfrak{A}$, $M' \in ind_1 \mathfrak{A}$ *or* $M \in \Pi_\downarrow N$, $M' \in \Pi_\downarrow N'$ *for* $N' \not\simeq N$, *then* $\mathrm{Hom}_{\mathfrak{A}}(M, M') \cup \mathrm{Hom}_{\mathfrak{A}}(M', M) \subseteq \mathrm{rad}^\infty_{\mathfrak{A}}$. *Moreover, if* $M = \Pi_* N \otimes_{\mathbf{k}[\mathbb{Z}]} J$ *and* $M' = \Pi_* N \otimes_{\mathbf{k}[\mathbb{Z}]} J'$ *for* $N \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$, *then* $\mathrm{Hom}_{\mathfrak{A}}(M, M') = 1 \otimes \mathrm{Hom}_{\mathbf{k}[\mathbb{Z}]}(J, J') + \mathrm{rad}^\infty_{\mathfrak{A}}(M, M')$.
  (c) *If* $N \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$, *there is a representation* $\overline{N} \in rep(\mathfrak{A}, \mathbf{k}[T])$ *such that* $\Pi_* N \simeq \overline{N} \otimes_{\mathbf{k}[T]} \mathbf{k}[\mathbb{Z}]$ *and* $\overline{N} \otimes_{\mathbf{k}[T]} J_{0,m} \in ind_0 \mathfrak{A}$ *for each* m, *where* $J_{0,m} = \mathbf{k}[T]/(T^m)$. (So to speak, the representations from $ind_1 \mathfrak{A}$ are deformations of those from $ind_0 \mathfrak{A}$.)

Here $\mathrm{rad}_{\mathfrak{A}}$ denotes the radical of the category $rep(\mathfrak{A})$. (Note that it is always locally finite dimensional.)

Proof. We know that if $\widetilde{\mathfrak{A}}$ is wild then $\mathfrak{A}$ is also wild (cf. Proposition 6.1). Suppose that $\widetilde{\mathfrak{A}}$ is tame and let $M \in \mathfrak{A}$-*mod* be any finite dimensional $\mathfrak{A}$-module. We are going to prove that M is isomorphic to a module either from $ind_0 \mathfrak{A}$ or from $ind_1 \mathfrak{A}$. Without loss of generality we may suppose that M is *sincere*, i.e., $M(x) \neq 0$ for all $x \in \mathrm{Ob}\, \mathcal{A}$.

Now use the induction by $||M||$. If $||M|| = 0$, M is a trivial representation: $M \simeq S_x$ for some $x \in \operatorname{Ob}\mathcal{A}$. Then $M \simeq \Pi_* S_y \in ind_0\,\mathfrak{A}$ where y is a preimage of x in $\operatorname{Ob}\widetilde{\mathcal{A}}$. Now suppose that the claim is true for all boxes and all representations M' with smaller values of $||M'||$. Choose a semi-free triangular system of generators $\widetilde{\Sigma}$ of $\widetilde{\mathfrak{A}}$ such that $\Sigma = \mathsf{G}\backslash\widetilde{\Sigma}$ is a semi-free triangular system of generators for $\mathfrak{A}$ (cf. Proposition 5.6). Then the system Σ contains either a minimal edge, or a minimal loop, or a superfluous arrow.

Let first $a : x \to y$ be a minimal edge. Then its preimage in $\widetilde{\Sigma}$ is a set $\{\, a_i : x_i \to y_i \,\}$ where all a_i are also minimal edges. Let $\mathcal{A}'$ be the subcategory of $\mathcal{A}$ generated by a and $\widetilde{\mathcal{A}}'$ be the subcategory of $\widetilde{\mathcal{A}}$ generated by all a_i. Denote by $\mathcal{B}'$ the trivial category with three objects x, y, xy, by $\widetilde{\mathcal{B}}'$ the trivial category with the objects $x_i, y_i, x_i y_i$, and consider the functors $F' : \mathcal{A}' \to \mathcal{B}'$ and $\widetilde{F}' : \widetilde{\mathcal{A}}' \to \widetilde{\mathcal{B}}'$ defined as in Example 3.2 (1). Then we are in the situation of Proposition 5.5. Hence, we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{A}} & \xrightarrow{\widetilde{F}} & \widetilde{\mathfrak{A}}^{\widetilde{F}} \\ \Pi\downarrow & & \downarrow\Theta \\ \mathfrak{A} & \xrightarrow{F} & \mathfrak{A}^{F} \end{array} \tag{2}$$

of Galois coverings with the group G. In this case the morphisms $\widetilde{F}$ and F induce equivalences of the representation categories $\widetilde{\mathfrak{A}}^{\widetilde{F}}\text{-}mod \simeq \widetilde{\mathfrak{A}}\text{-}mod$ and $\mathfrak{A}^F\text{-}mod \simeq \mathfrak{A}\text{-}mod$ respectively. In particular, the box $\widetilde{\mathfrak{A}}^{\widetilde{F}}$ is tame. Moreover, if $M \simeq F^* M'$ then $||M'|| < ||M||$, as $M(x) \neq 0$ and $M(y) \neq 0$ (cf. Proposition 3.5). Hence, M' is isomorphic to a representation belonging either to $ind_0\,\mathfrak{A}^F$ or to $ind_1\,\mathfrak{A}^F$. But as the diagram above is commutative, $ind_0\,\mathfrak{A} \simeq F^*\, ind_0\,\mathfrak{A}^F$ and $ind_1\,\mathfrak{A} \simeq F^*\, ind_1\,\mathfrak{A}^F$. Thus, M is also isomorphic to a representation belonging either to $ind_0\,\mathfrak{A}$ or to $ind_1\,\mathfrak{A}$.

Quite the same observation is valid in the case when Σ contains a superfluous arrow or a minimal loop a such that its preimage in $\widetilde{\Sigma}$ consists of loops.

Suppose now that $a : x \to x$ is a minimal loop from Σ and its preimage contains an edge $a_0 : x_0 \to x_1$ with $x_0 \neq x_1$. Then $\Pi x_0 = \Pi x_1$, hence, $x_1 = g x_0$ for some $g \in \mathsf{G}$. Consider the cyclic group $\mathsf{C} = \langle\, g \,\rangle$ and put $x_k = g^k x$, $a_k = g^k a : x_k \to x_{k+1}$. As $g^k \neq 1$ for $k \neq 0$, all objects x_k and all edges a_k are pairwise distinct. Therefore, the subcategory $\mathcal{L} \subseteq \mathcal{A}$ generated by all these objects and arrows is a *line*, i.e., is of the shape:

$$\dots \xrightarrow{a_{-1}} x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \dots \xrightarrow{a_{k-1}} x_k \xrightarrow{a_k} x_{k+1} \xrightarrow{a_{k+1}} \dots \tag{3}$$

Moreover, as G acts transitively on the preimage of a, this preimage is a disjoint union of lines. In this case we say that *a is a loop lifting into lines*.

Suppose first that $M(a)$ is nilpotent: $M(a)^n = 0$. Let $\mathcal{A}'$ be the subcategory of $\mathcal{A}$ generated by a and $\widetilde{\mathcal{A}}'$ be the subcategory of $\widetilde{\mathcal{A}}$ generated by all preimages of a. We know that $\widetilde{\mathcal{A}}'$ is a disjoint union of lines. Hence, its indecomposable representations are in 1-1 correspondence with the connected finite parts of $\widetilde{\mathcal{A}}'$. Let $\mathcal{B}'$ be a trivial category with n objects x^m ($m = 1, 2, \dots n$). Define a functor $F' : \mathcal{A}' \to \mathcal{B}'$ mapping x into the direct sum $\bigoplus_{m=1}^n mx^m$ and a into the direct sum of Jordan cells $\bigoplus_{m=1}^n J_m(0)$, where $J_m(0)$ is considered as a morphism of mx^m to itself.

Let now $\widetilde{\mathcal{B}}'$ be a trivial category whose objects x^I are in 1-1 correspondence with finite connected parts $I \subseteq \widetilde{\mathcal{A}}'$ consisting at most of n objects. Define a functor $\widetilde{F}' : \widetilde{\mathcal{A}}' \to \widetilde{\mathcal{B}}'$ mapping an object y into the direct sum $\bigoplus_{I \ni y} x^I$ and a morphism $b : y \to gy$ into the morphism $\bigoplus_{I \ni y} x^I \to \bigoplus_{I \ni gy} x^I$ such that all its non-zero components are just $1 : x^I \to x^I$ for $I \supset \{\, y, gy \,\}$. Define also a functor $\widetilde{\mathcal{B}}' \to \mathcal{B}'$ mapping x^I to x^m where $m = |I|$.

Now we are again in the situation of Proposition 5.5. Hence, we can construct a commutative diagram of the form (2) of Galois coverings with the group G. In this case the image of F^* consists of all representations of $\mathfrak{A}$ which are nilpotent of degree n at the loop a. In particular, this image contains M: $M \simeq F^*M'$, where $M' \in \mathit{ind}\,\mathfrak{A}^F$ and $||M'|| < ||M||$ as $M(x) \neq 0$. Therefore, just as above, we can conclude that M is isomorphic to a module from $\mathit{ind}_0\,\mathfrak{A} \cup \mathit{ind}_1\,\mathfrak{A}$.

To get the whole claim we need the following lemma that will be proved in the next section.

Lemma 7.2. *In the situation of Theorem 7.1, suppose that a is a minimal loop in $\mathcal{A}$ lifting to lines and M is an indecomposable $\mathfrak{A}$-module. If $\widetilde{\mathfrak{A}}$ is tame then either $M(a)$ is nilpotent or M is isomorphic to such module M' that $M'(b) = 0$ for all arrows $b \in \Sigma$ except of a.*

The consideration above and this lemma imply that each $M \in \mathit{ind}\,\mathfrak{A}$ is isomorphic either to a module from $\mathit{ind}_0\,\mathfrak{A} \cup \mathit{ind}_1\,\mathfrak{A}$ or to a module M' such that $M'(b) = 0$ for all arrows $b \in \Sigma$ except of some minimal loop a lifting to lines, while $M'(a)$ is invertible. Then we may suppose that $M'(a) = J_m(\lambda)$, the Jordan cell of size $m \times m$ with the eigenvalue

λ. Take any line $\mathcal{L}$ from the preimage of a and consider the $\widetilde{\mathfrak{A}}$-module L such that $L(x_k) = \mathbf{k}$ for all objects $x_k \in \mathcal{L}$, $L(y) = 0$ for all other objects, $L(a_k) = 1$ for all arrows $a_k \in \mathcal{L}$ and $L(c) = 0$ for all arrows $c \notin \mathcal{L}$. Evidently, L is an indecomposable $\mathbb{Z}$-representation of $\widetilde{\mathfrak{A}}$ and $M' \simeq L \otimes_{\mathbf{k}[\mathbb{Z}]} \mathbf{k}[\mathbb{Z}]/(T-\lambda)^m$. In particular, $M' \in ind_1 \mathfrak{A}$.

Hence, we have proved claim 2a of the theorem. To prove claim 1 it only remains to prove the following result.

Proposition 7.3. *In the situation of Theorem 7.1, denote by $\mathcal{Z}$ the set of rational families $\left\{ \Pi_\downarrow N \mid N \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}} \right\}$. If $\widetilde{\mathfrak{A}}$ is tame, this family is locally finite.*

Proof. Let d be a dimension of representations of the box $\mathfrak{A}$. We prove that $\mathcal{Z}_\mathsf{d}$ is finite using induction by $m = ||\mathsf{d}||$. Again, we suppose that this dimension is sincere, i.e., $\mathsf{d}(x) \neq 0$ for all $x \in \operatorname{Ver}\mathfrak{A}$. If $m = 0$, $\widetilde{\mathfrak{A}}$ is so-trivial, thus $\mathcal{Z} = \emptyset$. Let our claim be true for all boxes and all dimensions d' with $||\mathsf{d}'|| < m$. Just as above, we should consider the following cases:

(i) $\mathfrak{A}$ has a minimal edge;
(ii) $\mathfrak{A}$ has a superfluous arrow;
(iii) $\mathfrak{A}$ has a minimal loop $a : x \to x$ lifting to lines;
(iv) $\mathfrak{A}$ only has minimal loops lifting to loops (neither minimal edges or loops lifting to lines, nor superfluous arrows).

Cases (i) and (ii) are easy: we use the same reduction process as above and get a commutative diagram (2) such that $\widetilde{F}^*$ induces, in particular, an equivalence $ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}^{\widetilde{F}} \simeq ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$. Moreover, given d, the set $D(F, \mathsf{d}) = \left\{ \operatorname{Dim} M \mid M \in rep(\mathfrak{A}^F) \text{ and } \operatorname{Dim} F^*M = \mathsf{d} \right\}$ is finite, and if $\mathsf{d}' \in D(F,\mathsf{d})$, then $||\mathsf{d}'|| < m$. So, the inductive supposition implies that our claim is valid.

In case (iii) Lemma 7.2, together with Lemma 6.6, implies that for every $N \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$ either $N(b) = 0$ for every solid arrow b such that $\Pi b \neq a$ or $\Pi_* N(a)$ is nilpotent. There is only one (up to G-shift) representation of the former type, namely, such that $N(x_k) = \mathbf{k}$ and $N(a_k) = 1$ for some line (x_k, a_k) with $\Pi a_k = a$. On the other hand, to the representations of the latter type one can apply the reduction of minimal lines up to degree $\mathsf{d}(x)$ and then use the induction argument as before.

In the remaining case (iv) we can repeat the arguments from [11] (cf. also [7]) to prove that if c is a minimal loop in $\widetilde{\mathfrak{A}}$ there is a polynomial $f(t) \in \mathbf{k}[t]$ such that for any $N \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$ with $\operatorname{Dim} \Pi_* N = \mathsf{d}$ either $f(N(c))$ is nilpotent or $N(b) = 0$ for every solid arrow $b \neq c$.

Certainly, there are no indecomposable $\mathbb{Z}$-representations of the latter type while to the former one we are able to apply the reduction of minimal loops c with $\Pi c = a : x \to x$ up to degree $\mathsf{d}(x)$. Indeed, we have to reduce λ-parts of c for every root λ of the polynomial $f(t)$, but, nevertheless, it gives us only finitely many possibilities for reductions as well as for dimensions of the reduced representations. Thus, the same inductive argument accomplishes the proof of this proposition and, therefore, of claim 1 of Theorem 7.1. □

Claim 2b follows from a bit more general statement. For a loop $a : x \to x$ denote by $J_k(a,\lambda)$ a representation mapping x to $m\mathbf{k}$, objects $y \neq x$ to 0, a to $J_k(\lambda)$, and arrows $b \neq a$ to 0.

Proposition 7.4. *Let $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ be a semi-free triangular box, $a : x \to x$ be an invertible minimal loop from $\mathcal{A}$. Suppose that for every $M \in ind\,\mathfrak{A}$, either $M(x) = 0$ or $M(a) = J_k(a,\lambda)$ for some k,λ. Then each homomorphism $\varphi \in \mathrm{Hom}_{\mathfrak{A}}(M,M') \cup \mathrm{Hom}_{\mathfrak{A}}(M',M)$ where $M, M' \in ind\,\mathfrak{A}$, $M'(x) \neq 0$ and $\varphi(\omega_x) = 0$ belongs to $\mathrm{rad}^{\infty}_{\mathfrak{A}}$. In particular, $\mathrm{Hom}_{\mathfrak{A}}(M,M') \cup \mathrm{Hom}_{\mathfrak{A}}(M',M) \subseteq \mathrm{rad}^{\infty}_{\mathfrak{A}}$ if either $M(a)$ and $M'(a)$ have distinct eigenvalues or $M(x) = 0$.*

Note that we need not even suppose $\mathfrak{A}$ to be tame!

Proof. Evidently, it is enough to factorize any morphism $\varphi : M \to M'$ (or $M' \to M$) as $\eta\varphi'$ (respectively, $\varphi'\eta$) where $\varphi' : M \to M''$ (respectively, $\varphi' : M'' \to M$) for an indecomposable module M'' which is non-isomorphic to M', $M''(x) \neq 0$ and $\varphi'(\omega_x) = 0$.

First consider the case when there are no solid arrows between x and $\mathrm{Supp}\,M$. In this case *any* choice of $\varphi(v)$ where v runs through all dotted arrows between $\mathrm{Supp}\,M$ and x defines an homomorphism $M \to M'$. Suppose that $M' = J_k(a,\lambda)$. Consider the matrix $T = (0\ \mathrm{I}_k)$, where 0 denotes the zero column, and its transposed $T^\top$. Let $M''(a) = J_{k+1}(a,\lambda)$. Then putting $\eta(\omega_x) = T$ and $\eta(\alpha) = 0$ for all dotted arrows α we obtain a homomorphism $\eta : M'' \to M'$. Now define $\varphi' : M \to M''$ putting $\varphi'(v) = T^\top\varphi(v)$. Then $\varphi = \eta\varphi'$ is the necessary factorization. The case when $\varphi : M' \to M$ is treated analogously.

Now use the induction by $m = ||M \oplus M'||$. Clearly, we may suppose that this direct sum is sincere. If $m = 1$ the module M is trivial: $M = S_y$ and there are no solid arrows between x and y. Hence, we can use the above consideration. Suppose that the claim is true for all smaller values of m. If there is a minimal edge, or a minimal loop $b \neq a$, or a superfluous arrow, we can diminish m using the corresponding reduction process. So the only case we have to consider is when a is the unique minimal arrow.

Choose a solid arrow b such that its differential contains the unique solid arrow a. Suppose first that $b : x \to y$ where $y \neq x$. Then $\partial b = \sum_{i=1}^{s} u_i f_i(a)$ for some Laurent polynomials $f_i \in \mathbf{k}[t, t^{-1}]$. Let $h(t) = \gcd(f_1, f_2, \dots, f_s)$. If $h = 1$ the arrow b is superfluous. Then we can reduce it and diminish the value of m. Otherwise, choose $\lambda \neq 0$ in $\mathbf{k}$ such that $h(\lambda) = 0$. Define $\mathfrak{A}$-module M putting $M(x) = M(y) = \mathbf{k}$, $M(z) = 0$ for all objects except of x, y, $M(a) = \lambda$, $M(b) = 1$, and $M(c) = 0$ for all arrows except of a, b. Then M is indecomposable and does not satisfy the conditions of the proposition. The case when $b : y \to x$ with $y \neq x$ is treated in the same way.

Now suppose that $b : x \to x$. Then $\partial b = \sum_{i=1}^{s} f_i(t_1, t_2, t_1^{-1}, t_2^{-1}) u_i$ for some Laurent polynomials f_i in two variables. Here, just as in [11], t_1 describes the left multiplication by a and t_2 the right one. Consider the ideal I generated by $f_1, f_2, \dots, f_s$. If $I = (1)$, the arrow b is superfluous and can be reduced. Otherwise, there is a pair (λ, μ) of non-zero elements from $\mathbf{k}$ such that $f_i(\lambda, \mu) = 0$ for all i. Define $\mathfrak{A}$-module M putting $M(x) = \mathbf{k}^2$, $M(y) = 0$ for $y \neq x$,

$$M(a) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad M(b) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and $M(c) = 0$ for all arrows except a, b. This module is again indecomposable and does not satisfy the conditions of the proposition. $\square$

Now we can prove 2b using the induction by $m = ||M \oplus M'||$. Note that $||M'|| \geq 1$ and $||M'|| = 1$ if and only if $M' = J_1(a, \lambda)$ for some invertible minimal loop a. In this case we can use Proposition 7.4. If there is a minimal edge, or a minimal loop lifting to loops, or a superfluous edge in $\mathfrak{A}$, we use the corresponding reduction process diminishing m. So we may suppose that the only minimal arrows in $\mathfrak{A}$ are minimal loops lifting to lines. Let $a : x \to x$ be one of them. If both $M(a)$ and $M'(a)$ are nilpotent, we can also use the reduction process. Hence, we may suppose that either M or M' is isomorphic to some of $J_k(a, \lambda)$. In view to Lemma 7.2, a satisfies the conditions of Proposition 7.4, that accomplishes the proof.

To prove 2c consider $N \in ind_{\mathbb{Z}} \widetilde{\mathfrak{A}}$. We may suppose that N is sincere. Use the induction by $m = ||\Pi_* N||$. If $m = 1$, then $\mathcal{A}$ consists of a unique vertex x and a unique loop $a : x \to x$. Hence, $\widetilde{\mathcal{A}}$ is indeed a line of the shape (3), $N(x_i) = \mathbf{k}$, $N(a_i) = 1$. Therefore, $\Pi_* N(x) = \mathbf{k}[\mathbb{Z}]$, $\Pi_*(a) = T$. In this case we can put $\overline{N}(x) = \mathbf{k}[T]$, $\overline{N}(a) = T$, and the claim becomes obvious. Suppose that it is true when $||\Pi_* N|| < m$. Again, if there is a minimal edge, a superfluous arrow or a minimal loop lifting to loops, we can make a reduction step diminishing m. If there is a minimal loop lifting to lines, then, as we have just proved, either $\Pi_* N$ is nilpotent on this loop or it is non-zero only on it. In the first case we can again use a reduction, while the second one coincides with the base of induction. □

It is also easy now to compare the Auslander-Reiten quivers of a box and of its covering. Remind that the *diagram* (or the *quiver*) of a locally finite dimensional category $\mathcal{C}$ is defined as the oriented graph, whose vertices coincide with those of $\mathcal{C}$ and there are $r(x,y)$ arrows with the source x and the target y where

$$r(x,y) = \dim_{\mathbf{k}} \operatorname{rad}(x,y)/\operatorname{rad}^2(x,y)\,.$$

The *Auslander-Reiten quiver* $\mathsf{AR}\,\mathfrak{A}$ of a box $\mathfrak{A}$ (in particular, of a category) is, by definition, the diagram of the category $\mathfrak{A}$-*mod* . Remind also that a *homogeneous tube* is, by definition, a connected component of an Auslander-Reiten quiver which contains no projective modules and has the shape:

$$M_1 \leftrightarrows M_2 \leftrightarrows M_3 \leftrightarrows \dots$$

In the situation of Theorem 7.1, denote by $\mathsf{AR}_0\,\mathfrak{A}$ and $\mathsf{AR}_1\,\mathfrak{A}$, respectively, the diagrams of the categories $ind_0\,\mathfrak{A}$ and $ind_1\,\mathfrak{A}$. Then this theorem immediately implies the following result concerning the Auslander-Reiten quivers.

Corollary 7.5. *In the situation of Theorem 7.1:*

- $\mathsf{AR}\,\mathfrak{A} = \mathsf{AR}_0\,\mathfrak{A} \sqcup \mathsf{AR}_1\,\mathfrak{A}$ *(a disjoint union);*
- $\mathsf{AR}_0\,\mathfrak{A} \simeq \mathsf{G} \setminus \mathsf{AR}\,\widetilde{\mathfrak{A}}$ *;*
- $\mathsf{AR}_1\,\mathfrak{A}$ *is a disjoint union of homogeneous tubes.*

Remark 7.6. As usually, any results valid for boxes can immediately be drawn to *bimodules* (cf. [11, 7]), which are indeed a partial case. Moreover, just as for algebras, any Galois covering of bimodules induces a degree preserving Galois covering of the corresponding boxes. So, one can just substitute in Theorem 7.1 and other results of Sections 6 and 7 “bimodules” for “boxes” crossing out the words “degree preserving.” On the contrary, at the moment we do not see any direct proof of these results which would not involve the tedious construction of quasi-triangular boxes from the next section.

8. *Proof of the Main Lemma*

This section is devoted to the proof of Lemmas 7.2 and 6.6. To do it we need to consider a bit more general situation.

Definition 8.1. A *quasi-triangular* box is a box $\mathfrak{A} = (\mathcal{A}, \mathcal{V})$ with a fixed section $\boldsymbol{\omega}$, a system of generators $\Sigma = \Sigma_0 \sqcup \Sigma_1 \sqcup \Sigma'$ and a set $\mathfrak{L}$ of lines from $\mathcal{A}$ satisfying the following conditions:

(1) Σ_0 is a semi-free system of generators for $\mathcal{A}$ and $\Sigma_1 \sqcup \Sigma'$ is a system of generators of the kernel $\overline{\mathcal{V}}$.
(2) Σ_1 is a free system of generators for a free sub-bimodule $\overline{\mathcal{V}}_1$ of $\overline{\mathcal{V}}$ and, for each arrow $u \in \Sigma'(x, y)$, either x or y lies on a line from $\mathfrak{L}$.
(3) The box $\mathfrak{L}^{-1}\mathfrak{A}$ is semi-free triangular with the normal section $\boldsymbol{\omega}$ and the semi-free system of generators $\Sigma_0 \cup \Sigma_1$.
(4) The image in $\mathfrak{L}^{-1}\mathfrak{A}$ of ∂a is zero for each arrow a belonging to any line $\mathcal{L} \in \mathfrak{L}$.
(5) Distinct lines from $\mathfrak{L}$ have no common objects and no object belonging to such a line is a marked vertex in $\mathcal{A}$ (in the sense of the definition of semi-free categories).
(6) There is a function $\kappa : \mathfrak{L} \to \mathbb{N}$ such that the set of its values is bounded and for each arrow a belonging to a line $\mathcal{L} \in \mathfrak{L}$

$$\partial a \in \sum_{\substack{x \in \mathcal{L}' \\ \kappa\mathcal{L}' < \kappa\mathcal{L}}} (\mathcal{A} 1_x \overline{\mathcal{V}} + \overline{\mathcal{V}} 1_x \mathcal{A}) .$$

Here $\mathfrak{L}^{-1}\mathcal{A}$ denotes the category obtained from $\mathcal{A}$ by inverting all arrows from all lines $\mathcal{L} \in \mathfrak{L}$ and $\mathfrak{L}^{-1}\mathfrak{A} = (\mathfrak{L}^{-1}\mathcal{A}, \mathfrak{L}^{-1}\mathcal{A} \otimes_{\mathcal{A}} \mathcal{V} \otimes_{\mathcal{A}} \mathfrak{L}^{-1}\mathcal{A})$.

We call Σ a *quasi-triangular system of generators* of $\mathfrak{A}$ and the lines from $\mathfrak{L}$ the *special lines*.

We will consider a special sort of actions of groups on quasi-triangular boxes.

Definition 8.2. Let a group G acts freely on a quasi-triangular box $\mathfrak{A}$ with a quasi-triangular system of generators $\Sigma = \Sigma_0 \cup \Sigma_1 \cup \Sigma'$ and the set of special lines $\mathfrak{L}$ as above. We say that G *acts well* if the following conditions hold:

(1) $g\Sigma_0 = \Sigma_0$ and $g\Sigma_1 = \Sigma_1$ for each $g \in \mathsf{G}$.
(2) $g\mathfrak{L} = \mathfrak{L}$ for each $g \in \mathsf{G}$.
(3) The stabilizer $\mathsf{St}\,\mathcal{L}$ acts transitively on $\mathcal{L}$ for each $\mathcal{L} \in \mathfrak{L}$.

Note that if $\mathfrak{A}$ is semi-free and the action is degree preserving, Proposition 5.6 imply that one can always suppose that G acts well. From now on we consider the situation described in the following definition.

Definition 8.3. Let a group G act well on a quasi-triangular box $\widetilde{\mathfrak{A}}$ with a quasi-triangular system of generators $\widetilde{\Sigma} = \widetilde{\Sigma}_0 \sqcup \widetilde{\Sigma}_1 \sqcup \widetilde{\Sigma}'$. Put

$\mathfrak{A} = \mathsf{G} \setminus \widetilde{\mathfrak{A}}$, $\widetilde{\mathfrak{A}}^* = \mathfrak{L}^{-1}\widetilde{\mathfrak{A}}$, $\Sigma_i = \Pi\widetilde{\Sigma}_i$ $(i = 0, 1)$, $\mathfrak{A}^* = (\Pi\mathfrak{L})^{-1}\mathfrak{A}$, and denote by ∂^* the differentials in $\widetilde{\mathfrak{A}}^*$ and $\mathfrak{A}^*$. We expand the action of G onto the box $\widetilde{\mathfrak{A}}^*$ in the obvious way and identify $\mathfrak{A}^*$ with $\mathsf{G} \setminus \widetilde{\mathfrak{A}}^*$. Call the images of arrows from special lines the *special loops*. An arrow $a \in \Sigma_0$ is called a *quasi-minimal edge* (respectively, *quasi-superfluous arrow*) if its image in $\mathfrak{A}^*$ is a minimal edge (respectively, a superfluous arrow). On the other hand, call a a *quasi-minimal loop* if either it is a special loop or its image in $\mathfrak{A}^*$ is a minimal loop. Note that for each object $x \in \operatorname{Ob} \mathcal{A}$ there is at most one special loop in $\Sigma_0(x, x)$. If there is one, call x a *special vertex*.

Note that $\mathfrak{A}^*$ is also a semi-free box. We are going to prove the following generalization of Lemma 7.2.

Lemma 8.4. *In the situation of Definition 8.3, suppose that* G *is torsion free,* $\widetilde{\mathfrak{A}}$ *is not wild, and* a *is a quasi-minimal loop from* Σ_0 *lifting to lines. For each* $M \in \operatorname{ind} \mathfrak{A}^*$ *either* $M(a)$ *is nilpotent or* $M \simeq J_k(a, \lambda)$ *for some* $k \in \mathbb{N}$ *and* $\lambda \in \mathbf{k} \setminus \{0\}$.

Proof. Again we use the induction by $m = ||M||$ supposing M sincere. Moreover, for a fixed value of m we use the induction on the number of arrows from Σ_0 which are not special loops. The cases when $m = 0$ or all arrows from Σ_0 are special loops are trivial.

Consider the case when a is not a special loop and $M(x)$ is not nilpotent. Then we can split $M(a)$ into nilpotent and invertible parts: $M(a) = A_0 \oplus A_1$. Let $A_0^n = 0$. Denote by $\mathcal{A}'$ the subcategory of $\mathcal{A}$ consisting of x and a and by $\widetilde{\mathcal{A}}'$ its preimage in $\widetilde{\mathcal{A}}$ (which is a set of disjoint lines). Let $\mathcal{B}'$ be the category consisting of $n + 1$ objects $x^0, x^1, \dots, x^n$ and one generating morphism $a^0 : x^0 \to x^0$. Define a functor $F : \mathcal{A}' \to \mathcal{B}'$ putting

$$\begin{aligned} F(x) &= x^0 \oplus x^1 \oplus 2x^2 \oplus \dots \oplus nx^n ; \\ F(a) &= a^0 \oplus J_1(0) \oplus J_2(0) \oplus \dots \oplus J_n(0) . \end{aligned}$$

Note that we do not suppose a^0 invertible unlike in the usual reduction of a minimal loop.

Now let $\widetilde{\mathcal{B}}'$ be the category consisting of the objects x^I and x_{i0} where I runs through all finite connected parts of $\widetilde{\mathcal{A}}'$ consisting of at most n objects and x_i runs through the objects of $\widetilde{\mathcal{A}}'$. The set of generating arrows of $\widetilde{\mathcal{B}}'$ is, by definition, $\{a_{i0}\}$ with a_i running through all preimages of a. Define a functor $\widetilde{F}' : \widetilde{\mathcal{A}}' \to \widetilde{\mathcal{B}}'$ mapping the object x_i into the direct sum $x_{i0} \oplus (\bigoplus_{I \ni x_i} x^I)$ and the morphism $a_i : x_i \to gx_i$ $(g \in \mathsf{G})$ into the direct sum $a_{i0} \oplus a'$, where the morphism $a' : \bigoplus_{I \ni x_i} x^I \to \bigoplus_{I \ni gx_i} x^I$ is constructed just as in the proof of Theorem 7.1.

Now we are again in the situation of Lemma 5.5. So we get a commutative diagram of the shape (2). The obtained category $\mathcal{B}$ contains the whole set $\mathfrak{L}$ of special lines from $\widetilde{\mathcal{A}}$. Define the set of special lines from $\mathcal{B}$ as $\mathfrak{L}^F = \mathfrak{L} \cup \mathfrak{L}'$, where $\mathfrak{L}'$ is the set of all lines consisting of the arrows a_{i0}. Evidently, all components of the images of arrows from $\widetilde{\Sigma}_0$ together with the arrows a_{i0} form a semi-free system of generators for $\widetilde{\mathcal{B}}$. The components of all arrows from $\widetilde{\Sigma}_1$ generate a free sub-bimodule of $\overline{\widetilde{\mathcal{V}}}^F$. Denote these sets by $\widetilde{\Sigma}_0^F$ and by $\widetilde{\Sigma}_1^F$ correspondingly. Consider the image of $\omega_i = \omega_{x_i}$ in $\mathcal{V}^F$. It is of the form:

$$\begin{pmatrix} \eta_{00} & \eta_{01} \\ \eta_{10} & \eta_{11} \end{pmatrix} .$$

Here $\eta_{\alpha\beta} \in \mathcal{V}^F(x_{i\beta}, x_{i\alpha})$, where $\alpha, \beta \in \{0, 1\}$ and $x_{i1} = \bigoplus_{x_i \in I} x^I$. Consider the arrow $a_i : x_i \to gx_i$. As a is quasi-minimal, the image of $\partial a_i = a_i\omega_i - g\omega_i a_i$ in $\mathfrak{L}^{-1}\widetilde{\mathfrak{A}}$ is zero. The same is true for all components of the image of $a_i\omega_i - g\omega_i a_i$ in $\widetilde{\mathfrak{A}}^F$. It gives us the following relations for the components $\eta_{\alpha\beta}$:

$$\begin{aligned} a_{i0}\eta_{00} &= (g\eta_{00})a_{i0} + \delta_{00} , \\ a_{i0}\eta_{01} &= (g\eta_{01})a_{i1} + \delta_{01} , \\ a_{i1}\eta_{10} &= (g\eta_{10})a_{i0} + \delta_{10} , \\ a_{i1}\eta_{11} &= (g\eta_{11})a_{i1} + \delta_{11} , \end{aligned}$$

where the images of all $\delta_{\alpha\beta}$ are zero in $\mathfrak{L}^{-1}\widetilde{\mathfrak{A}}^F$. As $a_{i1}^n = 0$ the second and the third of these equations imply that the images of $a_{i0}^n\eta_{01}$ and $(g\eta_{10})a_{i0}^d$ are zeros in $\mathfrak{L}^{-1}\widetilde{\mathfrak{A}}^F$. Hence, the images of all possible η_{01} and η_{10} are zeros in $(\mathfrak{L}^F)^{-1}\widetilde{\mathfrak{A}}^F$. Now we can put $\omega_{x_{i0}} = \eta_{00}$. Then the condition 4 of Definition 8.1 holds. Moreover, ∂a_i lies in the submodule generated by $\widetilde{\Sigma}'$, hence, the same is true for ∂a_{i0}. As all arrows from $\widetilde{\Sigma}'$ are incident to some lines from $\mathfrak{L}$, the condition 6 from Definition 8.1 holds if we put $\kappa\mathcal{L}' = k$ for $\mathcal{L}' \in \mathfrak{L}'$ with $k > \kappa(\mathcal{L})$ for all $\mathcal{L} \in \mathfrak{L}$.

At last, the conditions for η_{11} are just the same as in the reduction of minimal lines. Therefore, we can take for $\widetilde{\Sigma}_1^F$ the set of all components of elements from $F(\widetilde{\Sigma}_1)$ and of all components of all possible η_{11} having non-zero images in $\mathfrak{L}^{-1}\widetilde{\mathfrak{A}}^F$. For $(\widetilde{\Sigma}^F)'$ we can take the set of all components of elements from $F(\widetilde{\Sigma}')$ and all components of all possible η_{01} and η_{10}. Then the conditions 1–3 of Definition 8.1 also hold.

Hence, we obtain a commutative diagram of the shape (2) of quasi-triangular boxes. Moreover, the representation M belongs to the image of F^*: $M \simeq F^*M'$, where $M' \in (\mathfrak{A}^F)^*\text{-}mod$ and $||M'|| < ||M||$ whenever $M(a)$ is not invertible. Otherwise $n = 0$, hence, there are no new objects (and arrows) in $\widetilde{\mathfrak{A}}$, but there are new special loops in it. Therefore, we can use the inductive supposition.

Thus, we may suppose that a is a special loop. Denote by Σ'_0 the set of all arrows from Σ_0 which are not special loops. Choose an arrow $b \in \Sigma'_0$ with the minimal triangular height in $\mathfrak{A}^*$, $b : y \to z$. Consider first the case when neither y nor z is a special vertex. Then b is either a quasi-minimal edge, or a quasi-superfluous arrow, or a quasi-minimal (non-special) loop. In the first case its preimage in $\widetilde{\mathcal{A}}$ is a set $\{\, b_i : y_i \to z_i \,\}$, where all objects y_i, z_i are pairwise distinct and none of them lies on a special line. Then we can use the reduction of the edge b in $\mathfrak{A}$ and the reduction of the set of edges b_i in $\widetilde{\mathfrak{A}}$. As none of y_i, z_i lies on special lines this reduction commutes with the "localization" $\mathfrak{L}^{-1}$. Hence, we can use Lemma 5.5 and obtain a commutative diagram of the form (2), where $\widetilde{\mathfrak{A}}^F$ is also quasi-triangular with the same set of special lines. Moreover, as b is a minimal edge in $\mathfrak{A}^*$, F induces an equivalence $F^* : (\mathfrak{A}^F)^*\text{-}mod \simeq \mathfrak{A}^*\text{-}mod$. In particular, $M \simeq F^*M'$ and $||M'|| < ||M||$ (as M is sincere), so the claim follows from the inductive supposition.

The same observations work in the case when b is a quasi-superfluous arrow, a quasi-minimal loop lifting to loops, or a quasi-minimal loop lifting to lines and such that $M(b)$ is nilpotent. At last, if b is any quasi-minimal loop lifting to lines, we can repeat the reduction used above in the case when a was non-special. Hence, in all these cases the inductive arguments accomplish the proof.

Now consider the case when for each minimal $b : y \to z$ either y or z (maybe both) is a special vertex. Suppose first that y is a special vertex with the special loop $c : y \to y$ and z is not a special vertex. Then $\partial^* b$ can contain the unique solid arrow c. Let first $\partial^* b = \sum_{\in \mathbb{Z}} \lambda_i u_i c^i \neq 0$ with $\lambda_i \in \mathbf{k}$ and $u_i \in \Sigma_1$. Put $k = \min \{\, i \,|\, \lambda_i \neq 0 \,\}$. Then we can replace u_k in Σ_1 by $\sum_{i \geq k} \lambda_i u_i c^{i-k}$. After this b becomes quasi-superfluous. Otherwise $\partial^* b = 0$. Denote $b_0 : y_0 \to z_0$ some preimage of b. The vertex y_0 lies on a special line

$$\dots \xrightarrow{c_{-1}} y_0 \xrightarrow{c_0} y_1 \xrightarrow{c_1} y_2 \xrightarrow{c_2} \dots \tag{4}$$

As $\partial^* b_0 = 0$ too, it becomes zero after inverting a finite set of arrows, say, $S = \{\, c_i \,|\, i = -m, \dots m \,\}$. Consider the subcategory of $S^{-1}\widetilde{\mathcal{A}}$ generated by all c_i and b_0. Now $\partial c_i = 0$ and $\partial b_0 = 0$ in $S^{-1}\widetilde{\mathfrak{A}}$ and this category contains a line and one more arrow b_0. It is well-known that it is wild. Therefore, $\widetilde{\mathfrak{A}}$ is also wild. The same observations are valid when z is a special vertex and y is not.

Suppose now that $y \neq z$ are both special vertices with the special loops $c \in \Sigma_0(y,y)$ and $d \in \Sigma_0(z,z)$. There are special lines $\mathcal{L}_y$ (4) and $\mathcal{L}_z$:

$$\dots \xrightarrow{d_{-1}} z_0 \xrightarrow{d_0} z_1 \xrightarrow{d_1} z_2 \xrightarrow{d_2} \dots$$

with $\Pi(y_i) = y$, $\Pi(z_i) = z$, $\Pi(c_i) = c$ and $\Pi(d_i) = d$. We suppose that $\kappa(\mathcal{L}_x) \leq \kappa(\mathcal{L}_y)$. Then $\partial^* b = \sum_i f_i(X,Y)u_i$, where $u_i \in \Sigma_1$ and $f_i \in \mathbf{k}[X, X^{-1}, Y, Y^{-1}]$ are some Laurent polynomials. As in [11] we put $Xu = uc$ and $Yu = du$ for $u \in \mathcal{V}(x,y)$. Therefore, for appropriate powers of c and d we have $d^s \partial bc^t = \sum_i u_i g_i(X,Y) + v$, where $g_i \in \mathbf{k}[X,Y]$ and the image of v in $\mathfrak{A}^*$ is zero. Choose a preimage b_0 of b. Suppose that $b_0 : y_0 \to z_0$. Let first $\mathsf{St}\,\mathcal{L}_y \cap \mathsf{St}\,\mathcal{L}_z \neq \{1\}$, g be the generator of this intersection and $gb_0 : y_m \to z_n$. Then $D_s \partial b_0 C_t = \sum_j \alpha_j D_{k_j} v_j C_{l_j} + v_0$ for some $v_j \in \widetilde{\Sigma}_1$ and some element v_0 such that its image in $\widetilde{\mathfrak{A}}^*$ is zero. Here we denote for $v \in \widetilde{\mathcal{V}}(y_p, z_q)$

$$D_k v C_l = d_{q+k-1} \dots d_{q+1} d_q v c_{p-1} c_{p-2} \dots c_{p-l}\,.$$

Moreover, if $gv_j = v_{j'}$ then $l_{j'} = l_j + m$ and $k_{j'} = k_j - n$. Hence, each of the polynomials g_i is of the form $X^{k_i} Y^{l_i} h_i(X^m/Y^n)$ for some $h_i \in \mathbf{k}[T, T^{-1}]$. Denote by $d(T)$ the greatest common divisor of all h_i. If $d = 1$, the arrow b can be considered as quasi-superfluous. Suppose that $\deg d > 0$. Then all $g_i(X,Y)$ have a common divisor of the form $X^m - \lambda Y^n$ for some non-zero $\lambda \in \mathbf{k}$. Therefore, $D_s \partial b_0 C_t = \eta_1 C_m - \lambda D_n \eta_0 + v_0$ for some $\eta_0 \in \widetilde{\mathcal{V}}$ and $\eta_1 = g\eta_0$. We may suppose that $\lambda = 1$. Put $b_i = g^i b_0$. Then $D_s \partial b_i C_t = \eta_{i+1} C_m - D_n \eta_i + v_i$, where $\eta_i = g^i \eta_0$, $v_i = g^i v_0$.

We are going to prove that the box $\widetilde{\mathfrak{A}}$ is wild in contradiction with the suppositions of Lemma 8.4. To do it replace $\widetilde{\mathfrak{A}}$ by the factor $\mathfrak{C} = \widetilde{\mathfrak{A}}/\mathcal{I}$, where $\mathcal{I}$ is generated by all arrows from $\widetilde{\Sigma}_0$ except of c_i, d_i and b_0. We will prove that even $\mathfrak{C}$ is wild. Denote by $\mathfrak{C}^*$ the box obtained from $\mathfrak{C}$ by inverting all c_i and d_i. Remark that $\eta_{i+1} C_m - \lambda D_n \eta_i + v_i = 0$ in $\mathfrak{C}$ if $i \neq 0$. The kernel $\mathcal{U}$ of the box $\mathfrak{C}$ is generated by the images of the arrows from $\widetilde{\Sigma}_1 \cup \widetilde{\Sigma}'$. Let $\mathcal{F}$ be the free bimodule with the same generators and $\mathcal{R} \subseteq \mathcal{F}$ be the set of relations between these generators in $\mathcal{U}$. As the images of all v_i are zeros in $\mathfrak{C}^*$, we have $v_i = \sum_j p_{ij} r_j$ for some $r_j \in \mathcal{R}$ and some polynomials p_{ij} in $c_k, d_l, c_k^{-1}, d_l^{-1}$ and b_0. Just in the same way $\partial d_i = \sum_j q_{ij} r_j$ for some polynomials q_{ij}, while $\partial c_i = 0$ in $\mathfrak{C}$. Consider all polynomials p_{0j} and those of p_{ij} and q_{ij} which really contain b_0. There is only a finite number of them. Hence, they contain only a finite number of c_k^{-1} and d_l^{-1}. Let K be the maximal absolute value of the corresponding indices k and l. Consider now the box $\mathfrak{C}'$ obtained from $\mathfrak{C}$ by inverting c_k and d_k with $|k| \le K$. It is enough to show that $\mathfrak{C}'$ is wild. But in $\mathfrak{C}'$ we have:

$$\begin{aligned}
&\partial d_i = 0\,,\\
&D_s \partial b_0 C_t = \eta_1 C_m - D_n \eta_0\,,\\
&\eta_{i+1} C_m - D_n \eta_i + v_i = 0 \ \text{ for } \ i \ne 0
\end{aligned}$$

where $v_i = \sum_{ij} p'_{ij} r_j$ and p'_{ij} are polynomials in $c_k, c_k^{-1}, d_l, d_l^{-1}$. Therefore, there exist s_i, t_i such that $D_{s_i}(\eta_{i+1} C_m - D_n \eta_i) C_{t_i} = 0$. Denote by L the maximum of the set $\{\, s, t, s_i, t_i \mid -K-4 \le i \le K \,\}$.

Consider the following representation N of $\mathfrak{C}'$ over the free algebra $\mathbf{k}\langle X, Y\rangle$:

$$\mathcal{N}(y_i) = \begin{cases} \mathbf{k}\langle X,Y\rangle^2 & \text{if } -K-4-L \le i \le K \\ 0 & \text{otherswise} \end{cases}$$

$$\mathcal{N}(c_i) = \begin{cases} \mathbf{1}_2 & \text{if } -K-4-L \le i \le K-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{N}(z_i) = \begin{cases} \mathbf{k}\langle X,Y\rangle^5 & \text{if } -K \le i \le K+L \\ \mathbf{k}\langle X,Y\rangle^{5-j} & \text{if } i = K+L+j \text{ or } \\ & \quad i = -K-j \text{ for } 1 \le j \le 4 \\ 0 & \text{for all other values of } i \end{cases}$$

$$\mathcal{N}(d_i) = \begin{cases} \mathbf{1}_5 & \text{if } -K \le i \le K+L-1 \\ \begin{pmatrix} \mathbf{1}_{4+j} \\ 0 \end{pmatrix} & \text{if } i = -K-j \text{ for } 1 \le j \le 4 \\ \begin{pmatrix} \mathbf{1}_{4+j} & 0 \end{pmatrix} & \text{if } i = K+L+j-1 \text{ for } 1 \le j \le 4 \\ 0 & \text{for all other values of } i \end{cases}$$

$$\mathcal{N}(b_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & X \\ 1 & Y \end{pmatrix}$$

We claim that it is strict. Indeed, let M_1, M_2 be any $\mathbf{k}\langle X, Y\rangle$-modules, $N_j = \mathcal{N} \otimes_{\mathbf{k}\langle X,Y\rangle} M_j$ $(j = 1, 2)$, and $\varphi \in \mathrm{Hom}_{\mathfrak{C}'}(N_1, N_2)$. As all $N_j(c_i)$

are isomorphisms for $-K-4 \le i \le K-1$ and all $N_j(d_i)$ are monomorphisms for $-K-4 \le i \le K+L-1$, the relations for η_i give us that $\varphi(\eta_i) = 0$ for all i. Moreover, as $\partial c_i = \partial d_i = 0$ in $\mathfrak{C}'$, we obtain the following relations:

$$N_2(d_i)\varphi_i = \varphi_{i+1} N_1(d_i)$$

where $\varphi_i = \varphi(\omega_{z_i})$. In particular, all φ_i are equal for $-K \le i \le K$. Putting $i = -K - j$ with $1 \le j \le 4$ we obtain

$$\varphi_{i+1} = \begin{pmatrix} \varphi_i & \xi_i \\ 0 & \theta_i \end{pmatrix}$$

for some θ_i and ξ_i. Hence, φ_{-K} is an upper triangular 5×5 matrix. Analogous observations show that φ_K is a lower triangular 5×5 matrix. As they are equal, both of them as well as φ_0 are diagonal: $\varphi_0 = \mathrm{diag}(\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5)$. Note that

$$N_j(b_0) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \\ \mathbf{1} & M_j(X) \\ \mathbf{1} & M_j(Y) \end{pmatrix}.$$

As $N_2(b_0)\varphi(y_0) = \varphi_0 N_1(b_0)$, we get $\Phi_1 = \Phi_2 = \Phi_3 = \Phi_4 = \Phi_5 = \varphi(y_0)$. Denote this common value by Φ. Then $\Phi M_1(X) = M_2(X)\Phi$ and $\Phi M_1(Y) = M_2(Y)\Phi$. Hence, N is strict.

If $\mathsf{St}\,\mathcal{L}_y \cap \mathsf{St}\,\mathcal{L}_z = \{1\}$ the same observation shows that either b is quasi-superfluous or $\partial^* b = 0$. In the latter case the box $\widetilde{\mathfrak{A}}$ (even that obtained from $\widetilde{\mathfrak{A}}$ by inverting all arrows from $\mathcal{L}_y$) is wild just as in the case when only one end of b is special.

The remaining case, when $b : y \to y$ and y is a special vertex, is quite analogous (just as in [11] or [7]) and we omit it. □

Just in the same way (though much easier) we generalize Lemma 6.6 as follows.

Lemma 8.5. *Suppose that a group* G *acts well on a quasi-triangular box* $\widetilde{\mathfrak{A}}$, $\mathfrak{A} = \mathsf{G} \setminus \widetilde{\mathfrak{A}}$, *and* $N \in \mathit{ind}_{\mathbb{Z}} \widetilde{\mathfrak{A}}$. *Then:*

1. $\Pi_* N$ *is strict, i.e., is a rational family of* $\widetilde{\mathfrak{A}}$*-modules.*
2. *No module from* $\mathit{ind}_0 \mathfrak{A}$ *belongs to the family* $\Pi_* N$.
3. *If* $N' \in \mathit{ind}_{\mathbb{Z}} \widetilde{\mathfrak{A}}$ *is such that* $N' \not\simeq N^g$ *for all* $g \in \mathsf{G}$ *then no* $\widetilde{\mathfrak{A}}$*-module belonging to the family* $\Pi_* N$ *belongs to* $\Pi_* N'$.

Proof. We denote by S the stabilizer of N in G. Again we suppose that N is sincere and use the induction on the value $m = ||\Pi_* N||$; the latter being well defined as $\Pi_* N \in \mathit{rep}(\mathfrak{A}, \mathbf{k}[\mathbb{Z}])$. Here the case $m = 0$

is evidently impossible. If $m = 1$ then $\mathfrak{A}$ consists of one vertex x and one loop a, while $\widetilde{\mathfrak{A}}$ consists of a unique line (3). Therefore, the only possibility for N is:

$$N(x_k) = \mathbf{k}, \quad \text{and} \quad N(a_k) = 1.$$

Thus, $\Pi_* N(x) = \mathbf{k}[\mathbb{Z}] = \mathbf{k}[T, T^{-1}]$ and $\Pi_* N(a)$ is the multiplication by T, so all claims are obvious.

Suppose that this lemma is valid for all $\mathbb{Z}$-representations (of all boxes) with smaller values of m. Following the same reduction processes as in the proof of Lemma 8.4 we construct a morphism of quasi-triangular G-boxes $F : \widetilde{\mathfrak{A}} \to \widetilde{\mathfrak{B}}$ compatible with the action of G and a $\mathbb{Z}$-representation M of the box $\widetilde{\mathfrak{B}}$ such that $N \simeq F^* M$ and $||\Pi_* M|| < ||\Pi_* N||$. Then the inductive supposition accomplishes the proof. As this procedure is quite analogous to the abovementioned one, we only sketch it. The unique non-trivial case is when we have a minimal loop in $\mathfrak{A}$ lifting to lines. Consider the restriction of N onto one of these lines $\mathcal{L}$. It splits into a direct sum of the representations L_I described in Proposition 3.4. As N is S-invariant and all $N(x)$ are finite dimensional every interval I either is finite or coincides with $\mathbb{Z}$; moreover, the numbers of elements in all finite intervals I must be bounded and every I can only occur finitely many times. Hence, we can apply the same procedure as in the proof of Lemma 8.4, in the case of a loop lifting to lines. As the result, we obtain a commutative diagram of the shape (2) and a $\mathbb{Z}$-representation M of $\widetilde{\mathfrak{A}}^F$ such that $N \simeq \widetilde{F}^* N'$ and either $||\Theta M|| < ||\Pi N||$ or the restriction of N on $\mathcal{L}$ only contains direct summands of the form $L_{\mathbb{Z}}$. In the former case we can apply the inductive supposition. In the latter one $N(x)$ is invertible for any object $x \in \mathcal{L}$; thus, we can preserve $\widetilde{\mathfrak{A}}$ and just consider $\mathcal{L}$ as a new special line, so diminishing the number of arrows in $\widetilde{\mathfrak{A}}$ not belonging to special lines. □

9. *Main Theorem for Algebras*

Usual arguments (cf. [11, 7]) allows to transfer the main theorem for boxes to the case of finite dimensional algebras, or, further, of locally finite dimensional categories. Remind that the ground field is always supposed to be algebraically closed.

Theorem 9.1. *Let $\Pi : \widetilde{\Lambda} \to \Lambda$ be a Galois covering of locally finite dimensional categories with a torsion free Galois group G. Then:*

1. *The representation type of $\widetilde{\Lambda}$ is the same as that of Λ.*
2. *If $\widetilde{\Lambda}$ is tame (hence, Λ is also tame), then:*

(a) *ind* $\Lambda = ind_0\,\Lambda \sqcup ind_1\,\Lambda$.
(b) *If* $M \in ind_0\,\Lambda$, $M' \in ind_1\,\Lambda$ *or* $M \in \Pi_\downarrow N$, $M' \in \Pi_\downarrow N'$ *for* $N' \neq N$, *then* $\mathrm{Hom}_\Lambda(M,M') \cup \mathrm{Hom}_\Lambda(M',M) \subseteq \mathrm{rad}_\Lambda^\infty$. *Moreover, if* $M = \Pi_* N \otimes_{\mathbf{k}[\mathbb{Z}]} J$ *and* $M' = \Pi_* N \otimes_{\mathbf{k}[\mathbb{Z}]} J'$ *for* $N \in ind_{\mathbb{Z}}\,\widetilde{\Lambda}$, *then* $\mathrm{Hom}_\Lambda(M,M') = 1 \otimes \mathrm{Hom}_{\mathbf{k}[\mathbb{Z}]}(J,J') + \mathrm{rad}_\Lambda^\infty(M,M')$.

Remark 9.2. We suppose that the property (2c) of Theorem 7.1 also remains valid for finite dimensional algebras, but at the moment we do not see any proof of it.

Proof. As we are only interested in finite dimensional representations of Λ, we may suppose that it is finite dimensional, i.e., such that $\mathcal{S}k\,\Lambda$ has finitely many objects. Then $\widetilde{\Lambda}$ is evidently *locally bound*, i.e., for any object $x \in \mathcal{S}k\,\widetilde{\Lambda}$ the set $\left\{ y \in \mathrm{Ver}\,\widetilde{\Lambda} \mid \widetilde{\Lambda}(x,y) \oplus \widetilde{\Lambda}(y,x) \neq 0 \right\}$ is finite. (Of course, $\widetilde{\Lambda}$ is not finite dimensional itself if $\mathsf{G} \neq \{1\}$.)

Remind first the following result establishing the relation between representations of locally bound categories (in particular, of algebras) and those of boxes. For any locally bound category Λ define the category Λ-$\mathcal{R}es$ of Λ-*resolutions*, whose objects are homomorphisms $\varphi : P' \to RP''$ where P', P'' are projective Λ-modules, $R = \mathrm{rad}\,\Lambda$, and morphisms from φ to $\psi : Q' \to Q''$ are pairs (α, β) where $\alpha : P' \to Q'$, $\beta : P'' \to Q''$ such that $\beta\varphi = \psi\alpha$. Let Λ-*res* be its full subcategory consisting of such objects $\varphi : P' \to JP''$ that both P' and P'' are finite dimensional. There is a natural functor Coker : Λ-$\mathcal{R}es \to \Lambda$-$\mathcal{M}od$ putting an homomorphism φ to its cokernel. The following claim is well-known (cf. [11, 7]).

Proposition 9.3. *The functor* Coker *induces an equivalence between the factor-category* Λ-$\mathcal{R}es/\mathcal{I} \to \Lambda$-$\mathcal{M}od$ *where* $\mathcal{I}$ *is the ideal generated by the identity morphisms of the objects of the form* $0 : P \to 0$ *and the morphisms of the form* $(\gamma f, g\gamma)$ *where* γ *is a homomorphism* $P \to Q'$. *The same is true if we replace* Λ-$\mathcal{R}es$ *by* Λ-*res and* Λ-$\mathcal{M}od$ *by* Λ-*mod*.

Given a locally bound category Λ we can construct a box $\mathfrak{A}_\Lambda = \mathfrak{A} = (\mathcal{A}, \mathcal{V})$ in the following way (cf. [11] or [7]). Let $\mathcal{S} = \Lambda^\emptyset \oplus \Lambda^\emptyset$ where $\Lambda^\emptyset$ is the trivial part of Λ, $\mathcal{R}$ be the bimodule

$$(x,y) \mapsto \mathrm{D}R(y,x)$$

(D being the vector space duality) considered as $\Lambda^\emptyset$-bimodule. Denote by x' the object $x \oplus 0$ of $\mathcal{S}$ and x'' the object $0 \oplus x$. Consider $\mathcal{S}$-bimodules

$\mathcal{R}_{12}$, $\mathcal{R}_{11}$, $\mathcal{R}_{22}$ where

$$\begin{aligned}
\mathcal{R}_{12}(x', y'') &= \mathcal{R}(x, y), \\
\mathcal{R}_{12}(x', y') &= \mathcal{R}_{12}(x'', y'') = \mathcal{R}_{12}(x'', y') = 0, \\
\mathcal{R}_{11}(x', y') &= \mathcal{R}(x, y), \\
\mathcal{R}_{11}(x', y'') &= \mathcal{R}_{11}(x'', y'') = \mathcal{R}_{11}(x'', y') = 0, \\
\mathcal{R}_{22}(x'', y'') &= \mathcal{R}(x, y), \\
\mathcal{R}_{22}(x', y') &= \mathcal{R}_{22}(x', y'') = \mathcal{R}_{22}(x'', y') = 0.
\end{aligned}$$

Consider also homomorphisms of $\mathcal{S}$-bimodules

$$\begin{aligned}
\delta_{12} &: \mathcal{R}_{12} \to \mathcal{R}_{12} \otimes_{\mathcal{S}} \mathcal{R}_{11} \oplus \mathcal{R}_{22} \otimes_{\mathcal{S}} \mathcal{R}_{12}, \\
\delta_{11} &: \mathcal{R}_{11} \to \mathcal{R}_{11} \otimes_{\mathcal{S}} \mathcal{R}_{11}, \\
\delta_{22} &: \mathcal{R}_{22} \to \mathcal{R}_{22} \otimes_{\mathcal{S}} \mathcal{R}_{22}
\end{aligned}$$

which are dual to the multiplication mappings in $\operatorname{rad}\Lambda$. Put $\mathcal{A} = \mathcal{C}[\mathcal{R}_{12}]$ (the tensor category) and $\overline{\mathcal{V}} = \mathcal{A} \otimes_{\mathcal{S}} (\mathcal{R}_{11} \oplus \mathcal{R}_{22}) \otimes_{\mathcal{S}} \mathcal{A}$. Then the box $\mathfrak{A}$ is given by the category $\mathcal{A}$, the kernel $\overline{\mathcal{V}}$ and the differential ∂ induced by the mappings δ_{12}, δ_{22} and $-\delta_{11}$.

Remark that as the field $\mathbf{k}$ is algebraically closed the box $\mathfrak{A}_\Lambda$ is free, normal and triangular (cf. [11]).

We are able to construct a functor $\mathsf{P} : \mathfrak{A}\text{-}\mathcal{M}od \to \Lambda\text{-}\mathcal{R}es$ in the following way. Let M be an $\mathfrak{A}$-module. Put

$$\begin{aligned}
P'_M &= \bigoplus_{x \in \operatorname{Ob}\Lambda} \Lambda^x \otimes M(x'), \\
P''_M &= \bigoplus_{x \in \operatorname{Ob}\Lambda} \Lambda^x \otimes M(x'').
\end{aligned}$$

For each $x, y \in \operatorname{Ob}\Lambda$, M induces linear mappings $\varphi_{xy} : \mathcal{R}_{12}(x', y'') \to \operatorname{Hom}(M(x'), M(y''))$ and is uniquely defined by these mappings. But we can identify $\operatorname{Hom}(\mathcal{R}_{12}(x', y''), \operatorname{Hom}(M(x'), M(y'')))$ with $R(y, x) \otimes \operatorname{Hom}(M(x'), M(y''))$. As morphisms $y \to x$ are the same as those $\Lambda^x \to \Lambda^y$, φ_{xy} can be considered as an homomorphism $\Lambda^x \otimes M(x') \to \Lambda^y \otimes M(y'')$. Now define $\mathsf{P}(M) : P'_M \to P''_M$ as the homomorphism with the components φ_{xy}. One can easily check that P is indeed a functor from $\mathfrak{A}\text{-}\mathcal{M}od$ to $\Lambda\text{-}\mathcal{R}es$ and, moreover, is an equivalence of categories (cf. [11] or [7]).

Therefore, we obtain the following result.

Proposition 9.4. *Let Λ be a locally bound category. Then the composition $\mathsf{cok} = \operatorname{Coker}\mathsf{P} : \mathfrak{A}_\Lambda\text{-}\mathcal{M}od \to \Lambda\text{-}\mathcal{M}od$ is full and dense. Moreover, its restriction onto the full subcategory that consists of all $\mathfrak{A}$-modules* *without trivial summands of the form $S_{x'}$ $(x \in \mathrm{Ob}\,\Lambda)$ maps indecomposable representations into indecomposable and non-isomorphic to non-isomorphic ones and its kernel is contained in the radical of the category $\mathfrak{A}_\Lambda$-$\mathcal{M}od$. The same is true if we replace $\mathfrak{A}$-$\mathcal{M}od$ by $\mathfrak{A}$-mod and Λ-$\mathcal{M}od$ by Λ-mod.*

Corollary 9.5. *The representation type of the box $\mathfrak{A}_\Lambda$ coincides with that of the category Λ.*

The explicit construction described above leads also to the following result.

Corollary 9.6. *Suppose that a group G acts freely on a locally bound category $\widetilde{\Lambda}$ and $\Lambda = \mathsf{G} \setminus \widetilde{\Lambda}$. Then this action induces a free, degree preserving action of G on $\mathfrak{A}_{\widetilde{\Lambda}}$ and an isomorphism of boxes $\mathfrak{A}_\Lambda \simeq \mathsf{G} \backslash \mathfrak{A}_{\widetilde{\Lambda}}$. Moreover, the functors cok commutes with the direct image functors Π_*.*

Note that if $x = \Pi\widetilde{x}$ then $S_{x'} = \Pi_* S_{\widetilde{x}'}$. Thus, all objects from $ind\,\mathfrak{A}_{\widetilde{\Lambda}}$ lying in the kernel of cok are in $ind_0\,\mathfrak{A}_{\widetilde{\Lambda}}$. Therefore, claim *1* of Theorem 9.1 follows immediately from claim *1* of Theorem 7.1. Moreover,

$$ind_0\,\Lambda = \Pi_*(ind\,\widetilde{\Lambda}) = \Pi_*\,\mathsf{cok}\,ind\,\mathfrak{A}_{\widetilde{\Lambda}} = \mathsf{cok}\,\Pi_*\,ind\,\mathfrak{A}_{\widetilde{\Lambda}} = \mathsf{cok}\,ind_0\,\mathfrak{A}_\Lambda$$

and obviously

$$ind\,\Lambda = \mathsf{cok}\,ind\,\mathfrak{A}_\Lambda = \mathsf{cok}\,ind_0\,\mathfrak{A}_\Lambda \sqcup \mathsf{cok}\,ind_1\,\mathfrak{A}_\Lambda\,.$$

If $L \in ind_{\mathbb{Z}}\,\mathfrak{A}_{\widetilde{\Lambda}}$, then $\mathsf{St}\,L = \mathsf{St}\,\mathsf{cok}\,L$. As the action of G on Λ is free, $\mathsf{cok}\,\Pi_* L$ is torsion-free (hence, free) as $\mathbf{k}[\mathsf{S}]$-module. Thus, $L \in ind_{\mathbb{Z}}\,\widetilde{\Lambda}$. Moreover, as tensor product is left exact, $(\mathsf{cok}\,\Pi_* L) \otimes_{\mathbf{k}[\mathbb{Z}]} J \simeq \mathsf{cok}(\Pi_* L \otimes_{\mathbf{k}[\mathbb{Z}]} J)$, whence $\mathsf{cok}\,ind_1\,\mathfrak{A}_\Lambda = ind_1\,\Lambda$, which proves claim 2a. Claim 2b follows now from the corresponding claim for boxes and the fullness of the functor cok. □

Note also that such results as Lemma 6.5 and Proposition 7.3 also remain valid for the coverings of locally finite dimensional categories.

Again, Theorem 9.1 immediately implies the following result for the Auslander-Reiten quivers. (We denote by $\mathsf{AR}_0\,\Lambda$ and $\mathsf{AR}_1\,\Lambda$, respectively, the diagrams of the categories $ind_0\,\Lambda$ and $ind_1\,\Lambda$.)

Corollary 9.7. *In the situation of Theorem 9.1:*

- $\mathsf{AR}\,\Lambda = \mathsf{AR}_0\,\Lambda \sqcup \mathsf{AR}_1\,\Lambda$ *(a disjoint union);*
- $\mathsf{AR}_0\,\Lambda \simeq \mathsf{G} \setminus \mathsf{AR}\,\widetilde{\Lambda}$ *;*
- $\mathsf{AR}_1\,\Lambda$ *is a disjoint union of homogeneous tubes.*

Remark 9.8. Usually the Auslander-Reiten quiver has an additional structure, the "*Auslander-Reiten translation*," related to the *almost split sequences* (cf. [1]). Unfortunately, it only exists for rather specific classes of free boxes, mainly when the category $\mathcal{A}$ is finite dimensional (cf. [3]) or, the same, there are no solid cycles in the corresponding bigraph. If almost split sequences exist, they can be defined as exact sequences

$$0 \longrightarrow M \xrightarrow{\begin{pmatrix}\alpha_1 & \dots & \alpha_k\end{pmatrix}^\top} N_1 \oplus \dots \oplus N_k \xrightarrow{\begin{pmatrix}\beta_1 & \dots & \beta_k\end{pmatrix}} L \longrightarrow 0$$

such that:

- $M, L, N_1, \dots, N_k$ are all indecomposable;
- M is not injective and L is not projective;
- $\alpha_1, \alpha_2, \dots, \alpha_k$ generate $\mathrm{rad}_\Lambda(M, \_\_)$;
- $\beta_1, \beta_2, \dots, \beta_k$ generate $\mathrm{rad}_\Lambda(\_\_, L)$.

The functor Π_* maps projective (injective) modules to projective (injective) ones and all of them belong to $ind_0\,\Lambda$. Moreover, it maps $\mathrm{rad}_{\widetilde{\Lambda}}$ onto $\mathrm{rad}_\Lambda \cap ind_0\,\Lambda$ view to the property (2b) of Theorem 9.1 (or 7.1). Therefore, the functor Π_* maps Auslander-Reiten sequences from $\widetilde{\Lambda}$-*mod* to Auslander-Reiten sequences in Λ-*mod*. So, Corollary 9.7 (as well as 7.5) remains valid with respect to this additional structure too.

Note one special case when all finite dimensional representations of the covered algebra (or box) are actually the images of those of the covering.

Proposition 9.9. (cf. [8]) *Suppose that a torsion free group* G *acts on a semi-free box or on a locally finite dimensional category* $\widetilde{\mathfrak{A}}$, $\mathfrak{A} = \mathsf{G} \setminus \widetilde{\mathfrak{A}}$. *In the case of boxes we also suppose that this action is degree preserving. The following conditions are equivalent:*

(1) $ind_{\mathbb{Z}}\,\widetilde{\mathfrak{A}} = \emptyset$.

(2) $ind\,\mathfrak{A} = ind_0\,\mathfrak{A} \simeq \mathsf{G} \setminus \widetilde{\mathfrak{A}}$.

These conditions always hold if $\widetilde{\mathfrak{A}}$ *is* locally support finite, *i.e., for every vertex* x *the set*

$$\left\{\, y \in \mathrm{Ver}\,\widetilde{\mathfrak{A}} \mid \text{ there is } M \in ind\,\widetilde{\mathfrak{A}} \text{ such that } M(x) \neq 0,\ M(y) \neq 0 \,\right\}$$

is finite.

Note that here we also do not suppose tameness.

Proof. Certainly, we only have to prove this claim for boxes; then the case of algebras follows from Proposition 9.4 and Corollary 9.6.

$2 \Rightarrow 1$ follows from Lemma 6.6.

$1 \Rightarrow 2$. We prove, using induction on $m = ||M||$, that under condition 1 any representation $M \in ind\,\mathfrak{A}$ is actually in $ind_0\,\mathfrak{A}$. It is trivial if $m = 0$. Condition 1 implies that $\widetilde{\mathfrak{A}}$ has no minimal lines. So, $\mathfrak{A}$ has either a minimal edge, or a superfluous arrow, or a minimal loop lifting to loops. In all these case we can reduce it parallel to all its preimages, thus, diminishing m, just as in the proof of Theorem 7.1. One only has to remember that a *given* representation has a finite and fixed set of eigenvalues at every minimal loop.

The same observation shows that condition 1 holds if $\mathfrak{A}$ is locally support finite. □

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