H.A. Snitko (Pidstryhach Institute for Applied Problems of Mechanics and Mathematics of NAS of Ukraine, Lviv, Ukraine)

Determination of the minor coefficient in a parabolic equation in a free boundary domain

In the domain $\Omega_T = \{(x,t) : h_1(t) < x < h_2(t), 0 < t < T\}$, where $h_1 = h_1(t)$, $h_2 = h_2(t)$ are unknown functions, we consider the inverse problem for the parabolic equation

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + c(t)u + f(x,t), \quad (x,t) \in \Omega_T, \tag{1}$$

with unknown coefficient $c = c(t)$, initial condition

$$u(x,0) = \varphi(x), \quad x \in [h_1(0), h_2(0)], \tag{2}$$

boundary conditions

$$u(h_1(t),t) = \mu_1(t), \quad u(h_2(t),t) = \mu_2(t), \quad t \in [0,T], \tag{3}$$

and overdetermination conditions

$$\int\limits_{h_1(t)}^{h_2(t)} u(x,t)dx = \mu_3(t), \int\limits_{h_1(t)}^{h_2(t)} xu(x,t)dx = \mu_4(t), \int\limits_{h_1(t)}^{h_2(t)} x^2u(x,t)dx = \mu_5(t), \quad t \in [0,T], \tag{4}$$

where $h_1(0) = h_{01}$ is given.

By change of the variables $y = \dfrac{x - h_1(t)}{h_2(t) - h_1(t)}, t = t$ we reduce the problem (1)-(4) to the problem with unknown functions $(c(t), h_1(t), h_3(t), v(y,t))$, $h_3(t) = h_2(t) - h_1(t)$, $v(y,t) = u(yh_3(t) + h_1(t), t)$ in the domain $Q_T = \{(y,t) : 0 < y < 1, 0 < t < T\}$.

We establish conditions of local existence and uniqueness of solution to the problem (1)-(4).

Theorem. *Suppose that the following conditions hold:*

1) $a \in C^{2,0}(R \times [0,T]),\ b, f \in C^{1,0}(R \times [0,T]),\ \varphi \in C^1[h_{01}, \infty),\ \mu_i \in C^1[0,T],\ i = \overline{1,5};$

2) $0 < a_0 \le a(x,t) \le a_1,\ |a_x(x,t)| \le a_2,\ |b(x,t)| \le b_0,\ 0 < f(x,t) \le f_0,\ |f_x(x,t)| \le f_1,$ $(x,t) \in R \times [0,T],\ 0 < \varphi_0 \le \varphi(x) \le \varphi_1,\ x \in [h_{01}, \infty),\ \mu_i(t) > 0,\ i = \overline{1,3},\ t \in [0,T];$

3) $\varphi(h_{01}) = \mu_1(0),\ \varphi(h_2(0)) = \mu_2(0).$

Then we can indicate a number T_0, $0 < T_0 \le T$, *depending on the data that there exists a unique solution* $(c, h_1, h_3, v) \in C[0,T_0] \times (C^1[0,T_0])^2 \times C^{2,1}(Q_{T_0}) \cap C^{1,0}(\overline{Q}_{T_0})$, $h_3(t) > 0, t \in [0,T_0]$ *to the problem (1)-(4).*

The proof of the theorem is based on Shauder fixed-point theorem and the properties of Volterra integral equations of the second kind.