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Morera-type Theorems in the Unit Disk

Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the hyperbolic plane; group $G = \left\{ \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} : \ |a|^2 - |b|^2 = 1 \right\}$ acts on D by mappings $g\circ z = \frac{az+b}{\overline{b}z+\overline{a}}$. Iwasawa decomposition is of the form $G = KAN$, where $K = \mathbf{SO}(2)$, $A = \left\{ a_t = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} : t \in \mathbb{R} \right\}$, $N = \left\{ n_s = \begin{pmatrix} 1+is & -is \\ is & 1-is \end{pmatrix} : s \in \mathbb{R} \right\}$.

For a fixed compact $A \subset D$ consider

$$\int_{\partial(gA)} f(z)\, dz = 0, \quad \forall g \in G \tag{1}$$

The problem posed is to decide, whether any $f \in C(D)$ that satisfies (1), is necessarily a holomorphic function. It is not true in general case. From the results of M.L. Agranovsky [1] it follows, in particular, that for $f \in L^2(D)$ equation (1) implies holomorphicity of f. Precise condition for the case of cyrcle was found by V.V. Volchkov [2].

Let us define $K_\alpha = \{z = a_t n_s \circ 0 : 0 \le s \le 1,\ 0 \le t \le \alpha\}$, and $Q_\alpha = \{z = n_s a_t \circ 0 : 0 \le s \le \alpha,\ 0 \le t \le 1\}$ $(\alpha > 0)$. In the present work the author found the precise conditions for nonholomorphic function with vanishing integrals along $\partial(gK_\alpha)$ (or $\partial(gQ_\alpha)$) to exist (for some $\alpha > 0$, and all $g \in NA$).

For $l \in \mathbb{N}$, and $z \in D$ define $\eta_l(z) = \frac{l(1-|z|^2)+iz(1-\overline{z})}{l(1-|z|^2)+i(1-\overline{z})}$, $\lambda_l(z) = \frac{e^{2l}(1-|z|^2)-(1+|z|^2-2z)}{e^{2l}(1-|z|^2)+(1+|z|^2-2\overline{z})}$, $m(z) = 1 - |z|^2$, $M_f(z) = \int_{-1}^{1}\int_{-1}^{1} |(f\cdot m)(n_u a_v \circ z)|\, du dv$.

Theorem 1. Let $f \in C(D)$, $\forall z \in D$ $M_f(\eta_l(z)) = o(1/l)$ as $l \to +\infty$, $l \in \mathbb{N}$;

$$\int_{\partial(gK_{\alpha_1})} f(z)\, dz = \int_{\partial(gK_{\alpha_2})} f(z)\, dz = 0, \quad \forall g \in NA.$$

If $\alpha_1/\alpha_2 \notin \mathbb{Q}$, or $M_f(\eta_1(z)),\ M_f(z) = o(1)$ as $z \to 1$ along the geodesic lines, then $f(z)$ is holomorphic in D.

Theorem 2. Let $f \in C(D)$, $\forall z \in D$ $M_f(\lambda_l(z)) = o(1)$ as $l \to +\infty$, $l \in \mathbb{N}$;

$$\int_{\partial(gQ_{\alpha_1})} f(z)\, dz = \int_{\partial(gQ_{\alpha_2})} f(z)\, dz = 0, \quad \forall g \in NA.$$

If $\alpha_1/\alpha_2 \notin \mathbb{Q}$, or $M_f(z) = o(1)$ as $z \to 1$ along the horycicles, then $f(z)$ is holomorphic in D.

[1] Agranovsky M.L. // Soviet Math. Dokl. — 1978. — **19**.

[2] Volchkov V.V Integral Geometry and Convolution Equations. — Dordrecht. Boston. London: Kluwer Academic Publishers, 2003.