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Research Paper

Representations and cohomologies of the alternating group of degree 4 ☆

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We describe integral representations of the alternating group $\mathfrak{A}_4$, in particular, the Auslander-Reiten quiver of its 2-adic representations. Using these results we calculate Tate cohomologies of all $\mathfrak{A}_4$-lattices.

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Introduction

In the paper [7] Nazarova described 2-adic representations of the alternating group $\mathfrak{A}_4$ of order 4. Unfortunately, it is very difficult to use this description for other purposes, such as calculation of cohomologies. In this paper we propose another approach, analogous to the study of representations and cohomologies of the Klein 4-group in [5]. Namely, we use the technique of *Bäckström orders* from [8,9] and thus relate the description of 2-adic representations of $\mathfrak{A}_4$ with representations of a *valued graph* [1]. It allows to completely describe the Auslander-Reiten quiver of 2-adic representations. As the Auslander-Reiten transform in this case coincides with syzygy, it gives almost immediately the values of all Tate cohomologies of 2-adic $\mathfrak{A}_4$-lattices. Since 3-adic representations of $\mathfrak{A}_4$ are very simple, we also describe all integral representations and their cohomologies. Note that knowing cohomologies is important for applications, such as classification of crystallographic and Chernikov groups etc.

1. Representations. Local structure

Let $G = \mathfrak{A}_4$ be the alternating group of degree 4, N be its *Klein subgroup* $N \simeq \{1, a, b, c \mid a^2 = b^2 = 1,\, ab = ba = c\}$, $H = G/K \simeq \langle \sigma \mid \sigma^3 = 1 \rangle$. We denote by $A = \mathbb{Z}G$ the group ring of G and set $A_p = A \otimes \mathbb{Z}_p$, the p-adic completion of A, and $QA = A \otimes \mathbb{Q}$, the rational envelope of A. By A-lat (respectively A_p-lat) we denote the category of *A-lattices*, i.e. A-modules M such that, as a group, M is a free abelian group of finite rank (respectively, free $\mathbb{Z}_p$-module of finite rank). For an A-lattice M we also denote $QM = M \otimes \mathbb{Q}$ and $M_p = M \otimes \mathbb{Z}_p$. Note that $QM_p \simeq M \otimes \mathbb{Q}_p$.

The ring A can be considered as the *crossed product* $K * H$, where $K = \mathbb{Z}N$ and H naturally acts on K by conjugation. Here $QK \simeq \mathbb{Q}^4$ with the basis $\{e_1, e_a, e_b, e_c\}$, where

$$e_1 = \frac{1+a+b+c}{4},$$
$$e_a = \frac{1+a-b-c}{4},$$
$$e_b = \frac{1-a+b-c}{4},$$
$$e_c = \frac{1-a-b+c}{4}.$$

Under this identification, $K_p = \mathbb{Z}_p^4$ if $p \neq 2$ and K_2 embeds into $\mathbb{Z}_2^4$ so that a, b, c identify, respectively, with the elements $(1, 1, -1, -1), (1, -1, 1, -1)$ and $(1, -1, -1, 1)$. The action of H is trivial on the first component of QK and cyclically permutes the other three. Hence

$$QA \simeq QK * H \simeq \mathbb{Q}H \times \mathrm{Mat}(3, \mathbb{Q}) \simeq \mathbb{Q} \times \mathbb{Q}[\theta] \times \mathrm{Mat}(3, \mathbb{Q}),$$

where θ is a primitive cubic root of 1. If $p \notin \{2,3\}$, then

$$A_p \simeq \mathbb{Z}_p \times \mathbb{Z}_p[\theta] \times \mathrm{Mat}(3,\mathbb{Z}_p) \text{ (a maximal order in } QA_p),$$

and

$$A_3 \simeq \mathbb{Z}_3 H \times \mathrm{Mat}(3,\mathbb{Z}_3).$$

Therefore, all indecomposable A_p-lattices for $p \notin \{2,3\}$ are irreducible lattices $\mathbb{Z}_p$, $Z_p[\theta]$ and $I_p = \mathbb{Z}_p^3$, and for A_3 there is one more indecomposable lattice $\mathbb{Z}_3 H$.

The case $p = 2$ is quite different, since $\mathbb{Z}_2 K$ is no longer a maximal order. Recall that every group ring $R = \mathbb{Z}G$ is *Gorenstein*, i.e. $\mathrm{inj.dim}_R R = 1$. Therefore, all non-projective R_p-lattices are actually lattices over the overing $R^+ = \mathrm{End}_R(\mathrm{rad}\, R_p)$ [3]. As $\#(H)$ is invertible in $\mathbb{Z}_2$, for the crossed product $A_2 = K_2 {*} H$ we have that $\mathrm{rad}\, A_2 = (\mathrm{rad}\, \mathbb{Z}_2 K){*}H$ and $A^+ = K^+ * H$. Note that K^+ is a *Bäckström order* in the sense of [8]. It means that there is a hereditary order $\tilde{K}$ such that $\tilde{K} \supset K^+ \supset \mathrm{rad}\, \tilde{K} = \mathrm{rad}\, K^+$. In our case $\tilde{K} = \mathbb{Z}_2^4$ and $K^+ = \{(x_1, x_2, x_4, x_4) \mid x_1 \equiv x_2 \equiv x_3 \equiv x_4 \pmod 2\}$. As $\tilde{A} = \tilde{K} * H$ is also hereditary, A^+ is also a Bäckström order. Namely,

$$\tilde{A} \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2[\theta] \times \mathrm{Mat}(3,\mathbb{Z}_2).$$

One can easily see that A^+ embeds into $\tilde{A}$ as the subring of triples (x_1, x_2, x_3), where $x_1 \in \mathbb{Z}$, $x_2 \in \mathbb{Z}[\theta]$ and $x_3 = (\xi_{ij}) \in \mathrm{Mat}(3,\mathbb{Z}_2)$, such that $x_1 \equiv \xi_{11} \pmod 2$, $\xi_{12} \equiv \xi_{13} \equiv \xi_{21} \equiv \xi_{31} \equiv 0 \pmod 2$ and $\rho(x_2) \equiv \begin{pmatrix} \xi_{22} & \xi_{23} \\ \xi_{32} & \xi_{33} \end{pmatrix} \pmod 2$, where ρ denotes the regular representation of $\mathbb{Z}_2[\theta]$: $\rho(u + v\theta) = \begin{pmatrix} u & -v \\ v & u-v \end{pmatrix}$. We denote by L_1, L_2, L_3 the irreducible $\tilde{A}$-lattices belonging, respectively, to the components $\mathbb{Z}_2$, $\mathbb{Z}_2[\theta]$ and $\mathrm{Mat}(3,\mathbb{Z}_2)$, and by P_1 and P_2 the indecomposable projective A^+-lattices $P_i = A^+ e_i$, where $e_1 = \frac{1+\sigma+\sigma^2}{3}$ and $e_2 = 1 - e_1$. Note that the only indecomposable A_2-lattices that are not A^+-lattices are $B_i = A_2 e_i$. They are *bijective*, i.e. both projective and injective in the exact category A_2-lat.

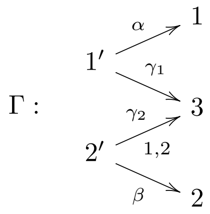
Recall [8], that representations of a Bäckström order are classified by representations of a *weighted graph* Γ in the sense of [1]. Namely, the vertices of Γ are in one-to-one correspondence with the simple components of semisimple algebras $\bar{A} = A^+/\,\mathrm{rad}\, A^+$ and $\bar{A}' = \tilde{A}/\,\mathrm{rad}\, \tilde{A}$. In our case

$$\bar{A} = (K^+/\,\mathrm{rad}\, K^+) * H \simeq \mathbb{F}_2 H \simeq \mathbb{F}_2 \times \mathbb{F}_4$$

and

$$\bar{A}' \simeq (\tilde{K}/\,\mathrm{rad}\, \tilde{K}) * H \simeq \mathbb{F}_2 * H \times \mathrm{Mat}(3,\mathbb{F}_2) \simeq \mathbb{F}_2 \times \mathbb{F}_4 \times \mathrm{Mat}(3,\mathbb{F}_2).$$

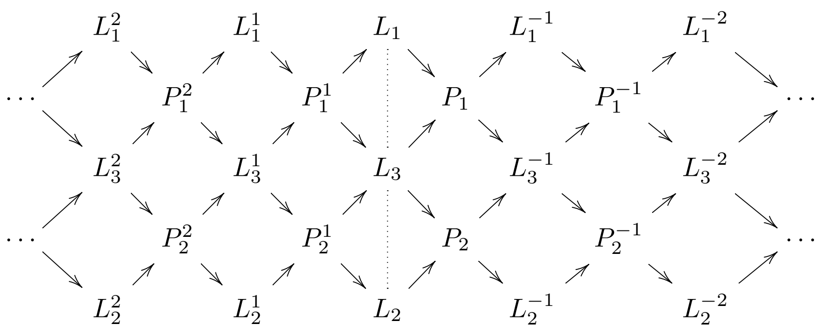
Hence, the corresponding graph Γ (with orientation) is of type $\tilde{F}_{41}$:



Here $1, 2, 3$ correspond, respectively, to the components $\mathbb{F}_2, \mathbb{F}_4$ and $\mathrm{Mat}(3, \mathbb{F}_2)$ of $\bar{A}'$, while $1'$ and $2'$ correspond, respectively, to the components $\mathbb{F}_2$ and $\mathbb{F}_4$ of $\bar{A}$. The weights of all arrows except γ_2 are $(1, 1)$, so we do note write them. In representations of Γ the arrows α, γ_1 correspond to matrices with entries from $\mathbb{F}_2$, the arrows γ_2, β correspond to matrices with entries from $\mathbb{F}_4$. The vector dimension of a representation M of Γ we denote by $\boxed{\begin{smallmatrix} & d_1 \\ d_{1'} & d_3 \\ d_{2'} & d_2 \end{smallmatrix}}$. Recall that the A^+-lattice M corresponding to a representation V of this graph is the preimage in $\tilde{M} = L_1^{d_1} \oplus L_3^{d_3} \oplus L_2^{d_2}$ of $\operatorname{Im} \varphi(V)$, where $\varphi(V) : V(1') \oplus V(2') \to V(1) \oplus V(3) \oplus V(2)$ is given by the matrix

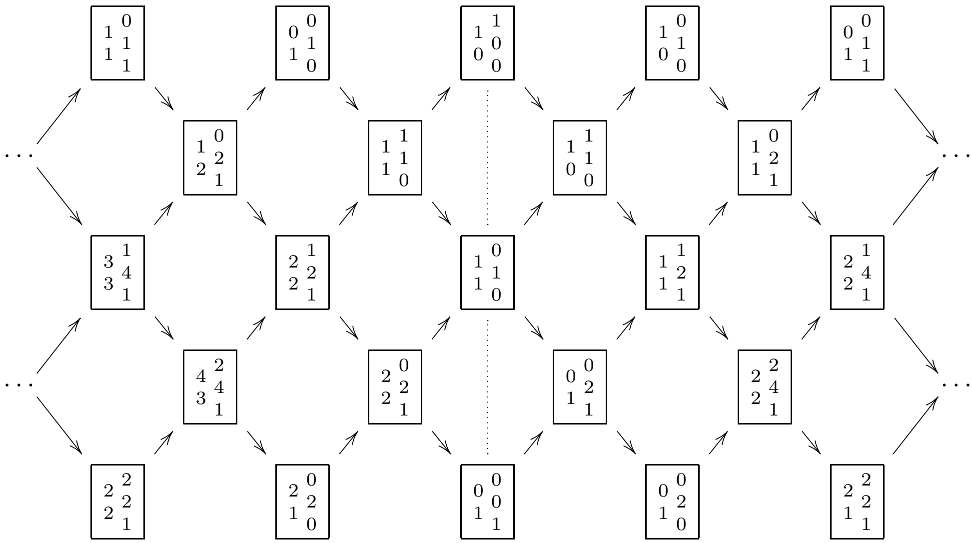
$$\begin{pmatrix} V(\alpha) & 0 \\ V(\gamma_1) & V(\gamma_2) \\ 0 & V(\beta) \end{pmatrix},$$

and we identify $V(1) \oplus V(2) \oplus V(3)$ with $\tilde{M}/2\tilde{M}$. The Auslander-Reiten quiver of the category of representations of graph Γ consists of preprojective, preinjective and regular components. When we pass to representations of the Bäckström order, we have to glue the preprojective and preinjective components into one component (we call it *principal*) [9]. Namely, we omit simple injective and simple projective modules and then add arrows from the remaining injective to the remaining projective modules. As a result, the principal component for the order A^+ becomes:



Here by M^k we denote the k-th Auslander-Reiten transform $\tau^k M$ of the lattice M over the ring A_2. As shown in [3], $M^k \simeq \Omega^k M$, the k-th syzyzy of M as of A_2-module, and P_i^1 are just the *injective A^+-lattices*, i.e. those dual to projective ones, or, the same, injective in the exact category A^+-lat. Actually, P_i is the unique minimal overmodule and P_i^1 is the unique maximal submodule of $B_i = A_2e_i$. Recall that B_i are the only indecomposable A_2-lattices which are not A^+-lattices. They are *bijective*, i.e. both projective and injective in A_2-lat. Note also that the other irreducible A^+-lattices are $L_1^{\pm1}$ (just as L_3 they belong to the component $\mathrm{Mat}(3,\mathbb{Z})$).

The dimensions of the corresponding representations of Γ are given in the next diagram.



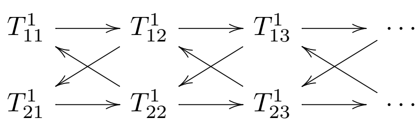
Note that the symmetry with respect to the central (dotted) column corresponds to the natural duality $M \mapsto M^* = \mathrm{Hom}_{\mathbb{Z}_2}(M, \mathbb{Z}_2)$ in the category of A_2-lattices arising from the anti-isomorphism $g \mapsto g^{-1}$ in the group ring.

The regular components for A^+-lattices are the same as those for the graph Γ. They consist of *homogeneous tubes* $\mathcal{T}^f$ corresponding to monic irreducible polynomials from $\mathbb{F}_2[t]$, except $t-1$ and t^2+t+1, and two *special tubes* $\mathcal{T}^1$ and $\mathcal{T}^\theta$. The homogeneous tubes are of the form

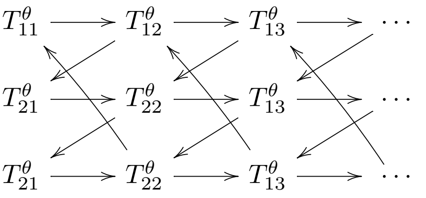
$$T_1^f \rightrightarrows T_2^f \rightrightarrows T_3^f \rightrightarrows \cdots.$$

In these components the Auslander-Reiten transform (or, the same, the syzygy) acts trivially. The dimension of the representation of Γ corresponding to T_k^f is $\boxed{\begin{smallmatrix} & kd \\ 2kd & \\ & 3kd \\ 2kd & \\ & kd \end{smallmatrix}}$, where $d = \deg f$.

The special tubes are of the forms:



and



The dimensions of the corresponding representations of Γ are:

$$\boxed{\begin{smallmatrix} & k \\ 2k & \\ & 3k \\ 2k & \\ & k \end{smallmatrix}} \qquad \text{for } T^1_{i,2k}\ \ (k>0)$$

$$\boxed{\begin{smallmatrix} & 1 \\ 1 & \\ & 1 \\ 1 & \\ & 1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & k \\ 2k & \\ & 3k \\ 2k & \\ & k \end{smallmatrix}} \qquad \text{for } T^1_{1,2k+1}$$

$$\boxed{\begin{smallmatrix} & 0 \\ 1 & \\ & 2 \\ 1 & \\ & 1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & k \\ 2k & \\ & 3k \\ 2k & \\ & k \end{smallmatrix}} \qquad \text{for } T^1_{2,2k+1}$$

$$\boxed{\begin{smallmatrix} & 2k \\ 4k & \\ & 6k \\ 4k & \\ & 2k \end{smallmatrix}} \qquad \text{for } T^\theta_{i,3k}\ \ (k>0)$$

$$\boxed{\begin{smallmatrix} & 0 \\ 0 & \\ & 2 \\ 2 & \\ & 1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & 2k \\ 4k & \\ & 6k \\ 4k & \\ & 2k \end{smallmatrix}} \qquad \text{for } T^\theta_{1,3k+1}$$

$$\boxed{\begin{smallmatrix} & 2 \\ 2 & \\ & 2 \\ 1 & \\ & 0 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & 2k \\ 4k & \\ & 6k \\ 4k & \\ & 2k \end{smallmatrix}} \qquad \text{for } T^\theta_{2,3k+1}$$

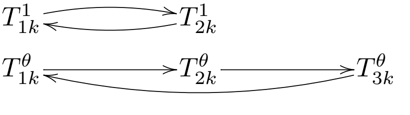
$$\boxed{\begin{smallmatrix} & 0 \\ 2 & \\ & 2 \\ 1 & \\ & 1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & 2k \\ 4k & \\ & 6k \\ 4k & \\ & 2k \end{smallmatrix}} \qquad \text{for } T^\theta_{3,3k+1}$$

$$\boxed{\begin{smallmatrix} & 2 \\ 2 & \\ & 4 \\ 3 & \\ & 1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & 2k \\ 4k & \\ & 6k \\ 4k & \\ & 2k \end{smallmatrix}} \qquad \text{for } T^\theta_{1,3k+2}$$

$$\boxed{\begin{smallmatrix} & 2 \\ 4 & \\ & 4 \\ 2 & \\ & 1 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & 2k \\ 4k & \\ & 6k \\ 4k & \\ & 2k \end{smallmatrix}} \qquad \text{for } T^\theta_{2,3k+2}$$

$$\boxed{\begin{smallmatrix} & 0 \\ 2 & \\ & 4 \\ 3 & \\ & 2 \end{smallmatrix}} + \boxed{\begin{smallmatrix} & 2k \\ 4k & \\ & 6k \\ 4k & \\ & 2k \end{smallmatrix}} \qquad \text{for } T^\theta_{3,3k+2}.$$

The Auslander-Reiten transform (or, the same, syzyzy) acts as follows:



2. Globalization

To describe indecomposable A-lattices, we use the following results of [6].

Proposition 2.1. *Let $M(p)$ be A_p-lattices given for all prime p.*

(1) *There is an A-lattice M such that $M_p \simeq M(p)$ for all p if and only if there is a QA-module V such that $QM(p) \simeq \mathbb{Q}_p \otimes_{\mathbb{Q}} V$ for all p. Then we say that all $M(p)$ are* of the same rational type.

(2) *Such lattice M is decomposable if and only if there are direct summands $N(p)$ of every $M(p)$ such that all $N(p)$ are of the same rational type. In particular, if M' is another lattice with the same localizations, M and M' decomposes simultaneously.*

They say that two A-lattices M and M' are *of the same genus* if $M_p \simeq M'_p$ for all p. As $A \subset 6\tilde{A}$, the following result follows from [2, Thm. 3.7].

Proposition 2.2. *If two A-lattices belong to the same genus, they are isomorphic.*

Note that if $p \notin \{2, 3\}$, for every QA-module V there is a unique A_p-lattice L such that $QL \simeq \mathbb{Q}_p \otimes_{\mathbb{Q}} V$. Therefore, an A-lattice is completely defined by its 2-adic and 3-adic localizations. If $p \in \{2, 3\}$, every QA_p-module is of the form $\mathbb{Q}_p \otimes_{\mathbb{Q}} V$, where V is a QA-module. V decomposes as $V \simeq \mathbb{Q}^{r_1} \oplus \mathbb{Q}[\theta]^{r_2} \oplus W^{r_3}$, where W is the unique simple $\mathrm{Mat}(3, \mathbb{Q})$-module. We write $\mathrm{rt}\, M_p = (r_1, r_2, r_3)$ and call $\mathrm{rt}\, M_p$ the *rational type* of M_p. Hence an A-lattice is defined by a pair M_2, M_3 of lattices over A_2 and A_3 which are of the same rational type.

Note that the unique indecomposable A_3-lattice which is not irreducible is the lattice $\Lambda = \mathbb{Z}_3 H$. The rational type of Λ is $(1, 1, 0)$. From now on, let M be an A-lattice of rational type (d_1, d_2, d_3) and $M_3 = \mathbb{Z}_3^{k_1} \oplus \mathbb{Z}_3[\theta]^{k_2} \oplus L_3^{k_3} \oplus \Lambda^k$, where L_3 is the irreducible $\mathrm{Mat}(3, \mathbb{Z}_3)$-lattice. Note that the dimension of the corresponding representation of the valued graph Γ is $\boxed{\begin{smallmatrix} & d_1 \\ d_{1'} & \\ & d_3 \\ d_{2'} & \\ & d_2 \end{smallmatrix}}$ for some $d_{1'}, d_{2'}$. Proposition 2.1 means that $k_1 + k = d_1$, $k_2 + k = d_2$ and $k_3 = d_3$. It implies a description of A-lattices M such that M_2 is indecomposable.

Theorem 2.3. *Let N be an indecomposable A_2-lattice,* $\mathrm{rt}\, N = (c_1, c_2, c_3)$ *and* $\tilde{c} = \min(c_1, c_2)$. *Denote by N^k $(0 \le k \le \tilde{c})$ the A-lattice such that $N_2^k \simeq N$ and $N_3^k \simeq \Lambda^k \oplus \mathbb{Z}_3^{c_1 - k} \oplus \mathbb{Z}_3[\theta]^{c_2 - k} \oplus L_3^{c_3}$. Every A-lattice M such that $M_2 \simeq N$ is isomorphic to one of N^k.*

Let now $M_2 \simeq \bigoplus_{i=1}^{s} N^i$, where $s > 1$, $\mathrm{rt}\, N^i = (c_{1i}, c_{2i}, c_{3i})$ and $\tilde{c}_i = \min(c_{1i}, c_{2i})$. The following result is obvious.

Proposition 2.4. *If $k \le \sum_{i=1}^{s} \tilde{c}_i$, then M decomposes as $\bigoplus_{i=1}^{s} M^i$, where $M_2^i \simeq N^i$ and $M_3^i \simeq \Lambda^k \oplus \mathbb{Z}_3^{c_{1i}-b_i} \oplus \mathbb{Z}_3[\theta]^{c_{2i}-b_i} \oplus L_3^{c_{3i}}$, where b_i are arbitrary integers such that $b_i \le \tilde{c}_i$ and $\sum_{i=1}^{s} b_i = k$.*

Thus from now on we suppose that $k > \sum_{i=1}^{s} \tilde{c}_i$.

Proposition 2.5. *If N is a direct summand of M_2 of rational type (c, c, c_3), then M has a direct summand M' such that $M'_2 \simeq N$.*

Therefore, if M is indecomposable, M_2 has no proper direct summands of rational type (c, c, c_3). In what follows we always suppose that this condition is satisfied.

Proposition 2.6. *If $c_{1i} < c_{2i}$ for all i or $c_{1i} > c_{2i}$ for all i, then M decomposes.*

Proof. Let $c_{1i} < c_{2i}$ for all i and c_{11} is the minimal among c_{1i}. Then M has a direct summand M' such that $M'_2 \simeq N^1$ and $M'_3 \simeq \Lambda^{c_{11}} \oplus \mathbb{Z}_3[\theta]^{c_{21}-c_{11}} \oplus L_3^{c_{13}}$. $\square$

Propositions 2.4-2.6 imply a description of indecomposable A-lattices M such that M_2 has two indecomposable components.

Proposition 2.7. *If M is indecomposable and $s = 2$, then, up to permutation of N^1 and N^2, $c_{11} < c_{21}$ and $c_{12} > c_{22}$. There are $c^+ - \tilde{c}$ such lattices, where $\tilde{c} = \tilde{c}_1 + \tilde{c}_2$ and $c^+ = \min(c_{11} + c_{12}, c_{21} + c_{22})$ corresponding to decompositions $M_3 \simeq \Lambda^k \oplus \mathbb{Z}_3^{c_{11}+c_{12}-k} \oplus \mathbb{Z}_3[\theta]^{c_{21}+c_{22}-k} \oplus L_3^{c_{31}+c_{32}}$, where $\tilde{c} < k \le c^+$.*

The description of 2-adic lattices shows that $|c_{1i} - c_{2i}| \le 2$. If all $|c_{1i} - c_{2i}| = 1$, Proposition 2.5 implies that M contains a direct summand M' such that $M'^2 \simeq N^i \oplus N^j$ for some $i \ne j$. The same holds if $c_{1i} - c_{2i} = c_{2j} - c_{1j} = 2$. Hence we can suppose now that there is one i such that $c_{1i} - c_{2i} = 2$ and $c_{2j} - c_{1j} = 1$ for all $j \ne i$ (or vice versa). Then Proposition 2.5 implies that if M is indecomposable, there are at most two such indices j. One immediately sees that the unique possibility with two such indices is when $M_3 \simeq \Lambda^k \oplus L_3^{c_3}$, where $k = \tilde{c}_1 + \tilde{c}_2 + 2$.

Thus, we have described all indecomposable A-lattices M with decomposable M_2.

Theorem 2.8. *Denote by*

- $\mathfrak{N}^1$ *be the set of indecomposable A_2-lattices such that $c_1 - c_2 = 1$,*
- $\mathfrak{N}^2$ *be the set of indecomposable A_2-lattices such that $c_1 - c_2 = 2$,*
- $\mathfrak{N}_1$ *be the set of indecomposable A_2-lattices such that $c_2 - c_1 = 1$,*
- $\mathfrak{N}_2$ *be the set of indecomposable A_2-lattices such that $c_2 - c_1 = 2$.*

Then the only possibilities for indecomposable A-lattices M such that M_2 is decomposable are the following:

(1) *$M_2 \simeq N^1 \oplus N^2$, where $N^1 \in \mathfrak{N}^1 \cup \mathfrak{N}^2$, $N^2 \in \mathfrak{N}^1 \cup \mathfrak{N}_2$, $M_3 \simeq \Lambda^{\tilde{c}_1+\tilde{c}_2+1} \oplus L_3^{c_3}$. We denote such M by $N^1 \bowtie N^2$.*
(2) *$M_2 \simeq N^1 \oplus N^2$, where $N^1 \in \mathfrak{N}^2$, $N@2 \in \mathfrak{N}_2$, $M_3 \simeq \Lambda^{\tilde{c}_1+\tilde{c}_2+2} \oplus L_3^{c_3}$. We denote such M by $N^1 \bowtie^2 N^2$.*
(3) *$M_2 \simeq N^1 \oplus N^2 \oplus N^3$, where $N^1 \in \mathfrak{N}^2$, $N^2, N^3 \in \mathfrak{N}_1$ or $N^1 \in \mathfrak{N}_2$, $N^2, N^3 \in \mathfrak{N}^1$, $M_3 \simeq \Lambda^{\tilde{c}_1+\tilde{c}_2+\tilde{c}_3+2} \oplus L_3^{c_3}$. We denote such M by $N^1 \bowtie (N^2 \oplus N^3)$.*

All lattices described in (1-3) are indecomposable and pairwise nonisomorphic.

Theorems 2.3 and 2.8 give a complete description of indecomposable A-lattices.

Note that the decomposition of an A-lattice M into a direct sum of indecomposables is far from being unique.

Example 2.9.

(1) If $N, N' \in \mathfrak{N}^1 \cup \mathfrak{N}^2$ and $L, L' \in \mathfrak{N}_1 \cup \mathfrak{N}_2$, then $N \bowtie L \oplus N' \bowtie L' \simeq N \bowtie L' \oplus N' \bowtie L$.
(2) Let $M \in \mathfrak{N}^2$, $N, L \in \mathfrak{N}_1$, $M' \in \mathfrak{N}_2$ and $N', L' \in \mathfrak{N}^1$. Then $M \bowtie (N \oplus L) \oplus M' \bowtie (N' \oplus L') \simeq M \bowtie^2 M' \oplus N \bowtie N' \oplus L \bowtie L'$. Hence even the number of indecomposable summands can differ in different decompositions.

3. Cohomology

We are going to calculate Tate cohomologies of G-lattices. As $\#(G) = 12$, for every G-module M the groups $\hat{H}^n(G, M)$ split into their 2-components $\hat{H}^n(G, M)_2$ and 3-components $\hat{H}^n(G, M)_3$. Moreover, if M is a lattice, $\hat{H}^n(G, M)_p \simeq \hat{H}^n(G, M_p)$. So we can consider 2-adic and 3-adic cases separately.

For the group $G = \mathfrak{A}_4$ the spectral sequence $E_2^{pq} = H^p(H, H^q(N, M)) \Rightarrow H^n(G, M)$ degenerates both in 2-adic and in 3-adic case. Namely, for 2-adic lattices $E_2^{pq} = 0$ if $p \neq 0$. So we obtain isomorphisms

$$H^n(G, M) \simeq H^n(N, M)^H.$$

For 3-adic lattices $E_2^{pq} = 0$ if $q \neq 0$, hence

$$H^n(G, M) \simeq H^n(C, M^N).$$

In the 3-adic case we have indecomposable lattices $\mathbb{Z}, \mathbb{Z}[\varepsilon], \mathbb{Z}H$ and I_3. Note that K acts trivially on $\mathbb{Z}_3, \mathbb{Z}_3[\varepsilon]$ and $\mathbb{Z}_3 H$ and has no fixed points on I_3. The quotient H is cyclic, so its cohomologies are periodic with period 2. Easy calculations give:

$$\hat{H}^n(G, \mathbb{Z}_3) = \begin{cases} \mathbb{F}_3 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd;} \end{cases}$$

$$\hat{H}^n(G, \mathbb{Z}_3[\varepsilon]) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \mathbb{F}_3 & \text{if } n \text{ is odd.} \end{cases}$$

The other indecomposable lattices are projective, hence have trivial Tate cohomologies.

For 2-adic lattices we use the following result analogous to [5, Lem. 2.2] and with analogous proof.

Lemma 3.1. *Let M be an indecomposable A^+-lattice corresponding to the representation V of the quiver Γ of dimension* $\boxed{\begin{smallmatrix} & d_1 \\ d_{1'} & \\ & d_3 \\ d_{2'} & \\ & d_2 \end{smallmatrix}}$. *If $M \not\simeq L_1$, then $\hat{H}^0(G, M) \simeq \mathbb{F}_2^{d_1}$.*

Proof. Recall that $\hat{H}^0(G, M) = M^G/\operatorname{tr} M$, where M^G is the set of invariants: $M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}$, and $\operatorname{tr} = \sum_{g\in G} g$. If we consider M as a sublattice between $\tilde{M} = \tilde{A}M = L_1^{d_1} \oplus L_2^{d_2} \oplus L_3^{d_3}$ and $\operatorname{rad} \tilde{M} = 2\tilde{M} = 2L_1^{d_1} \oplus 2L_2^{d_2} \oplus 2L_3^{d_3}$. Then $\tilde{M}^G = L_1^{d_1}$ and $M^G = \tilde{M}^G \cap M \supseteq 2L_1^{d_1}$. Let $\pi : \tilde{M} \to \tilde{M}^G$ be the projection. If $u \in M^G$ and $\pi(u) \notin 2\tilde{M}^G$, then $\tilde{M}^G = U \oplus N$, where $U = \mathbb{Z}_2 u$. Combining the restriction of π onto M with the projection $\tilde{M}^G \to U$, we obtain a homomorphism $\eta : M \to U$ such that $\eta\varepsilon = 1_U$, where $\varepsilon : U \to M$ is the embedding. Therefore, U is a direct summand of M and $M \simeq U \simeq L_1$, which is impossible. Hence $M^G = 2\tilde{M}_G$. On the other hand, $\pi(M) = \pi(\tilde{M})$, since the projection of A^+ onto the first component of QA_2 is maximal. Therefore $\operatorname{tr} M = \operatorname{tr} \tilde{M} = \operatorname{tr} \tilde{M}^G = 12\tilde{M}^G = 2M^G$, since $\operatorname{tr} \tilde{M} \subseteq \tilde{M}^G$. Thus $\hat{H}^0(G, M) = M^G/2M^G \simeq \mathbb{F}_2^{d_1}$. □

Note that the rational types of the lattices M and M^* are equal, hence $\hat{H}^0(M) \simeq \hat{H}^0(M^*)$. As also $H^n(G, M^*) \simeq H^{-n}(G, M)$ ([4, Prop. 3.2]), one immediately obtains by an obvious induction the following corollary.

Corollary 3.2. *The groups $\hat{H}^n(M)$ do not change when one replaces n by $-n$ or M by M^*.*

Having the Auslander-Reiten quiver, we only need to know $\hat{H}^0(G, M)$ for all indecomposable M, since $\hat{H}^n(G, M) \simeq \hat{H}^0(G, \tau^n M)$, as $\tau M = \Omega M$. Actually, for every representation V from the preinjective component of the Auslander-Reiten quiver there is a number $m|6$ such that $\dim \tau^m M = \dim M + q\omega$, where $\omega = \boxed{\begin{smallmatrix} & 1 \\ 2 & \\ & 3 \\ 2 & \\ & 1 \end{smallmatrix}}$ and $q \in \{1, 2\}$. Therefore, the value of d_1 just changes by q. It gives a simple procedure for calculation of cohomologies of lattices from the principal component. Here is the result of these calculations.

Theorem 3.3. *Let M be an idecomposable A^+-lattice from the principal component, namely, $M = M_0^r$, where $M_0 \in \{L_1, L_2, L_3, P_1, P_2\}$. Set $k = [|n+r|/m]$, $i = |n+r| - km$. Then $\hat{H}^n(G, M) \simeq \mathbb{F}_2^{qk+r_i}$, except the case when $M_0 = L_1$ and $i = 0$. The values m, q and r_i depend on M_0. Namely:*

If $M_0 = L_1$*, then* $m = 6$*,* $q = 1$ *and*

i	1	2	3	4	5
r_i	0	0	1	1	0

If $M_0 = L_2$*, then* $m = 6$*,* $q = 2$ *and*

i	0	1	2	3	4	5
r_i	0	0	2	0	2	2

If $M_0 = L_3$*, then* $m = 2$*,* $q = 1$ *and* $r = i$*.*
If $M_0 = P_1$*, then* $m = 3$*,* $q = 1$ *and*

i	1	2	3
r	1	0	1

If $M_0 = P_2$*, then* $m = 3$*,* $q = 2$ *and*

i	1	2	3
r	0	2	2

If $M \simeq L_1^r$ *and* $6|(n+r)$*, then* $\hat{H}^n(M) \simeq \mathbb{Z}/4\mathbb{Z}$*.*

For representations from tubes the situation is even easier, since the values of d_1, hence of $\hat{H}^0$, are given on page 148 after the description of tubes, and we know the action of τ. So we obtain the following result.

Theorem 3.4.

$$\hat{H}^n(G, T_k^f) \simeq \hat{H}^0(G, T_k^f) \simeq \mathbb{F}_2^{kd}, \text{ where } d = \deg f.$$

$$\hat{H}^{2n+r}(G, T^1_{i,2k+j}) \simeq \hat{H}^0(G, T^1_{i',j}) \simeq \mathbb{F}^{k+c}, \text{ where}$$

$$c = \begin{cases} 1 & \text{if } j = i' = 1, \\ 0 & \text{if } j = 0 \text{ or } i' = 0 \end{cases}$$

$$(i' \equiv i + r \pmod 2), \text{ and } i' \in \{0,1\}).$$

$$\hat{H}^{3n+r}(G, T^\theta_{i,3k+j}) \simeq \hat{H}^0(G, T^\theta_{i',j}) \simeq \mathbb{F}_2^{2k+c}, \text{ where}$$

$$c = \begin{cases} 2 & \text{if } j = 1,\ i' = 2 \text{ or } j = 2,\ i' \neq 0, \\ 0 & \text{otherwise} \end{cases}$$

$$(i' \equiv i + r \pmod 3), \text{ and } j' \in \{0,1,2\}).$$

Declaration of competing interest

Authors confirm that there are no financial or non-financial interests that are directly or indirectly related to the work submitted for publication.

Data availability

No data was used for the research described in the article.

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