TILTING, DEFORMATIONS AND REPRESENTATIONS OF LINEAR GROUPS OVER EUCLIDEAN ALGEBRAS

VIKTOR BEKKERT, YURIY DROZD, AND VYACHESLAV FUTORNY

Abstract. We consider the dual space of linear groups over Dynkinian and Euclidean algebras, i.e. finite dimensional algebras derived equivalent to the path algebra of Dynkin or Euclidean quiver. We prove that this space contains an open dense subset isomorphic to the product of dual spaces of full linear groups and, perhaps, one more (explicitly described) space. The proof uses the technique of bimodule categories, deformations and representations of quivers.

Introduction

Let G be a Lie group. Denote by $\hat{\mathsf{G}}$ its *dual space*, i.e. the space of irreducible unitary representations of G in Hilbert spaces. It is a topological (though often non-Hausdorff) space [12]; moreover, if G is of *type I* (or *domestic*), there is a natural measure μ_{G} (the *Plancherel measure*) on $\hat{\mathsf{G}}$, which inables a harmonic analysis on G [13]. The structure of $\hat{\mathsf{G}}$ is thoroughly investigated in two cases: that of *solvable* groups, when the *orbit method* [12] gives rather complete information, and that of *reductive* groups. Essentially less is known about the "*mixed*" groups, i.e. those which are neither solvable nor reductive. For instance, in the classical survey [25] only two rather simple examples of such groups are considered.

Certain explanation of this situation was given by Li Sun Gen [14]. Namely, he showed that if we deal with the groups of 2×2 block triangular matrices

$$\mathsf{G}(m,n)=\begin{pmatrix} A & B\\ 0 & C\end{pmatrix},\ A\in \mathrm{GL}(m),\ C\in \mathrm{GL}(n),\ B\in \mathrm{Mat}(m\times n),$$

then the problem of classification of unitary representations of these groups (for all values of m,n) is "*wild*," i.e. contains the problem of classification of all finite dimensional representations of all finitely generated algebras, which seems hopeless. Nevertheless, the same paper proposed a certain approach to this problem. Namely, it was shown there that the space $\hat{\mathsf{G}}(m,n)$ contains an open dense subset $\tilde{\mathsf{G}}$ isomorphic to $\widehat{\mathrm{GL}}(d)$, where $d=\gcd(m,n)$. Moreover, if G is any group of block triangular matrices (with any number of blocks),

2000 *Mathematics Subject Classification.* Primary 22E47, Secondary 16G20, 18E30.
Key words and phrases. Unitary representations, dual space, mixed Lie groups, linear groups, bimodule categories, derived categories, deformations, representations of quivers.

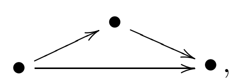
then $\hat{\mathsf{G}}$ contains an open dense subset $\tilde{\mathsf{G}}$ isomorphic to $\prod_{i=1}^{s} \widehat{\mathrm{GL}}(d_i)$ for some s and d_i.

These results were generalized by A. Timoshin and the author [22, 3]. The final result of [3] dealt with the *linear groups over Dynkinian algebras*. We call *linear groups* over an algebra $\mathbf{A}$ the groups $\mathrm{GL}(P, \mathbf{A}) = \mathrm{Aut}_{\mathbf{A}} P$, where P is a finitely generated projective $\mathbf{A}$-module. If P is free of rank r, then $\mathrm{GL}(P, \mathbf{A}) \simeq \mathrm{GL}(r, \mathbf{A})$, but in general case there are more possibilities. For instance, if $\mathbf{A}$ is the algebra of triangular matrices, we obtain as $\mathrm{GL}(P, \mathbf{A})$ the groups of block triangular matrices. A finite dimensional algebra $\mathbf{A}$ is called *Dynkinian* if it is *derived equivalent* to the path algebra of a *Dynkin quiver*, i.e. oriented graph such that its underlying non-oriented graph is a Dynkin scheme $\mathrm{A}_n, \mathrm{D}_n$ or $\mathrm{E}_{6,7,8}$. It was proved in [3] that if $\mathsf{G} = \mathrm{GL}(P, \mathbf{A})$, where $\mathbf{A}$ is a Dynkinain algebra over a locally compact field $\mathbb{K}$, then $\hat{\mathsf{G}}$ contains an open dense subset $\tilde{\mathsf{G}} \simeq \prod_{i=1}^{s} \mathrm{GL}(d_i, \mathbb{K})$. Using the results of [13], one can show that this isomorphism reflects the Plancherel measure and $\tilde{\mathsf{G}}$ is a support of this measure, so, in some sence, the obtained information is sufficient for the harmonic analysis. Note that all groups $\mathrm{GL}(P, \mathbf{A})$ are linear algebraic groups, so of type I, thus they fit the framework of [13]. The proof used rather delicate matrix calculations in terms of the so called *representations of boxes* and *small reduction algorithm*.

For linear groups over other algebras the picture becomes more complicated. In [23, 24] Timoshin considered the cases when $\mathbf{A}$ is the *Kronecker algebra*, i.e. the path algebra of the quiver

$$\bullet \rightrightarrows \bullet,$$

and the path algebra of the quiver of type $\tilde{\mathrm{A}}_2$, i.e.



as well as over derived equivalent algebras over the field $\mathbb{C}$ of complex numbers. He proved that in this case the dual space to a linear group $\mathrm{GL}(P, \mathbf{A})$ contains an open dense subset $\tilde{\mathsf{G}}$ isomorphic to

$$\prod_{i=1}^{s} \mathrm{GL}(d_i, \mathbb{C}) \times \mathbb{C}^{(m)}/\mathsf{S}_m \times (\mathbb{C}^{\times})^m,$$

where $\mathbb{C}^{(m)} = \{\, (\lambda_1, \lambda_2, \dots, \lambda_m) \mid \lambda_i \in \mathbb{C},\ \lambda_i \neq \lambda_j \text{ for } i \neq j \,\}$, S_m is the symmetric group naturally acting on $\mathbb{C}^{(m)}$, $\mathbb{C}^{(m)}/\mathsf{S}_m$ is the corresponding factor-space and $\mathbb{C}^{\times}$ is the multiplicative group of the field $\mathbb{C}$. The proof was based again on matrix calculations, in particular, on the small reduction algorithm (slightly generalized).

The aim of our paper is to generalize the last result to the linear groups over *Euclidean algebras*, i.e. those, which are derived equivalent to the path algebras of Euclidean (extended Dynkin) quivers of type $\tilde{\mathrm{A}}_n$, $\tilde{\mathrm{D}}_n$ or $\tilde{\mathrm{E}}_{6,7,8}$. Namely, we prove the following result (Theorem 5.2):

Theorem. *Let* $\mathbf{A}$ *be a Euclidean algebra,* $\mathsf{G} = \mathrm{GL}(P, \mathbf{A})$ *be a linear group over* $\mathbf{A}$*. There is a subset* $\tilde{\mathsf{G}} \subseteq \hat{\mathsf{G}}$*, which is open, dense and support of the Plancherel measure, such that*

$$\tilde{\mathsf{G}} \simeq \prod_{i=1}^{s} \widehat{\mathrm{GL}}(d_i, \mathbb{C}) \times \mathbb{X}^{(m)}/\mathsf{S}_m \times (\mathbb{C}^{\times})^m$$

for some values of $d_1, d_2, \dots, d_s$*,* m *and some cofinite subset* $\mathbb{X} \subseteq \mathbb{P}^1$*.*

The main distinction is that we use, instead of matrix calculations, the deformation technique, the *tilting theory* and the results on the representations of Euclidean quivers from [10, 21]. Since it simplifies the considerations, we also included a new simple proof of the result for Dynkinian algebras from [3]. Actually, we had to extend the deformation and tilting theories to the *bimodule categories*, in particular, to prove for them analogues of the well-known Riedtmann–Zwara theorem about deformations of modules [19, 26]. It is done in Sections 1 and 2. In section 3 we recall the facts concerning the Tits and the Euler forms and extend them to bimodule categories too. Section 4 contains the basic results on elements in general position in the bimodule categories of Dynkinian and Euclidean bimodules, and in Section 5 these results are applied to the proof of the main theorems about representations of linear groups over Dynkinian and Euclidean algebras. Here we p

Note that, though our proofs are essencially simpler than those from [3], they are not so constructive. To get explicit shape of representations from the subset $\tilde{\mathsf{G}}$, one still has to apply a sort of the small reduction algorithm. Our results then guarantee that one gets an answer after finitely many steps. Typical examples can de found in [3, 23, 24].

Acknowledgements. This investigation was accomplished during the visit of the second author to the University of São Paulo supported by Fapesp (processo 2007/05047-4). The second author was also partially supported by INTAS Grant 06-1000017-9093. The third author was partially supported by Fapesp (processo 2005/60337-2) and CNPq (processo 301743/2007-0).

1. Derived bimodule categtories

Let $\mathbf{A}$ be a ring. We denote by $\mathbf{A}$-Mod the category of all (left) $\mathbf{A}$-modules and by $\mathbf{A}$-mod the category of finitely generated (left) $\mathbf{A}$-modules. By $\mathbf{A}$-Proj we denote the subcategory of projective $\mathbf{A}$-modules and by $\mathbf{A}$-proj the subcategory of finitely generated projective $\mathbf{A}$-modules. Let $\mathbf{W}$ be an $\mathbf{A}$-bimodule. The *bimodule category* (or the *category of elements of the bimodule*, or of *matrices over the bimodule* $\mathbf{W}$) $\mathbf{W}$-El is defined as follows [2].

- The *objects* of $\mathbf{W}$-El are elements of groups $\mathbf{W}(P) = \mathrm{Hom}_{\mathbf{A}}(P, \mathbf{W} \otimes_{\mathbf{A}} P)$, where $P \in \mathbf{A}$-Proj; we call them *elements of the bimodule* $\mathbf{W}$.

- A *morphism* from $w \in \mathbf{W}(P)$ to $w' \in \mathbf{W}(P')$ is a homomorphism $\alpha : P \to P'$ such that $(1 \otimes \alpha)w = w'\alpha$. We denote the set of such morphisms by $\mathrm{Hom}_{\mathbf{W}}(w, w')$.
- The product of morphisms coincides with their compositions as maps.

The category $\mathbf{W}$-El is additive; moreover, it is *fully additive* (Karoubian), i.e. every its idempotent endomorphism in it splits. We denote by $\mathbf{W}$-el the full subcategory consisting of elements of the groups $\mathbf{W}(P)$ with $P \in \mathbf{A}$-proj.

The bimodule category is an *exact category* in the sense of [17] (we use the terminology of [6] or [11, Appendix A]). Namely, *conflations* (distinguished exact sequences) in $\mathbf{W}$-El are sequences $w_1 \xrightarrow{\alpha} w_2 \xrightarrow{\beta} w_3$, where $w_i \in \mathbf{W}(P_i)$, such that the underlying sequence $0 \to P_1 \xrightarrow{\alpha} P_2 \xrightarrow{\beta} P_3 \to 0$ is exact. Thus, $\alpha : w_1 \to w_2$ is an *inflation* if the underlying map $\alpha : P_1 \to P_2$ is a split monomorphism, and $\beta : w_2 \to w_3$ is a *deflation* if the underlying map $\beta : P_2 \to P_3$ is an epimorphism (automatically split, since P_3 is projective). Therefore, one can consider its *derived category* $\mathscr{D}(\mathbf{W}\text{-El})$ [16]. We denote by $\mathscr{C}(\mathbf{W}\text{-El})$ the category of complexes over $\mathbf{W}$-El and by $\mathscr{H}(\mathbf{W}\text{-El})$ its factor-category modulo homotopy. A complex $\mathcal{C} = (\mathcal{C}_n, d_n)$ is said to be *exact* if the underlying complex of projective $\mathbf{A}$-modules is exact and $\operatorname{Ker} d_n$ splits for every n (then also $\operatorname{Im} d_n$ splits for every n). Note that if this complex is *right bounded*, i.e. $\mathcal{C}_n = 0$ for $n \ll 0$, it is exact if and only if the underlying complex of modules is exact. The derived category $\mathscr{D}(\mathbf{W}\text{-El})$ is the localization of $\mathscr{H}(\mathbf{W}\text{-El})$ with respect to the full subcategory of exact complexes (it is a triangular and épaisse subcategory [16]). As usually, we denote by marks $^-$ and ^{b} the full subcategories of $\mathscr{C}(\mathbf{W}\text{-El})$, $\mathscr{H}(\mathbf{W}\text{-El})$ and $\mathscr{D}(\mathbf{W}\text{-El})$ consisting, respectively, of right bounded and two-sided bounded complexes. The full subcategories of $\mathscr{C}(\mathbf{W}\text{-El})$, $\mathscr{H}(\mathbf{W}\text{-El})$ and $\mathscr{D}(\mathbf{W}\text{-El})$ consisting of complexes over $\mathbf{W}$-el are denoted by $\mathscr{C}(\mathbf{W}\text{-el})$, $\mathscr{H}(\mathbf{W}\text{-el})$ and $\mathscr{D}(\mathbf{W}\text{-el})$ (with the marks $^-$ or ^{b}, if necessary). The set of morphisms $\mathcal{C} \to \mathcal{C}'$ in $\mathscr{D}(\mathbf{W}\text{-El})$ will be denoted by $\mathrm{Hom}_{\mathscr{D}\mathbf{W}}(\mathcal{C}, \mathcal{C}')$.

The most important case is that of *bipartite bimodules*. Recall that an $\mathbf{A}$-bimodule $\mathbf{W}$ is said to be *bipartite* if $\mathbf{A} = \mathbf{A}_1 \times \mathbf{A}_2$ and $\mathbf{W}$ is an $\mathbf{A}_2$-$\mathbf{A}_1$-bimodule, where the $\mathbf{A}$-action is defined as $(a_1, a_2)w = a_2 w$ and $w(a_1, a_2) = wa_1$. If P is a projective $\mathbf{A}$-module, then $P = P_1 \oplus P_2$, where P_i is a projective $\mathbf{A}_i$-bimodule, and $\mathbf{W}(P) = \mathbf{W}(P_2, P_1) = \mathrm{Hom}_{\mathbf{A}_2}(P_2, \mathbf{W} \otimes_{\mathbf{A}_1} P_1)$. A morphism $\alpha : w \to w'$, where $w \in \mathbf{W}(P_2, P_1)$, $w' \in \mathbf{W}(P_2', P_1')$ is then a pair (α_1, α_2), where $\alpha_i : P_i \to P_i'$, such that $(1 \otimes \alpha_1)w = w'\alpha_2$.

In what follows we mainly deal with the situation, when an $\mathbf{A}_2$-$\mathbf{A}_1$-bimodule arises as the *left dual* ${}^\wedge\mathbf{W} = \mathrm{Hom}_{\mathbf{A}_1}(\mathbf{W}, \mathbf{A}_1)$ of a $\mathbf{A}_1$-$\mathbf{A}_2$-bimodule $\mathbf{W}$. If $\mathbf{W}$ is finitely generated as $\mathbf{A}_1$-module, the category ${}^\wedge\mathbf{W}$-El can be interpreted as that of some modules over the ring $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$ of triangular matrices of the form $\begin{pmatrix} a_1 & w \\ 0 & a_2 \end{pmatrix}$, where $a_i \in \mathbf{A}_i$, $w \in \mathbf{W}$. Namely, there is an isomorphism of functors (on $\mathbf{A}$-Proj):

$$^\wedge\mathbf{W} \otimes_{\mathbf{A}_1} P_1 \simeq \mathrm{Hom}_{\mathbf{A}_1}(\mathbf{W}, P_1)$$

(mapping $f \otimes p$ to the homomorphism $w \mapsto f(w)p$, where $p \in P_1$, $f \in {}^\wedge\mathbf{W}$, $w \in \mathbf{W}$). Therefore,

$$^\wedge\mathbf{W}(P_2, P_1) \simeq \mathrm{Hom}_{\mathbf{A}_2}(P_2, \mathrm{Hom}_{\mathbf{A}_1}(^\wedge\mathbf{W}, P_1)) \simeq \mathrm{Hom}_{\mathbf{A}_1}(^\wedge\mathbf{W} \otimes_{\mathbf{A}_2} P_2, P_1).$$

It implies

Proposition 1.1. *Let* $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$*, where* $\mathbf{W}$ *is finitely generated as left* $\mathbf{A}_1$*-module. Then* $^\wedge\mathbf{W}$-El *is equivalent to the full subcategory* $\mathbf{A}$-Mod$^{\mathrm{p}}$ *of the category* $\mathbf{A}$-Mod *consisting of all modules* M *such that* e_iM $(i = 1, 2)$ *is a projective* $\mathbf{A}_i$*-module, where* $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Proof. Let $M \in \mathbf{A}$-Mod$^{\mathrm{p}}$, $P_i = e_iM$. The multiplication with $e_1\mathbf{A}e_2 \simeq \mathbf{W}$ induces a homomorphism $\phi : \mathbf{W} \otimes_{\mathbf{A}_2} P_2 \to P_1$ and the module M is uniquely defined by the triple (P_1, ϕ, P_2) (we usually write $M = (P_1, \phi, P_2)$), hence, by an element $f(M) \in {}^\wedge\mathbf{W}(P_2, P_1)$. Every homomorphism $\alpha : M \to M'$, where $M' = (P'_1, \phi', P'_2)$, maps P_i to P'_i, hence, defines a pair (α_1, α_2), $\alpha_i : P_i \to P'_i$, such that $\alpha_1\phi = \phi'(1 \otimes \alpha_2)$. One easily sees that the last condition is equivalent to the equality $(1 \otimes \alpha_1)f(M) = f(M')\alpha_2$, thus $(\alpha_1, \alpha_2) \in \mathrm{Hom}_{^\wedge\mathbf{W}}(f(M), f(M'))$. The inverse construction is immediate. □

The exact structure on $^\wedge\mathbf{W}$-El induces an exact structure on $\mathbf{A}$-Mod$^{\mathrm{p}}$. The *conflations* in $\mathbf{A}$-Mod$^{\mathrm{p}}$ are usual short exact sequences, the *deflations* are usual epimorphisms and the *inflations* are such monomorphisms $\alpha : M \to N$ that both restrictions $\alpha_i = \alpha|_{e_iM} : e_iM \to e_iN$ $(i = 1, 2)$ split. Suppose now that $\mathbf{W}$ is projective and finitely generated as left $\mathbf{A}_1$-module. Then the category $\mathbf{A}$-Mod$^{\mathrm{p}}$, hence $^\wedge\mathbf{W}$-El, contains enough projective objects. Indeed, in this case $\mathbf{A}$ itself belongs to $\mathbf{A}$-mod$^{\mathrm{p}}$, thus $\mathbf{A}$-Proj $\subseteq$ $\mathbf{A}$-Mod$^{\mathrm{p}}$ and a usual projective resolution of a module $M \in \mathbf{A}$-Mod$^{\mathrm{p}}$ is in fact its projective resolution in $\mathbf{A}$-Mod$^{\mathrm{p}}$. One easily sees that a module (P_1, ϕ, P_2) is projective if and only if ϕ is a split monomorphism. Moreover, the following result holds.

Proposition 1.2. *Let* $\mathbf{W}$ *be projective and finitely generated as left* $\mathbf{A}_1$*-module,* $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$*.*

1. pr.dim $M \le 1$ *for every* $M \in \mathbf{A}$-Mod$^{\mathrm{p}}$*.*
2. *If* gl.dim $\mathbf{A}_i \le n$ *for both* $i = 1, 2$*, then* gl.dim $\mathbf{A} \le n + 1$*.*

Proof. (1) Let $M = (P_1, \phi, P_2) \in \mathbf{A}$-Mod$^{\mathrm{p}}$. Choose an epimorphism $P' \to \mathrm{Cok}\,\phi$, where $P' \in \mathbf{A}_1$-Proj, and lift it to a homomorphism $P' \to P_1$. Then we get a projective resolution of M:

$$(1.1)\quad 0 \longrightarrow (\mathbf{W}\otimes_{\mathbf{A}} P_2, 0, 0) \xrightarrow{\left(\begin{pmatrix}-\phi\\ \mathbf{1}\end{pmatrix},\, 0\right)} \longrightarrow (P_1,0,0)\oplus(\mathbf{W}\otimes_{\mathbf{A}_2} P_2, \mathbf{1}, P_2) \xrightarrow{\left((\mathbf{1}\ \phi),\, \mathbf{1}\right)} (P_1,\phi,P_2) \longrightarrow 0.$$

We call (1.1) the *standard resolution* of M.

(2) Let N be any $\mathbf{A}$-module. Consider an exact sequence

$$(1.2)\qquad P(n-1) \xrightarrow{\alpha} P(n-2) \to \cdots \to P(1) \to P(0) \to N \to 0$$

with $P(k) \in \mathbf{A}\text{-Proj}$. It remains exact after multiplication by e_i $(i = 1, 2)$ Since $\operatorname{gl.dim}\mathbf{A}_i \le n$, $\operatorname{Ker}\alpha \in \mathbf{A}\text{-Mod}^{\mathrm{p}}$, hence it has a projective resolution $0 \to P(n+1) \to P(n) \to \operatorname{Ker}\alpha \to 0$. Combiming it with the sequence (1.2), we get a projective resolution of length $n+1$ for N. □

Corollary 1.3. *Let* $\mathbf{W}$ *be projective as* $\mathbf{A}_1$*-module. Then* $\mathscr{D}^-({}^\wedge\mathbf{W}\text{-El}) \simeq \mathscr{D}^-(\mathbf{A}\text{-Mod})$ *and* $\mathscr{D}^b({}^\wedge\mathbf{W}\text{-El}) \simeq \mathscr{D}^{\mathrm{per}}(\mathbf{A}\text{-Mod})$*. In particular, if* $\operatorname{gl.dim}\mathbf{A}_i < \infty$ *for both* $i = 1, 2$*, then* $\mathscr{D}^b({}^\wedge\mathbf{W}\text{-El}) \simeq \mathscr{D}^b(\mathbf{A}\text{-Mod})$.

Recall that $\mathscr{D}^{\mathrm{per}}(\mathbf{A}\text{-Mod})$ denotes the full subcategory of $\mathscr{D}(\mathbf{A}\text{-Mod})$ consisting of two-side bounded complexes of projective modules (and isomorphic objects). Actually, it is equivalent to $\mathscr{H}^b(\mathbf{A}\text{-Proj})$. If (and only if) $\operatorname{gl.dim}\mathbf{A} < \infty$, then $\mathscr{D}^{\mathrm{per}}(\mathbf{A}\text{-Mod}) = \mathscr{D}^b(\mathbf{A}\text{-Mod})$.

Proof. The first equivalence follows from the known fact that both categories are equivalent to $\mathscr{H}^-(\mathbf{A}\text{-Proj})$. Moreover, since all modules in $\mathbf{A}\text{-Mod}^{\mathrm{p}}$ are of finite projective dimension, $\mathscr{D}^b(\mathbf{W}) \simeq \mathscr{H}^b(\mathbf{A}\text{-Proj}) \simeq \mathscr{D}^{\mathrm{per}}(\mathbf{A}\text{-Mod})$. □

Corollary 1.4. *Let both rings* $\mathbf{A}_i$ $(i = 1, 2)$ *be left noetherian and* $\mathbf{W}$ *be finitely generated projective as* $\mathbf{A}_1$*-module. Then* $\mathscr{D}^-({}^\wedge\mathbf{W}\text{-el}) \simeq \mathscr{D}^-(\mathbf{A}\text{-mod})$ *and* $\mathscr{D}^b({}^\wedge\mathbf{W}\text{-el}) \simeq \mathscr{D}^{\mathrm{per}}(\mathbf{A}\text{-mod})$*. In particular, if* $\operatorname{gl.dim}\mathbf{A}_i < \infty$ *for both* $i = 1, 2$*, then* $\mathscr{D}^b({}^\wedge\mathbf{W}\text{-el}) \simeq \mathscr{D}^b(\mathbf{A}\text{-mod})$.

Proof. Under these conditions the ring $\mathbf{A}$ is also left noetherian, $\mathbf{A}\text{-proj} \subseteq \mathbf{A}\text{-mod}^{\mathrm{p}} \subseteq \mathbf{A}\text{-mod}$ and $\mathbf{A}\text{-mod}^{\mathrm{p}} \simeq {}^\wedge\mathbf{W}\text{-el}$. Therefore, the arguments of the preceding proof can be applied. □

Note that the dual to the category $\mathbf{W}$-el is equivalent to the category $\mathbf{W}^{\mathrm{op}}$-el, where $\mathbf{W}^{\mathrm{op}}$ denotes $\mathbf{W}$ considered as $\mathbf{A}^{\mathrm{op}}$-bimodule. Namely, if $P \in \mathbf{A}\text{-proj}$, set $\hat{P} = \operatorname{Hom}_{\mathbf{A}}(P, \mathbf{A}) \in \mathbf{A}^{\mathrm{op}}\text{-proj}$. Then

$$\mathbf{W}(P) = \operatorname{Hom}_{\mathbf{A}}(P, \mathbf{W}\otimes_{\mathbf{A}} P) \simeq \hat{P}\otimes_{\mathbf{A}}\mathbf{W}\otimes_{\mathbf{A}} P \simeq \operatorname{Hom}_{\mathbf{A}^{\mathrm{op}}}(\hat{P}, \mathbf{W}^{\mathrm{op}}\otimes_{\mathbf{A}^{\mathrm{op}}}\hat{P}),$$

and if we denote by w^{op} the image of w under this natural isomorphism, then $\operatorname{Hom}_{\mathbf{W}}(w, v) \simeq \operatorname{Hom}_{\mathbf{W}^{\mathrm{op}}}(v^{\mathrm{op}}, w^{\mathrm{op}})$. Obviously, this duality maps inflations to deflations and vice versa, so it is compatible with the exact structure.

2. Deformations in bimodule categories

From now on all rings are assumed to be finite dimesional algebras over an algebraically closed field $\Bbbk$; all bimodules $\mathbf{W}$ are $\Bbbk$-*central*, i.e. such that $\lambda w = w\lambda$ for every element $w \in \mathbf{W}$ and every scalar $\lambda \in \Bbbk$, and finite dimensional over $\Bbbk$. In this case all vector spaces $\mathrm{Hom}_{\mathbf{W}}(w, w')$, where $w, w' \in \mathbf{W}$-el, are also finite dimensional. It implies that the Krull–Schmidt theorem on unique decomposition holds in this category. For every $P \in \mathbf{A}$-proj, $\mathbf{W}(P)$ is an affine space over $\Bbbk$. The algebraic group $\mathrm{GL}(P, \mathbf{A})$ acts regularly on $\mathbf{W}(P)$: $g \cdot w = (1 \otimes g)wg^{-1}$, and its orbits coincide with isomorphism classes of elements from $\mathbf{W}(P)$ as objects of $\mathbf{W}$-el. Therefore, one can speak about *deformations* and *degenerations* of such elements. Namely, we say that w' is a *degeneration* of w if $w' \in \overline{\mathrm{GL}(P, \mathbf{A})w}$ (the orbit closure in Zariski topology); in this case we write $w \preccurlyeq w'$. It is useful to consider $\mathbf{W}(P)$ as a $\Bbbk$-scheme; the set $\mathbf{W}(P)(R)$ of its points in a $\Bbbk$-algebra R is

$$(\mathbf{W} \otimes R)(P \otimes R) \simeq \mathrm{Hom}_{\mathbf{A} \otimes R}(P \otimes R, \mathbf{W} \otimes_{\mathbf{A}} (P \otimes R)) \simeq \mathbf{W}(P) \otimes R,$$

where $\otimes$ denotes $\otimes_{\Bbbk}$. If $R \subseteq R'$, there is a natural embedding $\mathbf{W}(P)(R) \to \mathbf{W}(P)(R')$ and we identify the elements of $\mathbf{W}(P)(R)$ with their images in $\mathbf{W}(P)(R')$. The following result is an analogue of the well-known Riedmann–Zwara theorem on degenerations for finite dimensional modules [19, 26].

Theorem 2.1. *Let $w, w' \in \mathbf{W}(P)$. The following conditions are equivalent:*

(1) $w \preccurlyeq w'$.
(2) *There is an object $v \in \mathbf{W}$-el and a conflation $w' \to w \oplus v \to v$.*
(3) *There is an object $v \in \mathbf{W}$-el and a conflation $v \to w \oplus v \to w'$.*

Proof. Since $(\mathbf{W}\text{-el})^{\mathrm{op}} \simeq \mathbf{W}^{\mathrm{op}}$-el, and this duality is compatible with the exact structures, we only need to prove that (1) and (2) are equivalent. We follow the original proofs of Riedtmann [19] and Zwara [26].

(1)⇒(2). Let $w \preccurlyeq w'$. Then there is a discrete valuation algebra R with the maximal ideal $\mathfrak{m} = tR$, the residue field $R/\mathfrak{m} \simeq \Bbbk$, the field of fractions K, and an element $W \in \mathbf{W}(P)(R)$ such that

$$W \simeq w \;\text{ in } \mathbf{W}(P)(K),$$
$$W \bmod \mathfrak{m} \simeq w' \;\text{ in } \mathbf{W}(P) \simeq \mathbf{W}(R)/\mathfrak{m}\mathbf{W}(R).$$

(cf. the proof of [7, Theorem 1.2]). Choose an isomorphism $\tilde{\phi} : w \to W$ in $\mathbf{W}(P)(K)$, i.e. an automorphism $\tilde{\phi} : P \otimes K \to P \otimes K$ such that $(1 \otimes \tilde{\phi})w = W\tilde{\phi}$. Then $\tilde{\phi} = t^{-r_1}\phi$ and $\tilde{\phi}^{-1} = t^{-r_2}\phi'$ for some endomorphisms $\phi, \phi' : P \otimes R \to P \otimes R$ and some integers r_1, r_2. Thus $\phi\phi' = \phi'\phi = t^r\mathbf{1}$ for $r = r_1 + r_2$. Obviously, $\phi \in \mathrm{Hom}_{\mathbf{W}}(w, W)$ and $\phi' \in \mathrm{Hom}_{\mathbf{W}}(W, w)$. Let $R_m = R/\mathfrak{m}^m$. Denote by W_m and w_m repectively the images of W and w in $\mathbf{W}(P)(R_m)$. Since R_m is an m-dimensional vector space, $P \otimes R_m \simeq mP$ as $\mathbf{A}$-module, so the elements W_m and w_m can be considered as objects from $\mathbf{W}$-el. Moreover, $W_1 \simeq w'$ and $w_m \simeq mw$. The natural exact sequence $0 \to$

$\Bbbk \simeq R_1 \to R_{m+1} \to R_m \to 0$ induces a conflation $w' \simeq W_1 \to W_{m+1} \to W_m$. Thus, we have only to show that $W_{m+1} \simeq w \oplus W_m$ for some m.

There is an integer k and a homomorphism of $\mathbf{A}$-modules $\psi : P \otimes R \to P \otimes R$ such that $\psi\phi(t^k x) = t^k x$ for every $x \in P \otimes R$ (see [26, Lemma 3.4]), or, the same, $(\psi t^k)\phi = t^k \mathbf{1}$. Then

$$\begin{aligned} w\psi t^{k+r} &= w\psi t^k \phi\phi' = t^k w\phi' = t^k(1 \otimes \phi')W = \\ &= (1 \otimes \psi t^k)(1 \otimes \phi)(1 \otimes \phi')W = (1 \otimes \psi t^{k+r})W, \end{aligned}$$

so $\theta = \psi t^{k+r} \in \operatorname{Hom}_{\mathbf{W}}(W, w)$ and $\theta\phi = t^{k+r}\mathbf{1}$. Let Z be a complement of $\mathfrak{m}^{k+r}$ in R. Consider the $\mathbf{A}$-homomorphism $q : P \otimes R \to P \otimes R$ that maps $p \otimes x \mapsto x$ for $x \in P \otimes \mathfrak{m}^{k+r}$ and $p \otimes z \mapsto 0$ for $z \in Z$. One immediately sees that $q \in \operatorname{Hom}_{\mathbf{W}}(w, w)$ and $q\theta\phi = \mathbf{1}$. Therefore, w is a direct summand of W (in $\mathbf{W}$-El). More precisely, $P \otimes R = X \oplus Y$, where $X = \operatorname{Im}\phi,\ Y = \operatorname{Ker} q\theta$, and with respect to this decomposition $W = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ for some $y \in \mathbf{W}(Y)$, where $x \simeq w$ in $(\mathbf{W} \otimes R)$-el. Note that, since ϕ is a monomorphism of free R-modules of finite rank, $\dim_{\Bbbk} Y < \infty$, so $y \in \mathbf{W}$-el, and there is an integer m such that $P \otimes \mathfrak{m}^m \subseteq X$, so $P \otimes \mathfrak{m}^m \cap Y = 0$. Therefore, $W_m \simeq x_m \oplus y$ and $W_{m+1} \simeq x_{m+1} \oplus y$. Since $x_{m+1} \simeq w_{m+1} \simeq w \oplus w_m \simeq w \oplus x_m$, we get that $W_{m+1} \simeq w \oplus W_m$.

(2)⇒(1). Suppose now that there is a conflation $w' \xrightarrow{\alpha} w \oplus v \xrightarrow{\beta} v$, where $w, w' \in \mathbf{W}(P),\ v \in \mathbf{W}(Q)$. Then the sequence $0 \to P \xrightarrow{\alpha} P \oplus Q \xrightarrow{\beta} Q \to 0$ is exact and $w\alpha = (\alpha \otimes 1)w',\ v\beta = (1 \otimes \beta)w$. If $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$, $\beta = (\beta_1,\, \beta_2)$, it means that $\alpha_1 \in \operatorname{Hom}_{\mathbf{W}}(w', w)$, $\alpha_2 \in \operatorname{Hom}_{\mathbf{W}}(w', v)$, $\beta_1 \in \operatorname{Hom}_{\mathbf{W}}(w, v)$ and $\beta_2 \in \operatorname{Hom}_{\mathbf{W}}(v, v)$. Then $\beta_\lambda = (\beta_1,\, \beta_2 - \lambda\mathbf{1}_Q)$, where $\lambda \in \Bbbk$, also belongs to $\operatorname{Hom}_{\mathbf{W}}(w \oplus v, v)$. Let $U = \{\, \lambda \in \Bbbk \mid \beta_\lambda \text{ is surjective} \,\}$. It is an open neighbourhood of 0 in $\Bbbk$. Moreover, there is an open neighbourhood $V \subseteq U$ of 0 and regular maps $V \to \operatorname{Hom}_{\mathbf{A}}(P, P \oplus Q)$, $\lambda \mapsto \alpha_\lambda$, and $\alpha' : V \to \operatorname{Hom}_{\mathbf{A}}(P \oplus Q, P)$, $\lambda \mapsto \alpha'_\lambda$, such that $\alpha_\lambda = \operatorname{Ker}\beta_\lambda$ and $\alpha'_\lambda\alpha_\lambda = \mathbf{1}_P$ for all $\lambda \in V$. Then, for $w_\lambda = (1 \otimes \alpha'_\lambda)(w \oplus v)\alpha_\lambda$, the sequence $w_\lambda \xrightarrow{\alpha_\lambda} w \oplus v \xrightarrow{\beta_\lambda} v$ is a conflation. If $\beta_2 - \lambda\mathbf{1}_Q$ is invertible, it implies that $w_\lambda \simeq w$, and almost all $\lambda \in V$ satisfy this condition. On the other hand, $w_0 \simeq w'$. Thus $w \preccurlyeq w'$. □

Corollary 2.2. (1) *If $w \preccurlyeq w'$, $\dim \operatorname{Hom}_{\mathbf{W}}(w, z) \le \dim \operatorname{Hom}_{\mathbf{W}}(w', z)$ and $\dim \operatorname{Hom}_{\mathbf{W}}(z, w) \le \dim \operatorname{Hom}_{\mathbf{W}}(z, w')$ for all $z \in \mathbf{W}$-el.*

(2) *If $w \preccurlyeq w'$ and either $\dim \operatorname{Hom}_{\mathbf{W}}(w, w') = \dim \operatorname{Hom}_{\mathbf{W}}(w', w')$ or $\dim \operatorname{Hom}_{\mathbf{W}}(z, w) = \dim \operatorname{Hom}_{\mathbf{W}}(z, w')$ for all z, then $w \simeq w'$. Especially, if $\dim \operatorname{Hom}_{\mathbf{W}}(w', w') = \dim \operatorname{Hom}_{\mathbf{W}}(w, w)$, then $w \simeq w'$.*

(3) *If there is a non-split conflation $\xi : w' \xrightarrow{\alpha} w \xrightarrow{\beta} w''$, then $w \preccurlyeq w' \oplus w''$ and $\dim \operatorname{End}_{\mathbf{W}}(w) < \dim \operatorname{End}_{\mathbf{W}}(w' \oplus w'')$.*

Proof. (1) A conflation $w' \xrightarrow{\alpha} w \oplus v \xrightarrow{\beta} v$ induces exact sequences

(2.1) $0 \to \mathrm{Hom}_{\mathbf{W}}(v,z) \xrightarrow{\beta^*} \mathrm{Hom}_{\mathbf{W}}(w,z) \oplus \mathrm{Hom}_{\mathbf{W}}(v,z) \xrightarrow{\alpha^*} \mathrm{Hom}_{\mathbf{W}}(w',z)$

and

(2.1′)

$$0 \to \mathrm{Hom}_{\mathbf{W}}(z,w') \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathbf{W}}(z,w) \oplus \mathrm{Hom}_{\mathbf{W}}(z,v) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathbf{W}}(z,v),$$

which implies the necessary inequalities.

(2) If the first equality holds, then the map α^* in the exact sequence (2.1) for $z = w'$ is surjective, hence, there is a morphism $\alpha' : w \oplus v \to w'$ such that $\alpha'\alpha = \mathbf{1}$, so $w \oplus v \simeq w' \oplus v$ and $w \simeq w'$. We get the same if the second equality holds for $z = v$. At last, if $w \not\simeq w'$, then $\dim \mathrm{Hom}_{\mathbf{W}}(w,w) \leq \dim \mathrm{Hom}_{\mathbf{W}}(w,w') < \dim \mathrm{Hom}_{\mathbf{W}}(w',w')$.

(3) In this case there is a conflation $\xi' : w' \oplus w'' \xrightarrow{\alpha \oplus \mathbf{1}_{w''}} w \oplus w'' \xrightarrow{(\beta\ 0)} w''$, so $w \preccurlyeq w' \oplus w''$. If $\dim \mathrm{End}_{\mathbf{W}}(w) = \dim \mathrm{End}_{\mathbf{W}}(w' \oplus w'')$, then, as we have seen in the proof of (2), the conflation ξ' splits. Then ξ splits as well. □

Suppose now that $\mathbf{W}$ is an $\mathbf{A}_1$-$\mathbf{A}_2$-bimodule, $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$ and consider the equivalence ${}^\wedge\mathbf{W}$-el $\simeq \mathbf{A}$-mod$^{\mathrm{p}}$, $w \mapsto M(w)$. We denote by $\mathrm{rep}^{\mathrm{p}}(P_1,P_2)$ the set of all $\mathbf{A}$-modules M such that $e_iM \simeq P_i$ $(i = 1,2)$ or, the same, $M \simeq M(w)$ for $w \in {}^\wedge\mathbf{W}(P_2,P_1)$. Let $S_1, S_2, \dots, S_s$ be the set of isomorphism classes of simple $\mathbf{A}$-modules. Recall that the vector dimension $\mathbf{dim}\, M$ of an $\mathbf{A}$-module M is defined as the integer vector $(d_1, d_2, \dots, d_s)$, where $d_k = (M : S_k)$, the multiplicity of S_k in a Jordan–Hölder series of M. We denote by $\mathrm{rep}(\mathbf{d})$ (or $\mathrm{rep}(\mathbf{d},\mathbf{A})$, if necessary), the variety of $\mathbf{A}$-modules of vector dimension $\mathbf{d}$. Then $\mathrm{rep}^{\mathrm{p}}(P_1,P_2)$ is a subvariety of $\mathrm{rep}(\mathbf{d})$ with $\mathbf{d} = (\mathbf{d}_1, \mathbf{d}_2)$, where $\mathbf{d}_i = \mathbf{dim}\, P_i$. Obviously, $w \preccurlyeq w'$ if and only if $M(w) \preccurlyeq M(w')$.

Proposition 2.3. $\mathrm{rep}^{\mathrm{p}}(P_1,P_2)$ *is open in* $\mathrm{rep}(\mathbf{d})$ *and its closure* $\mathrm{rep}(P_1,P_2)$ *is an irreducible component of* $\mathrm{rep}(\mathbf{d})$*.*

Proof. Let $M \in \mathrm{rep}^{\mathrm{p}}(P_1,P_2)$, $\pi_i : \mathrm{rep}(\mathbf{d},\mathbf{A}) \to \mathrm{rep}(\mathbf{d}_i,\mathbf{A}_i)$ be the natural projections. As projective modules are *rigid* [5], there are neighbourhoods $U_i \subseteq \mathrm{rep}(\mathbf{d}_i,\mathbf{A}_i)$ of P_i such that $N \simeq P_i$ for every $N \in U_i, (i = 1,2)$. Then $\pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2)$ is a neighbourhood of M contained in $\mathrm{rep}^{\mathrm{p}}(P_1,P_2)$. Since $\mathrm{rep}^{\mathrm{p}}(P_1,P_2) \simeq \mathbb{A}^m$, where $m = \dim {}^\wedge\mathbf{W}(P_2,P_1)$, and hence it is irreducible, its closure in $\mathrm{rep}(\mathbf{d})$ is an irreducible component. □

Dealing with derived equivalences, it is more convenient to consider projective resolutions instead of modules. A degeneration theory for complexes of projective modules has been developed in [9]. In particular, an analogue of the Riedtmann–Zwara theorem for complexes has been proved there. Let $\Pi_1, \Pi_2, \dots, \Pi_s$ be the set of isomorphism classes of indecomposable projective $\mathbf{A}$-modules. Then every projective $\mathbf{A}$-module is isomoprhic to a module $P(p_1, p_2, \dots, p_s) = \bigoplus_{k=1}^s p_k\Pi_k$ for a uniquely defined integer vector $(p_1, p_2, \dots, p_s) \in \mathbb{N}^s$. Let $\mathbf{P} : n \mapsto \mathbf{P}_n$ be a function $\mathbb{Z} \to \mathbb{N}^s$. We denote by $\mathscr{P}(\mathbf{P})$ (or by $\mathscr{P}(\mathbf{A},\mathbf{P})$, if nesessary) the set of all complexes $\mathcal{P} = (\mathcal{P}_n, d_n)$ such that $\mathcal{P}_n = P(\mathbf{P}_n)$. If $\mathbf{P}$ is bounded, i.e. $\mathbf{P}_n = 0$ for almost all n, the set $\mathscr{P}(\mathbf{P})$ is an algebraic variety, a closed subvariety of $\prod_n \mathrm{Hom}_{\mathbf{A}}(\mathcal{P}_n, \mathcal{P}_{n-1})$. Isomorphism classes of complexes from $\mathscr{P}(\mathbf{P})$ are orbits of the group $\mathrm{GL}(\mathbf{P}) = \prod_n \mathrm{GL}(\mathcal{P}_n, \mathbf{A})$. Thus, the relation $\preccurlyeq$ is well defined for complexes from $\mathscr{P}(\mathbf{P})$: $\mathcal{P} \preccurlyeq \mathcal{P}'$ if $\mathcal{P}' \in \overline{\mathrm{GL}(\mathbf{P})\mathcal{P}}$. The following result is a special case of [9, Theorems 1,2].

Theorem 2.4. *Let $\mathbf{P} : \mathbb{Z} \to \mathbb{N}^s$ be bounded, $\mathcal{P}, \mathcal{P}' \in \mathscr{P}(\mathbf{P})$. Then $\mathcal{P} \preccurlyeq \mathcal{P}'$ if and only if there is a complex $\mathcal{Q} \in \mathscr{H}^-(\mathbf{A}\text{-proj})$ and a triangle*

$$\mathcal{P}' \to \mathcal{P} \oplus \mathcal{Q} \to \mathcal{Q} \to \mathcal{P}'[1].$$

If gl.dim $\mathbf{A} < \infty$, *one can choose $\mathcal{Q}$ bounded.*

Corollary 2.5. *Let $\mathcal{P}, \mathcal{P}'$ are two complexes from $\mathscr{P}(\mathbf{P})$, where $\mathbf{P}$ is bounded, such that $\mathbf{dim}\,\mathrm{H}_k(\mathcal{P}) = \mathbf{dim}\,\mathrm{H}_k(\mathcal{P}')$ for all k.*

1. *If $\mathcal{P} \preccurlyeq \mathcal{P}'$, then $\mathrm{H}_k(\mathcal{P}) \preccurlyeq \mathrm{H}_k(\mathcal{P}')$ for all k.*
2. *If $\mathbf{A}$ is* hereditary, *or, the same, Morita equivelent to a path algebra of a quiver* [4], *and $\mathrm{H}_k(\mathcal{P}) \preccurlyeq \mathrm{H}_k(\mathcal{P}')$ for all k, then $\mathcal{P} \preccurlyeq \mathcal{P}'$.*

Proof. (1) We may suppose that $\mathbf{P}_k = 0$ for $k < 0$. Consider a triangle $\mathcal{P}' \xrightarrow{\alpha} \mathcal{P} \oplus \mathcal{Q} \xrightarrow{\beta} \mathcal{Q}$. It induces an exact sequence of homologies

$$\begin{aligned} \dots \to \mathrm{H}_2(\mathcal{P}') &\xrightarrow{\mathrm{H}_2(\alpha)} \mathrm{H}_2(\mathcal{P}) \oplus \mathrm{H}_2(\mathcal{Q}) \xrightarrow{\mathrm{H}_2(\beta)} \mathrm{H}_2(\mathcal{Q}) \to \\ \to \mathrm{H}_1(\mathcal{P}') &\xrightarrow{\mathrm{H}_1(\alpha)} \mathrm{H}_1(\mathcal{P}) \oplus \mathrm{H}_1(\mathcal{Q}) \xrightarrow{\mathrm{H}_1(\beta)} \mathrm{H}_1(\mathcal{Q}) \to \\ \to \mathrm{H}_0(\mathcal{P}') &\xrightarrow{\mathrm{H}_0(\alpha)} \mathrm{H}_0(\mathcal{P}) \oplus \mathrm{H}_0(\mathcal{Q}) \xrightarrow{\mathrm{H}_0(\beta)} \mathrm{H}_0(\mathcal{Q}) \to 0 \end{aligned}$$

Since $\mathbf{dim}\,\mathrm{H}_0(\mathcal{P}) = \mathbf{dim}\,\mathrm{H}_0(\mathcal{P}')$, the map $\mathrm{H}_0(\alpha)$ is injective, hence, $\mathrm{H}_1(\beta)$ is surjective. Now, since $\mathbf{dim}\,\mathrm{H}_1(\mathcal{P}) = \mathbf{dim}\,\mathrm{H}_1(\mathcal{P}')$, the map $\mathrm{H}_1(\alpha)$ is injective, hence, $\mathrm{H}_0(\beta)$ is surjective, etc. Thus we get exact sequences

$$0 \to \mathrm{H}_k(\mathcal{P}') \to \mathrm{H}_k(\mathcal{P}) \oplus \mathrm{H}_k(\mathcal{Q}) \to \mathrm{H}_k(\mathcal{Q}) \to 0$$

for all k, whence the statement follows.

(2) Note that for projective modules $\mathcal{P}, \mathcal{P}'$ over a hereditary algebra $\dim \mathcal{P} = \dim \mathcal{P}'$ if and only if $\mathcal{P} \simeq \mathcal{P}'$. Recall also that every bounded complex $\mathcal{P}$ of projective modules over a hereditary algebra is isomorphic to the direct sum $\bigoplus_k \mathcal{P}^{(k)}[k]$, where $\mathcal{P}^{(k)}$ is the complex

$$0 \to \mathrm{Im}\, d_{k+1} \xrightarrow{\bar{d}_{k+1}} \mathrm{Ker}\, d_k \to 0 \quad (\mathrm{Ker}\, d_k \text{ at the 0-th place}),$$

where $\bar{d}_{k+1}$ is injective and $\mathrm{H}_0(\mathcal{P}^{(k)}) \simeq \mathrm{H}_k(\mathcal{P})$. If $\mathrm{H}_k(\mathcal{P}) \preccurlyeq \mathrm{H}_k(\mathcal{P}')$, there is an exact sequence

$$0 \to \mathrm{H}_k(\mathcal{P}') \to \mathrm{H}_k(\mathcal{P}) \oplus N \to N \to 0,$$

which induces an exact sequenece of complexes

$$0 \to \mathcal{P}'^{(k)} \to \tilde{\mathcal{P}} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is a projective resolution of N, $\tilde{\mathcal{P}}_i = \mathcal{P}'^{(k)}_i \oplus \mathcal{Q}_i$ and $\tilde{\mathcal{P}}$ is a projective resolution of $\mathrm{H}_k(\mathcal{P}) \oplus N$. Therefore, $\tilde{\mathcal{P}}$ is homotopy equivalent to $\mathcal{P}^{(k)} \oplus \mathcal{Q}$ and, since all their components are of the same dimensions, they are isomorphic (see [9, Lemma 1]). Thus $\mathcal{P}'^{(k)} \preccurlyeq \mathcal{P}^{(k)}$ for each k, which evidently implies that $\mathcal{P}' \preccurlyeq \mathcal{P}$. □

We denote by $\mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{C}, \mathcal{C}')$ the set of morphisms $\mathcal{C} \to \mathcal{C}'$ in the derived category $\mathscr{D}(\mathbf{A}\text{-Mod})$. Quite analogously to Corollary 2.2, one proves

Corollary 2.6. (1) *If $\mathcal{P} \preccurlyeq \mathcal{P}'$, $\dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{P}, \mathcal{Q}) \le \dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{P}', \mathcal{Q})$ and $\dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{Q}, \mathcal{P}) \le \dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{Q}, \mathcal{P}')$ for all $\mathcal{Q} \in \mathscr{D}(\mathbf{A}\text{-mod})$.*

(2) *If $\mathcal{P} \preccurlyeq \mathcal{P}'$ and either $\dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{P}, \mathcal{P}') = \dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{P}', \mathcal{P}')$ or $\dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{Q}, \mathcal{P}) = \dim \mathrm{Hom}_{\mathscr{D}\mathbf{A}}(\mathcal{Q}, \mathcal{P}')$ for all $\mathcal{Q}$, then $\mathcal{P} \simeq \mathcal{P}'$.*

(3) *If there is a non-split triangle $\mathcal{P}' \to \mathcal{P} \to \mathcal{P}'' \to \mathcal{P}[1]$, then $\mathcal{P} \preccurlyeq \mathcal{P}' \oplus \mathcal{P}''$ and $\dim \mathrm{End}_{\mathscr{D}\mathbf{A}}(\mathcal{P}) < \dim \mathrm{End}_{\mathscr{D}\mathbf{A}}(\mathcal{P}' \oplus \mathcal{P}'')$.*

3. Tits form and Euler form

Let $\mathbf{W}$ be an $\mathbf{A}$-bimodule. We define the *Tits form* of $\mathbf{W}$ as the bilinear form on the Grothendieck group $K_0(\mathbf{A}\text{-proj})$ of projective $\mathbf{A}$-modules

$$\langle\, [P], [P'] \,\rangle_{\mathbf{W}} = \dim \mathrm{Hom}_{\mathbf{A}}(P, P') - \dim \mathrm{Hom}_{\mathbf{A}}(P, \mathbf{W} \otimes_{\mathbf{A}} P').$$

The corresponding quadratic form $\mathrm{Q}_{\mathbf{W}}([P]) = \langle\, [P], [P] \,\rangle_{\mathbf{W}}$ is the classical Tits form, which equals the dimension of the group $\mathrm{GL}(P, \mathbf{A})$ minus the dimension of the affine space $\mathbf{W}(P)$, where this group acts so that the orbits are the isomorphism classes.

If $\mathbf{W}$ is a bipartite $\mathbf{A}_2$-$\mathbf{A}_1$-bimodule, $P = P_1 \oplus P_2$ and $P' = P'_1 \oplus P'_2$, where P_i, P'_i are $\mathbf{A}_i$-modules, then

$$\langle\, [P], [P'] \,\rangle_{\mathbf{W}} = \dim \mathrm{Hom}_{\mathbf{A}_1}(P_1, P'_1) + \dim \mathrm{Hom}_{\mathbf{A}_2}(P_2, P'_2) - \dim \mathbf{W}(P_2, P'_1).$$

In particular, if $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$ and we consider the Tits form of the dual bimodule ${}^{\wedge}\mathbf{W}$, then

$$\mathrm{Q}_{{}^{\wedge}\mathbf{W}}(P_1 \oplus P_2) = \dim \mathrm{GL}(P_1, \mathbf{A}_1) + \dim \mathrm{GL}(P_2, \mathbf{A}_2) - \dim \mathrm{rep}(P_1, P_2).$$

If, moreover, $\mathbf{W}$ is projective as left $\mathbf{A}_1$-bimodule and both $\mathbf{A}_i$ are of finite global dimension, the algebra $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$ is also of finite global dimension, so the Grothendieck group $K_0(\mathbf{A}\text{-proj})$ coincides with the Grothendieck group $K_0(\mathbf{A}\text{-mod})$ of all $\mathbf{A}$-modules. In this case the *Euler form* on this group can be defined as

$$\langle\, [M], [M'] \,\rangle_{\mathbf{A}} = \sum_k (-1)^k \dim \mathrm{Ext}^k_{\mathbf{A}}(M, M').$$

In particular, if P and P' are projective, $\langle\, [P, [P'] \,\rangle_{\mathbf{A}} = \dim \mathrm{Hom}_{\mathbf{A}}(P, P')$.

Proposition 3.1. *Let $\mathbf{W}$ be an $\mathbf{A}_1$-$\mathbf{A}_2$-bimodule projective as left $\mathbf{A}_1$-module, $M \in \mathrm{rep}^{\mathrm{p}}(P_1, P_2)$, $M' \in \mathrm{rep}^{\mathrm{p}}(P'_1, P'_2)$. Then $\langle\, [M], [M'] \,\rangle_{\mathbf{A}} = \langle\, [P], [P'] \,\rangle_{{}^{\wedge}\mathbf{W}}$, where $P = P_1 \oplus P_2$, $P' = P'_1 \oplus P'_2$.*

Proof. Consider the standard resolutions (1.1) $0 \to \mathcal{Q} \to \mathcal{P} \to M \to 0$ of M and $0 \to \mathcal{Q}' \to \mathcal{P}' \to M' \to 0$ of M'. Then $[M] = [\mathcal{P}] - [\mathcal{Q}]$ and $[M'] = [\mathcal{P}'] - [\mathcal{Q}']$ in $K_0(\mathbf{A}\text{-mod})$. Therefore,

$$\begin{aligned}
&\langle\, [M], [M']\,\rangle_{\mathbf{A}} = \langle\, [\mathcal{P}], [\mathcal{P}']\,\rangle_{\mathbf{A}} + \langle\, [\mathcal{Q}], [\mathcal{Q}']\,\rangle_{\mathbf{A}} - \langle\, [\mathcal{P}], [\mathcal{Q}']\,\rangle_{\mathbf{A}} - \langle\, [\mathcal{Q}], [\mathcal{P}']\,\rangle_{\mathbf{A}} = \\
&= \dim \operatorname{Hom}_{\mathbf{A}_1}(P_1, P_1') + \dim \operatorname{Hom}_{\mathbf{A}_2}(P_2, P_2') + \dim \operatorname{Hom}_{\mathbf{A}_1}(P_1, \mathbf{W} \otimes_{\mathbf{A}_1} P_2') + \\
&+ \dim \operatorname{Hom}_{\mathbf{A}_1}(\mathbf{W} \otimes_{\mathbf{A}_2} P_2, \mathbf{W} \otimes_{\mathbf{A}_2} P_2') - \dim \operatorname{Hom}_{\mathbf{A}_1}(P_1, \mathbf{W} \otimes_{\mathbf{A}_2} P_2') - \\
&- \dim \operatorname{Hom}_{\mathbf{A}_1}(\mathbf{W} \otimes_{\mathbf{A}_2} P_2, P_1') - \dim \operatorname{Hom}_{\mathbf{A}_1}(\mathbf{W} \otimes_{\mathbf{A}_2} P_2, \mathbf{W} \otimes_{\mathbf{A}_2} P_2') = \\
&= \dim \operatorname{Hom}_{\mathbf{A}_1}(P_1, P_1') + \dim \operatorname{Hom}_{\mathbf{A}_2}(P_2, P_2') - \dim \operatorname{Hom}_{\mathbf{A}_1}(\mathbf{W} \otimes_{\mathbf{A}_2} P_2, P_1') = \\
&= \langle\, [P], [P']\,\rangle_{^\wedge\mathbf{W}}, \text{ since } \operatorname{Hom}_{\mathbf{A}_1}(\mathbf{W} \otimes_{\mathbf{A}_2} P_2, P_1') \simeq {}^\wedge\mathbf{W}(P_2, P_1').
\end{aligned}$$

□

Corollary 3.2. *Under conditions of Proposition 3.1, the codimension of the orbit of M in $\operatorname{rep}(P_1, P_2)$ equals $\dim \operatorname{Ext}^1_{\mathbf{A}}(M, M)$. In particular, this orbit is open in $\operatorname{rep}(P_1, P_2)$ if and only if $\operatorname{Ext}^1_{\mathbf{A}}(M, M) = 0$.*

Proof. This codimension equals

$$\begin{aligned}
&\dim \operatorname{rep}(P_1, P_2) - (\dim \operatorname{GL}(P_1, A_1) + \dim \operatorname{GL}(P_2, A_2) - \dim \operatorname{Aut}_{\mathbf{A}} M) = \\
&= \dim {}^\wedge\mathbf{W}(P_1, P_2) - \dim \operatorname{GL}(P_1, A_1) - \dim \operatorname{GL}(P_2, A_2) + \dim \operatorname{End}_{\mathbf{A}} M = \\
&= -\langle\, [P], [P]\,\rangle_{^\wedge\mathbf{W}} + \dim \operatorname{End}_{\mathbf{A}}(M, M) = \dim \operatorname{Ext}^1(M, M),
\end{aligned}$$

since $\operatorname{pr.dim} M \le 1$ and

$$\langle\, [P], [P]\,\rangle_{^\wedge\mathbf{W}} = \langle\, [M], [M]\,\rangle_{\mathbf{A}} = \dim \operatorname{End}_{\mathbf{A}} M - \dim \operatorname{Ext}^1_{\mathbf{A}}(M, M).$$

□

Recall that analogous results hold for representations of quivers, or, the same, for modules over the path algebra $\mathbf{B} = \Bbbk\Delta$ of a quiver Δ [20]. If Δ_0 is the set of vertices and Δ_1 is the set of arrows of Δ, the *Tits form* is defined as

$$\langle\, [M], [M']\,\rangle_{\Delta} = \sum_{i \in \Delta_0} d_i d_i' - \sum_{a \in \Delta_1} d_{\sigma a} d_{\tau a}',$$

where, for any representation M of the quiver Δ, $d_i = \dim M(i)$ (thus $(d_1, d_2, \dots, d_s) = \mathbf{dim}\, M$, the dimension vector of M), σa is the *source* of the arrow a and τa is its *target*, i.e. $a : \sigma a \to \tau a$. The corresponding quadratic form is again of geometric meaning: if $\mathbf{d} = \dim M = (d_1, d_2, \dots, d_s)$, then

$$\mathrm{Q}_{\Delta}(\mathbf{d}) = \langle\, [M], [M]\,\rangle_{\Delta} = \dim \operatorname{GL}(\mathbf{d}, \Bbbk) - \dim \operatorname{rep}(\mathbf{d}, \mathbf{B}),$$

where $\operatorname{GL}(\mathbf{d}, \Bbbk) = \prod_i \operatorname{GL}(d_i, \Bbbk)$. The Tits form coincides with the *Euler form*: source

$$\langle\, [M], [M']\,\rangle_{\Delta} = \langle\, [M], [M']\,\rangle_{\mathbf{B}} = \dim \operatorname{Hom}_{\mathbf{B}}(M, M') - \dim \operatorname{Ext}^1_{\mathbf{B}}(M, M'),$$

and the codimenion in $\operatorname{rep}(\mathbf{d}, \mathbf{B})$ of the orbit of the representation $[M]$ equals $\dim \operatorname{Ext}^1_{\mathbf{B}}(M, M)$.

4. Tilting and deformations

Let $\mathbf{A}$ and $\mathbf{B}$ be derived equivalent finite dimensional $\Bbbk$-algebras. We suppose that they are of finite global dimension. Then it follows from [18] (see also [11]) that there is a bounded complex $\mathcal{T}=(\mathcal{T}_n,\partial_n)$ of finite dimensional projective $\mathbf{B}$-$\mathbf{A}$-bimodules such that the functor $\mathcal{T}\otimes^{\mathbb{L}}_{\mathbf{A}}\_$ is an equivalence $\mathscr{D}(\mathbf{A}\text{-mod})\to\mathscr{D}(\mathbf{B}\text{-mod})$ and $\mathbb{R}\mathrm{Hom}_{\mathbf{B}}(\mathcal{T},\_)$ is a quasi-inverse equivalence $\mathscr{D}(\mathbf{B}\text{-mod})\to\mathscr{D}(\mathbf{A}\text{-mod})$. We denote $\mathcal{T}M=\mathcal{T}\otimes_{\mathbf{A}}M$, $\mathcal{T}_kM=\mathcal{T}_k\otimes_{\mathbf{A}}M$ and $\mathcal{H}_kM=\mathrm{H}_k(\mathcal{T}M)$.

Lemma 4.1. *Let Z be a component of* $\mathrm{rep}(\mathbf{d},\mathbf{A})$ *for some vector dimension* $\mathbf{d}$. *There is an open subset $Z^{\mathcal{T}}\subseteq Z$ such that for all $M\in Z^{\mathcal{T}}$ and all k the vector dimensions $\mathbf{dim}\,\mathcal{H}_kM$ are the same and the smallest (componentwise) among $\mathbf{dim}\,\mathcal{H}_kN$ for $N\in Z$. Moreover, if $M,M'\in Z^{\mathcal{T}}$ and $M\preccurlyeq M'$, then $\mathcal{H}_kM\preccurlyeq\mathcal{H}_kM'$ for all k.*

Proof. The first assertion is evident: take for U the set of all $M\in Z$ such that $\mathbf{dim}\,\mathrm{Im}(\partial_k\otimes 1_M)$ is the biggest possible. If $M\preccurlyeq M'$, Theorems 2.1 and 2.4 show that $\mathcal{T}M\preccurlyeq\mathcal{T}M'$, so the second assertion follows from Corollary 2.5. □

Thus, one can define a map $Z_{\mathcal{T}}\to\mathrm{rep}(\mathbf{h}_k,\mathbf{B})$, where $\mathbf{h}_k=\mathbf{dim}\,\mathcal{H}_kM$. Moreover, for every $M\in Z_{\mathcal{T}}$ there is an open neighbourhood $U\ni M$ in $Z^{\mathcal{T}}$ such that this map can be considsered as a regular map $U\to\mathrm{rep}(\mathbf{h}_k,\mathbf{B})$.

Let also Z_0 be the *open sheet* of Z, i.e. the set of all $M\in Z$ such that $\dim_{\Bbbk}\mathrm{End}_{\mathbf{A}}M$ is minimal possible, or, the same, the dimension of the orbit of M is maximal possible (Z_0 is always open in Z). We denote $Z_0^{\mathcal{T}}=Z^{\mathcal{T}}\cap Z_0$ and call $Z_0^{\mathcal{T}}$ the $\mathcal{T}$*-sheet* of Z. Note that if $M'\in Z_0$ and $M\preccurlyeq M'$, then $M\simeq M'$. In particular, if $M\simeq M_1\oplus M_2\in Z_0$, then $\mathrm{Ext}^1(M_1,M_2)=0$. Therefore, the same is true for the object $\mathcal{T}M$ from $\mathscr{D}(\mathbf{B}\text{-mod})$. Note that the group $\mathrm{GL}(\mathbf{d},\Bbbk)$ acts on Z_0 and this action is closed.

We apply these considerations to the component $\mathrm{rep}(P_1,P_2)$ of $\mathrm{rep}(\mathbf{A})$, when $\mathbf{A}=\mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$, getting the $\mathcal{T}$-sheet $\mathrm{rep}_0^{\mathcal{T}}(P_1,P_2)$ and an open subset $\mathrm{rep^p}_0^{\mathcal{T}}(P_1,P_2)=\mathrm{rep^p}(P_1,P_2)\cap\mathrm{rep}_0^{\mathcal{T}}(P_1,P_2)$, which we call the $\mathcal{T}$*-sheet* of $\mathrm{rep^p}(P_1,P_2)$. The isomorphism $\mathrm{rep^p}(P_1,P_2)\simeq{}^\wedge\mathbf{W}(P_2,P_1)$ maps it to an open subset ${}^\wedge\mathbf{W}_0^{\mathcal{T}}(P_1,P_2)$, also called the $\mathcal{T}$*-sheet* of ${}^\wedge\mathbf{W}(P_2,P_1)$.

Suppose now that the algebra $\mathbf{A}=\mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$ is derived equivalent to a path algebra $\mathbf{B}=\Bbbk\Delta$ of a quiver Δ. Let $\mathbf{d}$ be a dimension of representations of $\mathbf{B}$. Then $\mathrm{rep}(\mathbf{d},\mathbf{B})$ is an affine space and there is an open subset $\mathrm{rep}^c(\mathbf{d})\subseteq\mathrm{rep}(\mathbf{d},\mathbf{B})$ such that every $N\in\mathrm{rep}^c(\mathbf{d})$ decomposes as $N\simeq\bigoplus_{i=1}^m N_i$, where the number of summands m and $\dim N_i=\mathbf{d}_i$ are common for all $N\in\mathrm{rep}^c(\mathbf{d})$, all modules N_i are *bricks* (or *schurian*), i.e. $\mathrm{End}_{\mathbf{A}}N_i=\Bbbk$, and $\mathrm{Ext}^1_{\mathbf{A}}(N_i,N_j)=0$ for $i\neq j$ [10, Proposition 3]. Recall also that every object $\mathcal{C}\in\mathscr{D}^b(\mathbf{B}\text{-mod})$ is isomorphic to the direct sum of shifted modules $\bigoplus_k H_k[k]$, where $H_k=\mathrm{H}_k(\mathcal{C})$. Especially, it is the case for $\mathcal{T}M$, where $M\in\mathbf{A}\text{-mod}$: it is isomorphic to $\bigoplus_k\mathcal{H}_kM[k]$.

Lemma 4.2. *Let* $\mathbf{A}$ *be a Dynkinian algebra,* Z_0 *be the open sheet of an irreducible component* Z *of the variety* $\mathrm{rep}(\mathbf{d},\mathbf{A})$ *for some dimenion* $\mathbf{d}$*. Then* Z_0 *consists of a unique isomorphim class of modules, and* $\mathrm{Ext}^1_{\mathbf{A}}(M,M)=0$ *for any* $M\in Z_0$*.*

Proof. It is known that if $\mathbf{B}=\Bbbk\Delta$, where Δ is a Dynkin quiver, N is an indecomposable $\mathbf{B}$-module, then $\mathrm{Ext}^1_{\mathbf{B}}(N,N)=0$ [21]. Therefore, it also holds for any indecomposbale object from $\mathscr{D}^b(\mathbf{B}\text{-mod})$, hence from $\mathscr{D}^b(\mathbf{A}\text{-mod})$ as well. In particular, $\mathrm{Ext}^1_{\mathbf{A}}(M_i,M_i)=0$ for every indecomposable direct summand of M. Since $\mathrm{Ext}^1_{\mathbf{A}}(M_i,M_j)=0$ for any two different direct summands, $\mathrm{Ext}^1_{\mathbf{A}}(M,M)=0$. Moreover, both $\mathbf{B}$ and $\mathbf{A}$ have finitely many isomorphism classes of indecomposable modules. Hence there are finitely many orbits in Z, so one of them is dense. Then it coincides with Z_0. $\square$

Recall that an $\mathbf{A}$-module M is called *partial tilting* [8] if $\mathrm{pr.dim}\, M\le 1$ and $\mathrm{Ext}^i_{\mathbf{A}}(M,M)=0$ for all $i\ne 0$. If, moreover, M generates $\mathcal{D}^b(\mathbf{A})$, it is called *tilting*. If M is tilting, $\tilde{\mathbf{A}}=(\mathrm{End}_{\mathbf{A}}\, M)^{\mathrm{op}}$, then $\mathscr{D}^b(\tilde{\mathbf{A}}\text{-mod})\simeq\mathscr{D}^b(\mathbf{A}\text{-mod})$ [8, Theorem III.2.10]. For any partial tilting module M there is a module M' such that $M\oplus M'$ is tilting [8, Lemma III.6.1].

Corollary 4.3. *Let* $\mathbf{A}=\mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$*, where* $\mathbf{W}$ *is a Dynkinian bimodule projective as* $\mathbf{A}_1$*-module and* w *belongs to the open sheet of* ${}^{\wedge}\mathbf{W}(P_1,P_2)$*. Then* $\mathrm{Ext}^i_{{}^{\wedge}\mathbf{W}}(w,w)=0$ *for all* $i>0$*. In particular,* $\mathbf{A}'=\mathrm{End}_{{}^{\wedge}\mathbf{W}}\, w$ *is a Dynkinian algebra.*

Proof. The first claim follows from Lemma 4.2 and Proposition 1.2 applied to the module $M\in\mathbf{A}\text{-mod}^{\mathrm{p}}$ corresponding to w. In particular, M is a partial tilting module, thus is a direct summand of a tilting module $\tilde{M}$. Then $\mathbf{A}'\simeq\mathrm{End}_{\mathbf{A}}\, M\simeq e\tilde{\mathbf{A}}e$, where $\tilde{\mathbf{A}}=\mathrm{End}_{\mathbf{A}}\, M$ and e is an idempotent from $\tilde{\mathbf{A}}$. Since $\mathscr{D}^b(\tilde{\mathbf{A}})\simeq\mathscr{D}^b(\mathbf{A})$, $\tilde{\mathbf{A}}$ is a Dynkinian algebra, hence so is also $\mathbf{A}'$. $\square$

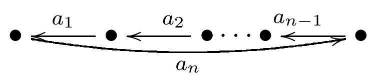
Let now $\mathbf{B}=\Bbbk\Delta$, where Δ is a Euclidean quiver. Recall that a $\mathbf{B}$-module M of vector dimension $\mathbf{d}=(d_1,d_2,\dots,d_s)$, as well as this dimension itself, is called *sincere*, if $d_i\ne 0$ for all i, or, equivalently, $\mathrm{Hom}_{\mathbf{B}}(P,M)\ne 0$ for every projective $\mathbf{B}$-module P. In Euclidean case the quadratic Euler form $\mathrm{Q}(\mathbf{d})=\langle\,\mathbf{d},\mathbf{d}\,\rangle_{\mathbf{B}}$ is non-negative, i.e. $\mathrm{Q}(\mathbf{d})\ge 0$ for every vector $\mathbf{d}$. Moreover, its kernel is one-dimensional, so there is a unique vector $\boldsymbol{\delta}$ with coprime positive integer coordinates such that $\mathrm{Q}(\boldsymbol{\delta})=0$. A non-negative integer vector $\mathbf{d}$ is a dimension of an indecomposable $\mathbf{B}$-module if and only if either $\mathrm{Q}(\mathbf{d})=1$ or $\mathrm{Q}(\mathbf{d})=0$. In the former case $\mathbf{d}$ is called a *real root*; then there is a unique indecomposable representation M of vector dimension $\mathbf{d}$. In the latter case $\mathbf{d}$ is called an *imaginary root*; there are infinitely many non-isomorphic indecomposable modules of vector dimension $\mathbf{d}$. Any imaginary root is sincere and equals $m\boldsymbol{\delta}$ for some m.

Let τ denotes the *Auslander–Reiten translation* in the category $\mathscr{D}^b(\mathbf{B}\text{-mod})$; it is an auto-equivalence of this category [8]. Then the category $\mathbf{B}$-ind consists of three disjoint parts:

preprojective part $\mathscr{P}$ consisting of modules $\tau^{-m}P$ $(m > 0)$, where τ is the Auslander–Reiten transform and P runs through indecomposable projective $\mathbf{B}$-modules;

preinjective part $\mathscr{I}$ consisting of modules $\tau^m I$ $(m > 0)$, where I runs through indecomposable injective $\mathbf{B}$-modules;

regular part $\mathscr{R}$, which is a disjoint union of *tubes* $\mathscr{R}_\lambda$ $(\lambda \in \mathbb{P}^1_{\Bbbk})$, where $\mathscr{R}_\lambda$ is equivalent to the category of finite dimensional representations R of a cyclic quiver $\mathbf{H}_n$:



such that $R(a_1a_2\dots a_n)$ is nilpotent. Here $n = n(\lambda)$ depends on λ and equals 1 for all $\lambda \in \mathbb{P}^1_{\Bbbk}$ except, possibly, 1, 2 or 3 points. We denote by $\mathbb{X}$ the subset $\{\, \lambda \in \mathbb{P}^1_{\Bbbk} \mid n(\lambda) = 1 \,\}$. Especially, every indecomposable module M such that $\mathbf{dim}\, M$ is an imaginary root belongs to $\mathscr{R}$.

More precisely, the indecomposable modules M such that $\mathbf{dim}\, M$ is an imaginary root can be described as follows [1, 15]. There is a locally free sheaf $\mathcal{F}$ over the projective line $\mathbb{P}^1_{\Bbbk}$ with $\mathbf{B}$-action on it such that the $\mathbf{B}$-module $\mathcal{F}(m,\lambda) = \mathcal{F}_\lambda/\mathfrak{m}_\lambda^m\mathcal{F}$, where $\lambda \in \mathbb{P}^1_{\Bbbk}$ and $\mathfrak{m}_\lambda$ is the maximal ideal of $\mathcal{O}_{\mathbb{P}^1,\lambda}$, is indecomposable of dimension $m\boldsymbol{\delta}$. Every indecomposable module from $\mathscr{R}_\lambda$ for $\lambda \notin \mathbb{X}$ is isomorphic to $\tau^k\mathcal{F}(m,\lambda)$ for some m and $0 \le k < n(\lambda)$, in particular, if $\lambda \in \mathbb{X}$, it is isomorphic to $\mathcal{F}(m,\lambda)$. The map $\lambda \mapsto \tau^k\mathcal{F}(m,\lambda)$ with fixed k induces a regular embedding $\mathbb{P}^1 \to \operatorname{rep}(m\boldsymbol{\delta})$. Note also that $\mathcal{F}(m,\lambda)$ is a brick if and only if $m = 1$.

Moreover, $\operatorname{Hom}_{\mathbf{B}}(M,N) = 0$ if either $M \in \mathscr{I}$, $N \in \mathscr{P} \cup \mathscr{R}$, or $M \in \mathscr{R}$, $N \in \mathscr{P}$, or $M \in \mathscr{R}_\lambda$, $N \in \mathscr{R}_\mu$, $\lambda \ne \mu$, while $\operatorname{Hom}_{\mathbf{B}}(M,N) \ne 0$ and $\operatorname{Ext}^1_{\mathbf{B}}(M,N) \ne 0$ if either $M \in \mathscr{P}$, $N \in \mathscr{R}_\lambda$ with $\lambda \in \mathbb{X}$, or $M \in \mathscr{R}_\lambda$, $N \in \mathscr{I}$ with $\lambda \in \mathbb{X}$, or $M, N \in \mathscr{R}_\lambda$ with $\lambda \in \mathbb{X}$.

Following [8], we denote by $\mathscr{C}[i] = \mathscr{P}[i] \cup \mathscr{I}[i+1]$; thus $\mathscr{D}^b(\mathbf{B}\text{-Mod}) = \bigcup_i(\mathscr{C}[i] \cup \mathscr{R}[i])$, and if $M \in \mathscr{R}_\lambda[i]$ with $\lambda \in \mathbb{X}$, N is an indecomposable object of $\mathscr{D}^b(\mathbf{B})$, then

$$\begin{aligned} \operatorname{Hom}_{\mathscr{D}\mathbf{B}}(M,N) \ne 0 \quad & \text{if and only if either } N \in \mathscr{C}[i+1] \\ & \text{or} \quad N \in \mathscr{R}_\lambda[j],\ j \in \{i, i+1\}; \\ \operatorname{Hom}_{\mathscr{D}\mathbf{B}}(N,M) \ne 0 \quad & \text{if and only if either } N \in \mathscr{C}[i] \\ & \text{or} \quad N \in \mathscr{R}_\lambda[j],\ j \in \{i, i-1\}. \end{aligned}$$

We also need the following simple lemma.

Lemma 4.4. *Let* $\mathbf{B}$ *be a hereditary algebra of Euclidean type,* $M \in \mathbf{B}$*-ind and* $\mathbf{d} = \dim M$*.*

(1) *If M is not a brick or $\operatorname{Ext}^1_{\mathbf{B}}(M,M) \neq 0$, then*

$$\operatorname{Hom}_{\mathbf{B}}(N,M) \neq 0 \neq \operatorname{Ext}^1_{\mathbf{B}}(M,N) \quad \text{for} \quad N \in \mathscr{P};$$

$$\operatorname{Hom}_{\mathbf{B}}(M,N) \neq 0 \neq \operatorname{Ext}^1_{\mathbf{B}}(N,M) \quad \text{for} \quad N \in \mathscr{I}.$$

(2) *If M is a brick and $\mathbf{d}$ is a real root, then $\operatorname{Ext}^1_{\mathbf{B}}(M,M) = 0$.*

(3) *If $\dim M$ is an imaginary root, then for every $\mathbf{B}$-module N*

$$\dim \operatorname{Hom}_{\mathbf{B}}(M,N) + \dim \operatorname{Hom}_{\mathbf{B}}(N,M) = \dim \operatorname{Ext}^1_{\mathbf{B}}(M,N) + \dim \operatorname{Ext}^1_{\mathbf{B}}(N,M).$$

Proof. (1) Since every proper subgraph of an Euclidean graph is a Dynkin one, every non-sincere $\mathbf{B}$-module is a brick without self-extensions. Thus M is sincere, hence $\operatorname{Hom}_{\mathbf{B}}(P,M) \neq 0$ for every projective P. Since $\tau^m M$ satisfies the same conditions, $\operatorname{Hom}_{\mathbf{B}}(N,M) \neq 0$ for every preprojective N. It remains to note that $\operatorname{Ext}^1_{\mathbf{B}}(M,N) \simeq \operatorname{Hom}_{\mathbf{B}}(\tau^{-1}N,M)^*$, where V^* denotes the dual vector space to V (see [8, Proposition 1.4.10]), and use the duality to get the assertion for preinjective modules.

(2) Follows from the formula $\mathsf{Q}(\mathbf{d}) = \dim \operatorname{Hom}_{\mathbf{B}}(M,M) - \dim \operatorname{Ext}^1_{\mathbf{B}}(M,M)$.

(3) Since $\dim M$ belongs to the kernel of the Euler form $\mathsf{Q}_{\mathbf{B}}$,

$$\langle\, M,N \,\rangle_{\mathbf{B}} + \langle\, N,M \,\rangle_{\mathbf{B}} = \dim \operatorname{Hom}_{\mathbf{B}}(M,N) + \dim \operatorname{Hom}_{\mathbf{B}}(N,M) - \\ - \dim \operatorname{Ext}^1_{\mathbf{B}}(M,N) - \dim \operatorname{Ext}^1_{\mathbf{B}}(N,M) = 0.$$

□

Lemma 4.5. *Let $f : X \to Y$ be a morphism of irreducible algebraic varieties, G and H be connected algebraic groups acting respectively on X and Y so that $\mathsf{G}x = \mathsf{G}x'$ if and only if $\mathsf{H}f(x) = \mathsf{H}f(x')$. Suppose also that $\dim \operatorname{Stab} x = \dim \operatorname{Stab} f(x)$ and $\operatorname{codim} \mathsf{G}x = \operatorname{codim} \mathsf{H}f(x)$ for all $x \in X$. Then $\mathsf{H}f(X)$ contains an open dense subset of Y.*

Proof. Obviously, we may replace X by an open subset X_0 such that the dimension $\dim \operatorname{Stab} x = s$ is minimal possible for all points $x \in X_0$, hence, also $\operatorname{codim} \mathsf{G}x = c$ is constant for $x \in X_0$. Choose an orbit $\mathsf{G}x_0$ with $x_0 \in X_0$ and a subvariety $Z \subseteq X_0$ of dimension $c = \operatorname{codim} \mathsf{G}x$ that intersects $\mathsf{G}x_0$ in finitely many points. Then there is an open subset $X' \subseteq X_0$ such that $Z \cap \mathsf{G}x$ is finite for every $x \in X'$. Therefore $\mathsf{H}y \cap f(Z)$ is finite for every $y \in f(X')$, and $\dim f(Z) = \dim Z$. It implies that $\dim \mathsf{H}f(Z) = \dim \mathsf{H} - s + c = \dim \mathsf{H}y + c = \dim Y$ for any $y \in f(Z)$. Hence $\mathsf{H}f(Z)$ contains an open dense subset from Y. □

Corollary 4.6. *Let $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$, where the bimodule $\mathbf{W}$ is projective as left $\mathbf{A}_1$-module, such that $\mathbf{A}$ is derived equivalent to a quiver algebra $\mathbf{B} = \Bbbk\Delta$, and $\mathcal{T}$ be a bounded complex of projective $\mathbf{B}$-$\mathbf{A}$-bimodules establishing this equivalence. Let also $M \in \operatorname{rep}^{\mathrm{p}}{}^{\mathcal{T}}_0(P_1,P_2)$ and $\mathbf{dim}\, \mathcal{H}_k M = \mathbf{d}_k$. Then there are open dense subsets $U_k \subseteq \operatorname{rep}(\mathbf{d}_k)$ such that for every set $\{\, N_k \mid N_k \in U_k \,\}$ there is a module $M' \in \operatorname{rep}^{\mathrm{p}}{}^{\mathcal{T}}_0(P_1,P_2)$ with $\mathcal{H}_k M' \simeq N_k$.*

Proof. Obviously, we may suppose that there is a unique k such that $\mathbf{d}_k \neq 0$. We use Lemma 4.5 for $X = \operatorname{rep}^{\mathrm{p}}{}^{\mathcal{T}}_0(P_1,P_2)$ with the natural action of $\mathrm{GL}(P_1, \mathbf{A}_1) \times \mathrm{GL}(P_2, \mathbf{A}_2)$ and $Y = \mathrm{rep}(\mathbf{d}_k)$ with the natural action of $\mathrm{GL}(\mathbf{d}_k, \Bbbk)$. One only has to note that in both cases $\dim \mathrm{Stab}\, M = \dim \mathrm{Hom}(M, M)$ and the codimension of the orbit equals $\dim \mathrm{Ext}^1(M, M)$, so they are presereved under the equivalence of the derived categtories. □

In particular, suppose that $\mathbf{B}$ is tame, $M \in \mathrm{rep}^{\mathrm{p}}{}_0^{\mathcal{T}}(P_1, P_2)$ and $\mathcal{H}_k M \simeq \bigoplus_{i=1}^m \mathcal{F}(1, \lambda_i)$ for some tuple $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{P}^1(m)$. Then there is an cofinite subset $\Lambda \subseteq \mathbb{P}^1$ such that all direct sums $\bigoplus_{i=1}^m \mathcal{F}(1, \mu_i)$, where $(\mu_1, \mu_2, \dots, \mu_m) \in \Lambda^{(m)}$, are isomorphic to $\mathcal{T}_k M'$ for some $M \in \mathrm{rep}^{\mathrm{p}}{}_0^{\mathcal{T}}(P_1, P_2)$.

Lemma 4.7. *Let* $\mathbf{B}$ *be a tame hereditary algebra,* N *be a partial tilting object from* $\mathscr{D}^b(\mathbf{B})$ *such that all direct summands of* N *are regular. Then* $\mathrm{End}_{\mathbf{B}} N \simeq \prod_{i=1}^k \mathbf{B}_i$*, where each* $\mathbf{B}_i$ *is the path algebra of a quiver of type* A_{n_i} *for some* i*.*

Proof. Since all regular modules belong to tubes and there are no nonzero morphisms or extensions between modules from different tubes, we may suppose that there is a unique tube $\mathscr{T}$ such that all direct summands of N belong to its shifts. Moreover, we may suppose that $\mathrm{Hom}_{\mathbf{B}}(N', N'') \neq 0$ or $\mathrm{Hom}_{\mathbf{B}}(N'', N') \neq 0$ for any nontrivial decomposition $N = N' \oplus N''$. If $\mathscr{T} \simeq \mathrm{rep}(\mathbf{H}_r)$, an easy explicite calculation shows that if $\mathrm{Hom}_{\mathscr{D}\mathbf{B}}(N, N[k]) = 0$ for $k \neq 0$, then $\mathrm{End}_{\mathbf{B}} N$ is isomorphic to a quiver algebra of type A_n with $n < r$. □

Theorem 4.8. *Let* $\mathbf{A} = \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$*, where* $\mathbf{W}$ *is a Euclidean bimodule projective as* $\mathbf{A}_1$*-module. Every space* $\mathrm{rep}^{\mathrm{p}}(P_1, P_2)$ *contains an open dense subset* U *such that every module* $M \in U$ *is either partial tilting or splits as* $M' \oplus M''$*, where* M'' *is a fixed partial tilting module,* $\mathrm{End}_{\mathbf{A}} M''$ *is a Dynkinian algebra,* $\mathrm{Hom}_{\mathbf{A}}(M', M'') = \mathrm{Hom}_{\mathbf{A}}(M'', M') = 0$*, and* $\mathcal{T} M'$ *runs through* $\bigoplus_{i=1}^m \mathcal{F}(1, \lambda_i)$ *with* $(\lambda_1, \lambda_2, \dots, \lambda_m) \in \mathbb{X}^{(m)}$ *for some cofinite subset* $\mathbb{X} \subseteq \mathbb{P}^1$*.*

Proof. Consider the open subset $\mathrm{rep}_0^{\mathrm{p}} = \mathrm{rep}^{\mathrm{p}}(P_1, P_2) \cap \mathrm{rep}_0(P_1, P_2)$. Let $M \in \mathrm{rep}_0^{\mathrm{p}}$, $M = \bigoplus_{i=1}^n M_i$ with $M_i \in \mathbf{A}\text{-ind}$. Then $\mathrm{Hom}_{\mathscr{D}\mathbf{A}}(M_i, M_j[k]) = 0$ for $i \neq j$ and $k \neq 0$. Let $\mathbf{B}$ be the quiver algebra derived equivalent to $\mathbf{A}$, N_i be the object of the derived category $\mathscr{D}^b(\mathbf{B}\text{-mod})$ corresponding to M_i, so also $\mathrm{Hom}_{\mathscr{D}\mathbf{B}}(N_i, N_j[k]) = 0$ for $i \neq j,\ k \neq 0$. Lemma 4.4 implies that if $N_i \in \mathscr{C}[m]$ for some i, m, then all N_j are bricks without self-extensions, hence the same is true for M_j and M is a partial tilting module.

Suppose now that all N_i are shifts of regular modules. Let $\dim N_i$ be imaginary roots for $1 \leq i \leq m$ and real roots for $m < i \leq n$; $M' = \bigoplus_{i=1}^m M_i$, $M'' = \bigoplus_{i=m+1}^n M_i$. Lemma 4.4 (3) implies that $\mathrm{Hom}_{\mathbf{A}}(M', M'') = \mathrm{Hom}_{\mathbf{A}}(M'', M') = 0$ and M'' is a partial tilting (in particular, rigid) modle. Moreover, Lemma 4.7 implies that $\mathrm{End}_{\mathbf{A}} M''$ is a Dynkinian algebra. Since N_i are bricks, $N_i \simeq \mathcal{F}(1, \lambda_i)[k]$ $(1 \leq i \leq m)$ for some $k \in \mathbb{Z}$, $\lambda_i \in \mathbb{P}^1$ and $\lambda_i \neq \lambda_j$ for $1 \leq i < j \leq m$ (it follows from [8, Proposition IV.7.1] that k is common for all i). Then Corollary 4.6 implies that there is a cofinite subset

$\mathbb{X} \subseteq \mathbb{P}^1$ and an open subset $U \subseteq \operatorname{rep}^{\mathrm{p}}(P_1, P_2)$ such that $\mathcal{T}M$ runs through $\mathcal{T}M'' \oplus \bigoplus_{i=1}^m \mathcal{F}(1, \lambda_i)$, where $(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{X}^{(m)}$. □

5. Representations of linear groups over algebras

Let now $\mathbf{A}$ be a finite dimensional algebra over the field $\mathbb{C}$ of complex numbers, $\mathsf{G} = \mathrm{GL}(P, \mathbf{A})$ be a linear group over $\mathbf{A}$, where $P \in \mathbf{A}$-proj, $\hat{\mathsf{G}}$ be the dual space of G, i.e. the space of its irreducible unitary representations and μ_{G} be the *Plancherel measure* on $\hat{\mathsf{G}}$ [12, 13]. We call a subset $\tilde{\mathsf{G}} \subseteq \hat{\mathsf{G}}$ *fat* if it is open, dense and support of the Plancherel measure, i.e. $\mu_{\mathsf{G}}(\hat{\mathsf{G}} \setminus \tilde{\mathsf{G}}) = 0$.

Suppose that there is an idempotent $e \in \mathbf{A}$ such that $(1-e)\mathbf{A}e = 0$ and set $\mathbf{A}_1 = e\mathbf{A}e$, $\mathbf{A}_2 = (1-e)\mathbf{A}(1-e)$, $\mathbf{W} = e\mathbf{A}(1-e)$, $P_1 = eP$, $P_2 = (1-e)P$. Then $\mathbf{A} \simeq \mathbf{A}_1[\mathbf{W}]\mathbf{A}_2$ and $\mathsf{G} \simeq \mathsf{H} \ltimes \mathsf{N}$, where $\mathsf{H} = \mathrm{GL}(P_1, \mathbf{A}_1) \times \mathrm{GL}(P_2, \mathbf{A}_2)$, $\mathsf{N} \simeq \mathbf{W}(P_1, P_2)$, hence,

$$\begin{aligned}\hat{\mathsf{N}} = \mathrm{Hom}_{\mathbb{C}}(\mathsf{N}, \mathbb{C}) &\simeq \mathrm{Hom}_{\mathbb{C}}(\hat{P}_1 \otimes_{\mathbf{A}_1} \mathbf{W} \otimes_{\mathbf{A}_2} P_2, \mathbb{C}) \simeq \\ &\simeq \mathrm{Hom}_{\mathbf{A}_1}(\hat{P}_1, \mathrm{Hom}_{\mathbb{C}}(\mathbf{W} \otimes_{\mathbf{A}_2} P_2, \mathbb{C})) \simeq \\ &\simeq \mathrm{Hom}_{\mathbf{A}_1}(\hat{P}_1, \mathrm{Hom}_{\mathbf{A}_2}(P_2, \mathbf{W}^*)) \simeq \mathbf{W}^*(\hat{P}_1, \hat{P}_2).\end{aligned}$$

The natural action of the group H on the space $\hat{\mathsf{N}}$ coincides with its action on $\mathbf{W}^*(\hat{P}_1, \hat{P}_2)$. Recall that, by the Mackey's theorem [12, 13], there is a surjection $\pi : \hat{\mathsf{G}} \to \hat{\mathsf{N}}/\mathsf{H}$ such that $\pi^{-1}(\mathsf{H}\chi) \simeq \hat{\mathsf{H}}_\chi$, where $\mathsf{H}_\chi = \{\, h \in \mathsf{H} \mid h\chi = \chi \,\}$. Moreover, π is continuous and compatible with the Plancherel measure; especially, if $\mu_{\mathsf{N}}(X) = 0$ for a subset $X \subseteq \hat{\mathsf{N}}$ and $\bar{X}$ is the image of X in $\hat{\mathsf{N}}/\mathsf{H}$, then $\mu_{\mathsf{G}}(\pi^{-1}\bar{X}) = 0$. Note that $\dim \mathsf{H}_w < \dim \mathsf{G}$ whenever $\mathsf{N} \neq 0$.

In particular, if $\mathbf{A}$ is derived equivalent to a hereditary algebra, it follows from [8, Lemma IV.1.10] that there is an idempotent $e \in \mathbf{A}$ such that $e\mathbf{A}e = \mathbb{C}$ and $(1-e)\mathbf{A}e = 0$. Then $\mathbf{A}_1 = \mathbb{C}$, so $\mathbf{W}$ is projective over $\mathbf{A}_1$ and all results from the preceding sections can be applied to this situation. It gives the following results. Moreover, if $\mathbf{A}$ is not semisimple, e can be so chosen that $\mathbf{W} \neq 0$; otherwise $\mathrm{GL}(P, \mathbf{A}) \simeq \prod_i \mathrm{GL}(d_i, \Bbbk)$ for every P.

Theorem 5.1 (cf. [3])**.** *Let $\mathbf{A}$ be a Dynkinian algebra, $\mathsf{G} = \mathrm{GL}(P, \mathbf{A})$ be a linear group over $\mathbf{A}$. There is a fat subset $\tilde{\mathsf{G}} \subseteq \hat{\mathsf{G}}$ such that $\tilde{\mathsf{G}} \simeq \prod_{i=1}^s \widehat{\mathrm{GL}}(d_i, \mathbb{C})$ for some values of $d_1, d_2, \ldots, d_s$.*

Proof. We use that above notations. Consider the open sheet $Z \subseteq {}^\wedge\mathbf{W}(P_1, P_2)$. Lemma 4.2 implies that it consists of a unique H-orbit and if $w \in Z$, then $\mathbf{A}' = \mathrm{End}_{{}^\wedge\mathbf{W}}\, w$ is a Dynkinian algebra. Therefore, H_w is again a linear group over a Dynkinian algebra, and $\dim \mathsf{H}_w < \dim \mathsf{G}$ if $\mathbf{A}$ is not semisimple. Let $\hat{\mathsf{G}}' = \pi^{-1}(Z)$; then $\hat{\mathsf{G}}' \simeq \hat{\mathsf{H}}_w$. An easy induction accomplishes the proof. □

Theorem 5.2. *Let* $\mathbf{A}$ *be a Euclidean algebra,* $\mathsf{G} = \mathrm{GL}(P, \mathbf{A})$ *be a linear group over* $\mathbf{A}$*. There is a fat subset* $\tilde{\mathsf{G}} \subseteq \hat{\mathsf{G}}$ *such that*

$$\tilde{\mathsf{G}} \simeq \prod_{i=1}^{s} \widehat{\mathrm{GL}}(d_i, \mathbb{C}) \times \mathbb{X}^{(m)}/\mathsf{S}_m \times (\mathbb{C}^\times)^m$$

for some values of $d_1, d_2, \dots, d_s$*,* m *and some cofinite subset* $\mathbb{X} \subseteq \mathbb{P}^1$*.*

Proof. Let $Z_0^{\mathcal{T}}$ be the $\mathcal{T}$-sheet of $^\wedge]\mathbf{W}(P_1, P_2)$. Theorem 4.8 implies that either $Z_0^{\mathcal{T}}$ consists of a unique orbit and $\mathrm{End}_{^\wedge\mathbf{W}} = \mathbf{A}'$ is also a Euclidean or Dynkinian algebra, or $Z_0^{\mathcal{T}}$ contains an open dense subset U such that, for every $w \in U$, $\mathrm{End}_{^\wedge\mathbf{W}}\, w \simeq \mathbf{A}' \times (\mathbb{C}^\times)^m$, where $\mathbf{A}'$ is a Dynkinian algebra and $U \simeq \mathbb{X}^{(m)}$ for some cofinite subswet $\mathbb{X} \subseteq \mathbb{P}^1$. Moreover, in the latter case two elements $\mathbf{x}, \mathbf{x}'$ from $\mathbb{X}^{(m)}$ belong to the same H-orbit if and only if they only differ by the ordering of its component, i.e. belong to the same orbit of S_m. Again an easy induction, together with Theorem 5.1 accomplishes the proof. □

References

[1] P. Donovan and M. R. Freislich. The representation theory of finite graphs and associated algebras. Carleton Math. Lecture Notes 5 (1973).

[2] Y. A. Drozd. Matrix problems and categories of matrices. Zapiski Nauchn. Semin. LOMI, 28 (1972), 144–153.

[3] Y. A. Drozd. Matrix problems, small reduction and representations of a class of mixed Lie groups. Representations of Algebras and Related Topics. London Math. Soc. Lecture Note Ser. 168 (1992), 225–249.

[4] Y. A. Drozd and V. V. Kirichenko. Finite Dimensional Algebras. Vyscha Shkola, Kiev, 1980. English edition: Springer–Verlag, 1994.

[5] P. Gabriel. Finite representation type is open. Springer Lect. Notes Math. 488 (1975), 132–155.

[6] P. Gabriel and A. V. Roiter. Algebra VIII: Representations of Finite-Dimensional Algebras. Encyclopaedia of Mathematical Sciences. 73. Springer–Verlag, 1992.

[7] F. Grunewald and J. O'Halloran. A characterization of orbit closure and applications. J. Algebra 116 (1988), 163–175.

[8] D. Happel. Triangulated Categories in the Representation Theory of Finite Dimensional Algebras. Cambridge University Press, 1988.

[9] B. T. Jensen, Xiuping Su and A. Zimmermann. Degenerations for derived categories. J. Pure Appl. Algebra 198 (2005), 281–295.

[10] V. Kac. Infinite root systems, representations of graphs and invariant theory. II. J. Algebra 78, (1982) 141–162.

[11] B. Keller. Chain complexes and stable categories. Manus. Math. 67 (1990), 379–417.

[12] A. A. Kirillov. Elements of the Theory of Representations. Nauka, Moscow, 1972. English translation: Springer–Verlag, 1976.

[13] A. Kleppner and R. L. Lipsman. The Plancherel formula for group extensions. Ann. Sci. École Norm. Sup. (4) 5 (1972), 459–516.

[14] Li Sun Gen. Representations of Lie groups of step matrices. Ukrain. Mat. Zh. 39, No. 6 (1987), 789–791.

[15] L. A. Nazarova. Representations of quivers of infinite type. Izv. Akad. Nauk SSSR, Ser. Mat. 37 (1973), 752–791.

[16] A. Neeman. The derived category of an exact category. J. Algebra 135 (1990), 388–394.

[17] D. Quillen. Higher algebraic K-theory I. Springer Lect. Notes Math. 341 (1973), 85–147.

[18] J. Rickard. Derived equivalences as derived functors, J. London Math. Soc. 43 (1991), 436–456.

[19] C. Riedtmann. Degenerations for representations of quivers with relations, Ann. Sci. École Norm. Sup. 19 (1986), 275–301.

[20] C. W. Ringel. Rational invariants of tame quivers. Invent. Math. 58 (1980), 217–239.

[21] C. W. Ringel. Tame Algebras and Integral Quadratic Forms. Springer Lecture Notes in Math. 1099 (1984).

[22] A. S. Timoshin. Representations of zigzag network subgroups. Ukrain. Mat. Zh. 41, No. 3 (1989), 398–400.

[23] A. S. Timoshin. Representations of linear groups over Kronecker algebra. Visnyk (Proc.) Kiev University, No. 3, (2002), 60—64.

[24] A. S. Timoshin. Representations of linear groups over $\tilde{A}_2$-algebras. Algebra Discrete Math. No. 3 (2004), 135–143.

[25] D. P. Zhelobenko and A. I. Shtern. Representations of Lie Groups. Nauka, Moscow, 1983.

[26] G. Zwara. Degenerations of finite-dimensional modules are given by extensions. Comp. Math. 121 (2000), 205–218.

Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal de Minas Gerais, Belo Horizonte, Brazil

E-mail address: bekkert@mat.ufmg.br

Institute of Mathematics, National Academy of Sciences of Ukraine, Kiev, Ukraine

E-mail address: drozd@imath.kiev.ua

Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo, Brazil

E-mail address: futorny@ime.usp.br